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INTRODUCTIO

IN ANALYSIN

INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, *& Academiæ Imperialis Scientiarum* PETROPOLITANÆ
Socio.

TOMUS PRIMUS.

LATIUS SUB ÆGIDELUCEBIT

LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.
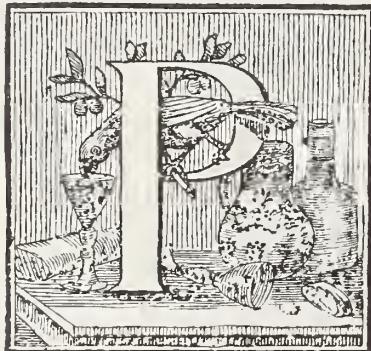
JEAN JACQUES DORTOUS
DE MAIRAN.

ILLUSTRISSIMO VIRO

JOHANNI JACOBO

DORTOUS DE MAIRAN,

UNI EX XLVIRIS

ACADEMIÆ GALLICÆ,

REGIÆ ETIAM SCIENTIARUM

PARISIENSIS,

IN QUA SECRETARII PERPETUI MUNUS NUPER

ABDICAVIT,

*   NEC

ALIARUM BENE MULTARUM,

LONDINENSIS, PETROPOLITANÆ,

&c.

SOCIETATUM, ACADEMIARUMVE

SOCIO DIGNISSIMO.

MARCUS-MICHAEL BOUSQUET.

VIR ILLUSTRISSIME,

Atronos Euleriano scripto quærere necesse neutiquam esse, Mathematicarum Disciplinarum cultoribus satis constat. Sciunt utique illi, varias earum partes novis eum luminibus sic illustrasse, ut indè meritò clarissimi

simi rerum in his abstrusissimarum inter-
pretis locum sit consequutus. Quem quin
egregiè tueatur, immò tollat se altiùs quo-
que opere isthoc, nemo dubitabit, certior
hisce factus, indulsisse Te mihi, ut illu-
strissimo nomini Tuo dicatum publicè
prodiret. Pertinere autem hoc in me col-
latum beneficium ad Auctoris decus pro-
be intelligens, Ipse, ut eo uterer, lubens
concessit ; & cum in rem meam faciat
omnimodò, quî neglexissem?

Ab his equidem, quibus Libros inscri-
bunt, sibi nescio quid ideò deberi, pleri-
que tacitè constituunt; acceptaque bene-
ficia quodammodo remunerari, ut sese
ferè nexu liberent omni. Ego verò secus
sentio. Mihi certè merum est beneficium
patroni, quòd scriptoris aut excusoris

* 2   opera

opera id genus honore condecorari patiantur. Hac mente utique Tibi, VIR ILLUSTRISSIME, animi gratissimi summæque observantiæ professionem hisce publicam excipias, rogo.

Paratum promtumque semper juvandis litterarum studiis qui Te novit, & notus vel hoc nomine es cuicumque in Republica doctorum Europæ totius non hospiti, plurimis officiis meæ etiam conditionis homines à Te affectos fuisse statuat necesse est. Nempe, tanquam Tibi uni esset injunctum curare, ut floreant humanum ingenium illustrantes scientiæ omnes, hominumque in usus adinventæ artes, ad singulis inservientium artifices etiam Te demittere dignaris, vel ab illa sublimium rerum perscrutatione, Cælive ipsius

fius Tibi *tam nota regione, ut quæ huc-usque mentes hominum metu complebant Phænomena minùs intellecta, per* Te *jam grato tantùm admirationis sensu contemplentur, earumque causas habeant perspectas.*

Hinc ille veluti ex condicto Academiarum Orbis eruditi concursus, ut adlectum Te cætui suo consequerentur, ornamento aliàs carituro insigni, quo cæteras nollent præ se frui. Hinc inprimis Illustrissimæ Parisiensis de Te judicium, cum ageretur de successore sufficiendo in locum emeriti Fontenellii, Viri, cujus ex ore calamoque fluere Scientiarum Artiumque omnium exquisitiores divitiæ, elegantiæque universæ perpetuo visæ sunt, & videbuntur dum sani sensus quicquam hu-

mano

mano ingenio erit. Tibi, *scilicet*, Commentariorum *Academiæ* conscribendorum provincia, cui præfectus ille erat, demandabatur continuò; quam, ut ornare diutiùs voluisses, docti omnes optabant: hoc uno minùs dolentes Te aliter censuisse, quòd aliis Tibi *magis placituris*, profuturisque nihilominus litteris in universum eruditionis ingeniive thesauros impenderes. *Quod ut ad ultimas usque metas hominum vitæ positas incolumis, florens, atque beatus præstes, omni votorum contentione precor. Vale!*

Dabam *Lausannæ* die 1. Aprilis
Anni Æræ Dionyf. 1748.

PRÆFATIO.

Æpenumero animadverti, maximam difficultatum partem, quas Matheseos cultores in addiscenda Analysi infinitorum offendere solent, inde oriri, quod, Algebra communi vix apprehensa, animum ad illam sublimiorem artem appellant; quo fit, ut non solum quasi in limine subsistant, sed etiam perversas ideas illius infiniti, cujus notio in subsidium vocatur, sibi forment. Quanquam autem Analysis infinitorum non perfectam Algebræ communis, omniumque artificiorum adhuc inventorum cognitionem requirit; tamen plurimæ extant quæstiones, quarum evolutio discentium animos ad sublimiorem scientiam præparare valet, quæ tamen in communibus Algebræ elementis, vel omittuntur, vel non satis accurate pertractantur. Hanc ob rem non dubito, quin ea, quæ in his libris congessi, hunc defectum abunde supplere queant. Non solum enim operam dedi, ut eas res, quas Analysis infini-

infinitorum abfolute requirit, uberius atque diftinctius exponerem, quam vulgo fieri folet; fed etiam fatis multas quæftiones enodavi, quibus Lectores fenfim & quafi præter expectationem ideam infiniti fibi familiarem reddent. Plures quoque quæftiones per præcepta communis Algebræ hic refolvi, quæ vulgo in Analyfi infinitorum tractantur : quo facilius deinceps utriufque Methodi fummus confenfus eluceat.

Divifi hoc Opus in duos Libros, in quorum priori, quæ ad meram Analyfin pertinent, fum complexus : in pofteriori vero, quæ ex Geometria funt fcitu neceffaria, explicavi, quoniam Analyfis infinitorum ita quoque tradi folet, ut fimul ejus applicatio ad Geometriam oftendatur. In utroque autem prima Elementa prætermifi, eaque tantum exponenda duxi, quæ alibi, vel omnino non, vel minus commode tractata, vel ex diverfis principiis petita reperiuntur.

In primo igitur Libro, cum univerfa Analyfis infinitorum circa quantitates variabiles earumque Functiones verfetur, hoc argumentum de Functionibus inprimis fufius expofui; atque Functionum tam transformationem, quam refolutionem & evolutionem per feries infinitas demonftravi. Complures enumeravi Functionum fpecies, quarum in Analyfi fublimiori præcipue ratio eft habenda. Primum eas diftinxi in algebraicas & tranfcendentes; quarum illæ per operationes in Algebra communi ufitatas ex quantitatibus variabilibus formantur, hæ vero vel per alias rationes componuntur, vel ex iifdem operationibus infinities repetitis efficiuntur. Algebraicarum functionum primaria fubdivifio fit in rationales & irrationales, priores docui cum in partes fimpliciores, tum in factores refolvere ; quæ operatio in Calculo integrali maximum adjumentum affert ; pofteriores vero, quemadmodum idoneis fubftitutionibus ad formam rationalem perduci queant oftendi. Evolutio autem per feries infinitas ad utrumque genus æque pertinet, atque etiam ad Functiones

ctiones tranfcendentes fumma cum utilitate applicari folet;
at quantopere doctrina de feriebus infinitis Analyfin fubli-
miorem amplificaverit, nemo eft qui ignoret. Nonnulla
igitur adjunxi Capita, quibus plurium ferierum infinita-
rum proprietates, atque fummas fum fcrutatus; quarum
quædam ita funt comparatæ, ut fine fubfidio Analyfis
infinitorum vix inveftigari poffe videantur. Hujufmodi
feries funt, quarum fummæ exprimuntur, vel per Loga-
rithmos vel Arcus circulares: quæ quantitates cum fint
tranfcendentes, dum per quadraturam Hyperbolæ & Cir-
culi exhibentur, maximam partem demum in Analyfi in-
finitorum tractari funt folitæ. Poftquam autem a potef-
tatibus ad quantitates exponentiales effem progreffus, quæ
nil aliud funt nifi poteftates, quarum exponentes funt va-
riabiles; ex earum converfione maxime naturalem ac fœ-
cundam Logarithmorum ideam fum adeptus: unde non
folum ampliffimus eorum ufus fponte eft confecutus, fed
etiam ex ea cunctas feries infinitas, quibus vulgo iftæ
quantitates repræfentari folent, elicere licuit: hincque adeo
facillimus fe prodidit modus Tabulas Logarithmorum con-
ftruendi. Simili modo in contemplatione Arcuum circu-
larium fum verfatus; quod quantitatum genus, etfi a Lo-
garithmis maxime eft diverfum, tamen tam arcto vincu-
lo eft connexum, ut dum alterum imaginarium fieri vi-
detur, in alterum tranfeat. Repetitis autem ex Geome-
tria quæ de inventione Sinuum & Cofinuum Arcuum
multiplorum ac fubmultiplorum traduntur, ex Sinu vel
Cofinu cujufque Arcus expreffi Sinum Cofinumque Arcus
minimi & quafi evanefcentis, quo ipfo ad feries infinitas
fum deductus: unde, cum Arcus evanefcens Sinui fuo fit
æqualis, Cofinus vero radio, quemvis Arcum cum fuo
Sinu & Cofinu ope ferierum infinitarum comparavi. Tum
vero tam varias expreffiones cum finitas tum infinitas pro
hujus generis quantitatibus obtinui, ut ad earum naturam
perfpiciendam Calculo infinitefimali prorfus non amplius
* *
effet

effet opus. Atque quemadmodum Logarithmi peculiarem Algorithmum requirunt, cujus in univerfa Analyfi fummus extat ufus, ita quantitates circulares ad certam quoque Algorithmi normam perduxi; ut in calculo æque commode ac Logarithmi & ipfæ quantitates algebraicæ tractari poffent. Quantum autem hinc utilitatis ad refolutionem difficillimarum quæftionum redundet, cum nonnulla Capita hujus Libri luculenter declarant, tum ex Analyfi infinitorum plurima fpecimina proferri poffent, nifi jam fatis effent cognita, & indies magis multiplicarentur. Maximum autem hæc inveftigatio attulit adjumentum ad Functiones fractas in factores reales refolvendas; quod argumentum, cum in Calculo integrali fit prorfus neceffarium, diligentius enucleavi. Series poftmodum infinitas, quæ ex hujufmodi Functionum evolutione nafcuntur, & quæ recurrentium nomine innotuerunt, examini fubjeci; ubi earum tam fummas quam terminos generales, aliasque infignes proprietates exhibui: & quoniam ad hæc refolutio in factores manuduxit, ita viciffim, quemadmodum producta ex pluribus, imo etiam infinitis, factoribus conflata per multiplicationem in feries explicentur, perpendi. Quod negotium non folum ad cognitionem innumerabilium ferierum viam aperuit, fed quia hoc modo feries in producta ex infinitis factoribus conftantia refolvere licebat, fatis commodas inveni expreffiones numericas, quarum ope Logarithmi Sinuum, Cofinuum, & Tangentium facillime fupputari poffunt. Præterea quoque ex eodem fonte folutiones plurium quæftionum, quæ circa partitionem numerorum proponi poffunt, derivavi; cujufmodi quæftiones fine hoc fubfidio vires Analyfeos fuperare videantur. Hæc tanta materiarum diverfitas in plura volumina facile excrefcere potuiffet; fed omnia, quantum fieri potuit, tam fuccincte propofui, ut ubique fundamentum clariffime quidem explicaretur, uberior vero amplificatio induftriæ Lectorum relinqueretur; quo habeant,

quibus

quibus vires fuas exerceant, finefque Analyfeos ulterius promoveant. Neque enim vereor profiteri, in hoc Libro non folum multa plane nova contineri; fed etiam fontes effe detectos, unde plurima infignia inventa adhuc hauriri queant.

Eodem inftituto fum ufus in altero Libro, ubi, quæ vulgo ad Geometriam fublimiorem referri folent, pertractavi. Antequam autem de Sectionibus Conicis, quæ alias fere folæ hunc locum occupant, agerem; Theoriam Linearum Curvarum in genere ita propofui, ut ad fcrutationem naturæ quarumvis Linearum Curvarum cum utilitate adhiberi poffet. Ad hoc nullum aliud fubfidium affero, præter æquationem, qua cujufque Lineæ Curvæ natura exprimitur; ex eaque cum figuram, tum primarias proprietates deducere doceo: id quod potiffimum in Sectionibus Conicis præftitiffe mihi fum vifus; quæ antehac vel fecundum folam Geometriam vel per Analyfin quidem, fed nimis imperfecte ac minus naturaliter, tractari funt folitæ. Ex æquatione fcilicet generali pro Lineis fecundi ordinis primum earum proprietates generales explicavi, tum eas in genera feu fpecies fubdivifi; refpiciendo utrum habeant ramos in infinitum excurrentes, an vero tota Curva finito fpatio includatur. Priori autem cafu infuper difpiciendum erat, quot fint rami in infinitum excurrentes, & cujus naturæ fint finguli; an habeant Lineas rectas afymtotas, an minus. Sicque obtinui tres confuetas Sectionum Conicarum fpecies; quarum prima eft Ellipfis, tota in fpatio finito contenta; fecunda autem Hyperbola, quæ quatuor habet ramos infinitos ad duas rectas afymptotas convergentes; tertia vero fpecies prodiit Parabola duos habens ramos infinitos afymtotis deftitutos. Simili porro ratione Lineas tertii ordinis fum perfecutus, quas, poft expofitas earum proprietates generales, divifi in fedecim genera; ad eaque omnes feptuaginta duas fpecies N e w t o n i revocavi. Ipfam vero methodum ita clare defcripfi, ut pro

* * 2   quovis

quovis Linearum ordine sequente divisio in genera facilli-
me institui queat; cujus negotii periculum quoque feci in
Lineis quarti ordinis. His deinde, quæ ad ordines Linea-
rum pertinent, expeditis, reversus sum ad generales omnium
Linearum affectiones eruendas. Explicavi itaque metho-
dum definiendi tangentes curvarum, earum normales,
atque etiam ipsam curvaturam, quæ per radium osculi
æstimari solet : quæ etsi nunc quidem plerumque Calculo
differentiali absolvuntur, tamen idem per solam commu-
nem Algebram hic præstiti, ut deinceps transitus ab Ana-
lysi finitorum ad Analysin infinitorum eo facilior reddatur.
Perpendi etiam curvarum puncta flexus contrarii, cus-
pides, puncta duplicia, ac multiplicia ; modumque ex-
posui hæc omnia ex æquationibus sine ulla difficultate de-
finiendi. Interim tamen non nego, has quæstiones multo
facilius Calculi differentialis ope enodari posse. Attigi
quoque controversiam de cuspide secundi ordinis, ubi am-
bo arcus in cuspidem coeuntes curvaturam in eandem par-
tem vertunt ; eamque ita composuisse mihi videor, ut
nullum dubium amplius superesse possit. Denique adjun-
xi aliquot Capita, in quibus Lineas Curvas, quæ datis
proprietatibus gaudeant, invenire docui ; pluraque tandem
Problemata circa singulares Circuli sectiones soluta dedi.
Quæ cum sint ea ex Geometria, quæ ad Analysin infini-
torum addiscendam maximum adminiculum afferre viden-
tur, Appendicis loco ex Stereometria Theoriam solidorum
eorumque superficierum per Calculum proposui, & quem-
admodum cujusque superficiei natura per æquationem in-
ter tres variabiles exponi queat, ostendi. Hinc, superfi-
ciebus instar linearum in ordines digestis, secundum di-
mensionum quas variabiles in æquatione constituunt nu-
merum, in primo ordine solam superficiem planam con-
tineri ostendi. Superficies vero secundi ordinis, ratione
habita partium in infinitum expansarum, in sex genera
divisi ; similique modo pro ceteris ordinibus divisio institui
poterit.

poterit. Contemplatus fum quoque interfectiones duarum fuperficierum; quæ cum plerumque fint curvæ non
in eodem plano fitæ, quemadmodum æquationibus comprehendi queant, monftravi. Tandem etiam pofitionem
planorum tangentium, atque rectarum, quæ ad fuperficies fint normales, determinavi.

De cetero, cum non paucæ res hic occurrant ab aliis
jam tractatæ, veniam rogare me oportet, quod non ubique honorificam mentionem eorum, qui ante me in eodem
genere elaborarunt, fecerim. Cum enim mihi propofitum
effet omnia quam breviffime pertractare, Hiftoria cujufque Problematis magnitudinem operis non mediocriter
auxiffet. Interim tamen pleræque quæftiones, quæ alibi
quoque folutæ reperiuntur, hic folutiones ex aliis principiis fint nactæ; ita ut non exiguam partem mihi vindicare
poffem. Spero autem cum ifta, tum ea potiffimum, quæ
prorfus nova hic proferuntur, plerifque, qui hoc ftudio
delectantur, non ingrata effe futura.

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INTRODUCTIO

IN

ANALYSIN INFINITORUM.

LIBER PRIMUS,

Continens

Explicationem de Functionibus quantitatum va-
riabilium ; earum resolutione in Factores, at-
que evolutione per Series infinitas : una cum
doctrina de Logarithmis, Arcubus circularibus,
eorumque Sinubus & Tangentibus; pluribus-
que aliis rebus, quibus Analysis infinitorum non
mediocriter adjuvatur.

LIBER PRIMUS.

CAPUT PRIMUM.

DE FUNCTIONIBUS IN GENERE.

1.

Uantitas conſtans eſt quantitas de-
terminata, perpetuo eumdem valo-
rem ſervans.

Ejuſmodi quantitates ſunt nu-
meri cujuſvis generis, quippe qui
eumdem, quem ſemel obtinue-
runt, valorem conſtanter conſer-
vant: atque ſi hujuſmodi quanti-
tates conſtantes per characteres
indicare convenit, adhibentur lit-
terae Alphabethi initiales *a*, *b*, *c*, &c. In Analyſi quidem com-
muni, ubi tantum quantitates determinatae conſiderantur, hæ
litteræ Alphabethi priores quantitates cognitas denotare ſolent,
poſteriores vero quantitates incognitas; at in Analyſi ſublimiori
hoc diſcrimen non tantopere ſpectatur, cum hic ad illud quan-
titatum diſcrimen præcipue reſpiciatur, quo aliæ conſtantes,
aliæ vero variabiles ſtatuuntur.

A 2　　　　　　　2. *Quan-*

2. *Quantitas variabilis est quantitas indeterminata seu universalis, quæ omnes omnino valores determinatos in se complectitur.*

Cum ergo omnes valores determinati numeris exprimi queant, quantitas variabilis omnes numeros cujusvis generis involvit. Quemadmodum scilicet ex ideis individuorum formantur ideæ specierum & generum ; ita quantitas variabilis est genus, sub quo omnes quantitates determinatæ continentur. Hujusmodi autem quantitates variabiles per litteras Alphabethi postremas z, y, x, &c. repræsentari solent.

3. *Quantitas variabilis determinatur, dum ei valor quicunque determinatus tribuitur.*

Quantitas ergo variabilis innumerabilibus modis determinari potest, cum omnes omnino numeros ejus loco substituere liceat. Neque significatus quantitatis variabilis exhauritur, nisi omnes valores determinati ejus loco fuerint substituti. Quantitas ergo variabilis in se complectitur omnes prorsus numeros, tam affirmativos quam negativos, tam integros quam fractos, tam rationales quam irrationales & transcendentes. Quinetiam cyphra & numeri imaginarii a significatu quantitatis variabilis non excluduntur.

4. *Functio quantitatis variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, & numeris seu quantitatibus constantibus.*

Omnis ergo expressio analytica, in qua præter quantitatem variabilem z omnes quantitates illam expressionem componentes sunt constantes, erit Functio ipsius z: Sic $a + 3z$; $az - 4zz$; $az + b \sqrt{(aa - zz)}$; c^z; &c. sunt Functiones ipsius z.

5. *Functio ergo quantitatis variabilis ipsa erit quantitas variabilis.*

Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc Functio innumerabiles valores determinatos induet ; neque ullus valor determinatus excipietur, quem Functio induere nequeat, cum quantitas variabilis quoque valores imaginarios involvat. Sic etsi hæc Functio $\sqrt{(9 - zz)}$, numeris realibus loco z substituendis, nunquam valorem ternario majorem recipere potest ; tamen ipsi z valores imaginarios tribuendo

tribuendo ut $5\sqrt{} - 1$, nullus aſſignari poterit valor determinatus quin ex formula $\sqrt{(9 - zz)}$ elici queat. Occurrunt autem non-nunquam Functiones tantum apparentes, quæ, utcunque quantitas variabilis varietur, tamen uſque eumdem valorem retinent, ut z°; 1^z; $\frac{aa - az}{a - z}$, quæ, etſi ſpeciem Functionis mentiuntur, tamen revera ſunt quantitates conſtantes.

6. Præcipuum Functionum diſcrimen in modo compoſitionis, quo ex quantitate variabili & quantitatibus conſtantibus formantur, poſitum eſt.

Pendet ergo ab Operationibus quibus quantitates inter ſe componi & permiſceri poſſunt: quæ Operationes ſunt Additio & Subtractio; Multiplicatio & Diviſio: Evectio ad Poteſtates & Radicum Extractio; quo etiam Reſolutio Æquationum eſt referenda. Præter has Operationes, quæ algebraïcæ vocari ſolent, dantur complures aliæ tranſcendentes, ut Exponentiales, Logarithmicæ, atque innumerabiles aliæ, quas Calculus integralis ſuppeditat.

Interim ſpecies quædam Functionum notari poſſunt; ut multipla $2z$; $3z$; $\frac{3}{5}z$; az; &c. & Poteſtates ipſius z, ut z^2; z^3; $z^{\frac{1}{2}}$; z^{-1}; &c. quæ, uti ex unica operatione ſunt deſumtæ, ita expreſſiones quæ ex operationibus quibuſcunque naſcuntur, Functionum nomine inſigniuntur.

7. Functiones dividuntur in Algebraïcas & Tranſcendentes; illæ ſunt, quæ componuntur per operationes algebraïcas ſolas, hæ vero in quibus operationes tranſcendentes inſunt.

Sunt ergo multiplæ ac Poteſtates ipſius z Functiones algebraïcæ; atque omnes omnino expreſſiones, quæ per operationes algebraïcas ante memoratas formantur, cujuſmodi eſt

$$\frac{a + bz^n - c\sqrt{(2z - zz)}}{aaz - 3bz^3}.$$ Quin-etiam Functiones algebraïcæ ſæpenumero nequidem explicite exhiberi poſſunt, cujuſmodi Functio ipſius z eſt Z, ſi definiatur per hujuſmodi æquationem;

$$Z^5 = azz\,Z^3 - bz^4\,Z^2 + cz^3\,Z - 1.$$ Quanquam enim hæc

A 3 æquatio

LIB. I. æquatio resolvi nequit; tamen constat Z æquari expressioni cuipiam ex variabili z & constantibus compositæ; ac propterea fore Z Functionem quamdam ipsius z. Cæterum de Functionibus transcendentibus notandum est, eas demum fore transcendentes, si operatio transcendens non solum ingrediatur, sed etiam quantitatem variabilem afficiat. Si enim operationes transcendentes tantum ad quantitates constantes pertineant, Functio nihilominus algebraïca est censenda: uti si c denotet circumferentiam Circuli, cujus radius sit $= 1$, erit utique c quantit.: transcendens, verumtamen hæ expressiones $c + z$; $c z^2$; $4 z^c$ &c. erunt Functiones algebraïcæ ipsius z. Parvi quidem est momenti dubium quod a quibusdam movetur, utrum ejusmodi expressiones z^c Functionibus algebraïcis annumerari jure possint, necne; quinetiam Potestates ipsius z, quarum exponentes sint numeri irrationales, uti $z^{\sqrt{2}}$ nonnulli maluerunt Functiones interscendentes quam algebraïcas appellare.

8. *Functiones algebraïcæ subdividuntur in Rationales & Irrationales: illæ sunt, si quantitas variabilis in nulla irrationalitate involvitur; hæ vero, in quibus signa radicalia quantitatem variabilem afficiunt.*

In Functionibus ergo rationalibus aliæ operationes præter Additionem, Subtractionem, Multiplicationem, Divisionem, & Evectionem ad Potestates, quarum exponentes sint numeri integri, non insunt: erunt adeo $a + z$; $a - z$; az; $\dfrac{aa + zz}{a + z}$; $az^3 - bz^5$; &c. Functiones rationales ipsius z. At hujusmodi expressiones $\sqrt{z}$; $a + \sqrt{(aa - zz)}$; $\sqrt[3]{(a - 2z + zz)}$: $\dfrac{aa - z\sqrt{(aa + zz)}}{a + z}$ erunt Functiones irrationales ipsius z.

Hæ commode distinguntur in Explicitas & Implicitas.

Explicitæ sunt, quæ per signa radicalia sunt evolutæ, cujusmodi exempla modo sunt data. Implicitæ vero Functiones irrationales sunt quæ ex resolutione æquationum ortum habent. Sic Z erit Functio irrationalis implicita ipsius z, si per hujusmodi

æquationem $z^7 = az Z^2 - bz^5$ definiatur; quoniam va-
lorem explicitum pro Z, admiſſis etiam ſignis radicalibus, ex-
hibere non licet; propterea quod Algebra communis nondum
ad hunc perfectionis gradum eſt evecta.

9. *Functiones rationales denuo ſubdividuntur in Integras & Frac-*
tas.

In illis neque z uſquam habet exponentes negativos, neque
expreſſiones continent fractiones, in quarum denominatores
quantitas variabilis z ingrediatur: unde intelligitur Functiones
fractas eſſe, in quibus denominatores z continentes, vel expo-
nentes negativi ipſius z occurrant. Functionum integrarum
hæc ergo erit Formula generalis: $a + bz + cz^2 + dz^3 + ez^4$
$+ fz^5 +$ &c. nulla enim Functio ipſius z integra excogitari
poteſt, quæ non in hac expreſſione contineatur. Functiones
autem fractæ omnes, quia plures fractiones in unam cogi poſ-
ſunt, continebuntur in hac Formula:

$$\frac{a + bz + cz^2 + dz^3 + ez^4 + fz^5 + \text{\&c.}}{\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \zeta z^5 + \text{\&c.}}$$

ubi notandum eſt quantitates conſtantes a, b, c, d, &c.
α, β, γ, δ, &c. ſive ſint affirmativæ, ſive negativæ, ſive in-
tegræ ſive fractæ, ſive rationales ſive irrationales, ſive etiam
tranſcendentes, naturam Functionum non mutare.

10. *Deinde potiſſimum tenenda eſt Functionum diviſio in Unifor-*
mes ac Multiformes.

Functio autem uniformis eſt, quæ ſi quantitati variabili z
valor determinatus quicunque tribuatur, ipſa quoque unicum
valorem determinatum obtineat. Functio autem Multiformis
eſt, quæ, pro unoquoque valore determinato in locum variabi-
lis z ſubſtituto, plures valores determinatos exhibet. Sunt
igitur omnes Functiones rationales, ſive integræ ſive fractæ,
Functiones uniformes; quoniam ejuſmodi expreſſiones, quicun-
que valor quantitati variabili tribuatur, non niſi unicum valorem
præbent. Functiones autem irrationales omnes ſunt multiformes;
propterea quod ſigna radicalia ſunt ambigua, & geminum valorem
involvunt. Dantur autem quoque inter Functiones tranſcenden-

tes, & uniformes, & multiformes : quin-etiam habentur Func-
tiones infinitiformes; cujufmodi eft Arcus Circuli Sinui z ref-
pondens; dantur enim Arcus circulares innumerabiles qui om-
nes eumdem habeant Sinum.   Denotent autem hæ litteræ P,
Q, R, S, T &c.   fingulæ Functiones uniformes ipfius z.

11. *Functio biformis ipfius* z *eft ejufmodi Functio, quæ pro quo-*
vis ipfius z *valore determinato, geminum valorem præbeat.*

Hujufmodi Functiones radices quadratæ exhibent, ut $\sqrt{(2z}$
$+zz)$: quicunque enim valor pro z ftatuatur expreffio
$\sqrt{(2z+zz)}$ duplicem habet fignificatum, vel affirmativum vel
negativum.   Generatim vero Z erit Functio biformis ipfius z,
fi determinetur per æquationem quadraticam $Z^2 - PZ +$
$Q = 0$: fi quidem P & Q fuerint Functiones uniformes ipfius
z. Erit namque $Z = \frac{1}{2}P \pm \sqrt{(\frac{1}{4}P^2 - Q)}$; ex quo
patet cuique valori determinato ipfius z duplicem valorem de-
terminatum ipfius Z refpondere. Hic autem notandum eft,
vel utrumque valorem Functionis Z effe realem, vel utrum-
que imaginarium.   Tum vero erit femper, uti conftat ex na-
tura æquationum, binorum valorum ipfius Z fumma $= P$,
ac productum $= Q$.

12. *Functio triformis ipfius* z *eft, quæ pro quovis ipfius* z *valo-*
re, tres valores determinatos exhibet.

Hujufmodi Functiones ex refolutione æquationum cubicarum
originem trahunt. Si enim fuerint P, Q, & R Functiones
uniformes, fitque $Z^3 - PZ^2 + QZ - R = 0$, erit Z
Functio triformis ipfius z; quia pro quolibet valore determina-
to ipfius z triplicem valorem obtinet. Tres ifti ipfius Z va-
lores unicuique valori ipfius z refpondentes, vel erunt omnes
reales, vel unicus erit realis, dum bini reliqui funt imaginarii.
Cæterum conftat horum trium valorum fummam perpetuo effe
$= P$; fummam factorum ex binis effe $= Q$, & productum
ex omnibus tribus effe $= R$.

13. *Functio quadriformis ipfius* z *eft, quæ pro quovis ipfius* z
valore quatuor valores determinatos exhibet.

Hujufmodi Functiones ex refolutione æquationum biquadra-
ticarum

ticarum nafcuntur. Quod fi enim P, Q, R, & S denotent
Functiones uniformes ipfius z, fueritque $Z^4 — P Z^3 + Q Z^2$
$— R Z + S = 0$, erit Z Functio quadriformis ipfius z; eo
quod cuique valori ipfius z quadruplex valor ipfius Z refpon-
det. Quatuor horum valorum ergo, vel omnes erunt reales,
vel duo reales duoque imaginarii, vel omnes quatuor erunt
imaginarii. Ceterum perpetuo fumma horum quatuor valorum
ipfius Z eft $= P$, fumma factorum ex binis $= Q$, fumma
factorum ex ternis $= R$, ac productum omnium $= S$. Si-
mili autem modo comparata eft ratio Functionum quinquefor-
mium & fequentium.

14. *Erit ergo Z Functio multiformis ipfius z, quæ, pro quovis*
valore ipfius z, tot exhibet valores quot numerus n continet unitates;
fi Z definiatur per hanc æquationem $Z^n — P Z^{n-1} +$
$Q Z^{n-2} — R Z^{n-3} + S Z^{n-4} — \&c. = 0.$
Ubi quidem notandum eft n effe oportere numerum inte-
grum; atque perpetuo, ut dijudicari poffit quam multiformis fit
Functio Z ipfius z, æquatio, per quam Z definitur, reduci
debet ad rationalitatem; quo facto exponens maximæ potefta-
tis ipfius Z indicabit quæfitum valorum numerum cuique ipfius
z valori refpondentium. Deinde quoque tenendum eft litte-
ras P, Q, R, S, &c. denotare debere Functiones uniformes
ipfius z: fi enim aliqua earum jam effet Functio multiformis,
tum Functio Z multo plures præbitura effet valores unicuique
valori ipfius z refpondentes, quam quidem numerus dimenfio-
num ipfius Z indicaret. Semper autem, fi qui valores ipfius
fuerint imaginarii, eorum numerus erit par; unde intelligitur,
fi fuerit n numerus impar, perpetuo unum ad minimum valorem
ipfius Z fore realem: contra autem fieri poffe, fi numerus n
fuerit par, ut nullus prorfus valor ipfius Z fit realis.

15. *Si Z ejufmodi fuerit Functio multiformis ipfius z ut perpetuo*
nonnifi unicum valorem exhibeat realem; tum Z Functionem uni-
formem ipfius z mentietur, ac pleramque loco Functionis uniformis
ufurpari poterit.

Euleri *Introduct. in Anal. infin. parv.* B Ejuf-

Ejufmodi Functiones erunt $\sqrt[3]{}\,P$, $\sqrt[5]{}\,P$, $\sqrt[7]{}\,P$, &c. quippe quæ
perpetuo nonnifi unicum valorem realem præbent, reliquis om-
nibus exiftentibus imaginariis, dummodo P fuerit Functio uni-
formis ipfius z. Hanc ob rem hujufmodi expreffio $P^{\frac{m}{n}}$, quo-
ties n fuerit numerus impar, Functionibus uniformibus annume-
rari poterit; five m fuerit numerus par five impar.   Quod fi
autem n fuerit numerus par, tum $P^{\frac{m}{n}}$ vel nullum habebit
valorem realem, vel duos; ex quo ejufmodi expreffiones
$P^{\frac{m}{n}}$, exiftente n numero pari, eodem jure Functionibus bifor-
mibus accenferi poterunt: fiquidem fractio $\frac{m}{n}$ ad minores ter-
minos non fuerit reducibilis.

16. *Si fuerit* y *Functio quæcunque ipfius* z; *tum viciffim* z *erit
Functio ipfius* y.

Cum enim y fit Functio ipfius z, five uniformis five multi-
formis; dabitur æquatio, qua y per z & conftantes quantitates
definitur.   Ex eadem vero æquatione viciffim z per y & con-
ftantes definiri poterit; unde quoniam y eft quantitas variabilis,
z æquabitur expreffioni ex y & conftantibus compofitæ, erit-
que adeo Functio ipfius y.   Hinc quoque patebit quam mul-
tiformis Functio futura fit z ipfius y: fierique poteft ut,
etiamfi y fuerit Functio uniformis ipfius z, tamen z futura fit
Functio multiformis ipfius y.   Sic fi y ex hac æquatione per z
definiatur; $y^3 = a\,y\,z - b\,z\,z$; erit utique y Functio trifor-
mis ipfius z, contra vero z Functio tantum biformis ipfius y.

17. *Si fuerint* y *&* x *Functiones ipfius* z, *erit quoque* y *Fun-
ctio ipfius* x, *& viciffim* x *Functio ipfius* y.

Cum enim fit y Functio ipfius z, erit quoque z Functio ipfius
y: fimilique modo erit etiam z Functio ipfius x.   Hanc ob rem
Functio ipfius y æqualis erit Functioni ipfius x; ex qua æquatio-
ne & y per x & viceverfa x per y definiri poterit: quocirca
manifeftum eft effe y Functionem ipfius x, atque x Functionem

ipfius

ipfius *y*. Sæpiffime quidem has Functiones explicite exhibere
non licet ob defectum Algebræ; interim tamen nihilo minus,
quafi omnes æquationes refolvi poffent, hæc Functionum reci-
procatio perfpicitur. Ceterum per methodum in Algebra tra-
ditam, ex datis binis æquationibus, quarum altera continet *y*
& *z*, altera vero *x* & *z*, per eliminationem quantitatis *z* for-
mabitur una æquatio relationem inter *x* & *y* exprimens.

18 *Species denique quædam Functionum peculiares funt notanda;*
fic Functio par ipfius z *eft, quæ eundem dat valorem, five pro* z
ponatur valor determinatus + k *five* — k.

Hujufmodi ergo Functio par ipfius *z* erit *z z*; five enim po-
natur $z = + k$, five $z = - k$, eundem valorem præbebit
expreffio *z z*, nempe $zz = + kk$. Simili modo Functiones
pares ipfius *z* erunt hæ ipfius *z* poteftates z^4, z^6, z^8, & ge-
neratim omnis poteftas z^m, fi fuerit *m* numerus par, five af-
firmativus five negativus. Quin etiam cum $z^{\frac{m}{n}}$ mentiatur
Functionem ipfius *z* uniformem, fi *n* fit numerus impar, per-
fpicuum eft $z^{\frac{m}{n}}$ fore Functionem parem ipfius *z*, fi *m* fuerit
numerus par, *n* vero numerus impar. Hanc ob rem, expref-
fiones ex hujufmodi poteftatibus utcunque compofitæ præbebunt
Functiones pares ipfius *z*; fic Z erit Functio par ipfius *z*, fi
fuerit $Z = a + b z^2 + c z^4 + d z^6 + \&c.$ item fi fuerit Z
$= \dfrac{a + b z^2 + c z^4 + d z^6 + \&c.}{a + \beta z^2 + \gamma z^4 + \delta z^6 + \&c.}$; Similique modo exponen-
tes fractos ipfius *z* introducendo, erit Z Functio par ipfius *z* fi
fuerit $Z = a + b z^{\frac{2}{3}} + c z^{\frac{2}{5}} + d z^{\frac{4}{7}}$ &c. vel $Z = a +$
$b z^{-\frac{2}{3}} + c z^{-\frac{4}{3}} + d z^{-\frac{2}{5}} + \&c.$ vel $Z =$
$\dfrac{a + b z^{\frac{2}{7}} + c z^{-\frac{4}{5}} + d z^{\frac{8}{3}}}{a + \beta z^{\frac{2}{3}} + \gamma z^{-\frac{2}{5}} + \delta z^{\frac{4}{7}}}$. Cujufmodi expreffiones, cum om-
nes fint Functiones uniformes ipfius *z*, appellari poterunt Fun-
ctiones pares uniformes ipfius *z*.

19. *Functio multiformis par ipsius z. est, quæ etiam si pro quo-vis valore ipsius z plures exhibeat valores determinatos, tamen eos-dem valores præbet, sive ponatur z $=+$ k, sive z $=-$ k.*

Sit Z ejusmodi Functio multiformis par ipsius z; quoniam natura Functionis multiformis exprimitur per æquationem inter Z & z, in qua Z tot habeat dimensiones, quot varios valores complectatur; manifestum est Z fore Functionem multiformem parem, si in æquatione naturam ipsius Z exprimente quantitas variabilis z ubique pares habeat dimensiones. Sic, si fuerit $Z^2 = a z Z^4 + b z^2$, erit Z Functio biformis par ipsius z; sin autem sit $Z^3 - a z^2 Z^2 + b z^4 Z - c z^3 = 0$, erit Z Func-tio triformis par ipsius z; atque generatim, si P, Q, R, S &c. denotent Functiones uniformes pares ipsius z, erit Z Functio biformis par ipsius z si sit $Z^2 - P Z + Q = 0$. At z erit Functio triformis par ipsius z si sit $Z^3 - P Z^2 + Q Z - R = 0$, & ita porro.

20 *Functio ergo, sive uniformis sive multiformis, par ipsius z erit ejusmodi expressio ex quantitate variabili z & constantibus con-flata in qua ubique numerus dimensionum ipsius z sit par.*

Hujusmodi ergo Functiones, præter uniformes quarum exem-pla ante sunt allata, erunt hæ expressiones $a + \sqrt{(bb - zz)}$; $a z z + \sqrt[3]{(a^6 z^4 - b z^2)}$ item $a z^{\frac{2}{3}} + \sqrt[3]{(z^2 + \sqrt{(a^4 - z^4)})}$ &c.

Unde patet Functiones pares ita definiri posse, ut dicantur esse Functiones ipsius z z.

Si enim ponatur $y = z z$, fueritque Z Functio quæcunque ipsius y; restituto ubique $z z$ loco y, erit Z ejusmodi Functio ipsius z, in qua z ubique parem habeat dimensionum nume-rum. Excipiendi tamen sunt ii casus, quibus in expres-sione ipsius Z occurrunt $\sqrt{y}$: ac hujusmodi aliæ formæ, quæ, facto $y = z z$ signa radicalia amittunt. Quamvis enim sit $y + \sqrt{a y}$ Functio ipsius y, tamen posito $y = z z$, eadem expres-sio non erit Functio par ipsius z; cum fiat $y + \sqrt{a y} = z z + z \sqrt{a}$. Exclusis ergo his casibus, definitio ultima Functio-num

num parium erit bona, atque ad ejufmodi Functiones forman- C A P. I.
das idonea.

21. *Functio impar ipfius z eft ejufmodi Functio, cujus valor, fi loco z ponatur — z, fit quoque negativus.*

Hujufmodi Functiones ergo impares erunt omnes poteftates ipfius z, quarum exponentes funt numeri impares, ut z^1, z^3, z^5, z^7; &c. item z^{-1}, z^{-3}, z^{-5}; &c. tum vero etiam $z^{\frac{m}{n}}$ erit Functio impar, fi ambo numeri, m & n fuerint numeri impares. Generatim vero omnis expreffio ex hujusmodi poteftatibus compofita erit Functio impar ipfius z; cujufmodi funt, $az + bz^3 : az + az^{-1}$; item $z^{\frac{1}{3}} + az^{\frac{3}{5}} + bz^{-\frac{5}{3}}$; &c. Harum autem Functionum natura & inventio ex Functionibus paribus facilius perfpicietur.

22. *Si Functio par ipfius z multiplicetur per z vel per ejufdem Functionem imparem quamcunque, productum erit Functio impar ipfius z.*

Sit P Functio par ipfius z, quæ idcirco manet eadem fi loco z ponatur — z; quod fi ergo in producto Pz, ponatur — z loco z, prodibit — Pz; unde Pz erit Functio impar ipfius z. Sit jam P Functio par ipfius z, & Q functio impar ipfius z; atque ex Definitione patet fi loco z ponatur — z, valorem ipfius P manere eundem, at valorem ipfius Q abire in fui negativum — Q; quare productum PQ, pofito — z loco z, abibit in — PQ, hoc eft in fui negativum; eritque ideo PQ Functio impar ipfius z. Sic cum fit $a + \sqrt{(aa + zz)}$ functio par, & z^3 Functio impar ipfius z, erit productum $az^3 + z^3 \sqrt{(aa + zz)}$ Functio impar ipfius z; fimilique modo $z \times \frac{a + bzz}{\alpha + 6zz} = \frac{az + bz^3}{\alpha + 6zz}$ Functio impar ipfius z. Ex his vero etiam intelligitur, fi duarum Functionum P & Q, quarum altera P eft par, altera Q, impar, altera per alteram dividatur, quotum fore Functionem imparem; erit ergo $\frac{P}{Q}$ itemque $\frac{Q}{P}$ Functio impar ipfius z.

B 3

23. *Si*

23. *Si Functio impar per Functionem imparem vel multiplicetur, vel dividatur; quod resultat erit Functio par.*

Sint Q & S Functiones impares ipsius z; ita ut, posito — z loco z, Q abeat in — Q, & S in — S; atque perspicuum est tam productum $Q\,S$, quam quotum $\frac{Q}{S}$ eundem valorem retinere, etiamsi pro z ponatur — z; ideoque esse utrumque Functionem parem ipsius z. Manifestum itaque porro est cujusque Functionis imparis quadratum esse Functionem parem; cubum vero Functionem imparem; biquadratum iterum Functionem parem, atque ita porro.

24. *Si fuerit* y *Functio impar ipsius* z; *erit vicissim* z *Functio impar ipsius* y.

Cum enim sit y Functio impar ipsius z; si ponatur — z loco z, abibit y in — y. Quod si ergo z per y definiatur, necesse est ut posito — y loco y, quoque z abeat in — z; eritque ideo z Functio impar ipsius y. Sic quia, posito $y = z^3$, est y Functio impar ipsius z; erit quoque, ex æquatione $z^3 = y$ seu $z = y^{\frac{1}{3}}$, z Functio impar ipsius y. Et quia si fuerit $y = az + bz^3$, est y Functio impar ipsius z, erit vicissim, ex æquatione $bz^3 + az = y$, valor ipsius z per y expressus Functio impar ipsius y.

25. *Si natura Functionis* y *per ejusmodi æquationem definiatur, in cujus singulis terminis numerus dimensionum, quas* y *&* z *occupant conjunctim, sit vel par ubique, vel impar; tum erit* y *Functio impar ipsius* z.

Quod si enim in ejusmodi æquatione ubique loco z scribatur — z; simulque — y loco y; omnes æquationis termini vel manebunt iidem, vel fient negativi, utroque vero casu æquatio manebit e dem. Unde patet — y eodem modo per — z determinatum iri, quo $+y$ per $+z$ determinatur; & hanc ob rem, si loco z ponatur — z, valor ipsius y abibit in — y, seu y erit Functio impar ipsius z. Sic si fuerit vel $yy = ayz + bzz + c$; vel $y^3 + ayyz = byzz + cy + dz$, ex utraque æquatione y erit Functio impar ipsius z.

26. *Si* Z *fuerit Functio ipsius* z, *&* Y *Functio ipsius* y, *atque* Y *eodem modo definiatur per variabilem* y *& constantes, quo* Z *definitur per variabilem* z *& constantes; tum hæ Functiones* Y *et* Z *vocantur Functiones similes ipsarum* y *&* z.

Si scilicet fuerit $Z = a + bz + cz^2$, & $Y = a + by + cy^2$, erunt Z & Y Functiones similes ipsarum z & y, similique modo in multiformibus, si fuerit $Z^3 = azz\,Z + b$ & $Y^3 = ayy\,Y + b$; erunt Z & Y Functiones similes ipsarum z & y. Hinc sequitur, si Y & Z fuerint hujusmodi Functiones similes ipsarum y & z; tum si loco z scribatur y, Functionem Z abituram esse in Functionem Y. Solet hæc similitudo etiam hoc modo verbis exprimi, ut Y talis Functio dicatur ipsius y, qualis Functio sit Z ipsius z. Hæ locutiones perinde occurrent, sive quantitates variabiles z & y a se invicem pendeant, sive secus: sic qualis Functio est $ay + by^3$ ipsius y, talis Functio erit $a(y+n) + b(y+n)^3$ ipsius $y+n$, existente scilicet $z = y + n$: tum qualis Functio est $\dfrac{a + bz + czz}{a + \mathcal{b}z + \gamma zz}$ ipsius z, talis Functio erit $\dfrac{azz + bz + c}{azz + \mathcal{b}z + \gamma}$ ipsius $\dfrac{1}{z}$; posito $y = \dfrac{1}{z}$. Atque ex his luculenter perspicitur ratio similitudinis Functionum, cujus per universam Analysin sublimiorem uberrimus est usus. Ceterum hæc in genere de natura Functionum unius variabilis sufficere possunt; cum plenior expositio in applicatione sequente tradatur.

CAPUT II.

De transformatione Functionum.

27. FUnctiones in alias formas transmutantur, *vel loco quantitatis variabilis aliam introducendo, vel eandem quantitatem variabilem retinendo.*

Quod si eadem quantitas variabilis servatur, Functio proprie mutari non potest. Sed omnis transformatio consistit in
alio

alio modo eandem Functionem exprimendi , quemadmodum ex Algebra conftat eandem quantitatem per plures diverfas formas exprimi poffe. Hujufmodi transformationes funt , fi loco hujus Functionis $2 - 3z + zz$ ponatur $(1 - z)(2 - z)$, vel $(a + z)^3$ loco $a^3 + 3aaz + 3azz + z^3$, vel $\frac{a}{a - z} +$ $\frac{a}{a + z}$ loco $\frac{2aa}{aa - zz}$; vel $\sqrt{(1 + zz)} + z$ loco $\frac{1}{\sqrt{(1 + zz)} - z}$; quæ expreffiones, etfi forma differunt , tamen revera congruunt. Sæpe numero autem harum plurium formarum idem fignificantium una aptior eft ad propofitum efficiendum quam reliquæ , & hanc ob rem formam commodiffimam eligi oportet.

Alter transformationis modus , quo loco quantitatis variabilis z alia quantitas variabilis y introducitur , quæ quidem ad z datam teneat relationem , per fubftitutionem fieri dicitur ; hocque modo ita uti convenit , ut Functio propofita fuccinctius & commodius exprimatur , uti fi ifta propofita fuerit ipfius z Functio , $a^4 - 4a^3z + 6aazz - 4az^3 + z^4$; fi loco $a - z$ ponatur y, prodibit ifta multo fimplicior ipfius y Functio y^4 : &, fi habeatur hæc Functio irrationalis $\sqrt{(aa + zz)}$ ipfius z, fi ponatur $z = \frac{aa - yy}{2y}$, ifta Functio per y expreffa fiet rationalis $= \frac{aa + yy}{2y}$. Hunc autem transformationis modum in fequens Caput differam , hoc Capite illum , qui fine fubftitutione procedit , expofiturus.

28. *Functio integra ipfius z fæpenumero commode in fuos factores refolvitur , ficque in productum transformatur.*

Quando Functio integra hoc pacto in factores refolvitur, ejus natura multo facilius perfpicitur; cafus enim ftatim innotefcunt , quibus Functionis valor fit $= 0$. Sic hæc ipfius z Functio $6 - 7z + z^3$ transformatur in hoc productum $(1 - z)(2 - z)(3 + z)$ ex quo ftatim liquet Functionem propofitam tribus cafibus fieri $= 0$; fcilicet fi $z = 1$, & $z = 2$, & $z = -3$, quæ proprietates ex forma $6 - 7z + z^3$ non tam facile intelliguntur. Iftiufmodi Factores , in quibus variabiles z

nulla

nulla occurrit poteſtas, vocantur Factores ſimplices, ut diſtinguantur a Factoribus compoſitis, in quibus ipſius z ineſt quadratum vel cubus, vel alia poteſtas altior. Erit ergo in genere $f + g z$ forma Factorum ſimplicium, $f + g z + h z z$ forma Factorum duplicium; $f + g z + h z z + i z^3$ forma Factorum triplicium, & ita porro. Perſpicuum autem eſt Factorem duplicem duos complecti valores ſimplices, Factorem triplicem tres ſimplices, & ita porro. Hinc Functio ipſius z integra, in qua exponens ſummæ poteſtatis ipſius z eſt $= n$, continebit n Factores ſimplices; ex quo ſimul, ſi qui Factores fuerint vel duplices vel triplices, &c. numerus Factorum cognoſcetur.

29. *Factores ſimplices Functionis cujuſcunque integræ* Z *ipſius* z *reperiuntur, ſi Functio* Z *nihilo æqualis ponatur, atque ex hac æquatione omnes ipſius* z *radices inveſtigentur: ſingulæ enim ipſius* z *radices dabunt totidem Factores ſimplices Functionis* Z.

Quod ſi enim ex æquatione $Z = 0$, fuerit quæpiam radix $z = f$, erit $z - f$ diviſor, ac proinde Factor Functionis Z, ſic igitur inveſtigandis omnibus radicibus æquationis $Z = 0$, quæ ſint $z = f$, $z = g$, $z = h$; &c., Functio Z reſolvetur in ſuos Factores ſimplices, atque transformabitur in productum $Z = (z - f)(z - g)(z - h)$ &c.: ubi quidem notandum eſt ſi ſummæ poteſtatis ipſius z in Z non fuerit coefficiens $= + 1$, tum productum $(z - f)(z - g)$ &c. inſuper per illum coefficientem multiplicari debere. Sic ſi fuerit $Z = A z^n + B z^{n-1} + C z^{n-2} + $ &c. erit $Z = A. (z - f).(z - g).(z - h)$. &c. At ſi fuerit $Z = A + Bz + Cz^2 + Dz^3 + Ez^4 + $ &c. atque æquationis $Z = 0$ radices z repertæ ſint; f; g; h; i; &c. erit $Z = A(1 - \frac{z}{f})(1 - \frac{z}{g})(1 - \frac{z}{h})$. &c. Ex his autem viciſſim intelligitur, ſi Functionis Z Factor fuerit $z - f$, ſeu $1 - \frac{z}{f}$; tum valorem Functionis in nihilum abire, ſi loco z ponatur f. Facto enim $z = f$, unus Factor $z - f$, ſeu $1 - \frac{z}{f}$, Functionis Z, ideoque ipſa Functio Z evaneſcere debet.

30. *Factores simplices ergo erunt vel reales, vel imaginarii; &, si Functio* Z *habeat Factores imaginarios eorum nnmerus semper erit par.*

Cum enim Factores simplices nascantur ex radicibus æquationis $Z = 0$, radices reales præbebunt Factores reales, & imaginariæ imaginarios; in omni autem æquatione numerus radicum imaginariarum semper est par : quamobrem Functio Z, vel nullos habebit Factores imaginarios, vel duos, vel quatuor, vel sex, &c. Quod si Functio Z duos tantum habeat Factores imaginarios, eorum productum erit reale, ideoque præbebit Factorem duplicem realem. Sit enim $P =$ producto ex omnibus Factoribus realibus, erit productum duorum Factorum imaginariorum $= \dfrac{Z}{P}$; hincque reale. Simili modo si Functio Z habeat quatuor, vel sex, vel octo &c. Factores imaginarios; erit eorum productum semper reale : nempe æquale quoto, qui oritur, si Functio Z dividatur per productum omnium Factorum realium.

31. *Si fuerit* Q *productum reale ex quatuor Factoribus simplicibus imaginariis, tum idem hoc productum* Q *resolvi poterit in duos Factores duplices reales.*

Habebit enim Q ejusmodi formam $z^4 + Az^3 + Bz^2 + Cz + D$; quæ si negetur in duos Factores duplices reales resolvi posse, resolubilis erit statuenda in duos Factores duplices imaginarios; qui hujusmodi formam habebunt $zz - 2(p+q\sqrt{-1})z + r + s\sqrt{-1}$. & $zz - 2(p-q\sqrt{-1})z + r - s\sqrt{-1}$; aliæ enim formæ imaginariæ concipi non possunt, quarum productum fiat reale, nempe $= z^4 + Az^3 + Bz^2 + Cz + D$. Ex his autem Factoribus imaginariis duplicibus sequentes emergent quatuor Factores simplices imaginarii ipsius Q,

I. $z - (p + q\sqrt{-1}) + \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$

II. $z - (p + q\sqrt{-1}) - \sqrt{(pp + 2pq\sqrt{-1} - qq - r - s\sqrt{-1})}$

III. $z - (p - q\sqrt{-1}) + \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

IV. $z - (p - q\sqrt{-1}) - \sqrt{(pp - 2pq\sqrt{-1} - qq - r + s\sqrt{-1})}$

Horum

Horum Factorum multiplicentur primus ac tertius in se invicem, CAP. II.
posito brevitatis gratia, $t = pp - qq - r$, & $u = 2pq - s$; erit-
que horum Factorum productum $= zz - (2p - \sqrt{2t + 2\sqrt{(tt + uu)}})z$
$+ pp + qq - p\sqrt{2t + 2\sqrt{(tt + uu)}} + \sqrt{(tt + uu)}$; quod uti-
$$+ q\sqrt{-2t + 2\sqrt{(tt + uu)}}$$
que est reale. Simili autem modo productum ex Factoribus se-
cundo & quarto erit reale nempe $= zz - (2p + \sqrt{2t + 2\sqrt{(tt + uu)}})z$
$+ pp + qq + p\sqrt{2t + 2\sqrt{(tt + uu)}} + \sqrt{(tt + uu)}$.
$$+ q\sqrt{-2t + 2\sqrt{(tt + uu)}}$$
Quocirca productum propositum Q, quod in duos Factores
duplices reales resolvi posse negabatur, nihilo minus actu in duos
Factores duplices reales est resolutum.

32. *Si Functio integra* Z *ipsius* z *quotcunque habeat Factores sim-
plices imaginarios, bini semper ita conjungi possunt, ut eorum pro-
ductum fiat reale.*

Quoniam numerus radicum imaginariarum semper est par,
sit is $= 2n$; ac primo quidem patet productum harum radicum
imaginariarum omnium esse reale. Quod si ergo duæ tantum
radices imaginariæ habeantur, erit earum productum utique rea-
le; sin autem quatuor habeantur Factores imaginarii, tum, uti
vidimus, eorum productum resolvi potest in duos Factores du-
plices reales formæ $fzz + gz + h$. Quanquam autem eundem
demonstrandi modum ad altiores potestates extendere non licet,
tamen extra dubium videtur esse positum eandem proprietatem
in quotcunque Factores imaginarios competere ; ita ut semper
loco $2n$ Factorum simplicium imaginariorum induci queant n Fac-
tores duplices reales. Hinc omnis Functio integra ipsius z re-
solvi poterit in Factores reales vel simplices vel duplices. Quod
quamvis non summo rigore sit demonstratum, tamen ejus veri-
tas in sequentibus magis corroborabitur, ubi hujus generis Func-
tiones $a + bz^n$; $a + bz^n + cz^{2n}$; $a + bz^n + cz^{2n} + dz^{3n}$ &c.
actu in istiusmodi Factores duplices reales resolventur.

33. *Si Functio integra* Z, *posito* $z = a$ *induat valorem* A, *& posito* $z = b$, *induat valorem* B; *tum, loco* z *valores medios inter* a *&* b *ponendo, Functio* Z *quosvis valores medios inter* A *&* B *accipere potest.*

Cum enim Z sit Functio uniformis ipsius z, quicunque valor realis ipsi z tribuatur, Functio quoque Z hinc valorem realem obtinebit. Cum igitur Z, priore casu $z = a$, nanciscatur valorem A; posteriore casu $z = b$, autem, valorem B; ab A ad B transire non poterit, nisi per omnes valores medios transeundo. Quod si ergo æquatio $Z - A = 0$ habeat radicem realem, simulque $Z - B = 0$ radicem realem suppeditet; tum æquatio quoque $Z - C = 0$ radicem habebit realem; si quidem C intra valores A & B contineatur. Hinc si expressiones $Z - A$ & $Z - B$ habeant Factorem simplicem realem, tum expressio quæcunque $Z - C$ Factorem simplicem habebit realem, dummodo C intra valores A & B contineatur.

34. *Si in Functione integra* Z *exponens maximæ ipsius* z *potestatis fuerit numerus impar* $2n + 1$, *tum ea Functio* Z *unicum ad minimum habebit Factorem simplicem realem.*

Habebit scilicet Z hujusmodi formam $z^{2n+1} + \alpha z^{2n} + \beta z^{2n-1} + \gamma z^{2n-2} + $ &c. in qua si ponatur $z = \infty$, quia valores singulorum terminorum præ primo evanescunt, fiet $Z = (\infty)^{2n+1} = \infty$; ideoque $Z - \infty$ Factorem simplicem habebit realem nempe $z - \infty$. Sin autem ponatur $z = -\infty$, fiet $Z = (-\infty)^{2n+1} = -\infty$, ideoque habebit $Z + \infty$ Factorem simplicem realem $z + \infty$. Cum igitur tam $Z - \infty$, quam $Z + \infty$ habeat Factorem simplicem realem; sequitur etiam $Z + C$ habiturum esse Factorem simplicem realem, siquidem C contineatur intra limites $+\infty$ & $-\infty$; hoc est si C fuerit numerus realis quicunque, sive affirmativus, sive negativus. Hanc ob rem, facto $C = 0$, habebit quoque ipsa Functio Z Factorem simplicem realem $z - c$; atque quantitas c

continebitur intra limites $+ \infty$ & $- \infty$, eritque idcirco vel
quantitas affirmativa, vel negativa, vel nihil.

35. *Functio igitur integra* Z, *in qua exponens maximæ potestatis ipsius z est numerus impar, vel unum habebit Factorem simplicem realem, vel tres, vel quinque, vel septem &c.*

Cum enim demonstratum sit Functionem Z certo unum habere Factorem simplicem realem $z - c$; ponamus eam præterea unum Factorem habere $z - d$, atque dividatur Functio Z, in qua maxima ipsius z potestas sit z^{2n+1}, per $(z - c) \cdot (z - d)$, erit quoti maxima potestas $= z^{2n-1}$, cujus exponens, cum sit numerus impar, indicat denuo ipsius Z dari Factorem simplicem realem. Si ergo Z plures uno habeat Factores simplices reales, habebit vel tres, vel (quoniam eodem modo progredi licet) quinque, vel septem, &c. Erit scilicet numerus Factorum simplicium realium impar, & quia numerus omnium Factorum simplicium est $= 2n + 1$, erit numerus Factorum imaginariorum par.

36. *Functio integra* Z, *in qua exponens maximæ potestatis ipsius z est numerus par* $2n$, *vel duos habebit Factores simplices reales vel quatuor, vel sex, vel &c.*

Ponamus ipsius Z constare Factorum simplicium realium numerum imparem $2m + 1$; si ergo per horum omnium productum dividatur Functio Z, quoti maxima potestas erit $= z^{2n - 2m - 1}$, ejusque ideo exponens numerus impar; habebit ergo Functio Z præterea unum certo Factorem simplicem realem, ex quo numerus omnium Factorum simplicium realium ad minimum erit $= 2m + 2$, ideoque par; ac numerus Factorum imaginariorum pariter par. Omnis ergo Functionis integræ Factores simplices imaginarii sunt numero pares; quemadmodum quidem jam ante statuimus.

37. *Si in Functione integra* Z *exponens maximæ potestatis ipsius z fuerit numerus par, atque terminus absolutus, seu constans, signo — affectus, tum Functio* Z *ad minimum duos habet Factores simplices reales.*

Functio

Functio ergo Z, de qua hic sermo est, hujusmodi formam habebit $z^{2n} \pm \alpha z^{2n-1} \pm 6 z^{2n-2} \pm \ldots \ldots \pm$ $\nu z - A$. Si jam ponatur $z = \infty$, fiet, uti supra vidimus, $Z = \infty$; atque, si ponatur $z = 0$, fiet $Z = -A$. Habebit ergo $Z - \infty$ Factorem realem $z - \infty$, & $Z + A$ Factorem $z - 0$: unde cum 0 contineatur intra limites $-\infty$ & $+A$, sequitur $Z + 0$ habere Factorem simplicem realem $z - c$, ita ut c contineatur intra limites 0 & ∞. Deinde, cum posito $z = -\infty$, fiat $Z = \infty$, ideoque $Z - \infty$ Factorem habeat $z + \infty$, & $Z + A$ Factorem $z + 0$, sequitur quoque $Z + 0$ Factorem simplicem realem habere $z + d$; ita ut d intra limites 0 & ∞ contineatur; unde constat propositum. Ex his igitur perspicitur si Z talis fuerit Functio, qualis hic est descripta, æquationem $Z = 0$, duas ad minimum habere debere radices reales, alteram affirmativam, alteram negativam. Sic æquatio hæc $z^4 + \alpha z^3 + 6 z^2 + \gamma z - a a = 0$, duas habet radices reales, alteram affirmativam, alteram negativam.

38. *Si in Functione fracta, quantitas variabilis z tot vel plures habeat dimensiones in numeratore, quam in denominatore; tum ista Functio resolvi poterit in duas partes, quarum altera est Functio integra, altera fracta; in cujus numeratore quantitas variabilis z pauciores habeat dimensiones quam in denominatore.*

Si enim exponens maximæ potestatis ipsius z minor fuerit in denominatore quam in numeratore; tum numerator per denominatorem dividatur more solito, donec in quoto ad exponentes negativos ipsius z perveniatur; hoc ergo loco abrupta divisionis operatione quotus constabit ex parte integra atque fractione, in cujus numeratore minor erit dimensionum numerus ipsius z quam in denominatore; hic autem quotus Functioni propositæ est æqualis. Sic, si hæc proposita fuerit Functio fracta $\frac{1+z^4}{1+zz}$, ea per divisionem ita resolvetur.

$$zz + 1) \; z^4 + 1 \; (\; zz - 1 + \frac{2}{1+zz}$$

$$z^4 + zz$$
$$\overline{} zz + 1$$
$$\overline{} zz - 1$$
$$+ \; 2$$

eritque $\dfrac{1 + z^4}{1 + zz} = zz - 1 + \dfrac{2}{1+zz}$. Hujufmodi Func-
tiones fractæ, in quibus quantitas variabilis z tot vel plures ha-
bet dimenfiones in numeratore quam in denominatore, ad
fimilitudinem Arithmeticæ vocari poffunt fractiones fpuriæ, vel
Functiones fractæ fpuriæ, quo diftinguantur a Functionibus frac-
tis genuinis, in quarum numeratore quantitas variabilis z pau-
ciores habet dimenfiones quam in denominatore. Functio ita-
que fracta fpuria refolvi poterit in Functionem integram, & Func-
tionem fractam genuinam; hæcque refolutio per vulgarem di-
vifionis operationem abfolvetur.

39. *Si denominator Functionis fractæ duos habeat Factores inter*
fe primos; tum ipfa Functio fracta refolvetur in duas fractiones,
quarum denominatores fint illis binis Factoribus refpective æquales.

Quanquam hæc refolutio ad Functiones fractas fpurias æque
pertinet atque ad genuinas, tamen eam ad genuinas potiffi-
mum accomodabimus. Refoluto autem denominatore hujuf-
modi Functionis fractæ in duos Factores inter fe primos, ipfa
Functio refolvetur in duas alias Functiones fractas genuinas,
quarum denominatores fint illis binis Factoribus refpective æqua-
les; hæcque refolutio, fi quidem fractiones fint genuinæ, uni-
co modo fieri poteft; cujus rei veritas ex exemplo clarius quam
per ratiocinium perfpicietur. Sit ergo propofita hæc Functio
fracta $\dfrac{1 - 2z + 3zz - 4z^3}{1 + 4z^4}$, cujus denominator $1 + 4z^4$ cum fit
æqualis huic producto $(1 + 2z + 2zz)(1 - 2z + 2zz)$,
fractio propofita in duas fractiones refolvetur, quarum alterius
denominator erit $1 + 2z + 2zz$, alterius $1 - 2z + 2zz$: ad
quas inveniendas, quia funt genuinæ, ftatuantur numeratores
illius $= \alpha + 6z$, hujus $= \gamma + \delta z$, eritque per hypothefin

I —

$$\frac{1-2z+3zz-4z^3}{1+4z^4} = \frac{\alpha+\mathfrak{C}z}{1+2z+2zz} + \frac{\gamma+\delta z}{1-2z+2zz} : \text{ad-}$$

dantur actu hæ duæ fractiones, eritque summæ

Numerator   Denominator

$$
\begin{aligned}
&+\alpha -2\alpha z + 2\alpha zz \\
&\qquad +\mathfrak{C}z -2\mathfrak{C}zz + 2\mathfrak{C}z^3 \qquad\qquad 1+4z^4 \\
&+\gamma +2\gamma z + 2\gamma zz \\
&\qquad +\delta z + 2\delta zz + 2\delta z^3
\end{aligned}
$$

Cum ergo denominator æqualis fit denominatori fractionis pro-
pofitæ, numeratores quoque æquales reddi debent: quod, ob
tot litteras incognitas $\alpha, \mathfrak{C}, \gamma, \delta$, quot funt termini æquales
efficiendi, utique fieri, idque unico modo poterit : nancifci-
mur fcilicet has quatuor æquationes

I. $\alpha + \gamma = 1$ III. $2\alpha - 2\mathfrak{C} + 2\gamma + 2\delta = 3$
II. $-2\alpha + \mathfrak{C} + 2\gamma + \delta = -2$ IV. $2\mathfrak{C} + 2\delta = -4$.

Hinc ob $\alpha + \gamma = 1$, & $\mathfrak{C} + \delta = -2$; æquationes II. &
III. dabunt $\alpha - \gamma = 0$ & $\delta - \mathfrak{C} = \frac{1}{2}$; ex quibus fit

$\alpha = \frac{1}{2}$; $\gamma = \frac{1}{2}$; $\mathfrak{C} = \frac{-5}{4}$; $\delta = \frac{-3}{4}$; ideoque fractio

propofita $\dfrac{1-2z+3zz-4z^3}{1+4z^4}$, transformatur in has duas

$$\frac{\frac{1}{2} - \frac{5}{4}z}{1+2z+2zz} + \frac{\frac{1}{2} - \frac{3}{4}z}{1-2z+2zz}.$$ Simili autem modo facile per-
fpicietur refolutionem femper fuccedere debere : quoniam fem-
per tot litteræ incognitæ introducuntur, quot opus eft ad nu-
meratorem propofitum eliciendum.   Ex doctrina vero fractio-
num communi intelligitur hanc refolutionem fuccedere non pof-
fe, nifi ifti denominatoris Factores fuerint inter fe primi.

40. *Functio igitur fracta* $\dfrac{M}{N}$ *in tot fractiones fimplices formæ*

$\dfrac{A}{p-qz}$ *refolvi poterit, quot Factores fimplices habet denominator*
N *inter fe inæquales.*

Repræ-

Repræfentat hic fractio $\frac{M}{N}$ Functionem quamcunque fractam genuinam, ita ut M & N fint Functiones integræ ipfius z, atque fumma poteftas ipfius z in M minor fit quam in N. Quod fi ergo denominator N in fuos Factores fimplices refolvatur, hique inter fe fuerint inæquales, expreffio $\frac{M}{N}$ in tot fractiones refolvetur, quot Factores fimplices in denominatore N continentur; propterea quod quifque Factor abit in denominatorem fractionis partialis. Si ergo $p - qz$ fuerit Factor ipfius N, is erit denominator fractionis cujufdam partialis, &, cum in numeratore hujus fractionis numerus dimenfionum ipfius z minor effe debeat quam in denominatore $p - qz$, numerator neceffario erit quantitas conftans. Hinc ex unoquoque Factore fimplici $p - qz$ denominatoris N nafcetur fractio fimplex $\frac{A}{p - qz}$; ita ut fumma omnium harum fractionum fit æqualis fractioni propofitæ $\frac{M}{N}$.

E X E M P L U M.

Sit, exempli caufa, propofita hæc Functio fracta $\frac{1 + zz}{z - z^3}$; quia Factores fimplices denominatoris funt z, $1 - z$, & $1 + z$, ifta Functio refolvetur in has tres fractiones fimplices $\frac{A}{z} + \frac{B}{1 - z} + \frac{C}{1 + z} = \frac{1 + zz}{z - z^3}$; ubi numeratores conftantes A, B, & C definire oportet. Reducantur hæ fractiones ad communem denominatorem, qui erit $z - z^3$; atque numeratorum fumma æquari debebit ipfi $1 + zz$, unde ifta æquatio oritur:

$$A + Bz - Azz = 1 + zz = 1 + 0z + zz.$$
$$+ Cz + Bzz$$
$$- Czz$$

Lib. I. quæ totidem comparationes præbet, quot funt litteræ inco-
gnitæ A, B, C; erit fcilicet ,

I°. $A = 1$.

II°. $B + C = 0$.

III°. $- A + B - C = 1$:

Hinc fit $B - C = 2$; & porro $A = 1$; $B = 1$ & $C = - 1$. Functio ergo propofita $\frac{1 + zz}{z - z^3}$ refolvitur in hanc formam $\frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$. Simili autem modo intelligitur, quotcunque habuerit denominator N Factores fimplices inter 'fe inæquales, femper fractionem $\frac{M}{N}$ in totidem fractiones fimplices refolvi. Sin autem aliquot Factores fuerint æquales inter fe, tum alio modo poft-explicando refolutio inftitui debet.

41. *Cum igitur quilibet Factor fimplex denominatoris* N *fuppeditet fractionem fimplicem pro refolutione Functionis propofitæ* $\frac{M}{N}$; *oftendendum eft quomodo ex Factore fimplice denominatoris* N *cognito, fractio fimplex refpondens reperiatur.*

Sit $p - qz$ Factor fimplex ipfius N, ita ut fit $N = (p - qz) S$; atque S Functio integra ipfius z; ponatur fractio ex Factore $p - q z$ orta $= \frac{A}{p - qz}$, & fit fractio ex altero Factore denominatoris S oriunda $= \frac{P}{S}$, ita ut, fecundum §. 39., futurum fit $\frac{M}{N} = \frac{A}{p - qz} + \frac{P}{S} = \frac{M}{(p - qz) S}$; hinc erit $\frac{P}{S} = \frac{M - AS}{(p - qz) S}$; quæ fractiones cum congruere debeant, neceffe eft ut $M - AS$ fit divifibile per $p - qz$; quoniam Functio integra P ipfi quoto æquatur. Quando vero $p - q z$ Divifor exiftit ipfius $M - AS$, hæc expreffio pofito $z = \frac{p}{q}$ evanefcit. Ponatur ergo ubique loco z hic valor conftans $\frac{p}{q}$ in M

&

& S, erit $M - AS = 0$, ex quo fiet $A = \frac{M}{S}$; hocque ergo
modo reperitur numerator A fractionis quæfitæ $\frac{A}{p - qz}$; atque fi
ex fingulis denominatoris N Factoribus fimplicibus, dummodo
fint inter fe inæquales, hujufmodi fractiones fimplices formen-
tur, harum fractionum fimplicium omnium fumma erit æqualis
Functioni propofitæ $\frac{M}{N}$.

E X E M P L U M.

Sic, fi in Exemplo præcedente $\frac{1 + zz}{z - z^3}$, ubi eft $M = 1 + zz$,
& $N = z - z^3$, fumatur z pro Factore fimplice, erit $S = 1 - zz$, atque fractionis fimplicis $\frac{A}{z}$ hinc ortæ erit numerator
$A = \frac{1 + zz}{1 - zz} = 1$ pofito $z = 0$, quem valorem z obtinet fi
ipfe hic Factor fimplex z nihilo æqualis ponatur. Simili modo
fi pro denominatoris Factore fumatur $1 - z$, ut fit $S = z + zz$
erit $A = \frac{1 + zz}{z + zz}$, facto $1 - z = 0$, unde erit $A = 1$, &
ex Factore $1 - z$ nafcitur fractio $\frac{1}{1 - z}$. Tertius denique Fa-
ctor $1 + z$, ob $S = z - zz$, & $A = \frac{1 + zz}{z - zz}$, pofito $1 + z = 0$, feu $z = -1$, dabit $A = -1$, & fractionem fim-
plicem $= \frac{-1}{1 + z}$. Quare per hanc regulam reperitur $\frac{1 + zz}{z - z^3}$
$= \frac{1}{z} + \frac{1}{1 - z} - \frac{1}{1 + z}$, ut ante.

42. *Functio fracta hujus forma* $\frac{P}{(p - qz)^n}$, *cujus numerator*
P *non tantam ipfius z poteftatem involvit quantam denominator*
$(p - qz)^n$, *tranfmutari poteft in hujufmodi fractiones partiales*

A

$\dfrac{A}{(p-qz)^n} + \dfrac{B}{(p-qz)^{n-1}} + \dfrac{C}{(p-qz)^{n-2}} + \ldots + \dfrac{K}{p-qz}$;

quarum omnium numeratores sint quantitates constantes.

Quoniam maxima potestas ipsius z in P minor est quam z^n, erit z^{n-1}, ideoque P hujusmodi habebit formam :

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \ldots\ldots\ldots + \varkappa z^{n-1}$$

existente terminorum numero $= n$, cui æquari debet numerator summæ omnium fractionum partialium, postquam singulæ ad eundem denominatorem $(p-qz)^n$ fuerint perductæ : qui numerator propterea erit $= A + B(p-qz) + C(p-qz)^2 + D(p-qz)^3 + \ldots\ldots + K(p-qz)^{n-1}$. Hujus maxima ipsius z potestas est, ut ibi, z^{n-1}, atque tot habentur litteræ incognitæ A, B, C, $\ldots\ldots K$, (quarum numerus est $= n$,) quot sunt termini congruentes reddendi. Quamobrem litteræ constantes A, B, C, &c. ita definiri poterunt, ut fiat Functio fracta genuina $\dfrac{P}{(p-qz)^n} = \dfrac{A}{(p-qz)^n} +$

$\dfrac{B}{(p-qz)^{n-1}} + \dfrac{C}{(p-qz)^{n-2}} + \dfrac{D}{(p-qz)^{n-3}} + \ldots\ldots :$

$+ \dfrac{K}{p-qz}$. Ipsa autem horum numeratorum inventio mox facilis aperietur.

43. *Si Functionis fractæ* $\dfrac{M}{N}$ *denominator* N *Factorem habeat* $(p-qz)^2$, *sequenti modo fractiones partiales ex hoc Factore oriundæ reperientur.*

Cujusmodi fractiones partiales ex singulis Factoribus denominatoris simplicibus, qui alios sibi æquales non habeant, oriantur, ante est ostensum : nunc igitur ponamus duos Factores inter se esse æquales, seu, iis conjunctis, denominatoris N Factorem esse $(p-qz)^2$. Ex hoc ergo Factore per §. præced. duæ nascentur fractiones partiales hæ $\dfrac{A}{(p-qz)^2} + \dfrac{B}{p-qz}$. Sit autem

tem

tem $N = (p - qz)^2 S$, eritque $\frac{M}{N} = \frac{M}{(p-qz)^2 S} = \frac{A}{(p-qz)^2}$

$+ \frac{B}{p-qz} + \frac{P}{S}$, denotante $\frac{P}{S}$ omnes fractiones simplices jun-

ctim sumptas ex denominatoris Factore S ortas. Hinc erit $\frac{P}{S}$

$= \frac{M - AS - B(p-qz)S}{(p-qz)^2 S}$, & $P = \frac{M - AS - B(p-qz)S}{(p-qz)^2}$

$=$ Functioni integræ. Debet ergo $M - AS - B(p-qz)S$
divisibile esse per $(p-qz)^2$: sit primum divisibile per $p-qz$,
atque tota exptessio $M - AS - B(p-qz)S$ evanescet,
posito $p - qz = 0$, seu $z = \frac{p}{q}$; ponatur ergo ubique $\frac{p}{q}$

loco z, eritque $M - AS = 0$, ideoque $A = \frac{M}{S}$; scilicet fra-

ctio $\frac{M}{S}$, si loco z ubique ponatur $\frac{p}{q}$, dabit valorem ipsius A

constantem. Hoc invento quantitas $M - AS - B(p-qz)S$

etiam per $(p-qz)^2$ divisibilis esse debet, seu $\frac{M - AS}{p - qz} -$

BS denuo per $p - qz$ divisibile esse debet. Posito ergo ubi-

que $z = \frac{p}{q}$ erit $\frac{M - AS}{p - qz} = BS$, ideoque $B = \frac{M - AS}{(p - qz)S}$

$= \frac{1}{p-qz}\left(\frac{M}{S} - A\right)$, ubi notandum est, cum $M - AS$ di-
visibile sit per $p - qz$, hanc divisionem prius institui debere,
quam loco z substituatur $\frac{p}{q}$. Vel ponatur $\frac{M - AS}{p - qz} = T$,

eritque $B = \frac{T}{S}$ posito $z = \frac{p}{q}$; inventis ergo numratori-
bus A & B, erunt fractiones partiales ex denominatoris N

Factore $(p-qz)^2$ ortæ hæ $\frac{A}{(p - qz)^2} + \frac{B}{p - qz}$.

E X E M P L U M I.

Sit hæc proposita Functio fracta $\frac{1 - zz}{zz(1+zz)}$ erit, ob de-

nomina-

nominatoris Factorem quadratum zz; $S = 1 + zz$ & $M =$ $1 - zz$. Sint fractiones partiales ex zz ortæ $\dfrac{A}{zz} + \dfrac{B}{z}$, erit

$A = \dfrac{M}{S} = \dfrac{1 - zz}{1 + zz}$, posito Factore $z = 0$; hincque $A = 1$. Tum erit $M - AS = - 2zz$ quod divisum per Factorem simplicem z, dabit $T = - 2z$, hincque $B = \dfrac{T}{S} = \dfrac{-2z}{1 + zz}$, posito $z = 0$; unde erit $B = 0$; atque ex Factore denominatoris zz orietur unica hæc fractio partialis $\dfrac{1}{zz}$.

Exemplum II.

Sit hæc proposita Functio fracta $\dfrac{z^3}{(1-z)^2(1+z^4)}$, cujus, ob denominatoris Factorem quadratum $(1-z)^2$, fractiones partiales sint $\dfrac{A}{(1-z)^2} + \dfrac{B}{1-z}$. Erit ergo $M = z^3$ & $S =$ $1 + z^4$; ideoque $A = \dfrac{M}{S} = \dfrac{z^3}{1+z^4}$, posito $1 - z = 0$, seu $z = 1$: unde fit $A = \dfrac{1}{2}$. Prodibit ergo $M - AS =$ $z^3 - \dfrac{1}{2} - \dfrac{1}{2}z^4 = - \dfrac{1}{2} + z^3 - \dfrac{1}{2}z^4$, quod per $1-z$ divisum dat $T = - \dfrac{1}{2} - \dfrac{1}{2}z - \dfrac{1}{2}zz + \dfrac{1}{2}z^3$; ideoque $B = \dfrac{T}{S} = \dfrac{-1 - z - zz + z^3}{2 + 2z^4}$, posito $z = 1$; ita ut sit $B = -\dfrac{1}{2}$; fractiones ergo partiales quæsitæ sunt $\dfrac{1}{2(1-z)^2}$ $- \dfrac{1}{2(1-z)}$.

44. *Si Functionis fractæ* $\dfrac{M}{N}$ *denominator* N *Factorem habeat* $(p - qz)^3$ *sequenti modo fractiones partiales ex hoc Factore oriundæ* $\dfrac{A}{(p-qz)^3} + \dfrac{B}{(p-qz)^2} + \dfrac{C}{p-qz}$ *reperientur.*

Ponatur

Ponatur $N = (p - qz)^3 S$, fitque fractio ex Factore S orta Cap. II.
$= \frac{P}{S}$, erit $P = \frac{M - AS - B(p - qz)S - C(p - qz)^2 S}{(p - qz)^3}$
$=$ Functioni integræ. Numerator ergo $M - AS -$
$B(p - qz)S - C(p - qz)^2 S$ ante omnia divifibilis effe
debet per $(p - qz)$; unde is, pofito $p - qz = o$, feu $z =$
$\frac{p}{q}$, evanefcere debet, eritque adeo $M - AS = o$, ideoque
$A = \frac{M}{S}$, pofito $z = \frac{p}{q}$. Invento hoc pacto A erit $M -$
AS divifibile per $p - qz$ ponatur ergo $\frac{M - AS}{p - qz} = T$, àt-
que $T - BS - C(p - qz)S$ adhuc per $(p - qz)^2$ erit di-
vifibile; fiet ergo $= o$, pofito $p - qz = o$; ex quo prodit
$B = \frac{T}{S}$ pofito $z = \frac{p}{q}$. Sic autem invento B erit $T - BS$
divifibile per $p - qz$. Hanc ob rem, pofito $\frac{T - BS}{p - qz} = V$,
fupereft ut $V - CS$ divifibile fit per $p - qz$; eritque ergo
$V - CS = o$, pofito $p - qz = o$, atque $C = \frac{V}{S}$, pofi-
to $z = \frac{p}{q}$. Inventis ergo hoc modo numeratoribus A, B, C,
fractiones partiales ex denominatoris N Factore $(p - qz)^3$ or-
tæ erunt $\frac{A}{(p - qz)^3} + \frac{B}{(p - qz)^2} + \frac{C}{p - qz}$.

E X E M P L U M.

Sit propofita hæc fracta Functio $\frac{zz}{(1 - z)^3(1 + zz)}$, ex cu-
jus denominatoris Factore cubico $(1 - z)^3$ oriantur hæ fractio-
nes partiales: $\frac{A}{(1 - z)^3} + \frac{B}{(1 - z)^2} + \frac{C}{1 - z}$. Erit er-
go $M = zz$ & $S = 1 + zz$; unde fit primum $A = \frac{zz}{1 + zz}$
pofito

posito $1 - z = 0$ feu $z = 1$; ex quo prodit $A = \frac{1}{2}$. Jam

ponatur $T = \frac{M - AS}{1 - z}$, erit $T = \frac{\frac{1}{2}zz - \frac{1}{2}}{1 - z} = -\frac{1}{2} -$

$\frac{1}{2}z$; unde oritur $B = \frac{-\frac{1}{2} - \frac{1}{2}z}{1 + zz}$, pofito $z = 1$, ita ut fit

$B = -\frac{1}{2}$. Ponatur porro $V = \frac{T - BS}{1 - z} = \frac{T + \frac{1}{2}S}{1 - z}$; erit

$V = \frac{-\frac{1}{2}z + \frac{1}{2}zz}{1 - z} = -\frac{1}{2}z$; unde fit $C = \frac{V}{S} = \frac{-\frac{1}{2}z}{1 + zz}$

pofito $z = 1$, ita ut fit $C = -\frac{1}{4}$. Quo circa fractiones

partiales ex denominatoris Factore $(1 - z)^3$ ortæ erunt

$$\frac{1}{2(1 - z)^3} - \frac{1}{2(1 - z)^2} - \frac{1}{4(1 - z)}.$$

45. *Si Functionis fractæ* $\frac{M}{N}$ *denominator* N *Factorem habeat*

$(p - qz)^n$; *fractiones partiales hinc ortæ* $\frac{A}{(p - qz)^n} +$

$\frac{B}{(p - qz)^{n-1}} + \frac{C}{(p - qz)^{n-2}} + \ldots + \frac{K}{p - qz}$ *fequenti*

modo invenientur.

Ponatur denominator $N = (p - qz)^n Z$, atque, ratioci-
nium ut ante inftituendo, reperietur ut fequitur:

Primo $A = \frac{M}{Z}$, pofito $z = \frac{p}{q}$. Ponatur $P = \frac{M - AZ}{p - qz}$

Secundo $B = \frac{P}{Z}$, pofito $z = \frac{p}{q}$. Ponatur $Q = \frac{P - BZ}{p - qz}$

Tertio $C = \frac{Q}{Z}$, pofito $z = \frac{p}{q}$. Ponatur $R = \frac{Q - CZ}{p - qz}$

Quarto $D = \frac{R}{Z}$, pofito $z = \frac{p}{q}$. Ponatur $S = \frac{R - DZ}{p - qz}$

Quinto $E = \frac{S}{Z}$, pofito $z = \frac{p}{q}$. &c.

Hoc ergo modo fi definiantur finguli numeratores conftan-
tes

tes A, B, C, D, $\&c.$ invenientur omnes fractiones partiales,
quæ ex denominatoris N Factore $(p - qz)^n$ nascuntur.

E X E M P L U M.

Sit proposita ista Functio fracta $\dfrac{1 + zz}{z^5 (1 + z^3)}$ ex cujus deno-
minatoris Factore z^5 nascantur hæ fractiones partiales $\dfrac{A}{z^5} + \dfrac{B}{z^4}$
$+ \dfrac{C}{z^3} + \dfrac{D}{z^2} + \dfrac{E}{z}$. Ad quarum numeratores constantes inve-
niendos, erit $M = 1 + zz$ atque $Z = 1 + z^3$; &
$\dfrac{p}{q} = 0$. Sequens ergo calculus ineatur.

Primum est $A = \dfrac{M}{Z} = \dfrac{1 + zz}{1 + z^3}$, posito $z = 0$; ergo $A = 1$.

Ponatur $P = \dfrac{M - AZ}{z} = \dfrac{zz - z^3}{z} = z - zz$. Eritque se-
cundo $B = \dfrac{P}{Z} = \dfrac{z - zz}{1 + z^3}$, posito $z = 0$; ergo $B = 0$.

Ponatur $Q = \dfrac{P - BZ}{z} = \dfrac{z - zz}{z} = 1 - z$; eritque ter-
tio $C = \dfrac{Q}{Z} = \dfrac{1 - z}{1 + z^3}$, posito $z = 0$: ergo $C = 1$.

Ponatur $R = \dfrac{Q - CZ}{z} = \dfrac{-z - z^3}{z} = -1 - zz$;
erit quarto $D = \dfrac{R}{Z} = \dfrac{-1 - zz}{1 + z^5}$, posito $z = 0$; ex quo
fit $D = -1$.

Ponatur $S = \dfrac{R - DZ}{z} = \dfrac{-zz + z^3}{z} = -z + zz$; erit
quinto $E = \dfrac{S}{Z} = \dfrac{-z + zz}{1 + z^3}$, posito $z = 0$; unde fit $E = 0$.

Quo circa fractiones partiales quæsitæ erunt hæ:
$\dfrac{1}{z^5} + \dfrac{0}{z^4} + \dfrac{1}{z^3} - \dfrac{1}{z^2} + \dfrac{0}{z}$.

46. Quæcunque ergo propoſita fuerit Functio rationalis fracta $\frac{M}{N}$ *, ea ſequenti modo in partes reſolvetur, atque in formam ſimpliciſſimam tranſmutabitur.*

Quærantur denominatoris N omnes Factores ſimplices five reales five imaginarii; quorum qui fibi pares non habeant, feorfim tractentur & ex unoquoque per §. 41, fractio partialis eruatur. Quod fi idem Factor ſimplex bis vel pluries occurrat, ii conjunctim fumantur atque ex eorum producto, quod erit poteſtas formæ $(p - q z)^n$, quærantur fractiones partiales convenientes per §. 45. Hocque modo cum ex ſingulis Factoribus ſimplicibus denominatoris erutæ fuerint fractiones partiales, tum harum omnium aggregatum æquabitur Functioni propofitæ $\frac{M}{N}$, nifi fuerit ſpuria; fi enim fuerit ſpuria, pars integra infuper extrahi atque ad iftas fractiones partiales inventas adjici debebit, quo prodeat valor Functionis $\frac{M}{N}$ in forma ſimpliciſſima expreſſus. Perinde autem eſt five fractiones partiales ante extractionem partis integræ, five poſt quærantur. Eædem enim ex ſingulis denominatoris N Factoribus prodeunt fractiones partiales, five adhibeatur ipſe numerator M, five idem quocunque denominatoris N multiplo vel auctus vel minutus; id quod regulas datas contemplanti facile patebit.

EXEMPLUM.

Quæratur valor Functionis $\frac{1}{z^3(1-z)^2(1+z)}$ in forma ſimpliciſſima expreſſus. Sumatur primum Factor denominatoris foľtarius $1+z$, qui dat $\frac{p}{q} = -1$. erit $M = 1$ & $z = z^3 - 2z^4 + z^5$. Hinc ad fractionem $\frac{A}{1+z}$ inveniendam, erit $A = \frac{1}{z^3 - 2z^4 + z^5}$, poſito $z = -1$; ideoque fit $A =$

$—\frac{1}{4}$, atque ex Factore $1 + z$ oritur hæc fractio partialis $\frac{—1}{4(1+z)}$. Jam fumatur Factor quadratus $(1 — z)^2$ qui dat $\frac{p}{q} = 1$. $M = 1$, & $Z = z^3 + z^4$; pofitis èrgo fractioni-

bus partialibus hinc ortis $\frac{A}{(1-z)^2} + \frac{B}{1-z}$, erit $A =$

$\frac{1}{z^3 + z^4}$, pofito $z = 1$; ergo $A = \frac{1}{2}$; fiat $P = \dfrac{M — \frac{1}{2} Z}{1 — z}$

$= \dfrac{1 — \frac{1}{2} z^3 — \frac{1}{2} z^4}{1 — z} = 1 + z + zz + \frac{1}{2} z^3$; eritque $B =$

$\frac{P}{Z} = \dfrac{1 + z + zz + \frac{1}{2} z^3}{z^3 + z^4}$, pofito $z = 1$; ergo $B =$

$\frac{7}{4}$ & fractiones partiales quæfitæ $\frac{1}{2(1-z)^2} + \frac{7}{4(1-z)}$.

Denique tertius Factor cubicus z^3 dat $\frac{p}{q} = 0$; $M = 1$; & $Z = 1 — z — zz + z^3$. Pofitis ergo fractionibus partialibus his $\frac{A}{z^3} + \frac{B}{z^2} + \frac{C}{z}$; erit primum. $A = \frac{M}{N} = \frac{1}{1 — z — zz + z^3}$

pofito $z = 0$; ergo $A = 1$. Ponatur $P = \frac{M — Z}{z} = 1 + z — zz$, erit $B = \frac{P}{Z}$, pofito $z = 0$; ergo $B = 1$. Pona-

tur $Q = \frac{P — Z}{z} = 2 — zz$; erit $C = \frac{Q}{Z}$, pofito $z = 0$;

ergo $C = 2$. Hanc ob rem Functio propofita $\frac{1}{z^3(1-z)^2(1+z)}$

in hanc formam refolvitur $\frac{1}{z^3} + \frac{1}{z^2} + \frac{2}{z} + \frac{1}{2(1-z)^2}$

$+ \frac{7}{4(1-z)} — \frac{1}{4(1+z)}$: nulla enim pars integra infuper acce-

dit, quia fractio propofita non eft fpuria.

CAPUT

CAPUT III.

De transformatione Functionum per substitutionem.

46. **S**Si fuerit y *Functio quacunque ipsius* z, *atque* z *definiatur per novam variabilem* x, *tum quoque* y *per* x *definiri poterit.*

Cum ergo antea y fuisset Functio ipsius *z*, nunc nova quantitas variabilis *x* inducitur, per quam utraque priorum *y* & *z* definiatur. Sic, si fuerit $y = \frac{1-zz}{1+zz}$, atque ponatur $z = \frac{1-x}{1+x}$; hoc valore loco z substituto, erit $y = \frac{2x}{1+xx}$. Sumpto ergo pro *x* valore quocunque determinato, ex eo reperiuntur valores determinati pro *z* & *y*, sicque invenitur valor ipsius *y* respondens illi valori ipsius *z* qui simul prodiit. Uti si sit $x = \frac{1}{2}$, fiet $z = \frac{1}{3}$, & $y = \frac{4}{5}$; reperitur autem quoque $y = \frac{4}{5}$, si in $\frac{1-zz}{1+zz}$, cui expressioni y æquatur, ponatur $z = \frac{1}{3}$.

Adhibetur autem hæc novæ variabilis introductio ad duplicem finem: vel enim hoc modo irrationalitas, qua expressio ipsius *y* per *z* data laborat, tollitur; vel quando ob æquationem altioris gradus, qua relatio inter *y* & *z* exprimitur, non licet Functionem explicitam ipsius z ipsi *y* æqualem exhibere, nova variabilis *x* introducitur, ex qua utraque *y* & *z* commode definiri queat: unde insignis substitutionum usus jam satis elucet, ex sequentibus vero multo clarius perspicietur.

47. *Si fuerit* y $= \sqrt{(a+bz)}$; *nova variabilis* x *per quam utraque* z *&* y *rationaliter exprimatur, sequenti modo invenietur.*

Quoniam tam z quam y debet esse Functio rationalis ipsius *x*; perspicuum est hoc obtineri si ponatur $\sqrt{(a+bz)} = bx$: Fiet enim primo $y = bx$; & $a + bz = bbxx$; hincque $z = bxx - \frac{a}{b}$. Quocirca utraque quantitas y & z per Functionem rationalem ipsius *x* exprimitur; scilicet cum sit $y = \sqrt{(a+bz)}$ fiat $z = bxx - \frac{a}{b}$; erit $y = bx$.

48. *Si fuerit* $y = (a + bz)^{m:n}$; *nova variabilis* x, *per quam tam* y *quam* z *rationaliter exprimatur, sic reperietur.*

Ponatur $y = x^m$, fietque $(a + bz)^{m:n} = x^m$ ideoque

$(a + bz)^{1:n} = x$: ergo $a + bz = x^n$ & $z = \dfrac{x^n - a}{b}$. Sic

ergo utraque quantitas y & z rationaliter per x definietur, ope

scilicet substitutionis $z = \dfrac{x^n - a}{b}$, quæ præbet $y = x^m$. Quam-

vis igitur neque y per z, neque viciffim z per y rationaliter
exprimi poffit; tamen utraque reddita est Functio rationalis no-
væ quantitatis variabilis x per fubftitutionem introductæ, fcopo
fubftitutionis omnino convenienter.

49. *Si fuerit* $y = (\dfrac{a + bz}{f + gz})^{m:n}$; *requiritur nova quantitas va-
riabilis* x *per quam utraque* y *& z rationaliter exprimatur.*

Manifestum primo est fi ponatur $y = x^m$, quæfito fatisfieri;

erit enim $(\dfrac{a + bz}{f + gz})^{m:n} = x^m$, ideoque $\dfrac{a + bz}{f + gz} = x^n$; ex qua

æquatione elicitur $z = \dfrac{a - fx^n}{gx^n - b}$; quæ fubftitutio præbet $y = x^m$.

Hinc quoque intelligitur fi fuerit $(\dfrac{\alpha + \varepsilon y}{\gamma + \delta y})^n = (\dfrac{a + bz}{f + gz})^m$,

tam y quam z rationaliter per x expreffum iri, fi utraque formula

ponatur $= x^{mn}$; reperietur enim $y = \dfrac{\alpha - \gamma x^m}{\delta x^m - \varepsilon}$ & $z = \dfrac{a - fx^n}{gx^n - b}$:

qui cafus nil habent difficultatis.

50. *Si fuerit* $y = \sqrt{((a + bz)(c + dz))}$; *fubftitutio idonea
invenietur, qua* y *& z rationaliter exprimuntur, hoc modo.*

Ponatur $\sqrt{((a + bz)(c + dz))} = (a + bz)x$, facile enim
perfpicitur hinc valorem rationalem pro z effe proditurum;
quia valor ipfius z per æquationem fimplicem determinatur. Erit

LIB. I. ergo $c + dz = (a + bz) xx$, hincque $z = \frac{c - axx}{bxx - d}$. Quare

porro fiet $a + bz = \frac{bc - ad}{bxx - d}$; & ob $y = \surd((a+bz)(c+dz))$

$= (a+bz) x$ habebitur $y = \frac{(bc - ad) x}{bxx - d}$. Functio ergo

irrationalis $y = \surd((a+bz)(c+dz))$ ad rationalitatem per-

ducitur ope fubftitutionis $z = \frac{c - axx}{bxx - d}$, quippe quæ dabit

$y = \frac{(bc - ad) x}{bxx - d}$. Sic, fi fuerit $y = \surd(aa - zz) = \surd((a+z)$

$(a - z))$; ob $b = +1$; $c = a$, $d = -1$, ponatur $z =$

$\frac{a - axx}{1 + xx}$, eritque $y = \frac{2ax}{1 + xx}$. Quoties ergo quantitas poft

fignum $\surd$ habuerit duos Factores fimplices reales, hoc modo
reductio ad rationalitatem abfolvetur; fin autem Factores bini
fimplices fuerint imaginarii, fequenti modo uti præftabit.

51. *Sit* $y = \surd(p + qz + rzz)$; *atque requiritur fubftitutio
idonea pro z facienda, ut valor ipfius y fiat rationalis.*

Pluribus modis hoc fieri poteft, prout p & q fuerint quan-
titates vel affirmativæ vel negativæ. Sit primo p quantitas
affirmativa, ac ponatur aa pro p; etiamfi enim p non fit qua-
dratum, tamen irrationalitas quantitatum conftantium præfens
negotium non turbat. Sit igitur

I. $y = \surd(aa + bz + czz)$; ac ponatur $\surd(aa + bz + czz)$
$= a + xz$; erit $b + cz = 2ax + xxz$; unde fit $z =$

$\frac{b - 2ax}{xx - c}$: tum vero erit $y = a + xz = \frac{bx - axx - ac}{xx - c}$; ubi z &

y funt Functiones rationales ipfius x. Sit jam

II. $y = \surd(aazz + bz + c)$; ac ponatur $\surd(aazz + bz + c)$
$= az + x$; erit $bz + c = 2axz + xx$, & $z = \frac{xx - c}{b - 2ax}$. Tum

autem fit $y = az + x = \frac{-ac + bx - axx}{b - 2ax}$.

III. Si fuerint p & r quantitates negativæ; tum, nifi fit
$qq > 4pr$, valor ipfius y femper erit imaginarius. Quod fi au-
tem fuerit $qq > 4pr$; expreffio $p + qz + rzz$ in duos Factores
refolvi

refolvi poterit, qui cafus ad §. præced. reducitur. Sæpenume-
ro autem commodius ad hanc formam reducitur, $y = \sqrt{(aa}$
$+ (b + cz)(d + ez))$; pro qua ad rationalitatem perdu-
cenda ponatur $y = a + (b + cz)x$, eritque $d + ez = 2ax$
$+ bxx + cxxz$; unde fit $z = \dfrac{d - 2ax - bxx}{cxx - e}$, & $y =$
$\dfrac{- ae + (cd - be)x - acxx}{cxx - e}$. Interdum commodius fieri

poteft reductio ad hanc formam, $y = \sqrt{(aazz + (b + cz)}$
$(d + ez))$. Tum ponatur $y = az + (b + cz)x$; erit $d + ez$
$= 2axz + bxx + cxxz$ & $z = \dfrac{bxx - d}{e - 2ax - cxx}$, atque

$y = \dfrac{- ad + (be - cd)x - abxx}{e - 2ax - cxx}$.

EXEMPLUM.

Si habeatur ifta ipfius z Functio irrationalis $y = \sqrt{(- 1 + 3z}$
$- zz)$; quæ cum reduci queat ad hanc formam $y = \sqrt{(1 - 2}$
$+ 3z - zz) = \sqrt{(1 - (1 - z)(2 - z))}$; ponatur $y =$
$1 - (1 - z)x$, erit $- 2 + z = - 2x + xx - xxz$ &
$z = \dfrac{2 - 2x + xx}{1 + xx}$. Deinde eft $1 - z = \dfrac{- 1 + 2x}{1 + xx}$ &
$y = 1 - (1 - z)x = \dfrac{1 + x - xx}{1 + xx}$. Atque hi funt fere
cafus, quos Algebra indeterminata, feu methodus *Diophantæa*,
fuppeditat; neque alios cafus in his tractatis non comprehen-
fos per fubftitutionem rationalem ad rationalitatem reducere li-
cet. Quocirca ad alterum fubftitutionis ufum monftrandum
progredior.

52. *Si* y *ejufmodi fuerit Functio ipfius* z *ut fit* $ay^{\alpha} + bz^{\varsigma} + cy^{\gamma}$
$z^{\delta} = 0$, *invenire novam variabilem* x, *per quam valores ipfa-*
rum y & z *explicite affignari queant.*

Quoniam refolutio æquationum generalis non habetur, ex
æquatione propofita $ay^{\alpha} + bz^{\varsigma} + cy^{\gamma} z^{\delta} = 0$ neque y per
z neque

z neque viciſſim z per y exhiberi poteſt. Quo igitur huic incommodo remedium afferatur; ponatur $y = x^m z^n$, eritque $a x^{\alpha m} z^{\alpha n} + b z^{\mathsf{G}} + c x^{\gamma m} z^{\gamma n + \delta} = 0$. Determinetur nunc exponens n ita ut ex hac æquatione valor ipſius z definiri queat: quod tribus modis præſtari poteſt.

I. Sit $\alpha n = \mathsf{G}$; ideoque $n = \dfrac{\mathsf{G}}{\alpha}$; erit, æquatione per $z^{\alpha n} = z^{\mathsf{G}}$ diviſa, $a x^{\alpha m} + b + c x^{\gamma m} z^{\gamma n - \mathsf{G} + \delta} = 0$; unde oritur $z = \left(\dfrac{- a x^{\alpha m} - b}{c x^{\gamma m}} \right)^{\frac{1}{\gamma n - \mathsf{G} + \delta}}$, ſive

$$z = \left(\dfrac{- a x^{\alpha m} - b}{c x^{\gamma m}} \right)^{\frac{\alpha}{\mathsf{G}\gamma - \alpha\mathsf{G} + \alpha\delta}}, \, \&$$

$$y = x^m \left(\dfrac{- a x^{\alpha m} - b}{c x^{\gamma m}} \right)^{\frac{\mathsf{G}}{\mathsf{G}\gamma - \alpha\mathsf{G} + \alpha\delta}}$$

II. Sit $\mathsf{G} = \gamma n + \delta$ ſeu $n = \dfrac{\mathsf{G} - \delta}{\gamma}$; erit, æquatione per z^{G} diviſa, $a x^{\alpha m} z^{\alpha n - \mathsf{G}} + b + c x^{\gamma m} = 0$; unde oritur

$$z = \left(\dfrac{- b - c x^{\gamma m}}{a x^{\alpha m}} \right)^{\frac{1}{\alpha n - \mathsf{G}}} = \left(\dfrac{- b - c x^{\gamma m}}{a x^{\alpha m}} \right)^{\frac{\gamma}{\alpha\mathsf{G} - \alpha\delta - \mathsf{G}\gamma}},$$

atque $y = x^m \left(\dfrac{- b - c x^{\gamma m}}{a x^{\alpha m}} \right)^{\frac{\mathsf{G} - \delta}{\alpha\mathsf{G} - \alpha\delta - \mathsf{G}\gamma}}.$

III. Sit $\alpha n = \gamma n + \delta$, ſeu $n = \dfrac{\delta}{\alpha - \gamma}$; erit, æquatione per $z^{\alpha n}$ diviſa, $a x^{\alpha m} + b z^{\mathsf{G} - \alpha n} + c x^{\gamma m} = 0$; unde oritur

$$z = \left(\dfrac{- a x^{\alpha m} - c x^{\gamma m}}{b} \right)^{\frac{1}{\mathsf{G} - \alpha n}} =$$

$$\left(\frac{- a\, x^{\alpha m} - c\, x^{\gamma m}}{b} \right)^{\overline{\dfrac{\alpha - \gamma}{\alpha \mathit{6} - \mathit{6}\gamma - \alpha \delta}}}; \text{ atque}$$

$$y = x^m \left(\frac{- a\, x^{\alpha m} - c\, x^{\gamma m}}{b} \right)^{\overline{\dfrac{\delta}{\alpha \mathit{6} - \mathit{6}\gamma - \alpha \delta}}}.$$

Tribus igitur diverfis modis erutæ funt Functiones ipfius x, quæ ipfis z & y funt æquales. Præterea vero pro m numerum pro lubitu fubftituere licet cyphra excepta; ficque formulæ ad commodiffimam expreffionem reduci poterunt.

E X E M P L U M.

Exprimatur natura Functionis y per hanc æquationem $y^3 + z^3 - c\,y\,z = 0$; atque quærantur Functiones ipfius x ipfis y & z æquales. Erit ergo $a = -1$; $b = -1$; $\alpha = 3$; $\mathit{6} = 3$; $\gamma = 1$; & $\delta = 1$. Hinc primus modus dabit, pofito $m = 1$,

$$z = \left(\frac{x^3 + 1}{c\,x} \right)^{-1} \quad \& \quad y = x \left(\frac{x^3 + 1}{c\,x} \right)^{-1}, \text{ five } z = \frac{c\,x}{1 + x^3} \quad \&$$

$$y = \frac{c\,x\,x}{1 + x^3};$$ quarum expreffionum utraque adeo eft rationalis.

Secundus modus vero dabit hos valores:

$$z = \left(\frac{c\,x - 1}{x^3} \right)^{1:3}, \quad \& \quad y = x \left(\frac{c\,x - 1}{x^3} \right)^{2:3}, \text{ five}$$

$$z = \frac{1}{x} \sqrt[3]{(c\,x - 1)}, \quad \& \quad y = \frac{1}{x} \sqrt[3]{(c\,x - 1)^2}.$$

Tertius modus ita rem expediet ut fit

$$z = (c\,x - x^3)^{2:3}, \quad \& \quad y = x (c\,x - x^3)^{1:3}.$$

53. *Hinc a pofteriori intelligitur cujufmodi æquationes, quibus valor Functionis* y *per* z *determinatur, hoc modo novam variabilem* x *introducendo refolvi queant.*

Ponamus enim refolutione jam inftituta prodiiffe has deter-

    mina-

minationes $z = \left(\dfrac{a x^\alpha + b x^6 + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.} \right)^{p \, : \, r}$, atque $y = x$

$\left(\dfrac{a x^\alpha + b x^6 + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.} \right)^{q \, : \, r}$; eritque $y^p = x^p z^q$; & hinc

$x = y z^{- q : p}$. Cum igitur fit $z^{r : p} = \dfrac{a x^\alpha + b x^6 + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.}$, fi

loco x ejus valorem $y z^{- q : p}$ fubftituamus; prodibit ifta æ-

quatio $z^{r : p} = \dfrac{a y^\alpha z^{- \alpha q : p} + b y^6 z^{- 6 q : p} + c y^\gamma z^{- \gamma q : p} + \&c.}{A + B y^\mu z^{- \mu q : p} + C y^\nu z^{- \nu q : p} + \&c.}$;

quæ reducitur ad hanc $A z^{r : p} + B y^\mu z^{(r - \mu q) : p} + C y^\nu$ $z^{(r - \nu q) : q} + \&c. = a y^\alpha z^{- \alpha q : p} + b y^6 z^{- 6 q : p}$ $+ c y^\gamma z^{- \gamma q : p} + \&c.$ quæ multiplicata per $z^{\alpha q : p}$ tranfibit

in hanc: $A z^{(\alpha q + r) : p} + B y^\mu z^{(\alpha q - \mu q + r) : p} + C y^\nu z^{(\alpha q - \nu q + r) : p}$ $+ \&c. = a y^\alpha + b y^6 z^{(\alpha q - 6 q) : p} + c y^\gamma z^{(\alpha q - \gamma q) : p} + \&c.$

Ponatur $\dfrac{\alpha q + r}{p} = m$ & $\dfrac{\alpha q - 6 q}{p} = n$: fiet $p = \alpha - 6$;

$q = n$, & $r = \alpha m - 6 m - \alpha n$: atque nafcetur ifta æquatio:

$A z^m + B y^\mu z^{m - \mu n : (\alpha - \beta)} + C y^\nu z^{m - \nu n : (\alpha - \beta)} + \&c. =$ $a y^\alpha + b y^6 z^n + c y^\gamma z^{(\alpha - \gamma) n : (\alpha - \beta)} + \&c.$ quæ propterea ita refolvetur ut fit:

$z = \left(\dfrac{a x^\alpha + b x^6 + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.} \right)^{\frac{\alpha - 6}{\alpha m - 6 m - \alpha n}}$ &

$y = x \left(\dfrac{a x^\alpha + b x^6 + c x^\gamma + \&c.}{A + B x^\mu + C x^\nu + \&c.} \right)^{\frac{n}{\alpha m - 6 m - \alpha n}}$.

Vel ponatur $\dfrac{\alpha q + r}{p} = m$, & $\dfrac{\alpha q - \mu q + r}{p} = n$, erit $m - n$

$=$

$= \frac{\mu q}{p}$; & $\frac{q}{p} = \frac{m - n}{\mu}$, atque $\frac{r}{p} = m - \frac{\alpha m + \alpha n}{\mu}$. Hinc

fit $p = \mu$; $q = m - n$; & $r = \mu m - \alpha m + \alpha n$; atque hæc æquatio refultabit :

$$Az^m + By^\mu z^n + Cy^\nu z^{\mu m - \nu(m - n):\mu} + \&c. = ay^\alpha$$
$$+ by^\epsilon z^{(\alpha - \epsilon)(m - n):\mu} + cy^\gamma z^{(\alpha - \gamma)(m - n):\mu}$$
$$+ \&c. \text{ quæ ita refolvetur ut fit :}$$

$$z = \left(\frac{ax^\alpha + bx^\epsilon + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{\frac{\mu}{m - \alpha m + \alpha n}} \&$$

$$y = x \left(\frac{ax^\alpha + bx^\epsilon + cx^\gamma + \&c.}{A + Bx^\mu + Cx^\nu + \&c.} \right)^{\frac{m - n}{\mu m - \alpha m + \alpha n}}$$

54. *Si* y *ita pendeat a* z *ut fit* $ayy + byz + czz + dy + ez = 0$, *fequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, erit divifione per z facta :
$axxz + bxz + cz + dx + e = 0$, ex qua reperitur
$z = \frac{-dx - e}{axx + bx + c}$, & $y = \frac{-dxx - ex}{axx + bx + c}$.

At vero ad formam propofitam reduci poteft hæc æquatio inter y & z: $ayy + byz + czz + dy + ez + f = 0$ diminuendo vel augendo utramque variabilem certa quadam quantitate conftante, unde & hæc æquatio per novam variabilem x rationaliter explicari poteft.

55. *Si* y *ita pendeat a* z, *ut fit* $ay^3 + by^2z + cyz^2 + dz^3 + eyy + fyz + gzz = 0$; *fequenti modo tam* y *quam* z *rationaliter per novam variabilem* x *exprimi poterit.*

Ponatur $y = xz$, & facta fubftitutione tota æquatio per zz dividi poterit: prodibit autem $ax^3z + bxxz + cxz + dz + exx + fx + g = 0$. Unde oritur $z = \frac{-exx - fx - g}{ax^3 + bxx + cx + d}$

ex quo erit $y = \frac{-ex^3 - fxx - gx}{ax^3 + bxx + cx + d}$.

Ex his cafibus facile intelligitur quemadmodum æquationes altiorum graduum, quibus y per z definitur, comparatæ effe debeant, ut hujufmodi refolutio locum habere queat. Ceterum hi cafus in fuperioribus formulis §. 53. continentur: at, quia formulæ generales non tam facile ad hujufmodi cafus fæpius occurrentes accommodantur, vifum eft horum aliquos feorfim evolvere.

56. *Si* y *ita pendeat a* z *ut fit* $ayy + byz + czz = d$ *hoc modo utraque quantitas* y *&* z *per novam variabilem* x *exprimetur.*

Ponatur $y = xz$, eritque $(axx + bx + c)zz = d$, ideoque $z = \sqrt{\dfrac{d}{axx + bx + c}}$ & $y = x\sqrt{\dfrac{d}{axx + bx + c}}$.

Simili modo fi fuerit, $ay^3 + by^2z + cyz^2 + dz^3 = ey + fz$; pofito $y = xz$, tota æquatio per z divifa dabit $(ax^3 + bxx + cx + d)zz = ex + f$; unde oritur $z = \sqrt{\dfrac{ex+f}{ax^3 + bxx + cx + d}}$; & $y = x\sqrt{\dfrac{ex+f}{ax^3 + bxx + cx + d}}$. Hi autem cafus aliique fimiles refolutiones admittentes comprehenduntur in fequente paragrapho.

57. *Si* y *ita pendeat a* z *ut fit* $ay^m + by^{m-1}z + cy^{m-2}z^2 + dy^{m-3}z^3 + \&c. = \alpha y^n + 6y^{n-1}z + \gamma y^{n-2}z^2 + \delta y^{n-3}z^3 + \&c.$ *Sequenti modo tam* z *quam* y *commode per novam variabilem* x *exprimetur.*

Sit $y = xz$, atque facta fubftitutione tota æquatio dividi poterit per z^n, fiquidem exponens m fit major quam n; eritque $(ax^m + bx^{m-1} + cx^{m-2} + \&c.)z^{m-n} = \alpha x^n + 6x^{n-1} + \gamma x^{n-2} + \&c.$ unde obtinebitur

$$z =$$

$$z = \left(\frac{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{a x^m + b x^{m-1} + c x^{m-2} + d x^{m-3} + \&c.} \right)^{1:(m-n)} \&$$

$$y = x \left(\frac{\alpha x^n + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3} + \&c.}{a x^m + b x^{m-1} + c x^{m-2} + d x^{m-3} + \&c.} \right)^{1:(m-n)}$$

Hæc scilicet resolutio locum habet, si in æquatione naturam Functionis y per z exprimente, duplex tantum ubique occurrit dimensionum ab y & z sumptarum numerus; uti in casu tractato in singulis terminis numerus dimensionum vel est m vel n.

58. *Si in æquatione inter* y *&* z *triplicis generis dimensiones occurrant, quarum summa tantum superet mediam, quantum hæc media infimam, ope resolutionis æquationis quadratæ variabiles* y *&* z *per novam* x *exprimi poterunt.*

Si enim ponatur $y = x z$, divisione per minimam ipsius z potestatem facta, valor ipsius z per x, ope extractionis radicis quadratæ exhibebitur, id quod ex sequentibus exemplis erit manifestum.

E X E M P L U M I.

Sit $a y^3 + b y^2 z + c y z z + d z^3 = 2 e y y + 2 f y z + 2 g z z + h y + i z$; ac ponatur $y = x z$: erit, divisione per z facta, $(a x^3 + b x x + c x + d) z z = 2 (e x x + f x + g) z + h x + i$; ex qua sequens ipsius z obtinebitur valor:

$$z = \frac{e x x + f x + g \pm \sqrt{((e x x + f x + g)^2 + (a x^3 + b x x + c x + d)(h x + i))}}{a x^3 + b x x + c x + d}$$

quo invento erit $y = x z$.

E X E M P L U M II.

Sit $y^5 = 2 a z^3 + b y + c z$; ac, posito $y = x z$, erit $x^5 z^4 = 2 a z z + b x + c$; ex qua reperitur $z z = \frac{a \pm \sqrt{(a a + b x^6 + c x^5)}}{x^5}$; &

$$z = \frac{\sqrt{(a \pm \sqrt{(a a + b x^6 + c x^5)})}}{x x \sqrt{x}} \quad \& \quad y = \frac{\sqrt{(a \pm \sqrt{(a a + b x^6 + c x^5)})}}{x \sqrt{x}}.$$

E X E M-

E X E M P L U M   III.

Sit $y^{10} = 2ayz^6 + byz^3 + cz^4$, in qua cum dimensiones sint 10, 7, & 4, ponatur $y = xz$; atque æquatio per z^4 divisa abibit in hanc : $x^{10} z^6 = 2axz^3 + bx + c$ seu $z^6 = \dfrac{2axz^3 + bx + c}{x^{10}}$;

unde invenitur $z^3 = \dfrac{ax \pm x \sqrt{(aa + bx^9 + cx^8)}}{x^{10}}$; ideoque

erit $z = \dfrac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^3}$; atque $y =$

$\dfrac{\sqrt[3]{(a \pm \sqrt{(aa + bx^9 + cx^8)})}}{x^2}$. Ex quibus exemplis usus hujusmodi substitutionum abunde perspicitur.

C A P U T   I V.

De explicatione Functionum per series infinitas.

59. CUm Functiones fractæ atque irrationales ipsius z non in forma integra $A + Bz + Cz^2 + Dz^3 +$ &c. continentur, ita ut terminorum numerus sit finitus, quæri solent hujusmodi expressiones in infinitum excurrentes, quæ valorem cujusvis Functionis sive fractæ sive irrationalis exhibeant. Quin etiam natura Functionum transcendentium melius intelligi censetur, si per ejusmodi formam, etsi infinitam, exprimantur. Cum enim natura Functionis integræ optime perspiciatur, si secundum diversas potestates ipsius z explicetur, atque adeo ad formam $A + Bz + Cz^2 + Dz^3 +$ &c. reducatur, ita eadem forma aptissima videtur ad reliquarum Functionum omnium indolem menti repræsentandam, etiamsi terminorum numerus sit revera infinitus. Perspicuum autem est nullam Functionem non integram ipsius z per numerum hujusmodi terminorum $A + Bz + Cz^2 +$ &c. finitum exponi posse; eo ipso enim

Functio

Functio foret integra; num vero per hujufmodi terminorum feriem infinitam exhiberi poffit, fi quis dubitet, hoc dubium per ipfam evolutionem cujufque Functionis tolletur. Quo autem hæc explicatio latius pateat, præter poteftates ipfius z exponentes integros affirmativos habentes, admitti debent poteftates quæcunque. Sic dubium erit nullum quin omnis Functio ipfius z in hujufmodi expreffionem infinitam tranfmutari poffit:

$$A z^{\alpha} + B z^{\mathsf{G}} + C z^{\gamma} + D z^{\delta} + \&c.$$ denotantibus exponentibus α, G, γ, δ, &c. numeros quofcunque.

60. *Per divifionem autem continuam intelligitur fractionem* $\dfrac{a}{\alpha + \mathsf{G} z}$ *refolvi in hanc feriem infinitam* $\dfrac{a}{\alpha} - \dfrac{a \mathsf{G} z}{\alpha^2} + \dfrac{a \mathsf{G}^2 z^2}{\alpha^3} - \dfrac{a \mathsf{G}^3 z^3}{\alpha^4} + \dfrac{a \mathsf{G}^4 z^4}{\alpha^5} - \&c.$, *quæ, cum quilibet terminus ad fequentem habeat rationem conftantem* $1 : \dfrac{\mathsf{G} z}{\alpha}$, *vocatur feries geometrica.*

Poteft vero quoque hæc feries ita inveniri, ut ipfa initio pro incognita habeatur : ponatur enim $\dfrac{a}{\alpha + \mathsf{G} z} = A + B z + C z^2 + D z^3 + E z^4 + \&c.$ atque ad æqualitatem producendam quærantur coëfficientes A, B, C, D, &c. Erit ergo $a = (\alpha + \mathsf{G} z)(A + B z + C z^2 + D z^3 + \&c.)$, & multiplicatione actu peracta fiet

$$a = \alpha A + \alpha B z + \alpha C z^2 + \alpha D z^3 + \alpha E z^4 + \&c.$$
$$\qquad + \mathsf{G} A z + \mathsf{G} B z^2 + \mathsf{G} C z^3 + \mathsf{G} D z^4 + \&c.$$

Quamobrem effe debet $a = \alpha A$, ideoque $A = \dfrac{a}{\alpha}$, & coëfficientium uniufcujufque poteftatis ipfius z fumma nihilo æqualis eft ponenda : unde prodibunt hæ æquationes,

$\alpha B + \mathsf{G} A = 0$ cognito ergo quovis coëfficiente
$\alpha C + \mathsf{G} B = 0$ facile reperitur fequens; fi enim
$\alpha D + \mathsf{G} C = 0$ fuerit coëfficiens termini cujufque $= P$
$\alpha E + \mathsf{G} D = 0$ & fequens $= Q$ erit $\alpha Q + \mathsf{G} P = 0$
     &c.   five $Q = \dfrac{\mathsf{G} P}{\alpha}$.

Cum

LIB. I. Cum igitur terminus primus A sit determinatus $= \frac{a}{\alpha}$ ex eo sequentes litteræ B, C, D, &c. definiuntur eodem modo, quo ex divisione sunt orti. Ceterum ex inspectione perspicitur in serie infinita pro $\frac{a}{\alpha + \mathfrak{b} z}$ inventa potestatis z^n coëfficientem fore $= \pm \frac{a \mathfrak{b}^n}{\alpha^{n+1}}$, ubi signum $+$ locum habet si n sit numerus par, signum $-$ autem si n sit numerus impar: seu coëfficiens erit $= \frac{a}{\alpha} \left(\frac{-\mathfrak{b}}{\alpha} \right)^n$.

61. *Simili modo ope divisionis continuatæ hæc Functio fracta* $\frac{a + b z}{\alpha + \mathfrak{b} z + \gamma z z}$ *in seriem infinitam converti potest.*

Cum autem divisio sit tædiosa, neque tam facile naturam seriei infinitæ ostendat, commodius erit seriem quæsitam fingere, atque modo ante tradito determinare. Sit igitur

$$\frac{a + b z}{\alpha + \mathfrak{b} z + \gamma z z} = A + B z + C z^2 + D z^3 + E z^4 + \&c.$$

multiplicetur utrinque per $\alpha + \mathfrak{b} z + \gamma z z$, atque fiet

$$a + b z = \alpha A + \alpha B z + \alpha C z^2 + \alpha D z^3 + \alpha E z^4 + \&c.$$
$$+ \mathfrak{b} A z + \mathfrak{b} B z^2 + \mathfrak{b} C z^3 + \mathfrak{b} D z^4 + \&c.$$
$$+ \gamma A z^2 + \gamma B z^3 + \gamma C z^4 + \&c.$$

Hinc erit $\alpha A = a$; $\alpha B + \mathfrak{b} A = b$; unde reperitur $A = \frac{a}{\alpha}$ & $B = \frac{b}{\alpha} - \frac{a \mathfrak{b}}{\alpha \alpha}$; reliquæ vero litteræ ex sequentibus æquationibus determinabuntur:

$\alpha C + \mathfrak{b} B + \gamma A = 0$ hinc ergo ex binis quibusque coëfficientibus contiguis sequens reperitur. Sic si duo coëfficientes contigui fuerint P, Q & sequens R, erit $\alpha R + \mathfrak{b} Q + \gamma P = 0$ seu $R = \frac{-\mathfrak{b} Q - \gamma P}{\alpha}$.

$\alpha D + \mathfrak{b} C + \gamma B = 0$

$\alpha E + \mathfrak{b} D + \gamma C = 0$

$\alpha F + \mathfrak{b} E + \gamma D = 0$

&c.

Cum igitur duæ litteræ primæ A & B jam sint inventæ sequentes C, D, E, F &c. omnes successive ex iis invenientur,

tur, sicque reperietur Series infinita $A + Bz + Cz^2 + Dz^3 + $&c.
Functioni fractæ propositæ $\dfrac{a + bz}{\alpha + \mathfrak{b}z + \gamma zz}$ æqualis.

EXEMPLUM.

Si fuerit proposita hæc fractio $\dfrac{1 + 2z}{1 - z - zz}$, huicque æqualis
statuatur Series $A + Bz + Cz^2 + Dz^3 + $ &c. ob $a = 1$;
$b = 2$; $\alpha = 1$; $\mathfrak{b} = - 1$; $\gamma = - 1$; erit $A = 1$; $B = 3$;
tum vero erit

$C = B + A$ quilibet ergo coëfficiens æqualis est sum-
$D = C + B$ mæ duorum præcedentium; quare si co-
$E = D + C$ gniti fuerint duo coëfficientes contigui
$F = E + D$ P & Q, erit sequens $R = P + Q$.
&c.

Cum igitur duo coëfficientes primi A & B sint cogniti, fractio
proposita $\dfrac{1 + 2z}{1 - z - zz}$ in hanc Seriem infinitam transmutatur
$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + $ &c., quæ nullo nego-
tio quousque libuerit continuari potest.

62. Ex his jam satis intelligitur indoles Serierum infinitarum,
in quas Functiones fractæ transmutantur; tenent enim ejusmodi
legem, ut quilibet terminus ex aliquot præcedentibus deter-
minari possit. Scilicet, si denominator fractionis propositæ fue-
rit $\alpha + \mathfrak{b}z$, atque Series infinita statuatur

$$A + Bz + Cz^2 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + Sz^{n+3} + \&c.;$$

quilibet coëfficiens Q ex præcedente P solo ita definietur ut
sit $\alpha Q + \mathfrak{b}P = 0$. Sin denominator fuerit trinomium $\alpha + \mathfrak{b}z + $
γzz, quilibet coëfficiens Seriei R ex duobus præcedenti-
bus Q & P ita definietur ut sit $\alpha R + \mathfrak{b}Q + \gamma P = 0$: simili
modo si denominator fuerit quadrinomium, ut $\alpha + \mathfrak{b}z + \gamma zz + $
δz^3, quilibet coëfficiens seriei S ex tribus antecedentibus
R, Q & P ita determinabitur, ut sit $\alpha S + \mathfrak{b}R + \gamma Q + \delta P = 0$,

Euleri *Introduct. in Anal. infin. parv.*   G   sicque

L I B. I. ficque de ceteris.   In his ergo Seriebus quilibet terminus determinatur ex aliquot antecedentibus fecundum legem quandam conftantem , quæ lex ex denominatore fractionis hanc Seriem producentis fponte apparet. Vocari autem hæ Series a Celeb. M O I V R Æ O , qui earum naturam maxime eft fcrutatus , folent *recurrentes* , propterea quod ad terminos antecedentes eft recurrendum , fi fequentes inveftigare velimus.

63. Ad harum porro Serierum formationem requiritur ut denominatoris terminus conftans α non fit $= 0$: cum enim inventus fit terminus Seriei primus $A = \frac{a}{\alpha}$, tum is , tum omnes fequentes fierent infiniti , fi effet $\alpha = 0$. Hoc ergo cafu exclufo , quem deinceps evolvam , Functio fracta in Seriem infinitam recurrentem tranfmutanda , hujufmodi habebit formam $\frac{a + b z + c z^2 + d z^3 + \&c.}{1 - \alpha z - 6 z^2 - \gamma z^3 - \delta z^4 - \&c.}$; ubi primum denominatoris terminum pono $= 1$, huc enim femper fractio reduci poteft , nifi is fit $= 0$; reliquos autem denominatoris terminos omnes tanquam negativos contemplor , ut Seriei hinc formatæ omnes termini fiant affirmativi.   Quod fi enim Series recurrens hinc orta ponatur $A + B z + C z^2 + D z^3 + E z^4 + \&c.$ coëfficientes ita determinabuntur ut fit

$$A = a$$
$$B = \alpha A + b$$
$$C = \alpha B + 6 A + c$$
$$D = \alpha C + 6 B + \gamma A + d$$
$$E = \alpha D + 6 C + \gamma B + \delta A + e$$
$$\&c.$$

Quilibet ergo coëfficiens æqualis eft aggregato ex multiplis aliquot præcedentium una cum numero quodam , quem numerator præbet. Nifi autem numerator in infinitum progrediatur , hæc additio mox ceffabit , atque quivis terminus fecundum legem conftantem ex aliquot præcedentibus determinabitur.   Ne ergo lex progreffionis ufquam turbetur conveniet

Functio-

Functionem fractam genuinam adhibere: fi enim fractio fpuria C A P.IV.
accipiatur, tum pars integra in ea contenta ad Seriem accedet,
atque in illis terminis, quos vel auget vel minuit, legem pro-
greffionis interrumpet. Exempli gratia hæc fractio fpuria

$\frac{1 + 2z - z^3}{1 - z - zz}$, præbebit hanc Seriem $1 + 3z + 4zz + 6z^3$
$+ 10z^4 + 16z^5 + 26z^6 + 42z^7 + $ &c. ubi a lege, qua quivis
coëfficiens eft fumma duorum præcedentium, terminus quartus
$6z^3$ excipitur.

64. Peculiarem contemplationem Series recurrentes meren-
tur, fi denominator fractionis, unde oriuntur, fuerit poteftas.
Sic, fi ista fractio $\frac{a + bz}{(1 - \alpha z)^2}$ in Seriem refolvatur, prodit

$$a \pm 2\alpha a_z \pm 3\alpha^2 a_{z^2} + 4\alpha^3 a_{z^3} \pm 5\alpha^4 a_{z^4} + \text{&c.}$$
$$\pm b + 2\alpha b + 3\alpha^2 b + 4\alpha^3 b$$

in qua coëfficiens poteftatis z^n erit $(n + 1)\alpha^n a + n\alpha^{n-1} b$. Erit
tamen hæc Series recurrens, quia quilibet terminus ex duobus
præcedentibus determinatur, cujus determinationis lex perfpi-
citur ex denominatore evoluto $1 - 2\alpha z + \alpha\alpha zz$. Si pona-
tur $\alpha = 1$ & $z = 1$, abit Series in progreffionem arithmeticam
generalem $a + (2a + b) + (3a + 2b) + (4a + 3b) + $ &c.
cujus differentiæ funt conftantes. Omnis ergo progreffio a-
rithmetica eft Series recurrens: fi enim fit
$A + B + C + D + E + F + $ &c. progreffio arithmetica, erit
$C = 2B - A; D = 2C - B; E = 2D - C$, &c.

65. Deinde hæc fractio $\frac{a + bz + czz}{(1 - \alpha z)^3}$ ob $\frac{1}{(1 - \alpha z)^3} = (1 - \alpha z)^{-3}$
$= 1 + 3\alpha z + 6\alpha^2 z^2 + 10\alpha^3 z^3 + 15\alpha^4 z^4 + $ &c. trans-
mutabitur in hanc Seriem infinitam:

$$a + 3\alpha a_z + 6\alpha^2 a + 10\alpha^3 a + 15\alpha^4 a + \text{&c.}$$
$$\pm b + 3\alpha b z^2 + 6\alpha^2 b z^3 + 10\alpha^3 b z^4$$
$$+ c + 3\alpha c + 6\alpha^2 c$$

in

LIB. I. in qua poteſtas z^n coëfficientem habebit $\frac{(n+1)(n+2)}{1.\quad 2}\, \alpha^n\, a +$

$\frac{n(n+1)}{1.2}\, \alpha^{n-1}\, b + \frac{(n-1)n}{1.\quad 2}\, \alpha^{n-2}\, c.$ Quod ſi autem

ponatur $\alpha = 1$ & $z = 1$, Series hæc abibit in progreſſionem generalem ſecundi ordinis, cujus differentiæ ſecundæ ſunt conſtantes. Deſignet $A + B + C + D + E +$ &c. hujuſmodi progreſſionem, erit ea ſimul Series recurrens, cujus quilibet terminus ex tribus antecedentibus ita determinatur ut ſit $D = 3C - 3B + A$; $E = 3D - 3C + B$; $F = 3E - 3D + C$ &c. Cum igitur terminorum in progreſſione arithmetica procedentium ſecundæ differentiæ quoque ſint æquales, nempe $= 0$, hæc proprietas quoque ad progreſſiones arithmeticas extenditur.

66. Simili modo hæc fractio $\frac{a + bz + czz + dz^3}{(1 - \alpha z)^4}$ dabit

Seriem infinitam, in qua poteſtas ipſius z quæcunque z^n hunc habebit coëfficientem $\frac{(n+1)(n+2)(n+3)}{1.\quad 2.\quad 3}\, \alpha^n\, a +$

$\frac{n(n+1)(n+2)}{1.\quad 2.\quad 3}\, \alpha^{n-1}\, b + \frac{(n-1)n(n+1)}{1.\quad 2.\quad 3}\, \alpha^{n-2}\, c +$

$\frac{(n-2)(n-1)n}{1.\quad 2.\quad 3}\, \alpha^{n-3}\, d$: poſito ergo $\alpha = 1$ & $z = 1$;

hæc Series in ſe complectetur omnes progreſſiones algebraicas tertii ordinis, quarum differentiæ tertiæ ſunt conſtantes: omnes ergo hujus ordinis progreſſiones, cujuſmodi ſit $A + B + C + D + E + F +$ &c. erunt ſimul recurrentes ex denominatore $1 - 4z + 6zz - 4z^3 + z^4$ ortæ ; unde erit $E = 4D - 6C + 4B - A$; $F = 4E - 6D + 4C - B$ &c., quæ proprietas ſimul in omnes progreſſiones inferiorum ordinum competit.

67. Hoc modo oſtendentur omnes progreſſiones algebraicas cujuscunque ordinis, quæ tandem ad differentias conſtantes deducunt, eſſe Series recurrentes, quarum lex definiatur ex denominatore $(1 - z)^n$, exiſtente n numero majore quam is, qui ordinem progreſſionis indicat. Cum igitur $a''' + (a+b)''' +$

$(a+2b)$

$(a + 2b)^{m} + (a + 3b)^{m} +$ &c. exhibeat progreffio-
nem ordinis m; erit ob naturam Serierum recurrentium

$$0 = a^{m} - \frac{n}{1}(a+b)^{m} + \frac{n(n-1)}{1.\ 2}(a+2b)^{m}$$

$$- \frac{n(n-1)(n-2)}{1.\ \ 2.\ \ 3}(a+3b)^{m} + \cdots\cdots\cdots \pm$$

$\frac{n}{1}(a+(n-1)b)^{m} \mp (a+nb)^{m}$; ubi figna fuperiora valent
fi n fit numerus par, inferiora autem fi n fit numerus impar.
Hæc ergo æquatio femper eft vera fi fuerit n numerus integer
major quam m. Hinc ergo intelligitur quam late pateat doc-
trina de Seriebus recurrentibus.

68. Si denominator fuerit poteftas non binomii fed multinomii,
natura Seriei quoque alio modo explicari poteft. Sit nempe hæc

fractio $\dfrac{1}{(1 - \alpha z - 6z^2 - \gamma z^3 - \delta z^4 - \&c.\ )^{m+1}}$ pro-

pofita, erit Series infinita hinc nata

$$1 + \frac{(m+1)}{1}\alpha z + \frac{(m+1)(m+2)}{1.\ \ 2}\alpha^2 + \frac{(m+1)(m+2)(m+3)}{1.\ \ 2.\ \ 3}\alpha^3$$

$$+ \frac{(m+1)}{1}6z^2 + \frac{(m+1)(m+2)}{1.\ \ 2}2\alpha 6z^3 + \&c.$$

$$+ \frac{(m+1)}{1}\gamma$$

Ad naturam hujus Seriei penitius infpiciendam, exponatur hæc
Series per litteras generales hoc modo:

$$1 + Az + Bz^2 + Cz^3 + \ldots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} +$$

$N z^{n} +$ &c., ac quilibet coëfficiens N ex tot procedentibus,
quot funt litteræ $\alpha. 6. \gamma. \delta.$ &c. ita determinabitur ut fit: N

$$= \frac{m+n}{n}\alpha M + \frac{2m+n}{n}6L + \frac{3m+n}{n}\gamma K + \frac{4m+n}{n}\delta I + \&c.$$

quæ lex continuationis etfi non eft conftans, fed ab exponen-
te poteftatis z pendet, tamen eidem Seriei alia convenit lex
progreffionis conftans, quam denominator evolutus præbet, na-

tu__æ

LIB. I. turæ Serierum recurrentium confentaneam. Illa vero lex non
conftans tantum locum habet fi numerator fractionis fuerit uni-
tas feu quantitas conftans ; fi enim quoque aliquot poteftates
ipfius z contineret, tum illa lex multo magis fieret complicata,
id quod poft tradita calculi differentialis principia facilius pa-
tebit.

69. Quoniam hactenus pofuimus primum denominatoris ter-
minum conftantem non effe $= 0$, ejufque loco unitatem col-
locavimus; nunc videamus cujufmodi Series oriantur, fi in
denominatore terminus conftans evanefcat. His cafibus ergo
Functio fracta hujufmodi formam habebit

$\frac{a + bz + czz + \&c.}{z(1 - \alpha z - 6zz - \&c.)}$, convertatur ergo, neglecto deno-

minatoris Factore z, reliqua fractio $\frac{a + bz + czz + \&c.}{1 - \alpha z - 6z^2 - \&c.}$ in Se-

riem recurrentem $A + Bz + Cz^2 + Dz^3 + \&c.$ atque

manifeftum eft fore $\frac{a + bz + czz + \&c.}{z(1 - \alpha z - 6z^2 - \gamma z^3 - \&c.)} = \frac{A}{z} + B + Cz$

$+ Dz^2 + Ez^3 + \&c.$ Simili modo erit $\frac{a + bz + cz^2 + \&c.}{z^2(1 - \alpha z - 6z^2 - \&c.)}$

$= \frac{A}{zz} + \frac{B}{z} + C + Dz + Ez^2 + \&c.$, atque generatim erit

$\frac{a + bz + cz^2 + \&c.}{z^m(1 - \alpha z - 6z^2 - \gamma z^3 - \&c.)} = \frac{A}{z^{-m}} + \frac{B}{z^{m-1}} + \frac{C}{z^{m-2}} + $

$\frac{D}{z^{m-3}} + \&c.$ quicunque numerus fuerit exponens m.

70. Quoniam per fubftitutionem loco z alia variabilis x in
Functionem fractam introduci, hocque pacto Functio fracta quæ-
vis in innumerabiles formas diverfas tranfmutari poteft; hoc
modo eadem Functio fracta infinitis modis per Series recurren-
tes explicari poterit. Sit fcilicet propofita hæc fractio $y =$

$\frac{1 + z}{1 - z - zz}$ & per Seriem recurrentem $y = 1 + 2z +$

$3z^2 + 5z^3 + 8z^4 + \&c.$: ponatur $z = \frac{1}{x}$ erit y

$=$

$$= \frac{xx + x}{xx - x - 1} = \frac{-x(1+x)}{1+x-xx}. \quad \text{Jam} \quad \frac{1+x}{1+x-xx} = 1 + \qquad \text{CAP.IV.}$$

$0x + xx - x^3 + 2z^4 - 3x^5 + 5x^6 - \&c.;$ unde

erit $y = -x + 0x^2 - x^3 + x^4 - 2x^5 + 3x^6 - 5x^7 +$

&c. Vel ponatur $z = \frac{1-x}{1+x}$, erit $y = \frac{-2-2x}{1-4x-xx}$; un-

de fit $y = -2 - 10x - 42xx - 178x^3 - 754x^4 - \&c.$

cujufmodi Series recurrentes pro y innumerabiles inveniri poffunt.

71. Functiones irrationales ex hoc theoremate univerfali

in Series infinitas transformari folent, quod fit $(P+Q)^{\frac{m}{n}}$

$$= P^{\frac{m}{n}} + \frac{m}{n} P^{\frac{m-n}{n}} Q + \frac{m(m-n)}{n.\ 2n} P^{\frac{m-2n}{n}} Q^2 +$$

$$\frac{m(m-n)(m-2n)}{n.\quad 2n.\quad 3n} P^{\frac{m-3n}{n}} Q^3 + \&c. : \text{hi enim}$$

termini, nifi fuerit $\frac{m}{n}$ numerus integer affirmativus, in infini-

tum excurrunt. Sic erit pro m & n numeros definitos fcri-

bendo.

$$(P+Q)^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{1}{2} P^{-\frac{1}{2}} Q - \frac{1.1}{2.4} P^{-\frac{3}{2}} Q^2 +$$

$$\frac{1.1.3}{2.4.6} P^{-\frac{5}{2}} Q^3 - \&c.$$

$$(P+Q)^{-\frac{1}{2}} = P^{-\frac{1}{2}} - \frac{1}{2} P^{-\frac{3}{2}} Q + \frac{1.3}{2.4} P^{-\frac{5}{2}} Q^2 -$$

$$\frac{1.3.5}{2.4.6} P^{-\frac{7}{2}} Q^3 + \&c.$$

$$(P+Q)^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{1}{3} P^{-\frac{2}{3}} Q - \frac{1.2}{3.6} P^{-\frac{5}{3}} Q^2 +$$

$$\frac{1.2.5}{3.6.19} P^{-\frac{8}{3}} Q^3 - \&c.$$

$(P$

$$(P+Q)^{-\frac{1}{3}} = P^{-\frac{1}{3}} - \frac{1}{3} P^{-\frac{4}{3}} Q + \frac{1.4}{3.6} P^{-\frac{7}{3}} Q^2 -$$

$$\frac{1.4.7}{3.6.9} P^{-\frac{10}{3}} Q^3 + \&c.$$

$$(P+Q)^{\frac{2}{3}} = P^{\frac{2}{3}} + \frac{2}{3} P^{-\frac{1}{3}} Q - \frac{2.1}{3.6} P^{-\frac{4}{3}} Q^2 +$$

$$\frac{2.1.4}{3.6.9} P^{-\frac{7}{3}} Q^3 - \&c.$$

$$\&c.$$

72. Hujufmodi ergo Serierum termini ita progrediuntur ut quilibet ex antecedente formari poffit: fit enim Seriei, quæ ex $(P+Q)^{\frac{m}{n}}$ nafcitur, terminus quilibet $= M P^{\frac{m-kn}{n}} Q^k$ erit fequens $= \frac{m-kn}{(k+1)n} M P^{\frac{m-(k+1)n}{n}} Q^{k+1}$. Notandum autem eft in quovis termino fequente exponentem ipfius P unitate decrefcere, contra vero exponentem ipfius Q unitate crefcere. Quo autem hæc facilius ad quemvis cafum accommodentur, forma generalis $(P+Q)^{\frac{m}{n}}$ ita exponi poteft $P^{\frac{m}{n}} (1+\frac{Q}{P})^{\frac{m}{n}}$: evoluta enim formula $(1+\frac{Q}{P})^{\frac{m}{n}}$ Serieque refultante per $P^{\frac{m}{n}}$ multiplicata, prodibit ipfa Series ante data. Tum vero fi m non folum numeros integros denotet, fed etiam fractos, pro n tuto unitas collocari poterit. Quibus factis, fi pro $\frac{Q}{P}$, quæ eft Functio ipfius z, ponatur Z, habebitur

$$(1+Z)^m = 1 + \frac{m}{1} Z + \frac{m(m-1)}{1.2} Z^2 + \frac{m(m-1)(m-2)}{1.2.3} Z^3$$

$+ \&c.$ Ad fequentes progreffionum leges autem obfervandas conveniet hanc formulæ generalis in Seriem converfionem notaffe

taffe $(1 + Z)^{m-1} = 1 + \frac{(m-1)}{1} Z + \frac{(m-1)(m-2)}{1.\quad2} Z^2 +$

$\frac{(m-1)(m-2)(m-3)}{1.\qquad2.\qquad3} Z^3 + \&c.$

73. Sit igitur primum $Z = \alpha z$, eritque $(1 + \alpha z)^{m-1}$
$= 1 + \frac{m-1}{1} \alpha z + \frac{(m-1)(m-2)}{1\qquad2} \alpha^2 z^2 +$

$\frac{(m-1)(m-2)(m-3)}{1.\qquad2.\qquad3} \alpha^3 z^3 + \&c.$ Scribatur pro hac
Serie ista forma generalis

$1 + A z + B z^2 + C z^3 + \ldots\ldots + M z^{n-1} + N z^n + \&c.$

atque quilibet coëfficiens N ex præcedente M ita determina-
bitur ut fit $N = \frac{m-n}{n} \alpha M$. Sic, pofito $n = 1$, cum fit

$M = 1$, erit $N = A = \frac{m-1}{1} \alpha$; tum facto $n = 2$, ob

$M = A = \frac{m-1}{1} \alpha$, erit $N = B = \frac{m-2}{2} \alpha M =$

$\frac{(m-1)(m-2)}{1.\qquad2} \alpha^2$; fimilique modo porro $C = \frac{m-3}{3} \alpha B$

$= \frac{(m-1)(m-2)(m-3)}{1.\qquad2.\qquad3} \alpha^3$, uti Series ante inventa
declarat. .

74. Sit $Z = \alpha z + \mathfrak{G} z z$, eritque $(1 + \alpha z + \mathfrak{G} z z)^{m-1}$
$= 1 + \frac{(m-1)}{1} (\alpha z + \mathfrak{G} z z) + \frac{(m-1)(m-2)}{1.\quad2} (\alpha z + \mathfrak{G} z z)^2 + \&c.$
Quod fi ergo termini fecundum poteftates ipfius z difponantur
erit $(1 + \alpha z + \mathfrak{G} z z)^{m-1} =$
$1 + \frac{(m-1)}{1} \alpha z + \frac{(m-1)(m-2)}{1.\quad2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1.\quad2.\quad3} \alpha^3 z^3 + \&c.$
$+ \quad \frac{(m-1)}{1} \quad \mathfrak{G} z^2 + \frac{(m-1)(m-2)}{1.\quad2} 2 \alpha \mathfrak{G} z^3 + \&c.$

Euleri Indroduct. in Anal. infin. parv.   **H**   Scri-

Scribatur pro hac Serie ista forma generalis :

$$1 + Az + B z^2 + Cz^3 + \ldots + Lz^{n-2} + Mz^{n-1} + Nz^n + \&c.$$

atque quilibet coëfficiens ex duobus antecedentibus ita definie-
tur ut fit $N = \frac{m-n}{n} \alpha M + \frac{2m-n}{n} \mathfrak{S} L$, unde omnes
termini ex primo, qui eft 1, definiri poterunt.   Erit nempe

$$A = \frac{m-1}{1} \alpha \,;$$

$$B = \frac{(m-2)}{2} \alpha A + \frac{(2m-2)}{2} \mathfrak{S}$$

$$C = \frac{(m-3)}{3} \alpha B + \frac{(2m-3)}{3} \mathfrak{S} A$$

$$D = \frac{(m-4)}{4} \alpha C + \frac{(2m-4)}{4} \mathfrak{S} B$$

$$\&c.$$

75. Si fuerit $Z = \alpha z + \mathfrak{S} z^2 + \gamma z^3$, erit $\big(1 + \alpha z +$
$\mathfrak{S} z^2 + \gamma z^3 \big)^{m-1} = 1 + \frac{(m-1)}{1} (\alpha z + \mathfrak{S} z^2 + \gamma z^3) +$
$\frac{(m-1)(m-2)}{1 . \quad 2} (\alpha z + \mathfrak{S} z^2 + \gamma z^3)^2 + \&c.$, quæ expref-
fio, fi omnes termini fecundum poteftates ipfius z ordinentur,
abibit in hanc Seriem :

$$1 + \frac{(m-1)}{1} \alpha z + \frac{(m-1)(m-2)}{1 . \quad 2} \alpha^2 z^2 + \frac{(m-1)(m-2)(m-3)}{1 . \quad 2 . \quad 3} \alpha^3 z^3$$
$$+ \frac{(m-1)}{1} \mathfrak{S} z^2 + \frac{(m-1)(m-2)}{1 . \quad 2} 2 \alpha \mathfrak{S} z^3 + \&c.$$
$$+ \frac{(m-1)}{1} \gamma z^3$$

cujus lex progreffionis ut melius patefcat, ponatur ejus loco
$$1 + Az + Bz^2 + Cz^3 + \ldots + Kz^{n-3} + Lz^{n-2} + Mz^{n-1} + Nz^{n},$$
cujus Seriei quilibet coëfficiens ex tribus antecedentibus ita deter-
minatur ut fit $N = \frac{(m-n)}{n} \alpha M + \frac{(2m-n)}{n} \mathfrak{S} L + \frac{(3m-n)}{n} \gamma K.$

Cum

Cum igitur primus terminus fit $= 1$, & antecedentes nulli, Cap.IV.
erit

$$A = \frac{m-1}{1} \alpha$$

$$B = \frac{(m-2)}{2} \alpha A + \frac{(2m-2)}{2} \mathfrak{c}$$

$$C = \frac{(m-3)}{3} \alpha B + \frac{(2m-3)}{3} \mathfrak{c} A + \frac{(3m-3)}{3} \gamma$$

$$D = \frac{(m-4)}{4} \alpha C + \frac{(2m-4)}{4} \mathfrak{c} B + \frac{(3m-4)}{4} \gamma A$$

$$E = \frac{(m-5)}{5} \alpha D + \frac{(2m-5)}{5} \mathfrak{c} C + \frac{(3m-5)}{5} \gamma B$$

&c.

76. Generaliter ergo fi ponatur $(1 + \alpha z + \mathfrak{c} z^2 + \gamma z^3 + \delta z^4 + \&c.)^{m-1} = 1 + A z + B z^2 + C z^3 + D z^4 + E z^5 + \&c.$, hujus Seriei finguli termini ita ex præcedentibus definientur, ut fit

$$A = \frac{m-1}{1} \alpha$$

$$B = \frac{(m-2)}{2} \alpha A + \frac{(2m-2)}{2} \mathfrak{c}$$

$$C = \frac{(m-3)}{3} \alpha B + \frac{(2m-3)}{3} \mathfrak{c} A + \frac{(3m-3)}{3} \gamma$$

$$D = \frac{(m-4)}{4} \alpha C + \frac{(2m-4)}{4} \mathfrak{c} B + \frac{(3m-4)}{4} \gamma A + \frac{(4m-4)}{4} \delta$$

$$E = \frac{(m-5)}{5} \alpha D + \frac{(2m-5)}{5} \mathfrak{c} C + \frac{(3m-5)}{5} \gamma B + \frac{(4m-5)}{5} \delta A + \frac{(5m-5)}{5} \varepsilon$$

&c.

quilibet fcilicet terminus per tot præcedentes determinatur, quot habentur litteræ a, $\mathfrak{c}$, γ, δ, &c. in Functione ipfius z cujus poteftas in Seriem convertitur. Ceterum ratio hujus legis convenit cum ea, quam fupra §. 68. ubi fimilem formam

$$(1 - \alpha z - \mathfrak{c} z^2 - \gamma z^3 - \&c.)^{-m-1}$$ in Seriem infini-

tam

LIB. I. tam refolvimus; fi enim loco *m* fcribatur —— *m* atque litteræ —— *α*, *϶*, *γ*, *δ*, &c. negative accipiantur, Series inventæ prorfus congruent. Interim hoc loco non licet rationem hujus progreffionis legis a priori demonftrare, id quod per principia calculi differentialis demum commode fieri poterit; interea ergo fufficiet veritatem per applicationem ad omnis generis exempla comprobaffe.

C A P U T V.

De Functionibus duarum pluriumve variabilium.

77. QUanquam plures hactenus quantitates variabiles fumus contemplati, tamen eæ ita erant comparatæ, ut omnes unius effent Functiones, unaque determinata reliquæ fimul determinarentur. Nunc autem ejufmodi confiderabimus quantitates variabiles, quæ a fe invicem non pendeant, ita ut quamvis uni determinatus valor tribuatur, reliquæ tamen nihilominus maneant indeterminatæ ac variabiles. Ejufmodi ergo quantitates variabiles, cujufmodi fint x, y, z, ratione fignificationis convenient, cum quælibet omnes valores determinatos in fe complectatur; at, fi inter fe comparentur maxime erunt diverfæ, cum, licet pro una z valor quicunque determinatus fubftituatur, reliquæ tamen x & y æque late pateant, atque ante. Difcrimen ergo inter quantitates variabiles a fe pendentes, & non pendentes in hoc verfatur, ut priori cafu, fi una determinetur, fimul reliquæ determinentur; pofteriori vero determinatio unius fignificationes reliquarum minime reftringat.

78. *Functio ergo duarum pluriumve quantitatum variabilium,* x, y, z, *eft expreffio quomodocunque ex his quantitatibus compofita.*

Ita erit $x^3 + xyz + az^3$ Functio quantitatum variabilium trium x, y, z. Hæc ergo Functio, fi una determinetur variabilis,

riabilis, puta *z*, hoc eft ejus loco conftans numerus fubftitua-
tur, manebit adhuc quantitas variabilis, fcilicet Functio ipfa-
rum *x* & *y*. Atque fi, præter *z*, quoque *y* determinetur, tum
erit adhuc Functio ipfius *x*. Hujufmodi ergo plurium varia-
bilium Functio non ante valorem determinatum obtinebit,
quam fingulæ quantitates variabiles fuerint determinatæ. Cum
igitur una quantitas variabilis infinitis modis determinari pof-
fit, Functio duarum variabilium, quia pro quavis determina-
tione unius infinitas determinationes fufcipere poteft, omnino
infinities infinitas determinationes admittet. Atque in Functio-
ne trium variabilium numerus determinationum erit adhuc in-
finities major; ficque porro crefcet pro pluribus variabilibus.

79. *Hujufmodi Functiones plurium variabilium perinde atque*
Functiones unius variabilis, commodiffime dividuntur in algebrai-
cas ac tranfcendentes.

Quarum illæ funt, in quibus ratio compofitionis in folis Al-
gebræ operationibus eft pofita; hæ vero, in quarum ͺormā-
tionem quoque operationes tranfcendentes ingrediuntur. In
his denuo fpecies notari poffent, prout operationes tranfcen-
dentes vel omnes quantitates variabiles implicant, vel aliquot,
vel tantum unicam. Sic ifta expreffio *zz* + *y* log. *z*, quia
Logarithmus ipfius *z* ineft, erit quidem Functio tranfcendens
ipfarum *y* & *z*, verum ideo minus tranfcendens eft putanda,
quod fi variabilis *z* determinetur, fuperfit Functio algebraica
ipfius *y*. Interim tamen non expedit hujufmodi fubdivifionibus
tractationem amplificari.

80. *Functiones deinde algebraicæ fubdividuntur in rationales*
& irrationales; rationales autem porro in integras ac fractas.

Ratio harum denominationum ex Capite primo jam abunde
intelligitur. Functio fcilicet rationalis omnino eft libera ab om-
ni irrationalitate quantitates variabiles, quarum Functio dici-
tur, afficiente; hæcque erit integra fi nullis fractionibus inqui-
netur, contra vero fracta. Sic Functionis integræ duarum va-
riabilium *y* & *z* hæc erit forma generalis: $\alpha + \varepsilon y + \gamma z + \delta y^2 +$
$\varepsilon yz + \zeta z^2 + \eta z^3 + \theta y^2 z + \iota yz^2 + \kappa z^3 +$ &c. Quod

.fi

L I B. I. fi ergo P & Q denotent hujufmodi Functiones integras, five duarum five plurium variabilium, erit $\frac{P}{Q}$ forma generalis Functionum fractarum. Functio denique irrationalis eft vel explicita, vel implicita; illa per figna radicalia jam penitus eft evoluta, hæc autem per æquationem irrefolubilem exhibetur: fic V erit Functio implicita irrationalis ipfarum y & z, fi fuerit $V^5 = (a y z + z^3) V^2 + (y^4 + z^4) V + y^5 + 2 a y z^3 + z^5$.

81. *Multiformitas deinde in his Functionibus æque notari debet, atque in iis, quæ ex unica variabili conftant.*

Sic Functiones rationales erunt uniformes, quia fingulis quantitatibus variabilibus determinatis, unicum valorem determinatum exibent.   Denotent P, Q, R, S, &c. Functiones rationales feu uniformes variabilium x, y, z, eritque V Functio biformis earundem variabilium, fi fuerit $V^2 - P V + Q = 0$; quicunque enim valores determinati quantitatibus x, y, & z tribuuntur, Functio V non unum fed duplicem perpetuo habebit valorem determinatum.   Simili modo erit V Functio triformis fi fuerit $V^3 - P V^2 + Q V - R = 0$: atque Functio quadriformis fi fuerit $V^4 - P V^3 + Q V^2 - R V + S = 0$: hocque modo ratio Functionum multiformium ulteriorum erit comparata.

82. Quemadmodum fi Functio unius variabilis z nihilo æqualis ponitur, quantitas variabilis z valorem confequitur determinatum vel fimplicem vel multiplicem; ita fi Functio duarum variabilium y & z nihilo æqualis ponitur, tum altera variabilis per alteram definitur, ejufque ideo Functio evadit, cum ante a fe mutuo non penderent.   Simili modo fi Functio trium variabilium x, y, z, nihilo æqualis ftatuatur, tum una variabilis per duas reliquas definitur, earumque Functio exiftit.   Idem evenit fi Functio non nihilo fed quantitati conftanti vel etiam alii Functioni æqualis ponatur; ex omni enim æquatione, quotcunque variabiles involvat, femper una variabilis per reliquas definitur earumque fit Functio; duæ autem æqua-
tiones

tiones diverſæ inter eaſdem variabiles ortæ binas per reliquas Cap. V.
definient, atque ita porro.

83. *Functionum autem duarum pluriumve variabilium diviſio*
maxime notatu digna eſt in homogeneas & heterogeneas.

Functio homogenea eſt per quam ubique idem regnat va-
riabilium numerus dimenſionum : Functio autem heterogenea
eſt , in qua diverſi occurrunt dimenſionum numeri.   Cenſetur
vero unaquæque variabilis unam dimenſionem conſtituere ; qua-
dratum uniuscujuſque atque productum ex duabus , duas ;
productum ex tribus variabilibus , ſive iiſdem ſive diverſis ,
tres & ita porro ; quantitates autem conſtantes ad dimenſio-
num numerationem non admittuntur.   Ita in his formulis αy ;
$6z$, unica dimenſio ineſſe dicitur ; in his vero αy^2 ; $6yz$; γz^2
duæ inſunt dimenſiones : in his αy^3 ; $6y^2z$; $\gamma y z^2$; δz^3 ,
tres; in his vero αy^4 ; $6y^3z$; $\gamma y^2 z^2$; $\delta y z^3$; εz^4 ; quatuor,
ſicque porro.

84. Applicemus primum hanc diſtinctionem ad Functiones
integras , atque duas tantum variabiles ineſſe ponamus , quo-
niam plurium par eſt ratio.

Functio igitur integra erit homogenea in cujus ſingulis terminis
idem exiſtit dimenſionum numerus.

Subdividentur ergo hujuſmodi Functiones commodiſſime ſe-
cundum numerum dimenſionum , quem variabiles in ipſis ubi-
que conſtituunt.   Sic erit $\alpha y + 6z$ forma generalis Functio-
num integrarum unius dimenſionis : hæc vero expreſſio αy^2
$+ 6yz + \gamma z^2$ erit forma generalis Functionum duarum dimen-
ſionum , tum forma generalis Functionum trium dimenſionum
erit : $\alpha y^3 + 6y^2z + \gamma y z^2 + \delta z^3$; quatuor dimenſionum vero
hæc : $\alpha y^4 + 6y^3z + \gamma y^2z^2 + \delta y z^3 + \varepsilon z^4$; & ita porro.   Ad
analogiam igitur erit quantitas conſtans ſola α Functio nullius
dimenſionis.

85. *Functio porro fracta erit homogenea , ſi ejus **Numerator** ac*
***Denominator** fuerint Functiones homogeneæ.*

Sic hæc Fractio $\dfrac{ayy + bzz}{av + 6z}$ erit Functio homogenea ipſa-

rum

LIB. I. rum y & z; numerus dimenſionum autem habebitur, ſi a nu-
mero dimenſionum Numeratoris ſubtrahatur numerus dimen-
ſionum Denominatoris: atque ob hanc rationem Fractio al-
lata erit Functio unius dimenſionis.   Hæc vero Fractio
$\frac{y^5 + z^5}{yy + zz}$ erit Functio trium dimenſionum.   Quando ergo in
Numeratore ac Denominatore idem dimenſionum numerus in-
eſt, tum Fractio erit Functio nullius dimenſionis, uti evenit in
hac Fractione $\frac{y^3 + z^3}{yyz}$, vel etiam in his $\frac{y}{z}$; $\frac{\alpha zz}{yy}$; $\frac{6y^3}{z^3}$.   Quod
ſi igitur in Denominatore plures ſint dimenſiones quam in
Numeratore, numerus dimenſionum Fractionis erit negativus:
ſic $\frac{y}{zz}$ erit Functio — 1 dimenſionis: $\frac{y+z}{y^4 + z^4}$ erit Functio —3
dimenſionum: $\frac{1}{y^5 + ayz^4}$ erit Functio — 5 dimenſionum, quia
in Numeratore nulla ineſt dimenſio.   Ceterum ſponte intelli-
gitur plures Functiones homogeneas, in quibus ſingulis idem
regnat dimenſionum numerus, ſive additas ſive ſubtractas præ-
bere Functionem quoque homogeneam ejuſdem dimenſionum
numeri.   Sic hæc expreſſio $\alpha y + \frac{6zz}{y} + \frac{\gamma v^4 - \delta z^4}{yyz + yzz}$ erit Fun-
ctio unius dimenſionis: hæc autem $\alpha + \frac{6y}{z} + \frac{\gamma zz}{yy} + \frac{yy + zz}{yy - zz}$
erit Functio nullius dimenſionis.

86. Natura Functionum homogenearum quoque ad expreſ-
ſiones irrationales extenditur.   Si enim fuerit P Functio quæ-
cunque homogenea, puta n dimenſionum, tum $\sqrt{P}$ erit Fun-
ctio $\frac{1}{2} n$ dimenſionum; $\sqrt[3]{P}$ erit Functio $\frac{1}{3} n$ dimenſionum,
& generatim $P^{\frac{\mu}{\nu}}$ erit Functio $\frac{\mu}{\nu} n$ dimenſionum. Sic $\sqrt{(yy + zz)}$
erit Functio unius dimenſionis; $\sqrt[3]{(y^9 + z^9)}$ erit Functio trium
dimenſionum: $(yz + zz)^{\frac{3}{4}}$ erit Functio $\frac{3}{2}$ dimenſionum: at-
que

que $\dfrac{yy + zz}{\sqrt{(y^4 + z^4)}}$ erit Functio nullius dimensionis. His ergo

cum præcedentibus conjunctis intelligetur hæc expressio $\dfrac{1}{y}$

$+ \dfrac{y\sqrt{(yy+zz)}}{z^3} - \dfrac{y}{\sqrt[3]{(y^6 - z^6)}} + \dfrac{y\sqrt{z}}{zz\sqrt{y} + \sqrt{(y^5 + z^5)}}$ esse

Functio homogenea — 1 dimensionis.

87. Utrum Functio irrationalis implicita sit homogenea nec-
ne, ex his facile colligi potest. Sit V hujusmodi Functio im-
plicita ac $V^3 + PV^2 + QV + R = 0$, existentibus P, Q &
R Functionibus ipsarum y & z. Primum igitur patet V Fun-
ctionem homogeneam esse non posse, nisi P, Q, & R sint
Functiones homogeneæ. Præterea vero si ponamus V esse
Functionem n dimensionum, erit V^2 Functio $2n$, & V^3 Fun-
ctio $3n$ dimensionum; cum igitur ubique idem debeat esse nu-
merus dimensionum, oportet, ut P sit Functio n dimensionum,
Q Functio $2n$ dimensionum, & R Functio $3n$ dimensionum. Si
ergo vicissim litteræ P, Q, R Functiones homogeneæ res-
pective n, $2n$, $3n$ dimensionum, hinc concludetur fore V
Functionem n dimensionum. Ita si fuerit $V^5 + (y^4 + z^4) V^3$
$+ ay^8 V - z^{10} = 0$ erit V Functio homogenea duarum di-
mensionum, ipsarum y & z.

88. *Si fuerit* V *Functio homogenea* n *dimensionum ipsarum* y
& z, *in eaque ponatur ubique* y $=$ u z, *Functio* V *abibit in pro-*
ductum ex potestate z^n *in Functionem quandam variabilis* u.

Per hanc enim substitutionem $y = uz$, in singulos termi-
nos tantæ inducentur potestates ipsius z, quantæ ante inerant
ipsius y. Cum igitur in singulis terminis dimensiones ipsarum
y & z conjunctim æquassent numerum n, nunc sola variabilis
z ubique habebit n dimensiones, ideoque ubique inerit ejus
potestas z^n. Per hanc ergo potestatem Functio V fiet divi-
sibilis & quotus erit Functio variabilem tantum u involvens.
Hoc primum patebit in Functionibus integris; si enim sit $V =$
$ay^3 + \beta y^2 z + \gamma yz^2 + \delta z^3$, posito $y = uz$, fiet $V = z^3$

$(\alpha u^3 + 6 u^2 + \gamma u + \delta)$. Deinde vero idem manifestum est in fractis: fit enim $V = \dfrac{\alpha y + 6 z}{y y + z z}$, nempe Functio $- 1$ dimensionis, facto $y = u z$ fiet $V = z^{-1} \left(\dfrac{\alpha u + 6}{u u + 1} \right)$. Neque etiam Functiones irrationales hinc excipiuntur, si enim sit $V = \dfrac{y + \sqrt{(y y + z z)}}{z \sqrt{(y^3 + z^3)}}$, quæ est Functio $- \dfrac{3}{2}$ dimensionum; posito $y = u z$, prodibit $V = z^{-\frac{3}{2}} \left(\dfrac{u + \sqrt{(u u + 1)}}{\sqrt{(u^3 + 1)}} \right)$.

Hoc itaque modo Functiones homogeneæ duarum tantum variabilium reducentur ad Functiones unius variabilis; neque enim potestas ipsius z, quia est Factor, Functionem illam ipsius u inquinat.

89. *Functio ergo homogenea* V *duarum variabilium* y *&* z *nullius dimensionis, posito* y $=$ u z *, transmutabitur in Functionem unicæ variabilis* u *puram.*

Cum enim numerus dimensionum sit nullus, Potestas ipsius z, quæ Functionem ipsius u multiplicabit, erit $z^0 = 1$; hocque casu variabilis z prorsus ex computo egredietur. Ita si fuerit $V = \dfrac{y + z}{y - z}$, facto $y = u z$, orietur $V = \dfrac{u + 1}{u - 1}$: atque in irrationalibus si sit $V = \dfrac{y - \sqrt{(y y - z z)}}{z}$ posito $y = u z$ erit $V = u - \sqrt{(u u - 1)}$.

90. *Functio integra homogenea duarum variabilium* y *&* z *, resolvi poterit in tot Factores simplices forma* α y $+ 6$ z *, quot habuerit dimensiones.*

Cum enim Functio sit homogenea, posito $y = u z$, transibit in productum ex z^n in Functionem quandam ipsius u integram, quæ Functio propterea in Factores simplices formæ $\alpha u + 6$ resolvi poterit. Multiplicentur singuli Factores hi per z, eritque uniuscujusque forma $\alpha u z + 6 z = \alpha y + 6 z$ ob $u z = y$. Propter multiplicatorem autem z^n, tot hujusmodi Factores nascentur quot exponens n contineat unitates; Factores autem

hi

hi fimplices erunt vel reales vel imaginarii, hoc eft coëfficien-
tes α, & 6 erunt vel reales vel imaginarii.

Ex hoc itaque fequitur Functionem duarum dimenfionum
$ayy + byz + czz$ duos habere Factores fimplices formæ αy
$+ 6z$; Functio autem $ay^3 + by^2z + cyz^2 + dz^3$ habebit tres
Factores fimplices formæ $\alpha y + 6z$; ficque porro Functionum
homogenearum integrarum, quæ plures habent dimenfiones,
natura erit comparata.

91. Quemadmodum ergo hæc expreffio $\alpha y + 6z$ continet
formam generalem Functionum integrarum unius dimenfionis -
ita $(\alpha y + 6z)(\gamma y + \delta z)$ erit forma generalis Functionum in,
tegrarum duarum dimenfionum: atque in hac forma $(\alpha y + 6z)$
$(\gamma y + \delta z)(\varepsilon y + \xi z)$ continebuntur omnes Functiones in-
tegræ trium dimenfionum, ficque omnes Functiones integræ
homogeneæ per producta ex tot hujufmodi Factoribus $\alpha y + 6z$
exhiberi poterunt, quot Functiones illæ contineant dimenfio-
nes. Ifti autem Factores eodem modo per refolutionem æqua-
tionum reperiuntur, quo fupra Factores fimplices Functionum
integrarum unius variabilis invenire docuimus. Ceterum hæc
proprietas Functionum homogenearum duarum variabilium non
extenditur ad Functiones homogeneas trium pluriumve varia-
bilium: forma enim generalis hujufmodi Functionum duarum
tantum dimenfionum, quæ eft $ayy + byz + cyx + dxy + exx +$
fzz generaliter non reduci poteft ad hujufmodi productum
$(\alpha y + 6z + \gamma x)(\delta y + \varepsilon z + \xi x)$; multoque minus Fun-
ctiones plurium dimenfionum ad hujufmodi producta revocari
poffunt.

92. Ex his, quæ de Functionibus homogeneis funt dicta,
fimul intelligitur, quid fit Functio heterogenea: in cujus fcili-
cet terminis non ubique idem dimenfionum numerus depre-
henditur. Poffunt autem Functiones heterogeneæ fubdividi
pro multiplicitate dimenfionum, quæ in ipfis occurrunt. Sic
Functio bifida erit, in qua duplex dimenfionum numerus
occurrit, eritque adeo aggregatum duarum Functionum homo-

genea-

LIB. I. genearum, quarum numeri dimensionum differunt; ita $y^5 +$ $2y^3z^2 + yy + zz$ erit Functio bifida, quia partim quinque, partim duas continet dimensiones. Functio autem trifida est, in qua tres diversi dimensionum numeri insunt, seu quæ in tres Functiones homogeneas distribui possunt, uti $y^6 + y^3z^2 + z^4 + y - z$.

Præterea autem dantur Functiones heterogeneæ fractæ vel irrationales tantopere permixtæ, quæ in Functiones homogeneas resolvi non possunt, cujusmodi sunt $\dfrac{y^3 + ayz}{by + zz}$,

$\dfrac{a + \sqrt{(yy + zz)}}{yy - bz}$.

93. Interdum Functio heterogenea ope substitutionis idoneæ, vel loco unius vel utriusque variabilis factæ, ad homogeneam reduci potest; quod quibus casibus fieri queat, non tam facile indicare licet. Sufficiet ergo exempla quædam attulisse, quibus ejusmodi reductio locum habet. Si scilicet hæc proposita sit Functio $y^5 + zzy + y^3z + \dfrac{z^3}{y}$; post levem attentionem apparebit, eam ad homogeneitatem perduci, posito $z =$ xx: prodibit enim $y^5 + x^4y + y^3xx + \dfrac{x^6}{y}$, Functio homogenea 5 dimensionum ipsarum x & y. Deinde hæc Functio $y +$ $y^2x + y^3xx + y^5x^4 + \dfrac{a}{x}$ ad homogeneitatem reducitur ponendo $x = \dfrac{1}{z}$, prodit enim Functio unius dimensionis $y +$ $\dfrac{yy}{z} + \dfrac{y^3}{zz} + \dfrac{y^5}{z^4} + az$. Multo difficiliores autem sunt casus, quibus non per tam simplicem substitutionem ad homogeneitatem pervenire licet.

94. Tandem inprimis notari meretur Functionum integrarum secundum ordines divisio satis usitata, secundum quam ordo definitur ex maximo dimensionum numero qui in Functione inest. Sic $xx + yy + zz + ay - aa$ est Functio secundi ordinis, quia duæ dimensiones occurrunt. Et $y^4 + yz^3 - ay^2z +$ $abyz -$

$abyz$ — $aayy$ + b^4 pertinet ad Functiones quarti ordinis. Ad C A P. V.
hanc divifionem potiffimum in doctrina de lineis curvis refpici ——
folet; unde adhuc una Functionum integrarum divifio comme-
moranda venit.

95. Supereft fcilicet divifio Functionum integrarum in com-
plexas atque incomp exas. Functio autem complexa eft, quæ
in Factores rationales refolvi poteft, feu quæ eft productum
ex duabus Functionibus pluribusve rationalibus; cujufmodi eft
y^4 — z^4 + $2az^3$ — $2byzz$ — $aazz$ + $2abzy$ — $bbyy$, quæ eft
productum ex his duabus Functionibus (yy + zz — az + by)
(yy — zz + az — by). Ita vidimus omnem Functionem in-
tegram homogeneam, quæ tantum duas variabiles complecta-
tur, effe Functionem complexam, quoniam tot Factores fim-
plices formæ ay + $6z$ habet, quot continet dimenfiones. Fun-
ctio igitur integra erit incomplexa, fi in Factores rationales re-
folvi omnino nequeat; uti yy + zz — aa, cujus nullos dari
Factores rationales facile intelligitur. Ex inquifitione Divi-
forum patebit, utrum Functio propofita fit complexa an in-
complexa.

CAPUT VI.

De Quantitatibus exponentialibus ac Logarithmis.

96. QUanquam notio Functionum tranfcendentium in cal-
culo integrali demum perpendetur, tamen ante-
quam co perveniamus, quafdam fpecies magis obvias, atque
ad plures inveftigationes aditum aperientes, evolvere conve-
niet. Primum ergo confiderandæ funt quantitates exponentia-
les, feu Poteftates, quarum Exponens ipfe eft quantitas varia-
bilis. Perfpicuum enim eft hujufmodi quantitates ad Functio-
nes algebraicas referri non poffe, cum in his Exponentes non
nifi conftantes locum habeant. Multiplices autem funt quan-

I 3 titates

LIB. I. titates exponentiales, prout vel folus Exponens eft quantitas variabilis , vel præterea etiam ipfa quantitas elevata; prioris generis eft a^z, hujus vero y^z ; quin etiam ipfe Exponens poteft effe quantitas exponentialis uti in his formis a^{a^z} ; a^{y^z} ; y^{a^z} ; x^{y^z}. Hujufmodi autem quantitatum non plura conftituemus genera, cum earum natura fatis clarè intelligi queat, fi primam tantum fpeciem a^z evolverimus.

97. Sit igitur propofita hujufmodi quantitas exponentialis a^z , quæ eft Poteftas quantitatis conftantis a, Exponentem habens variabilem z. Cum igitur ifte Exponens z, omnes numeros determinatos in fe complectatur, primum patet fi loco z omnes numeri integri affirmativi fucceffive fubftituantur, loco a^z hos prodituros effe valores determinatos a^1 ; a^2 ; a^3 ; a^4; a^5 ; a^6 ; &c. Sin autem pro z ponantur fucceffive numeri negativi — 1, — 2, — 3, &c. prodibunt $\frac{1}{a}$; $\frac{1}{a^2}$; $\frac{1}{a^3}$; $\frac{1}{a^4}$; &c. ac, fi fuerit $z = 0$, habebitur femper $a^0 = 1$. Quod fi loco z numeri fracti ponantur, ut $\frac{1}{2}$; $\frac{1}{3}$; $\frac{2}{3}$; $\frac{1}{4}$; $\frac{3}{4}$; &c. orientur ifti valores $\sqrt{a}$; $\sqrt[3]{a}$; $\sqrt[3]{aa}$; $\sqrt[4]{a}$; $\sqrt[4]{a^3}$; &c., qui in fe fpectati geminos plurefve induunt valores, cum radicum extractio femper valores multiformes producat. Interim tamen hoc loco valores tantum primarii, reales fcilicet atque affirmativi admitti folent; quia quantitas a^z tanquam Functio uniformis ipfius z fpectatur. Sic $a^{\frac{5}{2}}$ medium quendam tenebit locum inter a^2 & a^3, eritque ideo quantitas ejufdem generis; & quamvis valor $a^{\frac{5}{2}}$ fit æque $= - aa\sqrt{a}$, ac $= +aa\sqrt{a}$, tamen pofterior tantum in cenfum venit. Eodem modo res fe habet, fi Exponens z valores irrationales accipiat , quibus cafibus cum difficile fit numerum valorum involutorum concipere.

pere.

pere, unicus tantum realis confideratur. Sic $a^{\sqrt{7}}$ erit valor determinatus intra limites a^2 & a^3 comprehenfus.

98. Maxime autem valores quantitatis exponentialis a^z a magnitudine numeri conftantis a pendebunt. Si enim. fuerit $a = 1$, femper erit $a^z = 1$, quicunque valores Exponenti z tribuatur; fin autem fuerit $a > 1$, tum valor ipfius a^z eo erunt majores, quo major numerus loco z fubftituatur, atque adeo, pofito $z = \infty$, in infinitum excrefcunt; fi fuerit $z = 0$, fiet $a^z = 1$, &, fi fit $z < 0$ valores a^z fient unitate minores, quoad pofito $z = -\infty$ fiat $a^z = 0$. Contrarium evenit fi fit $a < 1$, verum tamen quantitas affirmativa; tum enim valores ipfius a^z decrefcent, crefcente z fupra 0; crefcent vero, fi pro z numeri negativi fubftituantur. Cum enim fit $a < 1$, erit $\frac{1}{a} > 1$; pofito ergo $\frac{1}{a} = b$; erit $a^z = b^{-z}$, unde pofterior cafus ex priori dijudicari poterit.

99. Si fit $a = 0$, ingens faltus in valoribus ipfius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus feu major nihilo, erit perpetuo $a^z = 0$: fi fit $z = 0$ erit $a^0 = 1$; fin autem fuerit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Sit enim $z = -3$; erit $a^z = 0^{-3} = \frac{1}{0^3} = \frac{1}{0}$, ideoque infinitum. Multo majores autem faltus occurrent, fi quantitas conftans a habeat valorem negativum, puta -2; tum enim ponendis loco z numeris integris valores ipfius a^z alternatim erunt affirmativi & negativi, ut ex hac Serie intelligitur

$$a^{-4}; a^{-3}; a^{-2}; a^{-1}; a^0; a^1; a^2; a^3; a^4; \text{&c.}$$

+

$+\frac{1}{16}$; $-\frac{1}{8}$; $+\frac{1}{4}$; $-\frac{1}{2}$; 1; -2; $+4$; -8; $+16$.

Præterea vero si Exponenti z valores tribuantur fracti, Potestas $a^z = (-2)^z$ mox reales mox imaginarios induet valores, erit enim $a^{\frac{1}{2}} = \sqrt{-2}$, imaginarium; at erit $a^{\frac{1}{3}} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, si Exponenti z tribuantur valores irrationales, Potestas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. His igitur incommodis numerorum negativorum loco a substituendorum commemoratis, statuamus a esse numerum affirmativum, & unitate quidem majorem, quia huc quoque illi casus, quibus a est numerus affirmativus unitate minor, facile reducuntur. Si ergo ponatur $a^z = y$, loco z substituendo omnes numeros reales, qui intra limites $+\infty$ & $-\infty$ continentur, y adipiscetur omnes valores affirmativos intra limites $+\infty$ & 0 contentos. Si enim sit $z = \infty$ erit $y = \infty$; si $z = 0$ erit $y = 1$, & si $z = -\infty$ fiet $y = 0$. Vicissim ergo quicunque valor affirmativus pro y accipiatur, dabitur quoque valor realis respondens pro z ita ut sit $a^z = y$; sin autem ipsi y tribueretur valor negativus, Exponens z valorem realem habere non poterit.

101. Si igitur fuerit $y = a^z$, erit y Functio quædam ipsius z, &, quemadmodum y a z pendeat, ex natura Potestatum facile intelligitur; hinc enim quicunque valor ipsi z tribuatur, valor ipsius y determinatur. Erit autem $yy = a^{2z}$; $y^3 = a^{3z}$: & generaliter erit $y^n = a^{nz}$; unde sequitur fore $\sqrt{y} = a^{\frac{1}{2}z}$; $\sqrt[3]{y} = a^{\frac{1}{3}z}$ & $\frac{1}{y} = a^{-z}$; $\frac{1}{yy} = a^{-2z}$; & $\frac{1}{\sqrt{y}} = a^{-\frac{1}{2}z}$, & ita porro. Præterea, si fuerit $v = a^x$ erit $vy = a^{x+z}$ & $\frac{v}{y} = a^{x-z}$, quorum subsidiorum beneficio eo facilius valor ipsius y ex dato valore ipsius z inveniri potest.

EXEM-

E X E M P L U M.

Si fuerit $a = 10$, ex Arithmetica, qua utimur, denaria in promtu erit valores ipsius y exhibere, si quidem pro z numeri integri ponantur. Erit enim $10^1 = 10$; $10^2 = 100$; $10^3 = 1000$; $10^4 = 10000$; & $10^0 = 1$; item $10^{-1} = \frac{1}{10} = 0,1$; $10^{-2} = \frac{1}{100} = 0,01$; $10^{-3} = \frac{1}{1000} = 0,001$: sin autem pro z Fractiones ponantur, ope radicum extractionis valores ipsius y indicari possunt: sic erit $10^{\frac{1}{2}} = \sqrt{10} = 3,162277$, &c.

102. Quemadmodum autem, dato numero a, ex quovis valore ipsius z reperiri potest valor ipsius y, ita vicissim, dato valore quocunque affirmativo ipsius y, conveniens dabitur valor ipsius z, ut sit $a^z = y$; iste autem valor ipsius z, quatenus tanquam Functio ipsius y spectatur, vocari solet LOGARI-THMUS ipsius y. Supponit ergo doctrina Logarithmorum numerum certum constantem loco a substituendum, qui propterea vocatur *basis* Logarithmorum; qua assumta erit Logarithmus cujusque numeri y Exponens Potestatis a^z, ita ut ipsa Potestas a^z aequalis sit numero illi y; indicari autem Logarithmus numeri y solet hoc modo ly. Quod si ergo fuerit $a^z = y$, erit $z = ly$: ex quo intelligitur, basin Logarithmorum, etiamsi ab arbitrio nostro pendeat, tamen esse debere numerum unitate majorem: hincque nonnisi numerorum affirmativorum Logarithmos realiter exhiberi posse.

103. Quicunque ergo numerus pro basi Logarithmica a accipiatur, erit semper $l\,1 = 0$; si enim in aequatione $a^z = y$, quae convenit cum hac $z = ly$, ponatur $y = 1$, erit $z = 0$. Deinde numerorum unitate majorum Logarithmi erunt affirmativi, pendentes a valore basis a, sic erit $la = 1$; $laa = 2$; $la^3 = 3$;

LIB. I. $la^4 = 4$; &c., unde a pofteriori intelligi poteft, quantus numerus pro bafi fit affumtus, fcilicet ille numerus, cujus Logarithmus eft $= 1$, erit bafis Logarithmica. Numerorum autem unitate minorum, affirmativorum tamen, Logarithmi erunt negativi; erit enim $l\,\frac{1}{a} = -1$; $l\frac{1}{aa} = -2$; $l\frac{1}{a^3} = -3$, &c.; numerorum autem negativorum Logarithmi non erunt reales, fed imaginarii, uti jam notavimus.

104. Simili modo fi fuerit $ly = z$; erit $lyy = 2z$; $ly^3 = 3z$; & generaliter $l\,y^n = nz$, feu $l\,y^n = nl\,y$, ob $z = ly$. Logarithmus igitur cujufque Poteftatis ipfius y æquatur Logarithmo ipfius y per Exponentem Poteftatis multiplicato; fic erit $l\sqrt{y} = \frac{1}{2}z = \frac{1}{2}ly$; $l\,\frac{1}{\sqrt{y}} = ly^{-\frac{1}{2}} = -\frac{1}{2}ly$; & ita porro; unde ex dato Logarithmo cujufque numeri inveniri poffunt Logarithmi quarumcunque ipfius Poteftatum. Sin autem jam inventi fint duo Logarithmi, nempe $ly = z$ & $lv = x$: cum fit $y = a^z$ & $v = a^x$ erit $luy = x + z = lv + ly$; hinc Logarithmus Producti duorum numerorum æquatur fummæ Logarithmorum Factorum; fimili vero modo erit $l\,\frac{y}{v} = z - x = ly - lv$; hincque Logarithmus Fractionis æquatur Logarithmo Numeratoris dempto Logarithmo Denominatoris, quæ regulæ inferviunt Logarithmis plurium numerorum inveniendis, ex cognitis jam aliquot Logarithmis.

105. Ex his autem patet aliorum numerorum non dari Logarithmos rationales, nifi Poteftatum bafeos a; nifi enim numerus alius b fuerit Poteftas bafis a, ejus Logarithmus numero rationali exprimi non poterit. Neque vero etiam Logarithmus ipfius b erit numerus irrationalis; fi enim foret $lb = \sqrt{n}$, tum effet $a^{\sqrt{n}} = b$; id quod fieri nequit, fi quidem numeri a & b rationales ftatuantur; folent autem imprimis numerorum rationa-

tionalium & integrorum Logarithmi defiderari, quia ex his Logarithmi Fractionum ac numerorum furdorum inveniri poffunt. Cum igitur Logarithmi numerorum, qui non funt Poteftates bafis *a*, neque rationaliter neque irrationaliter exhiberi queant, merito ad quantitates tranfcendentes referuntur, hincque Logarithmi quantitatibus tranfcendentibus annumerari folent.

106. Hanc ob rem Logarithmi numerorum vero tantum proxime per Fractiones decimales exprimi folent, qui eo minus à veritate difcrepabunt, ad quo plures figuras fuerint exacti. Atque hoc modo per folam radicis quadratæ extractionem cujufque numeri Logarithmus vero proxime determinari poterit. Cum enim, pofito $l\,y = z$ & $l\,v = x$, fit $l\,\sqrt{v\,y} = \frac{x+z}{2}$; fi numerus propofitus *b* contineatur intra limites a^2 & a^3, quorum Logarithmi funt 2 & 3, quæratur valor ipfius $a^{2\frac{1}{2}}$ feu $a^2\sqrt{a}$, atque *b* vel intra limites a^2 & $a^{2\frac{1}{2}}$ vel $a^{2\frac{1}{2}}$ & a^3 continebitur, utrumvis accidat, fumendo medio proportionali, denuo limites propiores prodibunt, hocque modo ad limites pervenire licebit, quorum intervallum data quantitate minus evadat, & quibufcum numerus propofitus *b* fine errore confundi poffit. Quoniam vero horum fingulorum limitum Logarithmi dantur, tandem Logarithmus numeri *b* reperietur.

EXEMPLUM.

Ponatur bafis Logarithmica $a = 10$, quod in tabulis ufu receptis fieri folet; & quæratur vero tantum proxime Logarithmus numeri 5; quia hic continetur intra limites 1 & 10 quorum Logarithmi funt 0 & 1; fequenti modo radicum extractio continua inftituatur, quoad ad limites à numero propofito 5 non amplius difcrepantes perveniatur.

$A =$

$A = 1,000000;$	$lA = 0,0000000$		fit
$B = 10,000000;$	$lB = 1,0000000;$	$C = \sqrt{AB}$
$C = 3,162277;$	$lC = 0,5000000;$	$D = \sqrt{BC}$
$D = 5,623413;$	$lD = 0,7500000;$	$E = \sqrt{CD}$
$E = 4,216964;$	$lE = 0,6250000;$	$F = \sqrt{DE}$
$F = 4,869674;$	$lF = 0,6875000;$	$G = \sqrt{DF}$
$G = 5,232991;$	$lG = 0,7187500;$	$H = \sqrt{FG}$
$H = 5,048065;$	$lH = 0,7031250;$	$I = \sqrt{FH}$
$I = 4,958069;$	$lI = 0,6953125;$	$K = \sqrt{HI}$
$K = 5,002865;$	$lK = 0,6992187;$	$L = \sqrt{IK}$
$L = 4,980416;$	$lL = 0,6972656;$	$M = \sqrt{KL}$
$M = 4,991627;$	$lM = 0,6982421;$	$N = \sqrt{KM}$
$N = 4,997242;$	$lN = 0,6987304;$	$O = \sqrt{KN}$
$O = 5,000052;$	$lO = 0,6989745;$	$P = \sqrt{NO}$
$P = 4,998647;$	$lP = 0,6988525;$	$Q = \sqrt{OP}$
$Q = 4,999350;$	$lQ = 0,6989135;$	$R = \sqrt{OQ}$
$R = 4,999701;$	$lR = 0,6989440;$	$S = \sqrt{OR}$
$S = 4,999876;$	$lS = 0,6989592;$	$T = \sqrt{OS}$
$T = 4,999963;$	$lT = 0,6989668;$	$V = \sqrt{OT}$
$V = 5,000008;$	$lV = 0,6989707;$	$W = \sqrt{TV}$
$W = 4,999984;$	$lW = 0,6989687;$	$X = \sqrt{WV}$
$X = 4,999997;$	$lX = 0,6989697;$	$Y = \sqrt{VX}$
$Y = 5,000003;$	$lY = 0,6989702;$	$Z = \sqrt{XY}$
$Z = 5,000000;$	$lZ = 0,6989700;$	

Sic ergo mediis proportionalibus fumendis tandem perventum est ad $Z = 5,000000$, ex quo Logarithmus numeri 5 quæsitus est $0,698970$, pofita bafi Logarithmica $= 10$. Quare erit proxime $10^{\frac{69897}{100000}} = 5$. Hoc autem modo computatus est canon Logarithmorum vulgaris à BRIGGIO & VLACQUIO, quamquam postea eximia inventa sunt compendia, quorum ope multo expeditius Logarithmi supputari possunt.

107. Dantur ergo tot diversa Logarithmorum systemata quot varii numeri pro basi a accipi possunt, atque ideo numerus systema-

tematum Logarithmicorum erit infinitus. Perpetuo autem in duobus fyftematis Logarithmi ejufdem numeri eandem inter fe fervant rationem. Sit bafis unius fyftematis $= a$, alterius $= b$, atque numeri n Logarithmus in priori fyftemate $= p$, in pofteriori $= q$; erit $a^p = n$ & $b^q = n$; unde $a^p = b^q$; ideoque $a = b^{\frac{q}{p}}$. Oportet ergo ut Fractio $\frac{q}{p}$ conf-tantem obtineat valorem, quicunque numerus pro n fuerit af-fumtus. Quod fi ergo pro uno fyftemate Logarithmi omnium numerorum fuerint computati, hinc facili negotio per regulam auream Logarithmi pro quovis alio fyftemate reperiri poffunt. Sic, cum dentur Logarithmi pro bafi 10, hinc Logarithmi pro quavis alia bafi, puta 2, inveniri poffunt; quæratur enim Loga-rithmus numeri n pro bafi 2, qui fit $= q$, cum ejufdem numeri n Logarithmus fit $= p$ pro bafi 10. Quoniam pro bafi 10 eft $l\,2 = 0, 3010300$, & pro bafi 2, eft $l\,2 = 1$, erit $0, 3010300 : 1 = p : q$ ideoque $q = \dfrac{p}{0, 3010300} = 3,$ $3219277. p$, fi ergo omnes Logarithmi communes multiplicen-tur per numerum $3, 3219277$, prodibit tabula Logarithmorum pro bafi 2.

108. *Hinc fequitur duorum numerorum Logarithmos in quocun-que fyftemate eandem tenere rationem*

Sint enim duo numeri M & N, quorum pro bafi a Loga-rithmi fint m & n, erit $M = a^m$ & $N = a^n$: hinc fiet $mn = M^n = N^m$, ideoque $M = N^{\frac{m}{n}}$; in qua æquatio-ne cum bafis a non amplius infit, perfpicuum eft Fractionem $\frac{m}{n}$ habere valorem à bafi a non pendentem. Sint enim pro alia bafi b numerorum eorundem M & N Logarithmi μ & ν, pari modo colligetur fore $M = N^{\frac{\mu}{\nu}}$. Erit ergo $N^{\frac{m}{n}} = N^{\frac{\mu}{\nu}}$, hincque $\frac{m}{n} = \frac{\mu}{\nu}$, feu $m : n = \mu : \nu$. Ita jam vidimus

in

LIB. I. in omni Logarithmorum fyftemate Logarithmos diverfarum ejuf-
dem numeri Poteftatum ut y^m & y^n tenere rationem Exponen-
tium $m : n$.

109. Ad canonem ergo Logarithmorum pro bafi quacun-
que a condendum fufficit numerorum tantum primorum Loga-
rithmos methodo ante tradita, vel alia commodiori, fupputaffe.
Cum enim Logarithmi numerorum compofitorum fint æquales
fummis Logarithmorum fingulorum Factorum, Logarithmi nu-
merorum compofitorum per folam additionem reperientur. Sic,
fi habeantur Logarithmi numerorum 3 & 5, erit $l\,15 = l\,3 +$
$l\,5$; $l\,45 = 2\,l\,3 + l\,5$. Atque, cum fupra pro bafi $a = 10$,
inventus fit $l\,5 = 0,6989700$, præterea autem fit $l\,10 = 1$
erit $l\,\dfrac{10}{5} = l\,2 = l\,10 - l\,5$, ideoque orietur $l\,2 = 1 -$
$0,6989700 = 0,3010300$; ex his autem numerorum pri-
morum 2 & 5 Logarithmis inventis reperientur Logarithmi o-
mnium numerorum ex his 2 & 5 compofitorum; cujufmodi funt
ifti 4, 8, 16, 32, 64, &c; 20, 40, 80, 25, 50; &c.

110. Tabularum autem Logarithmicarum ampliffimus eft
ufus in contrahendis calculis numericis, propterea quod ex ejuf-
modi tabulis non folum dati cujufque numeri Logarithmus, fed
etiam cujufque Logarithmi propofiti numerus conveniens repe-
riri poteft. Sic, fi c, d, e, f, g, h, denotent numeros quof-
cunque, citra multiplicationem reperiri poterit valor iftius ex-
preffionis $\dfrac{ccd \sqrt{e}}{f \sqrt[3]{gh}}$, erit enim hujus expreffionis Logarithmus
$= 2\,l\,c + l\,d + \dfrac{1}{2}\,l\,e - l\,f - \dfrac{1}{3}\,l\,g - \dfrac{1}{3}\,l\,h$, cui Lo-
garithmo fi quæratur numerus refpondens, habebitur valor quæ-
fitus. Inprimis autem inferviunt tabulæ Logarithmicæ digni-
tatibus atque radicibus intricatiffimis inveniendis, quarum ope-
rationum loco in Logarithmis tantum multiplicatio & divifio
adhibetur.

EXEM-

EXEMPLUM I.

Quæratur valor hujus Poteſtatis $2^{\frac{7}{12}}$: quoniam ejus Logarithmus eſt $= \frac{7}{12} \, l \, 2$, multiplicetur Logarithmus binarii ex tabulis qui eſt o, 3010300 per $\frac{7}{12}$ hoc eſt per $\frac{1}{2} + \frac{1}{12}$ erit,

$l \, 2^{\frac{7}{12}} =$ o, 1756008, cui Logarithmo reſpondet numerus 1, 498307, qui ergo proxime exhibet valorem $2^{\frac{7}{12}}$.

EXEMPLUM II.

Si numerus incolarum cujuſpiam provinciæ quotannis ſui parte trigeſima augeatur, initio autem in provincia habitaverint 100000 hominum, quæritur poſt 100 annos incolarum numerus. Sit brevitatis gratia initio incolarum numerus $= n$, ita ut ſit $n =$ 100000, anno elapſo uno erit incolarum numerus $= (1 + \frac{1}{30}) n$ $= \frac{31}{30} \, n$: poſt duos annos $= (\frac{31}{30})^2 \, n$: poſt tres annos $= (\frac{31}{30})^3 \, n$, hincque poſt centum annos $= (\frac{31}{30})^{100} \, n =$ $(\frac{31}{30})^{100}$ 100000 ; cujus Logarithmus eſt $= 100 \, l \frac{31}{30} +$ $l \,$ 100000. At eſt $l \frac{31}{30} = l \, 31 - l \, 30 =$ o, 014240439, unde 100 $l \frac{31}{30} =$ 1, 4240439, ad quem ſi addatur l 100000 $= 5$, erit Logarithmus numeri incolarum quæſiti $= 6$ 4240439, cui reſpondet numerus $=$ 2654874. Poſt centum ergo annos numerus incolarum fit plus quam vicies ſexies cum ſemiſſe major.

EXEMPLUM III.

Cum poſt diluvium à ſex hominibus genus humanum ſit pro-
pagatum, ſi ponamus ducentis annis poſt, numerum hominum
jam ad 1000000 excreviſſe, quæritur quanta ſui parte nume-
rus hominum quotannis augeri debuerit. Ponamus hoc tem-
pore numerum hominum parte ſua $\frac{1}{x}$ quotannis increviſſe, at-
que poſt ducentos annos prodierit neceſſe eſt numerus homi-

num $= (\frac{1+x}{x})^{200} 6 = 1000000$, unde fit $\frac{1+x}{x} =$

$(\frac{1000000}{6})^{\frac{1}{200}}$. Erit ergo $l \frac{1+x}{x} = \frac{1}{200} l \frac{1000000}{6} = \frac{1}{200}$.

5, 2218487 $= 0$, 0261092, ideoque $\frac{1+x}{x} = \frac{1061963}{1000000}$, &

1000000 $= 61963 x$, unde fit $x = 16$ circiter. Ad tantam
ergo hominum multiplicationem ſuffeciſſet, ſi quotannis deci-
ma ſexta ſui parte increverint; quæ multiplicatio ob longævam
vitam non nimis magna cenſeri poteſt. Quod ſi autem eadem
ratione per intervallum 400 annorum numerus hominum creſcere
perrexiſſet, tum numerus hominum ad 1000000. $\frac{1000000}{6} =$
166666666666 aſcendere debuiſſet, quibus ſuſtentandis uni-
verſus orbis terrarum nequaquam par fuiſſet.

EXEMPLUM IV.

*Si ſingulis ſeculis numerus hominum duplicetur, quæritur incre-
mentum annuum.* Si quotannis hominum numerum parte ſua
$\frac{1}{x}$ creſcere ponamus, & initio numerus hominum fuerit $= n$,

erit is poſt centum annos $= (\frac{1+x}{x})^{100} n$, qui cum eſſe de-

beat

beat $= 2n$, erit $\frac{1+x}{x} = 2^{\frac{1}{100}}$ & $l \frac{1+x}{x} = \frac{1}{100} l 2 =$

$0, 0030103$; hinc $\frac{1+x}{x} = \frac{10069555}{10000000}$; ergo $x =$

$\frac{10000000}{69555} = 144$, circiter; fufficit ergo fi numerus hominum

quotannis parte fua $\frac{1}{144}$ augeatur. Quam ob caufam maxime ridiculæ funt eorum incredulorum hominum objectiones, qui negant tam brevi temporis fpatio ab uno homine univerfam terram incolis impleri potuiffe.

111. Potiffimum autem Logarithmorum ufus requiritur ad ejufmodi æquationes refolvendas, in quibus quantitas incognita in Exponentem ingreditur. Sic, fi ad hujufmodi perveniatur æquationem $a^x = b$, ex qua incognitæ x valorem erui oporteat, hoc non nifi per Logarithmos effici poterit. Cum enim fit $a^x = b$ erit $l a^x = x l a = l b =$ ideoque $x = \frac{l b}{l a}$, ubi quidem perinde eft, quonam fyftemate Logarithmico utatur, cum in omni fyftemate Logarithmi numerorum a & b eandem inter fe teneant rationem.

Exemplum I.

Si numerus hominum quotannis centefima fui parte augeatur; quæritur poft quot annos numerus hominum fiat decuplo major. Ponamus hoc evenire poft x annos, & initio hominum numerum fuiffe $= n$, erit is ergo elapfis x annis $= \left(\frac{101}{100}\right)^x n$, qui cum æqualis fit $10n$, fiet $\left(\frac{101}{100}\right)^x = 10$; ideoque $x l \frac{101}{100} = l 10$ & $x = \frac{l 10}{l 101 - l 100}$. Prodibit itaque $x = \frac{10000000}{43214} = 231$. Poft annos ergo 231 fiet homi-

Euleri *Introduct. in Anal. infin. parv.*   L   num

num numerus, quorum incrementum annuum tantum centefimam partem efficit, decuplo major; hinc poſt 462 annos fiet centies, & poſt 693 annos millies major.

E x e m p l u m. II.

Qiidam debet 400000 florenos hac conditione ut quotannis uſuram 5 de centenis ſolvere teneatur; exſolvit autem ſingulis annis 25000 florenos: quæritur poſt quot annos debitum penitus extinguatur. Scribamus a pro debita ſumma 400000 fl. & b pro ſumma 25000 fl. quotannis ſoluta; debebit ergo elapſo uno anno $\frac{105}{100} a - b$; elapſis duobus annis $(\frac{105}{100})^2 a - \frac{105}{100} b - b$; elapſis tribus annis $(\frac{105}{100})^3 a - (\frac{105}{100})^2 b - \frac{105}{100} b - b$; hinc, poſito brevitatis cauſa, n pro $\frac{105}{100}$, elapſis x annis adhuc debebit $n^x a - n^{x-1} b - n^{x-2} b - n^{x-3} b - \ldots\ldots - b = n^x a - b(1 + n + n^2 + \ldots + n^{x-1})$. Cum igitur ſit ex natura progreſſionum geometricarum, $1 + n + n^2 + \ldots\ldots + n^{x-1} = \frac{n^x - 1}{n - 1}$, poſt x annos debitor adhuc debebit $n^x a - \frac{n^x b - b}{n - 1}$ flor., quod debitum nihilo æquale poſitum dabit hanc æquationem $n^x a = \frac{n^x b - b}{n - 1}$, ſeu $(n - 1) n^x a = n^x b - b$, ideoque $(b - na + a) n^x = b$ & $n^x = \frac{b}{b - (n-1)a}$, unde fit $x = \frac{lb - l(b - (n-1)a)}{ln}$ Cum jam ſit $a = 400000$, $b = 25000$, $n = \frac{105}{100}$, erit $(n - 1) a = 20000$ & $b - (n - 1) a = 5000$, atque annorum, quibus debitum penitus extinguitur, numerus $x =$

l

$$\frac{l\,25000 - l\,5000}{l\frac{105}{100}} = \frac{l\,5}{l\frac{21}{20}} = \frac{6989700}{211893};$$ erit ergo x aliquanto mi-

nor quam 33 ; fcilicet elapfis annis 33 non folum debitum ex-

tinguetur, fed creditor debitori reddere tenebitur $\frac{(n^{33}-1)b}{n-1}$

$$-n^{33}a = \frac{(\frac{21}{20})^{33} \cdot 5000 - 25000}{\frac{1}{20}} = 100000\,(\frac{21}{20})^{33} - 500000$$

flor. Quia vero eft $l\,\frac{21}{20} = 0,021189299\,1$, erit $l\,(\frac{21}{20})^{33} =$

$0,69924687$, & $l\,100000\,(\frac{21}{20})^{33} = 5,6992469$, cui refpon-

det hic numerus $500318,8$; unde creditor debitori poft 33 an-

nos reftituere debet $318\frac{4}{5}$ florenos.

112. Logarithmi autem vulgares fuper bafi $= 10$ extracti, praeter hunc ufum, quem Logarithmi in genere praeftant, in Arithmetica decimali ufu recepta fingulari gaudent commodo, atque ob hanc caufam prae aliis fyftematibus infignem afferunt utilitatem. Cum enim Logarithmi omnium numerorum, prae-ter denarii Poteftates, in Fractionibus decimalibus exhibeantur, numerorum inter 1 & 10 contentorum Logarithmi intra limi-tes 0 & 1, numerorum autem inter 10 & 100 contentorum Logarithmi inter limites 1 & 2, & ita porro, continebuntur. Conftat ergo Logarithmus quifque ex numero integro & Fra-ctione decimali ; & ille numerus integer vocari folet C H A-R A C T E R I S T I C A ; Fractio decimalis autem M A N T I S S A. Characteriftica itaque unitate deficiet a numero notarum, qui-bus numerus conftat ; ita Logarithmi numeri 78509 Characte-riftica erit 4, quia is ex quinque notis feu figuris conftat. Hinc ex Logarithmo cujufvis numeri ftatim intelligitur, ex quot figuris numerus fit compofitus. Sic numerus Logarithmo $7,5804631$ refpondens ex 8 figuris conftabit.

113. Si ergo duorum Logarithmorum Mantiffae conveniant, Characterifticae vero tantum difcrepent, tum numeri his Loga-

L 2 rithmis

Lib. I. rithmis refpondentes rationem habebunt, ut Poteftas denarii ad unitatem, ideoque ratione figurarum, quibus conftant, convenient. Ita horum Logarithmorum 4, 9130187 & 6 9130187 numeri erunt 81850 & 8185000; Logarithmo autem 3 9130187 conveniet 8185, & Logarithmo huic 0, 9130187 convenit 8, 185. Sola ergo Mantiffa indicabit figuras numerum componentes, quibus inventis, ex Characteriftica patebit, quot figuræ a finiftra ad integra referri debeant, reliquæ ad dextram vero dabunt Fractiones decimales. Sic, fi hic Logarithmus fuerit inventus 2, 7603429, Mantiffa indicabit has figuras 5758945, Characteriftica 2 autem numerum illi Logarithmo determinat, ut fit 575, 8945; fi Characteriftica effet 0, foret numerus 5, 758945; fin denuo unitate minuatur ut fit — 1, erit numerus refpondens decies minor, nempe 0, 5758945; & Characterifticæ — 2 refpondebit 0, 05758945 &c.: loco Characterifticarum autem hujufmodi negativarum — 1, — 2, — 3, &c. fcribi folent 9, 8, 7, &c., atque fubintelligitur hos Logarithmos denario minui debere. Hæc vero in manductionibus ad tabulas Logarithmorum fufius exponi folent.

EXEMPLUM.

Si hæc progreffio 2, 4, 16, 256, &c., *cujus quifque terminus eft quadratum præcedentis, continuetur ufque ad terminum vigefimum quintum; quæritur magnitudo hujus termini ultimi.* Termini hujus progreffionis per Exponentes ita commodius exprimuntur: 2^1, 2^2, 2^4, 2^8, &c. ubi patet Exponentes progreffionem geometricam conftituere, atque termini vigefimi quinti exponentem fore $2^{24} = 16777216$, ita ut ipfe terminus quæfitus fit $= 2^{16777216}$, hujus ergo Logarithmus erit $= 16777216. l2$. Cum ergo fit $l2 = 0, 30102999566398 1195$, erit numeri quæfiti Logarithmus $= 5050445, 25973367$, ex cujus Characteriftica patet numerum quæfitum more folito expreffum conftare ex 5050446 figuris. Mantiffa autem 259733675932 in tabu-

la

la Logarithmorum quæſita dabit figuras initiales numeri quæ-
ſiti, quæ erunt 181858. Quanquam ergo iſte numerus nullo
modo exhiberi poteſt, tamen affirmari poteſt eum omnino ex
5050446 figuris conſtare, atque figuras initiales ſex eſſe 181858,
quas dextrorſum adhuc 5050440 figuræ ſequantur, quarum
inſuper nonnullæ ex majori Logarithmorum canone definiri poſ-
ſent, undecim ſcilicet figuræ initiales erunt 18185852986.

C A P U T V I I.

De quantitatum exponentialium ac Logarithmorum per Series explicatione.

114. QUia eſt $a^° = 1$, atque creſcente Exponente ipſius
a ſimul valor Poteſtatis augetur, ſi quidem *a* eſt
numerus unitate major; ſequitur ſi Exponens infinite parum
cyphram excedat, Poteſtatem ipſam quoque infinite parum
unitatem eſſe ſuperaturam. Sit ω numerus infinite parvus, ſeu
Fractio tam exigua, ut tantum non nihilo ſit æqualis, erit

$a^\omega = 1 + \psi$, exiſtente ψ quoque numero infinite parvo. Ex
præcedente enim capite conſtat niſi ψ eſſet numerus infinite
parvus, neque ω talem eſſe poſſe. Erit ergo vel $\psi = \omega$, vel
$\psi > \omega$, vel $\psi < \omega$, quæ ratio utique a quantitate litteræ *a*
pendebit, quæ cum adhuc ſit incognita, ponatur $\psi = k\omega$,
ita ut ſit $a^\omega = 1 + k\omega$; &, ſumta *a* pro baſi Logarithmica,
erit $\omega = l(1 + k\omega)$.

E X E M P L U M.

Quo clarius appareat, quemadmodum numerus *k* pendeat a
baſi *a*, ponamus eſſe $a = 10$; atque ex tabulis vulgaribus
quæramus Logarithmum numeri quam minime unitatem ſupe-

L 3 rantis,

LIB. I. rantis, puta $1 + \frac{1}{1000000}$, ita ut fit $k \, \omega = \frac{1}{1000000}$; erit

$l(1 + \frac{1}{1000000}) = l\frac{1000001}{1000000} = 0,00000043429 = \omega$. Hinc,

ob $k \, \omega = 0,00000100000$, erit $\frac{1}{k} = \frac{43429}{100000}$ & $k =$

$\frac{100000}{43429} = 2,30258$: unde patet k effe numerum finitum pen-
dentem a valore bafis a. Si enim alius numerus pro bafi a
ftatuatur, tum Logarithmus ejufdem numeri $1 + k\omega$ ad prio-
rem datam tenebit rationem, unde fimul alius valor litteræ k
prodiret.

115. Cum fit $a^{\omega} = 1 + k\omega$, erit $a^{i\omega} = (1 + k\omega)^{i}$, qui-
cunque numerus loco i fubftituatur. Erit ergo $a^{i\omega} = 1 +$
$\frac{i}{1} k\omega + \frac{i(i-1)}{1.2} k^2 \omega^2 + \frac{i(i-1)(i-2)}{1.2.3} k^3 \omega^3 +$ &c.
Quod fi ergo ftatuatur $i = \frac{z}{\omega}$, & z denotet numerum quem-
cunque finitum, ob ω numerum infinite parvum, fiet i numerus
infinite magnus, hincque $\omega = \frac{z}{i}$, ita ut fit ω Fractio deno-
minatorem habens infinitum, adeoque infinite parva, qualis eft
affumta. Subftituatur ergo $\frac{z}{i}$ loco ω, eritque $a^z = (1 +$
$\frac{kz}{i})^i = 1 + \frac{1}{1} kz + \frac{1(i-1)}{1.2i} k^2 z^2 + \frac{1(i-1)(i-2)}{1.2i.3i} k^3 z^3 +$
$\frac{1(i-1)(i-2)(i-3)}{1.2i.3i.4i} k^4 z^4 +$ &c., quæ æquatio erit ve-
ra fi pro i numerus infinite magnus fubftituatur. Tum vero
eft k numerus definitus ab a pendens, uti modo vidimus.

116. Cum autem i fit numerus infinite magnus, erit $\frac{i-1}{i} = 1$;
patet enim quo major numerus loco i fubftituatur, eo propius
valorem Fractionis $\frac{i-1}{i}$ ad unitatem effe acceffurum, hinc fi
i fit

i fit numerus omni affignabili major, Fractio quoque $\frac{i-1}{i}$ CAP.VII.

ipfam unitatem adæquabit. Ob fimilem autem rationem erit

$\frac{i-2}{i} = 1$; $\frac{i-3}{i} = 1$; & ita porro; hinc fequitur fore

$\frac{i-1}{2i} = \frac{1}{2}$; $\frac{i-2}{3i} = \frac{1}{3}$; $\frac{i-3}{4i} = \frac{1}{4}$; & ita porro. His

igitur valoribus fubftitutis, erit $a^z = 1 + \frac{kz}{1} + \frac{k^2 z^2}{1.2} + \frac{k^3 z^3}{1.2.3} +$

$\frac{k^4 z^4}{1.2.3.4} +$ &c. in infinitum. Hæc autem æquatio fimul re-

lationem inter numeros a & k oftendit, pofito enim $z = 1$,

erit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \frac{k^4}{1.2.3.4} +$ &c., atque

ut a fit $= 10$, neceffe eft ut fit circiter $k = 2,30258$, uti

ante invenimus.

117. Ponamus effe $b = a^n$, erit, fumto numero a pro bafi

Logarithmica, $lb = n$. Hinc, cum fit $b^z = a^{nz}$, erit per Se-

riem infinitam $b^z = 1 + \frac{knz}{1} + \frac{k^2 n^2 z^2}{1.2} + \frac{k^3 n^3 z^3}{1.2.3} + \frac{k^4 n^4 z^4}{1.2.3.4} +$

&c., pofito vero lb pro n, erit $b^z = 1 + \frac{kz}{1} lb + \frac{k^2 z^2}{1.2} (lb)^2 +$

$\frac{k^3 z^3}{1.2.3} (lb)^3 + \frac{k^4 z^4}{1.2.3.4} (lb)^4 +$ &c.. Cognito ergo valore

litteræ k ex dato valore bafis a, quantitas exponentialis quæ-

cunque b^z per Seriem infinitam exprimi poterit, cujus termini

fecundum Poteftates ipfius z procedant. His expofitis often-

damus quoque quomodo Logarithmi per Series infinitas ex-

plicari poffint.

118. Cum fit $a^\omega = 1 + k\omega$, exiftente ω Fractione infinite

parva, atque ratio inter a & k definiatur per hanc æquatio-

nem $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} +$ &c., fi a fumatur pro

bafi Logarithmica, erit $\omega = l(1 + k\omega)$ & $i\omega = l(1 + k\omega)^i$.

Mani-

Lɪʙ. I. Manifeſtum autem eſt, quo major numerus pro i ſumatur, eo magis Poteſtatem $(1 + k\omega)^i$ unitatem eſſe ſuperaturam; atque ſtatuendo $i =$ numero infinito, valorem Poteſtatis $(1+k\omega)^i$ ad quemvis numerum unitate majorem aſcendere. Quod ſi ergo ponatur $(1 + k\omega)^i = 1 + x$, erit $l(1+x) = i\omega$, unde, cum ſit $i\omega$ numerus finitus, Logarithmus ſcilicet numeri $1 + x$, perſpicuum eſt, i eſſe debere numerum infinite magnum, alioquin enim $i\omega$ valorem finitum habere non poſſet.

119. Cum autem poſitum ſit $(1+k\omega)^i = 1 + x$, erit $1 + k\omega = (1 + x)^{\frac{1}{i}}$ & $k\omega = (1 + x)^{\frac{1}{i}} - 1$, unde fit $i\omega = \frac{i}{k}((1 + x)^{\frac{1}{i}} - 1)$. Quia vero eſt $i\omega = l(1+x)$, erit

$$l(1 + x) = \frac{i}{k}(1 + x)^{\frac{1}{i}} - \frac{i}{k},$$ poſito i numero infinite magno. Eſt autem $(1+x)^{\frac{1}{i}} = 1 + \frac{1}{i}x - \frac{1(i-1)}{i.\ 2i}x^2 + \frac{1(i-1)(2i-1)}{i.\ 2i.\ 3i}x^3 - \frac{1(i-1)(2i-1)(3i-1)}{i.\ 2i.\ 3i.\ 4i}x^4 + \&c.$

Ob i autem numerum infinitum, erit $\frac{i-1}{2i} = \frac{1}{2}$; $\frac{2i-1}{3i} = \frac{2}{3}$; $\frac{3i-1}{4i} = \frac{3}{4}$, &c.; hinc erit $i(1+x)^{\frac{1}{i}} = i + \frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$, & conſequenter $l(1 + x) = \frac{1}{k}(\frac{x}{1} - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.)$, poſita baſi Logarithmica $= a$ ac denotante k numerum huic baſi convenientem, ut ſcilicet ſit $a = 1 + \frac{k}{1} + \frac{k^2}{1.2} + \frac{k^3}{1.2.3} + \&c.$

120. Cum igitur habeamus Seriem Logarithmo numeri $1+x$ æqualem, ejus ope ex data baſi a definire poterimus valorem numeri

numeri k. Si enim ponamus $1 + x = a$, ob $la = 1$, erit

$$1 = \frac{1}{k} \left(\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c. \right),$$

hincque habebitur $k = \dfrac{a-1}{1} - \dfrac{(a-1)^2}{2} + \dfrac{(a-1)^3}{3} -$

$\dfrac{(a-1)^4}{4} + \&c.$, cujus ideo Seriei infinitæ valor, fi ponatur $a = 10$, circiter effe debebit $= 2,30258$; quanquam diffi-culter intelligi poteft effe $2,30258 = \dfrac{9}{1} - \dfrac{9^2}{2} + \dfrac{9^3}{3} -$

$\dfrac{9^4}{4} + \&c.$, quoniam hujus Seriei termini continuo fiunt majo-res, neque adeo aliquot terminis fumendis fumma vero propin-qua haberi poteft : cui incommodo mox remedium afferetur.

121. Quoniam igitur eft $l(1 + x) = \dfrac{1}{k} \left(\dfrac{x}{1} - \dfrac{x^2}{2} + \right.$

$\left. \dfrac{x^3}{3} - \&c. \right)$, erit, pofito x negativo, $l(1 - x) = - \dfrac{1}{k}$

$\left(\dfrac{x}{1} + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \&c. \right)$. Subtrahatur Series pofte-rior a priori, erit $l(1 + x) - l(1 - x) = l\dfrac{1+x}{1-x} = \dfrac{2}{k} \times$

$\left(\dfrac{x}{1} + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \&c. \right)$. Nunc ponatur $\dfrac{1+x}{1-x} = a$,

ut fit $x = \dfrac{a-1}{a+1}$, ob $la = 1$ erit $k = 2 \left(\dfrac{a-1}{a+1} + \right.$

$\left. \dfrac{(a-1)^3}{3(a+1)^3} + \dfrac{(a-1)^5}{5(a+1)^5} + \&c. \right)$, ex qua æquatione valor numeri k ex bafi a inveniri poterit. Si ergo bafis a ponatur $= 10$ erit $k = 2 \left(\dfrac{9}{11} + \dfrac{9^3}{3.11^3} + \dfrac{9^5}{5.11^5} + \dfrac{9^7}{7.11^7} + \&c. \right)$, cu-jus Seriei termini fenfibiliter decrefcunt, ideoque mox valorem pro k fatis propinquum exhibent.

122. Quoniam ad fyftema Logarithmorum condendum ba-fin a pro lubitu accipere licet, ea ita affumi poterit ut fiat $k = 1$. Ponamus ergo effe $k = 1$, eritque per Seriem fupra

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(116) inventam, $a = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$, qui termini , fi in fractiones decimales convertantur atque actu addantur , præbebunt hunc valorem pro $a =$ 2,718281828459045235236028, cujus ultima adhuc nota veritati eft confentanea.   Quod fi jam ex hac bafi Logarithmi conftruantur , ii vocari folent Logarithmi *naturales* feu *hyperbolici*, quoniam quadratura hyperbolæ per iftiufmodi Logarithmos exprimi poteft.   Ponamus autem brevitatis gratia pro numero hoc 2,718281828459 &c. conftanter litteram e, quæ ergo denotabit bafin Logarithmorum naturalium feu hyperbolicorum, cui refpondet valor litteræ $k = 1$; five hæc littera e quoque exprimet fummam hujus Seriei $1 + \frac{1}{1} + \frac{1}{1.2} +$

$\frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \&c.$ in infinitum.

123. Logarithmi ergo hyperbolici hanc habebunt proprietatem , ut numeri $1 + \omega$ Logarithmus fit $= \omega$, denotante ω quantitatem infinite parvam, atque cum ex hac proprietate valor $k = 1$ innotefcat, omnium numerorum Logarithmi hyperbolici exhiberi poterunt.   Erit ergo, pofita e pro numero fupra invento, perpetuo $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} + \&c.$ ipfi vero Logarithmi hyperbolici ex his Seriebus invenientur , quibus eft $l(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} -$

$\frac{x}{6} + \&c$, & $l\frac{1+x}{1-x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \frac{2x^9}{9} + \&c.$, quæ Series vehementer convergunt, fi pro x ftatuatur fractio valde parva : ita ex Serie pofteriori facili negotio inveniuntur Logarithmi numerorum unitate non multo majorum.   Pofito namque $x = \frac{1}{5}$, erit $l\frac{6}{4} = l\frac{3}{2} = \frac{2}{1.5} + \frac{2}{3.5^3} + \frac{2}{5.5^5} +$

$\frac{2}{7.5^7} + \&c.$, & facto $x = \frac{1}{7}$, erit $l\frac{4}{3} = \frac{2}{1.7} + \frac{2}{3.7^3} + \frac{2}{5.7^5} +$

$\frac{2}{7.7^7}$

$\frac{2}{7 \cdot 7^7} + $ &c., facto $x = \frac{1}{9}$, erit $l \frac{5}{4} = \frac{2}{1 \cdot 9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} + \frac{2}{7 \cdot 9^7} + $ &c.. Ex Logarithmis vero harum fractionum reperientur Logarithmi numerorum integrorum, erit enim ex natura Logarithmorum $l \frac{3}{2} + l \frac{4}{3} = l 2$; tum $l \frac{3}{2} + l 2 = l 3$; & $2 l 2 = l 4$; porro $l \frac{5}{4} + l 4 = l 5$; $l 2 + l 3 = l 6$; $3 l 2 = l 8$; $2 l 3 = l 9$; & $l 2 + l 5 = l 10$.

EXEMPLUM.

Hinc Logarithmi hyperbolici numerorum ab 1 usque ad 10 ita se habebunt, ut sit

$$l 1 = 0, 00000\ 00000\ 00000\ 00000\ 00000$$
$$l 2 = 0, 69314\ 71805\ 59945\ 30941\ 72321$$
$$l 3 = 1, 09861\ 22886\ 68109\ 69139\ 52452$$
$$l 4 = 1, 38629\ 43611\ 19890\ 61883\ 44642$$
$$l 5 = 1, 60943\ 79124\ 34100\ 37460\ 07593$$
$$l 6 = 1, 79175\ 94692\ 28055\ 00081\ 24773$$
$$l 7 = 1, 94591\ 01490\ 55313\ 30510\ 54639$$
$$l 8 = 2, 07944\ 15416\ 79835\ 92825\ 16964$$
$$l 9 = 2, 19722\ 45773\ 36219\ 38279\ 04905$$
$$l 10 = 2, 30258\ 50929\ 94045\ 68401\ 79914$$

Hi scilicet Logarithmi omnes ex superioribus tribus Seriebus sunt deducti, praeter $l 7$, quem hoc compendio sum assecutus. Posui nimirum in Serie posteriori $x = \frac{1}{99}$ sicque obtinui $l \frac{100}{98} = l \frac{50}{49} = 0, 02020270731751944840782 30$, qui subtractus a $l 50 = 2 l 5 + l 2 = 3, 91202300542814605861875 08$, relinquit $l 49$, cujus semissis dat $l 7$.

M 2

124. Ponatur Logarithmus hyperbolicus ipſius $1 + x$ ſeu $l(1 + x) = y$; erit $y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.$. Sumto autem numero a pro baſi Logarithmica, ſit numeri ejuſdem $1 + x$ Logarithmus $= v$; erit, ut vidimus, $v = \frac{1}{k}(x - \frac{xx}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \&c.) = \frac{y}{k}$; hincque $k = \frac{y}{v}$; ex quo commodiſſime valor ipſius k baſi a reſpondens ita definitur ut ſit æqualis cujuſvis numeri Logarithmo hyperbolico diviſo per Logarithmum ejuſdem numeri ex baſi a formati. Poſito ergo numero hoc $= a$, erit $v = 1$, hincque fit $k =$ Logarithmo hyperbolico baſis a. In ſyſtemate ergo Logarithmorum communium, ubi eſt $a = 10$, erit $k =$ Logarithmo hyperbolico ipſius 10, unde fit $k = 2,3025850929940456840179914$, quem valorem jam ſupra ſatis prope collegimus. Si ergo ſinguli Logarithmi hyperbolici per hunc numerum k dividantur, vel, quod eodem redit, multiplicentur per hanc fractionem decimalem $0,4342944819032518276511289$, prodibunt Logarithmi vulgares baſi $a = 10$ convenientes.

125. Cum ſit $e^z = 1 + \frac{z}{1} + \frac{z^2}{1.2} + \frac{z^3}{1.2.3} + \&c.$, ſi ponatur $a^y = e^z$, erit, ſumtis Logarithmis hyperbolicis, $y\,la = z$, quia eſt $le = 1$, quo valore loco z ſubſtituto, erit $a^y = 1 + \frac{y\,la}{1} + \frac{y^2(la)^2}{1.2} + \frac{y^3(la)^3}{1.2.3} + \&c.$, unde quælibet quantitas exponentialis ope Logarithmorum hyperbolicorum per Seriem infinitam explicari poteſt. Tum vero, denotante i numerum infinite magnum, tam quantitates exponentiales quam Logarithmi per poteſtates exponi poſſunt. Erit enim $e^z = (1 + \frac{z}{i})^i$, hincque $ay = (1 + \frac{y\,la}{i})^i$, deinde pro Logarithmis hyperbolicis habetur $l(1 + x) = i((1 + x)^{\frac{1}{i}} - 1)$. De cetero

tero Logarithmorum hyperbolicorum ufus in calculo integrali fufius demonftrabitur.

C A P U T V I I I.

De quantitatibus tranfcendentibus ex Circulo ortis.

126. POft Logarithmos & quantitates exponentiales confiderari debent Arcus circulares eorumque Sinus & Cofinus, quia non folum aliud quantitatum tranfcendentium genus conftituunt, fed etiam ex ipfis Logarithmis & exponentialibus, quando imaginariis quantitatibus involvuntur, proveniunt, id quod infra clarius patebit.

Ponamus ergo Radium Circuli feu Sinum totum effe $= 1$, atque fatis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non poffe, per approximationes autem inventa eft Semicircumferentia hujus Circuli effe $=$

3, 14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384466 $+$, pro quo numero, brevitatis ergo, fcribam π, ita ut fit $\pi =$ Semicircumferentiæ Circuli, cujus Radius $= 1$, feu π erit longitudo Arcus 180 graduum.

127. Denotante z Arcum hujus Circuli quemcunque, cujus Radium perpetuo affumo $= 1$; hujus Arcus z confiderari potiffimum folent Sinus & Cofinus. Sinum autem Arcus z in pofterum hoc modo indicabo, *fin. A. z*, feu tantum *fin. z*. Cofinum vero hoc modo *cof. A. z*, feu tantum *cof. z*. Ita, cum π fit Arcus 180°, erit *fin.* $0 \pi = 0$; *cof.* $0 \pi = 1$; & *fin.* $\frac{1}{2} \pi = 1$, *cof.* $\frac{1}{2} \pi = 0$; *fin.* $\pi = 0$; *cof.* $\pi = -1$;

fin. $\frac{3}{2} \pi = -1$; *cof.* $\frac{3}{2} \pi = 0$; *fin.* $2\pi = 0$; & *cof.* $2\pi = 1$.

Omnes ergo Sinus & Cofinus intra limites $+ 1$ & $- 1$ con-

tinen-

tinentur. Erit autem porro $cof. z = fin. (\frac{1}{2} \pi - z)$, & $fin. z = cof. (\frac{1}{2} \pi - z)$, atque $(fin. z)^2 + (cof. z)^2 = 1$. Præter has denominationes notandæ funt quoque hæ: *tang. z*, quæ denotat Tangentem Arcus z ; *cot. z* Cotangentem Arcus z ; conftatque effe $tang. z = \frac{fin. z}{cof. z}$ & $cot. z = \frac{cof. z}{fin. z} = \frac{1}{tang. z}$; quæ omnia ex Trigonometria funt nota.

128. Hinc vero etiam conftat fi habeantur duo Arcus y & z, fore $fin. (y+z) = fin. y. cof. z + cof. y. fin. z$, & $cof. (y+z) = cof. y. cof. z - fin. y. fin. z$, itemque $fin. (y - z) = fin. y. cof. z - cof. y. fin. z$ & $cof. (y - z) = cof. y. cof. z + fin. y. fin. z$.

Hinc loco y fubftituendo Arcus $\frac{1}{2} \pi$; π ; $\frac{3}{2} \pi$, &c., erit

$$fin. (\tfrac{1}{2} \pi + z) = + cof. z \qquad\qquad fin. (\tfrac{1}{2} \pi - z) = + cof. z$$

$$cof. (\tfrac{1}{2} \pi + z) = - fin. z \qquad\qquad cof. (\tfrac{1}{2} \pi - z) = + fin. z$$

$$fin. (\pi + z) = - fin. z \qquad\qquad fin. (\pi - z) = + fin. z$$

$$cof. (\pi + z) = - cof. z \qquad\qquad cof. (\pi - z) = - cof. z$$

$$fin. (\tfrac{3}{2} \pi + z) = - cof. z \qquad\qquad fin. (\tfrac{3}{2} \pi - z) = - cof. z$$

$$cof. (\tfrac{3}{2} \pi + z) = + fin. z \qquad\qquad cof. (\tfrac{3}{2} \pi - z) = - fin. z$$

$$fin. (2 \pi + z) = + fin. z \qquad\qquad fin. (2 \pi - z) = - fin. z$$

$$cof. (2 \pi + z) = + cof. z \qquad\qquad cof. (2 \pi - z) = + cof. z$$

Si ergo n denotet numerum integrum quemcunque , erit

$$\sin.(\tfrac{4n+1}{2}\pi+z) = + \cos.z \qquad \sin.(\tfrac{4n+1}{2}\pi - z) = + \cos.z$$

$$\cos.(\tfrac{4n+1}{2}\pi+z) = - \sin.z \qquad \cos.(\tfrac{4n+1}{2}\pi - z) = - \sin.z$$

$$\sin.(\tfrac{4n+2}{2}\pi+z) = - \sin.z \qquad \sin.(\tfrac{4n+2}{2}\pi - z) = + \sin.z$$

$$\cos.(\tfrac{4n+2}{2}\pi+z) = - \cos.z \qquad \cos.(\tfrac{4n+2}{2}\pi - z) = - \cos.z$$

$$\sin.(\tfrac{4n+3}{2}\pi+z) = - \cos.z \qquad \sin.(\tfrac{4n+3}{2}\pi - z) = - \cos.z$$

$$\cos.(\tfrac{4n+3}{2}\pi+z) = + \sin.z \qquad \cos.(\tfrac{4n+3}{2}\pi - z) = - \sin.z$$

$$\sin.(\tfrac{4n+4}{2}\pi+z) = + \sin.z \qquad \sin.(\tfrac{4n+4}{2}\pi - z) = - \sin.z$$

$$\cos.(\tfrac{4n+4}{2}\pi+z) = + \cos.z \qquad \cos.(\tfrac{4n+4}{2}\pi - z) = + \cos.z$$

Quæ formulæ veræ funt five n fit numerus affirmativus five negativus integer.

129. Sit $\sin.z = p$ & $\cos.z = q$ erit $pp + qq = 1$; & $\sin.y = m$; $\cos.y = n$; ut fit quoque $mm + nn = 1$; Arcuum ex his compofitorum Sinus & Cofinus ita fe habebunt.

$$\sin. \quad z = p$$
$$\sin.(y+z) = mq + np$$
$$\sin.(2y+z) = 2mnq + (nn - mm)p$$
$$\sin.(3y+z) = (3mn^2 - m^3)q + (n^3 - 3m^2n)p$$
$$\&c.$$

$$\cos. \quad z = q$$
$$\cos.(y+z) = nq - mp$$
$$\cos.(2y+z) = (nn - mm)q - 2mnp$$
$$\cos.(3y+z) = (n^3 - 3m^2n)q - (3mn^2 - m^3)p$$
$$\&c.$$

Arcus ifti z, $y + z$, $2y + z$, $3y + z$, &c., in arithmetica progreffione progrediuntur ; eorum vero tam Sinus quam Cofinus progreffionem recurrentem conftituunt, qualis ex denominatore $1 - 2nx + (mm + nn)xx$ oritur ; eft enim

$$\sin.$$

LIB. I. $\sin. (2y+z) = 2n \sin. (y +z) - (mm+nn)\sin. z$ sive

$\sin. (2y+z) = 2\cos. y. \sin.(y+z) - (\sin. z)$; atque simili modo

$\cos. (2y+z) = 2\cos. y. \cos.(y+z) - \cos. z$. Eodem modo erit porro

$\sin.(3y+z) = 2\cos. y. \sin.(2y+z) - \sin.(y+z)$,   &

$\cos.(3y+z) = 2\cos. y. \cos.(2y+z) - \cos.(y+z)$, itemque

$\sin.(4y+z) = 2\cos. y. \sin.(3y+z) - \sin.(2y+z)$,   &

$\cos.(4y+z) = 2\cos. y. \cos.(3y+z) - \cos.(2y+z)$ &c.

Cujus legis beneficio Arcuum in progressione arithmetica progredientium tam Sinus quam Cosinus quousque libuerit expedite formari possunt.

130. Cum sit $\sin. (y + z) = \sin. y.\cos. z + \cos. y \sin. z$ atque $\sin. (y - z) = \sin. y. \cos. z - \cos. y. \sin. z$, erit his expressionibus vel addendis vel subtrahendis :

$$\sin. y. \cos. z = \frac{\sin. (y + z) + \sin. (y - z)}{2}$$

$$\cos. y. \sin. z = \frac{\sin. (y + z) - \sin. (y - z)}{2}$$

Quia porro est $\cos. (y+z) = \cos. y. \cos. z - \sin. y. \sin. z$, atque $\cos. (y - z) = \cos. y. \cos. z + \sin. y. \sin. z$, erit pari modo

$$\cos. y. \cos. z = \frac{\cos.(y - z) + \cos. (y+z)}{2}$$

$$\sin. y. \sin. z = \frac{\cos.(y - z) - \cos. (y+z)}{2}.$$

Sit $y = z = \frac{1}{2} v$, erit ex his postremis formulis :

$$(\cos. \tfrac{1}{2} v)^2 = \frac{1+\cos. v}{2}, \& \cos. \tfrac{1}{2} v = \sqrt{\frac{1+\cos. v}{2}}$$

$$(\sin. \tfrac{1}{2} v)^2 = \frac{1-\cos. v}{2}, \& \sin. \tfrac{1}{2} v = \sqrt{\frac{1-\cos. v}{2}}$$

unde, ex dato Cosinu cujusque anguli reperiuntur ejus semissis Sinus & Cosinus.

131. Ponatur Arcus $y + z = a$, & $y - z = b$; erit $y = \frac{a + b}{2}$ & $z = \frac{a - b}{2}$, quibus in superioribus formulis substi-

tutis

tutis, habebuntur hæ æquationes, tanquam totidem Theore- C A P.
mata. V I I I.

$$\textit{fin. } a \; + \; \textit{fin. } b \; = \; 2 \; \textit{fin. } \frac{a+b}{2} \; \textit{cof. } \frac{a-b}{2}$$

$$\textit{fin. } a \; - \; \textit{fin. } b \; = \; 2 \; \textit{cof. } \frac{a+b}{2} \; \textit{fin. } \frac{a-b}{2}$$

$$\textit{cof. } a \; + \; \textit{cof. } b \; = \; 2 \; \textit{cof. } \frac{a+b}{2} \; \textit{cof. } \frac{a-b}{2}$$

$$\textit{cof. } b \; - \; \textit{cof. } a \; = \; 2 \; \textit{fin. } \frac{a+b}{2} \; \textit{fin. } \frac{a-b}{2}$$

ex his porro nafcuntur, ope divifionis, hæc Theoremata

$$\frac{\textit{fin. } a + \textit{fin. } b}{\textit{fin. } a - \textit{fin. } b} \; = \; \textit{tang. } \frac{a+b}{2} . \; \textit{cot. } \frac{a-b}{2} = \frac{\textit{tang. } \frac{a+b}{2}}{\textit{tang. } \frac{a-b}{2}}$$

$$\frac{\textit{fin. } a + \textit{fin. } b}{\textit{cof. } a + \textit{cof. } b} \; = \; \textit{tang. } \quad \frac{a+b}{2}$$

$$\frac{\textit{fin. } a + \textit{fin. } a}{\textit{cof. } b - \textit{cof. } a} \; = \; \textit{cot.. } \quad \frac{a-b}{2}$$

$$\frac{\textit{fin. } a - \textit{fin. } b}{\textit{cof. } a + \textit{cof. } b} \; = \; \textit{tang. } \quad \frac{a-b}{2}$$

$$\frac{\textit{fin. } a - \textit{fin. } a}{\textit{cof. } b - \textit{cof. } a} \; = \; \textit{cot. } \quad \frac{a+b}{2}$$

$$\frac{\textit{cof. } a + \textit{cof. } b}{\textit{cof. } b - \textit{cof. } a} \; = \; \textit{cot. } \quad \frac{a+b}{2} . \; \textit{cot. } \frac{a-b}{2}$$

Ex his denique deducuntur ifta Theoremata

$$\frac{\textit{fin. } a + \textit{fin. } b}{\textit{cof. } a + \textit{cof. } b} \; = \; \frac{\textit{cof. } b - \textit{cof. } a}{\textit{fin. } a - \textit{fin. } b} ;$$

$$\frac{\textit{fin. } a + \textit{fin. } b}{\textit{fin. } a - \textit{fin. } b} \times \frac{\textit{cof. } a + \textit{cof. } b}{\textit{cof. } b - \textit{cof. } a} \; = \; \left(\textit{cot. } \frac{a-b}{2} \right)^2$$

$$\frac{\textit{fin. } a + \textit{fin. } b}{\textit{fin. } a - \textit{fin. } b} \times \frac{\textit{cof. } b - \textit{cof. } a}{\textit{cof. } a + \textit{cof. } b} \; = \; \left(\textit{tang. } \frac{a+b}{2} \right)^2$$

132. Cum fit $(\textit{fin. } z)^2 + (\textit{cof. } z)^2 = 1$ erit, Factoribus
fumendis, $(\textit{cof. } z + \sqrt{-1} . \textit{fin. } z)(\textit{cof. } z - \sqrt{-1} . \textit{fin.} z) = 1$;
qui Factores, etfi imaginarii, tamen ingentem præftant ufum in
Arcubus combinandis & multiplicandis. Quæratur enim pro-
ductum horum Factorum $(\textit{cof. } z + \sqrt{-1} . \textit{fin. } z)(\textit{cof. } y. + \sqrt{-1} .$
$\textit{fin.} y.)$ ac reperietur $\textit{cof. } y. \; \textit{cof. } z - \textit{fin.} y. \textit{fin.} z + (\textit{cof.} y. \textit{fin.} z + \textit{fin.} y. \textit{cof.} z)$

Euleri Introduct. in Anal. infin. parv.   N $\sqrt{}$

$\sqrt{-1}$. Cum autem fit *cof. y. cof. z* — *fin. y. fin. z* $=$ *cof.* $(y+z)$ & *cof. y. fin. z* $+$ *fin. y. cof. z* $=$ *fin.* $(y+z)$ erit hoc productum $(cof. y. + \sqrt{-1}. fin. y.)(cof. z + \sqrt{-1}. fin. z) = cof. (y+z) + \sqrt{-1}. fin. (y+z)$

<div align="center">& fimili modo</div>

$(cof. y - \sqrt{-1}. fin. y)(cof. z - \sqrt{-1}. fin. z) = cof. (y+z) - \sqrt{-1}. fin. (y+z)$

<div align="center">item</div>

$(cof. x \pm \sqrt{-1}. fin. x)(cof. y \pm \sqrt{-1}. fin. y)(cof. z \pm \sqrt{-1}. fin. z) = cof. (x+y+z) \pm \sqrt{-1}. fin. (x+y+z).$

133. Hinc itaque fequitur fore $(cof. z \pm \sqrt{-1}. fin z)^2 = cof. 2z \pm \sqrt{-1}. fin. 2z$, & $(cof. z \pm \sqrt{-1}. fin. z)^3 = cof. 3 z \pm \sqrt{-1}. fin. 3 z.$

ideoque generaliter erit $(cof. z \pm \sqrt{-1}. fin. z)^n = cof. nz \pm \sqrt{-1}. fin. nz.$

Unde, ob fignorum ambiguitatem, erit

$$cof. nz = \left(\frac{(cof. z + \sqrt{-1}. fin. z)^n + (cof. z - \sqrt{-1}. fin. z)^n}{2} \right) \&$$

$$fin. nz = \frac{(cof. z + \sqrt{-1}. fin. z)^n - (cof. z - \sqrt{-1}. fin. z)^n}{2 \sqrt{-1}}$$

Evolutis ergo binomiis hifce erit per Series:

$$cof. nz = (cof. z)^n - \frac{n(n-1)}{1.2}(cof. z)^{n-2}(fin. z)^2 + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}(cof. z)^{n-4}(fin. z)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 2.\ 3.\ 4.\ 5.\ 6}(cof. z)^{n-6}(fin. z)^6 + \&c., \&$$

$$fin. nz = \frac{n}{1}(cof. z)^{n-1} fin. z - \frac{n(n-1)(n-2)}{1.\ 2.\ 3}(cof. z)^{n-3}(fin. z)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5}(cof. z)^{n-5}(fin. z)^5 - \&c.$$

134. Sit Arcus z infinite parvus, erit *fin.* $z = z$ & *cof.* z CAP.
$= 1$: fit autem n numerus infinite magnus, ut fit Arcus $n z$ VIII.

finitæ magnitudinis, puta, $nz = v$; ob *fin.* $z = z = \frac{v}{n}$ erit

cof. $v = 1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3.4.5.6} + $ &c. , &

fin. $v = v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3.4.5.6.7} + $ &c. Da-
to ergo Arcu v, ope harum Serierum ejus Sinus & Cofinus
inveniri poterunt; quarum formularum ufus quo magis pateat,
ponamus Arcum v effe ad quadrantem, feu 90°, ut m ad n, feu

effe $v = \frac{m}{n} \cdot \frac{\pi}{2}$; Quia nunc valor ipfius π conftat, fi is ubi-
que fubftituatur, prodibit

$$\text{fin. A. } \frac{m}{n} \, 90° =$$

$$+ \frac{m}{n} \cdot 1, 5707963267948966192313216916$$

$$- \frac{m^3}{n^3} \cdot 0, 6459640975062462536557565636$$

$$+ \frac{m^5}{n^5} \cdot 0, 0796926262461670451205055488$$

$$- \frac{m^7}{n^7} \cdot 0, 0046817541353186881006854632$$

$$+ \frac{m^9}{n^9} \cdot 0, 0001604411847873598218726605$$

$$- \frac{m^{11}}{n^{11}} \cdot 0, 0000035988432352120853404580$$

$$+ \frac{m^{13}}{n^{13}} \cdot 0, 0000000569217292196792681171$$

$$- \frac{m^{15}}{n^{15}} \cdot 0, 0000000006688035109811467224$$

$$+ \frac{m^{17}}{n^{17}} \cdot 0, 0000000000060669357311061950$$

$- \dfrac{m^{19}}{n^{19}}.$ 0, 00000000000000437706546731370

$+ \dfrac{m^{21}}{n^{21}}.$ 0, 00000000000000002571422892856

$- \dfrac{m^{33}}{n^{33}}.$ 0, 00000000000000000012538995403

$+ \dfrac{m^{35}}{n^{35}}.$ 0, 00000000000000000000051564550

$- \dfrac{m^{27}}{n^{27}}.$ 0, 0000000000000000000000181239

$+ \dfrac{m^{29}}{n^{29}}.$ 0, 00000000000000000000000000549

atque *cof.* A. $\dfrac{m}{n}$ 90°$=$

$+$ 1, 000000000000000000000000000000

$- \dfrac{m^{2}}{n^{2}}.$ 1, 2337005501361698273543113745

$+ \dfrac{m^{4}}{n^{4}}.$ 0, 2536695079010480136365633659

$- \dfrac{m^{6}}{n^{6}}.$ 0, 0208634807633529608730516364

$+ \dfrac{m^{8}}{n^{8}}.$ 0, 0009192602748394265802417158

$- \dfrac{m^{10}}{n^{10}}.$ 0, 0000252020423730606054810526

$+ \dfrac{m^{12}}{n^{12}}.$ 0, 0000004710874778818171503665

$- \dfrac{m^{14}}{n^{14}}.$ 0, 0000000063866030837918522408

$+ \dfrac{m^{16}}{n^{16}}.$ 0, 0000000000656596311497947230

$- \dfrac{m^{18}}{n^{18}}.$ 0, 0000000000005294400200734620

$+ \dfrac{m^{20}}{n^{20}}.$ 0, 0000000000000034377391790981

$- \dfrac{m^{22}}{n^{22}}.$ 0, 0000000000000000183599165212

$+$

$$+ \frac{m^{24}}{n^{24}}. \quad 0, \, 00000000000000000000820675327$$

$$- \frac{m^{26}}{n^{26}}. \quad 0, \, 000000000000000000003115285$$

$$+ \frac{m^{28}}{n^{28}}. \quad 0, \, 0000000000000000000000010165$$

$$- \frac{m^{30}}{n^{30}}. \quad 0, \, 00000000000000000000000000026$$

Cum igitur fufficiat Sinus & Cofinus angulorum ad 45° noffe, fractio $\frac{m}{n}$ femper minor erit quam $\frac{1}{2}$, hincque etiam ob Poteftates fractionis $\frac{m}{n}$, Series exhibitæ maxime convergent, ita ut plerumque aliquot tantum termini fufficiant, præcipue, fi Sinus & Cofinus non ad tot figuras defiderentur.

135. Inventis Sinibus & Cofinibus inveniri quidem poffunt Tangentes & Cotangentes, per analogias confuetas, at quia in hujufmodi ingentibus numeris multiplicatio & divifio vehementer eft molefta, peculiari modo eas exprimere convenit. Erit ergo

$$tang. \, v = \frac{\text{fin. } v}{\text{cof. } v} = \frac{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2 \; 3....7} + \&c.}{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.0....6} + \&c.}$$

$$\& \, cot. \, v = \frac{\text{cof. } v}{\text{fin. } v} = \frac{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3....6} + \&c.}{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2 \; 3....7} + \&c.}$$

fi

si jam sit Arcus $v = \frac{m}{n} 90°$ erit eodem modo quo ante

tang. A. $\frac{m}{n} 90° =$	cot. A. $\frac{m}{n} 90° =$
$+ \frac{2mn}{nn-mm}$. 0, 6366197723675	$+ \frac{n}{m}$ 0, 6366197723675
$+ \frac{m}{n}$. 0, 2975567820597	$- \frac{4mn}{4nn-mm}$. 0, 3183098861837
$+ \frac{m^3}{n^3}$. 0, 0186886502773	$- \frac{m}{n}$. 0, 2052888894145
$+ \frac{m^5}{n^5}$. 0, 0018424752034	$- \frac{m^3}{n^3}$. 0, 0065510747882
$+ \frac{m^7}{n^7}$. 0, 0001975800714	$- \frac{m^5}{n^5}$. 0, 0003450292554
$+ \frac{m^9}{n^9}$. 0, 0000216977245	$- \frac{m^7}{n^7}$. 0, 0000202791060
$+ \frac{m^{11}}{n^{11}}$. 0, 0000024011370	$- \frac{m^9}{n^9}$. 0, 0000012366527
$+ \frac{m^{13}}{n^{13}}$. 0, 0000002664132	$- \frac{m^{11}}{n^{11}}$. 0, 0000000764959
$+ \frac{m^{15}}{n^{15}}$. 0, 0000000295864	$- \frac{m^{13}}{n^{13}}$. 0, 0000000047597
$+ \frac{m^{17}}{n^{17}}$. 0, 0000000032867	$- \frac{m^{15}}{n^{15}}$. 0, 0000000002969
$+ \frac{m^{19}}{n^{19}}$. 0, 0000000003651	$- \frac{m^{17}}{n^{17}}$. 0, 0000000000185
$+ \frac{m^{21}}{n^{21}}$. 0, 0000000000405	$- \frac{m^{19}}{n^{19}}$. 0, 0000000000011
$+ \frac{m^{23}}{n^{23}}$. 0, 0000000000045	
$+ \frac{m^{25}}{n^{25}}$. 0, 0000000000005	

quarum Serierum ratio infra fusius exponetur.

136. Ex superioribus quidem constat, si cogniti fuerint omnium angulorum semirecto minorum Sinus & Cosinus, inde simul omnium angulorum majorum Sinus & Cosinus haberi. Verum si tantum angulorum 30° minorum habeantur

Sinus

Sinus & Cofinus, ex iis, per folam additionem & fubtra&tio- C A P.
nem, omnium angulorum majorum Sinus & Cofinus inveniri V I I I.

poffunt. Cum enim fit *fin.* 30° $= \frac{1}{2}$, erit, pofito $y = 30°$ ex
(130) *cof. z* $=$ *fin.* (30 $+ z$) $+$ *fin.* (30 $- z$); & *fin. z* $=$
cof. (30 $- z$) $-$ *cof.* (30 $+ z$), ideoque ex Sinibus & Co-
finibus angulorum z & 30 $- z$, reperiuntur *fin.* (30 $+ z$) $=$
cof. z $-$ *fin.* (30 $- z$) & *cof.* (30 $+ z$) $=$ *cof.* (30 $- z$) $-$ *fin. z*,
unde Sinus & Cofinus angulorum a 30° ad 60°, hincque omnes
majores definiuntur.

137. In Tangentibus & Cotangentibus fimile fubfidium ufu
venit. Cum enim fit *tang.* ($a + b$) $= \frac{tang.\ a\ +\ tang.\ b}{1\ -\ tang.\ a.\ tang.\ b}$, erit

tang. 2a $= \frac{2\ tang.\ a}{1\ -\ tang.\ a. tang.\ a}$, & *cot.* 2 a $= \frac{cot.\ a\ -\ tang.\ a}{2}$
unde ex Tangentibus & Cotangentibus Arcuum 30° minorum
inveniuntur Cotangentes ufque ad 60°.

Sit jam $a = 30 - b$ erit 2$a = 60 - 2b$ & *cot.* 2$a =$
tang. (30 $+ 2b$); erit ergo *tang.* (30 $+ 2 b$) $=$
$\frac{cot.\ (30\ -\ b)\ -\ tang.\ (30\ -\ b)}{2}$, unde etiam Tangentes Ar-
cuum 30° majorum obtinentur.

Secantes autem & Cofecantes ex Tangentibus per folam fub-
tra&ionem inveniuntur; eft enim *cofec. z* $=$ *cot.* $\frac{1}{2}$ z $-$ *cot. z*,

& hinc *fec. z* $=$ *cot.* (45° $- \frac{1}{2}$ z) $-$ *tang. z.* Ex his ergo
luculenter perfpicitur, quomodo canones Sinuum conftrui po-
tuerint.

138. Ponatur denuo in formulis §. 133, Arcus z infinite
parvus, & fit n numerus infinite magnus i, ut iz obti-
neat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$,

unde *fin. z* $= \frac{v}{i}$ & *cof. z* $= 1$; his fubftitutis fit *cof.* $v =$

(1 $+$

$$\frac{(1 + \frac{v\sqrt{-1}}{i})^i + (1 - \frac{v\sqrt{-1}}{i})^i}{2} \; ; \text{ atque } \textit{fin. } v =$$

$$\frac{(1 + \frac{v\sqrt{-1}}{i})^i - (1 - \frac{v\sqrt{-1}}{i})^i}{2\sqrt{-1}}. \quad \text{In Capite autem}$$

præcedente vidimus effe $(1 + \frac{z}{i})^i = e^z$, denotante e bafin Logarithmorum hyperbolicorum : fcripto ergo pro z partim $+ v \sqrt{-1}$ partim $- v \sqrt{-1}$ erit $\textit{cof. } v =$

$$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2} \quad \& \textit{fin.} v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur quomodo quantitates exponentiales imaginariæ ad Sinus & Cofinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = \textit{cof. } v + \sqrt{-1}.\textit{fin. } v$ & $e^{-v\sqrt{-1}} = \textit{cof. } v - \sqrt{-1}.\textit{fin. } v$.

139. Sit jam in iifdem formulis §. 130. n numerus infinite parvus, feu $n = \frac{1}{i}$, exiftente i numero infinite magno, erit $\textit{cof. } n z = \textit{cof. } \frac{z}{i} = 1$ & $\textit{fin. } nz = \textit{fin. } \frac{z}{i} = \frac{z}{i}$; Arcus enim evanefcentis $\frac{z}{i}$ Sinus eft ipfi æqualis, Cofinus vero $= 1$. His pofitis habebitur

$$1 = \frac{(\textit{cof. } z + \sqrt{-1}.\textit{fin. } z)^{\frac{1}{i}} + (\textit{cof. } z - \sqrt{-1}.\textit{fin. } z)^{\frac{1}{i}}}{2} \quad \&$$

$$\frac{z}{i} = \frac{(\textit{cof. } z + \sqrt{-1}.\textit{fin. } z)^{\frac{1}{i}} - (\textit{cof. } z - \sqrt{-1}.\textit{fin. } z)^{\frac{1}{i}}}{2\sqrt{-1}}. \quad \text{Su-}$$

mendis autem Logarithmis hyperbolicis fupra (125) oftendimus effe $l(1 + x) = i(1 + x)^{\frac{1}{i}} - i$, feu $y^{\frac{1}{i}} = 1 + \frac{1}{i} ly$,

pofito

pofito *y* loco 1 + *x*. Nunc igitur, pofito loco *y*, partim *cof. z +*
√ — 1. *fin. z* partim *cof. z* — √ — 1. *fin. z*, prodibit 1 =

$$ \frac{1 + \frac{1}{i} l\left(cof. z + \sqrt{-1} . fin. z\right) + 1 + \frac{1}{i} l\left(cof. z - \sqrt{-1} . fin. z\right)}{2} $$

= 1, ob Logarithmos evanefcentes, ita ut hinc nil fequa-
tur. Altera vero æquatio pro Sinu fuppeditat:

$$ \frac{z}{i} = \frac{\frac{1}{i} l\left(cof. z + \sqrt{-1} . fin. z\right) - \frac{1}{i} l\left(cof. z - \sqrt{-1} . fin. z\right)}{2\sqrt{-1}} $$

ideoque $z \dfrac{1}{2\sqrt{-1}} l \dfrac{cof.\ z + \sqrt{-1} . fin.\ z}{cof.\ z - \sqrt{-1} . fin.\ z}$, unde patet quem-
admodum Logarithmi imaginarii ad Arcus circulares revo-
centur.

140. Cum fit $\frac{fin. z}{cof. z} = tang.\ z$, Arcus *z* per fuam Tangen-

tem ita exprimetur ut fit $z = \dfrac{1}{2\sqrt{-1}} l \dfrac{1 + \sqrt{-1} . tang. z}{1 - \sqrt{-1} . tang. z}$. Su-

pra vero (§. 123) vidimus effe $l\frac{1 + x}{1 - x} = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} +$

$\frac{2x^7}{7} + $ &c.. Pofito ergo $x = \sqrt{-1} . tang. z$, fiet $z = \dfrac{tang.\ z}{1}$

$- \dfrac{(tang.\ z)^3}{3} + \dfrac{(tang.\ z)^5}{5} - \dfrac{(tang.\ z)^7}{7} + $ &c.. Si ergo po-

namus *tang. z* = *t*, ut fit *z* Arcus, cujus Tangens eft *t*, quem
ita indicabimus *A. tang. t*, ideoque erit *z* = *A tang. t*. Co-
gnita ergo Tangente *t* erit Arcus refpondens $z = \dfrac{t}{1} - \dfrac{t^3}{3} +$

$\dfrac{t^5}{5} - \dfrac{t^7}{7} + \dfrac{t^9}{9} - $ &c.. Cum igitur, fi Tangens *t* æque-

tur Radio 1, fiat Arcus *z* = Arcui 45° feu $z = \dfrac{\pi}{4}$, erit

$\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + $ &c., quæ eft Series a L E I B-
N I T Z I O primum producta, ad valorem Peripheriæ Circuli
exprimendum.

141. Quo autem ex hujufmodi Serie longitudo Arcus Cir-

culi expedite definiri poffit, perfpicuum eft pro Tangente *t* fractionem fatis parvam fubftitui debere. Sic ope hujus Seriei facile reperietur longitudo Arcus *z*, cujus Tangens *t* æquetur $\frac{1}{10}$, foret enim ifte Arcus $z = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} -$ &c., cujus Seriei valor per approximationem non difficulter in fractione decimali exhiberetur. At vero ex tali Arcu cognito nihil pro longitudine totius Circumferentiæ concludere licebit, cum ratio, quam Arcus, cujus Tangens eft $= \frac{1}{10}$, ad totam Peripheriam tenet, non fit affignabilis. Hanc ob rem ad Peripheriam indagandam, ejufmodi Arcus quæri debet, qui fit fimul pars aliquota Peripheriæ, & cujus Tangens fatis exigua commode exprimi queat. Ad hoc ergo inftitutum fumi folet Arcus 30°. cujus Tangens eft $= \frac{1}{\sqrt{3}}$, quia minorum Arcuum cum Peripheria commenfurabilium Tangentes nimis fiunt irrationales. Quare, ob Arcum $30° = \frac{\pi}{6}$, erit $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3 \cdot 3\sqrt{3}} + \frac{1}{5 \cdot 3^2 \sqrt{3}} -$ &c., & $\pi = \frac{2\sqrt{3}}{1} - \frac{2\sqrt{3}}{3 \cdot 3} + \frac{2\sqrt{3}}{5 \cdot 3^2} - \frac{2\sqrt{3}}{7 \cdot 3^3} +$ &c., cujus Seriei ope valor ipfius π ante exhibitus incredibili labore fuit determinatus.

142. Hic autem labor eo major eft, quod primum finguli termini fint irrationales, tum vero quifque tantum, circiter, triplo fit minor quam præcedens. Huic itaque incommodo ita occurri poterit: fumatur Arcus 45° feu $\frac{\pi}{4}$ cujus valor, etfi per Seriem vix convergentem $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c., exprimitur, tamen is retineatur, atque in duos Arcus *a* & *b* difpertiatur ut fit $a + b = \frac{\pi}{4} = 45°$. Cum igitur fit *tang.* $(a+b) = 1 = \frac{tang.\ a + tang.\ b}{1 - tang.\ a.\ tang.\ b}$ erit $1 - tang.a.\ tang.b =$ *tang.* a :

tang. a $+$ *tang. b* & *tang. b* $= \dfrac{\text{I} - \text{\textit{tang. a}}}{\text{I} + \text{\textit{tang. a}}}$. Sit nunc *tang. a* $=$ $\dfrac{\text{I}}{2}$, erit *tang. b* $= \dfrac{\text{I}}{3}$, hinc uterque Arcus *a* & *b* per Seriem rationalem multo magis, quam fuperior, convergentem exprimetur, eorumque fumma dabit valorem Arcus $\dfrac{\pi}{4}$; hinc itaque erit

$$\pi = 4 \cdot \left\{ \begin{array}{l} \dfrac{\text{I}}{\text{I}.2} - \dfrac{\text{I}}{3.2^3} + \dfrac{\text{I}}{5.2^5} - \dfrac{\text{I}}{7.2^7} + \dfrac{\text{I}}{9.2^9} - \&c. \\[2mm] \dfrac{\text{I}}{\text{I} \cdot 3} - \dfrac{\text{I}}{3.3^3} + \dfrac{\text{I}}{5.3^5} - \dfrac{\text{I}}{7.3^7} + \dfrac{\text{I}}{9.3^9} - \&c. \end{array} \right\}$$

hoc ergo modo multo expeditius longitudo femicircumferentiæ π inveniri potuiffet, quam quidem factum eft ope Seriei ante commemoratæ.

CAPUT IX.

De invefligatione Factorum trinomialium.

143. QUemadmodum Factores fimplices cujufque Functionis integræ inveniri oporteat, fupra quidem oftendimus hoc fieri per refolutionem æquationum. Si enim propofita fit Functio quæcunque integra $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, hujusque quærantur Factores fimplices formæ $p - qz$, manifeftum eft, fi $p - qz$ fuerit Factor Functionis $\alpha + \beta z + \gamma z^2 + \&c.$, tum, pofito $z = \dfrac{p}{q}$, quo cafu Factor $p - qz$ fit $= 0$, etiam ipfam Functionem propofitam evanefcere debere. Hinc $p - qz$ erit Factor vel divifor Functionis $\alpha + \beta z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$, fequitur fore hanc

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expref-

expreſſionem $\alpha + \frac{\mathfrak{b}p}{q} + \frac{\gamma p^2}{q^2} + \frac{\delta p^3}{q^3} + \frac{\varepsilon p^4}{q^4} +$ &c. $= 0$. Unde viciſſim, ſi omnes radices $\frac{p}{q}$ hujus æquationis eruantur, ſingulæ dabunt totidem Factores ſimplices Functionis integræ propoſitæ $\alpha + \mathfrak{b}z + \gamma z^2 + \delta z^3 +$ &c., nempe $p - qz$. Patet autem ſimul numerum Factorum hujuſmodi ſimplicium ex maxima Poteſtate ipſius z definiri.

144. Hoc autem modo plerumque difficulter Factores imaginarii eruuntur, quamobrem hoc Capite methodum peculiarem tradam, cujus ope ſæpenumero Factores ſimplices imaginarii inveniri queant. Quoniam vero Factores ſimplices imaginarii ita ſunt comparati, ut binorum productum fiat reale, hos ipſos Factores imaginarios reperiemus, ſi Factores inveſtigemus duplices, ſeu hujus formæ $p - qz + rzz$, reales quidem, ſed quorum Factores ſimplices ſint imaginarii. Quod ſi enim Functionis $\alpha + \mathfrak{b}z + \gamma z^2 + \delta z^3 +$ &c., conſtent omnes Factores reales duplices hujus formæ trinomialis $p - qz + rzz$, ſimul omnes Factores imaginarii habebuntur.

145. Trinomium autem $p - qz + rzz$ Factores ſimplices habebit imaginarios, ſi fuerit $4pr > qq$ ſeu $\frac{q}{2\sqrt{pr}} < 1$. Cum igitur Sinus & Coſinus Angulorum ſint unitate minores, formula $p - qz + rzz$ Factores ſimplices habebit imaginarios ſi fuerit $\frac{q}{2\sqrt{pr}} =$ Sinui vel Coſinui cujuspiam Anguli. Sit ergo $\frac{q}{2\sqrt{pr}} = coſ. A\phi$, ſeu $q = 2\sqrt{pr} . coſ. \phi$, atque trinomium $p - qz + rzz$ continebit Factores ſimplices imaginarios. Ne autem irrationalitas moleſtiam faceſſat, aſſumo hanc formam $pp - 2pqz. coſ. \phi + qqzz$, cujus Factores ſimplices imaginarii erunt hi, $qz - p(coſ. \phi + \sqrt{-1}. ſin. \phi)$ & $qz - p(coſ. \phi - \sqrt{-1}. ſin. \phi)$. Ubi quidem patet ſi fuerit $coſ. \phi = \pm 1$, tum ambos Factores, ob $ſin. \phi = 0$, fieri æquales & reales.

146. Propoſita ergo Functione integra $\alpha + \mathfrak{b}z + \gamma z^2 + \delta z^3 +$ &c.,

&c. , ejus Factores fimplices imaginarii eruentur , fi determi-
nentur litteræ p & q cum Angulo ϕ , ut hoc trinomium
$pp - 2pqz. \mathit{cof}. \phi + qqzz$ fiat Factor Functionis. Tum enim
fimul inerunt ifti Factores fimplices imaginarii $qz - p\,(\mathit{cof}. \phi +$
$\sqrt{} - 1.\mathit{fin}.\,\phi)$ & $qz - p\,(\mathit{cof}. \phi - \sqrt{} - 1.\mathit{fin}. \phi)$. Quam
ob rem Functio propofita evanefcet , fi ponatur tam $z = \frac{p}{q} \times$
$(\mathit{cof}. \phi + \sqrt{} - 1.\mathit{fin}. \phi)$ quam $z = \frac{p}{q}\,(\mathit{cof}. \phi - \sqrt{} - 1.\mathit{fin}. \phi)$.
Hinc, facta fubftitutione utraque , duplex nafcetur æquatio, ex
quibus tam fractio $\frac{p}{q}$ quam Arcus ϕ definiri poterunt.

147. Hæ autem fubftitutiones loco z faciendæ , etiamfi pri-
mo intuitu difficiles videantur , tamen per ea , quæ in Capite
præcedente funt tradita , fatis expedite abfolventur. Cum enim
fuerit oftenfum effe $(\mathit{cof}.\phi \pm \sqrt{} - 1.\mathit{fin}.\phi)^n = \mathit{cof}.\, n\phi \pm \sqrt{} - 1 \times$
$\mathit{fin}.\, n\,\phi$, fequentes formulæ loco fingularum ipfius z Potefta-
tum habebuntur fubftituendæ.

pro priori Factore	pro altero Factore
$z = \frac{p}{q}(\mathit{cof}. \; \phi + \sqrt{} - 1.\mathit{fin}. \; \phi)$	$z = \frac{p}{q}(\mathit{cof}. \; \phi - \sqrt{} - 1.\mathit{fin}. \; \phi)$
$z^2 = \frac{p^2}{q^2}(\mathit{cof}.2\phi + \sqrt{} - 1.\mathit{fin}.2\phi)$	$z^2 = \frac{p^2}{q^2}(\mathit{cof}.2\phi - \sqrt{} - 1.\mathit{fin}.2\phi)$
$z^3 = \frac{p^3}{q^3}(\mathit{cof}.3\phi + \sqrt{} - 1.\mathit{fin}.3\phi)$	$z^3 = \frac{p^3}{q^3}(\mathit{cof}.3\phi - \sqrt{} - 1.\mathit{fin}.3\phi)$
$z^4 = \frac{p^4}{q^4}(\mathit{cof}.4\phi + \sqrt{} - 1.\mathit{fin}.4\phi)$	$z^4 = \frac{p^4}{q^4}(\mathit{cof}.4\phi - \sqrt{} - 1.\mathit{fin}.4\phi)$
&c.	&c.

Ponatur brevitatis gratia $\frac{p}{q} = r$, factaque fubftitutione fequen-
tes duæ nafcentur æquationes.

$$0 = \begin{cases} \alpha + \mathfrak{b}r.\,\mathit{cof}. \phi \; + \gamma r^2.\,\mathit{cof}. 2\,\phi \; + \delta r^3.\,\mathit{cof}. 3\,\phi \; + \&c. \\ + \mathfrak{b}r\sqrt{} - 1.\mathit{fin}.\phi + \gamma r^2\sqrt{} - 1\,\mathit{fin}.2\phi + \delta r^3\sqrt{} - 1.\mathit{fin}.3\phi + \&c. \end{cases}$$

$$0 = \begin{cases} \alpha + \mathfrak{b}r.\,\mathit{cof}. \phi \; + \gamma r^2.\,\mathit{cof}. 2\,\phi \; + \delta r^3.\,\mathit{cof}. 3\,\phi \; + \&c. \\ - \mathfrak{b}r\sqrt{} - 1\,\mathit{fin}.\phi - \gamma r^2\sqrt{} - 1\,\mathit{fin}\,2\phi - \delta r^3\sqrt{} - 1.\mathit{fin}.3\phi - \&c. \end{cases}$$

148. Quod

148. Quod fi hæ duæ æquationes invicem addantur & fub-
trahantur , & pofteriori cafu per $2 \sqrt{-1}$ dividantur, prodibunt
hæ duæ æquationes reales :

$$0 == \alpha + \mathcal{E}r.cof.\phi + \gamma r^2.cof. 2\phi + \delta r^3.cof. 3\phi + \&c.$$

$$0 = \mathcal{E}r.fin.\phi + \gamma r^2.fin. 2\phi + \delta r^3.fin. 3\phi + \&c.$$

quæ ftatim ex forma Functionis propofitæ

$$\alpha + \mathcal{E}z + \gamma z^2 + \delta z^3 + \epsilon z^4 + \&c.$$

formari poffunt, ponendo primum pro unaquaque ipfius z po-
teftate $z^n = r^n cof. n\phi$, deinceps $z^n . = r^n fin. n\phi$. Sic enim
ob *fin.* $0\phi = 0$ & *cof.* $0\phi = 1.$, pro z^0 feu 1 in termino con-
ftanti priori cafu ponitur 1 , pofteriori autem 0. Si ergo ex
his duabus æquationibus definiantur incognitæ r & ϕ, ob $r =$
$\frac{p}{q}$, habebitur Factor Functionis trinomialis $pp — 2pqz. cof. \phi +$
$qqzz$, duos Factores fimplices imaginarios involvens.

149. Si æquatio prior multiplicetur per *fin.* $m\phi$; pofterior
per *cof.* $m\phi$, atque producta vel addantur vel fubtrahantur ,
prodibunt iftæ duæ æquationes :

$$0 = \alpha. fin.m\phi + \mathcal{E}r.fin.(m+1)\phi + \gamma r^2.fin.(m+2)\phi +$$
$$\delta r^3.fin.(m+3)\phi + \&c.$$

$$0 = \alpha.fin.m\phi + \mathcal{E}r.fin.(m+1)\phi + \gamma r^2.fin.(m-2)\phi +$$
$$\delta r^3.fin.(m-3)\phi + \&c.$$

Sin autem æquatio prior multiplicetur per *cof.* $m\phi$ & pofte-
rior per *fin.* $m\phi$, per additionem ac fubtractionem fequentes
emergent æquationes.

$$0 = \alpha. cof.m\phi + \mathcal{E}r.cof.(m-1)\phi + \gamma r^2. cof.(m-2)\phi +$$
$$\delta r^3.cof.(m-3)\phi + \&c.$$

$$0 = \alpha.cof.m\phi + \mathcal{E}r.cof.(m+1)\phi + \gamma r^2.cof.(m+2)\phi +$$
$$\delta r^3.cof.(m+3)\phi + \&c.$$

Hujuf-

Hujufmodi ergo duæ æquationes quæcunque conjunctæ deter-
minabunt incognitas r & ϕ; quod cum plerumque pluribus
modis fieri poffit, fimul plures Factores trinomiales obtinen-
tur, iique adeo omnes, quos Functio propofita in fe com-
plectitur.

150. Quo ufus harum regularum clarius appareat, quarum-
dam Functionum fæpius occurrentium Factores trinomiales hic
indagabimus, ut eos, quoties occafio poftulaverit, hinc depro-
mere liceat. Sit itaque propofita hæc Functio $a^n + z^n$, cujus
Factores trinomiales formæ $pp - 2pqz.cof. \phi + qq$ determinari
oporteat; pofito ergo $r = \frac{p}{q}$, habebuntur hæ duæ æquationes:

$o = a^n + r^n.cof.n.\phi$ & $o = r^n.fin.n\phi$, quarum pofte-
rior dat $fin.n\phi = o$; unde erit $n\phi$ Arcus vel hujus formæ
$(2k + 1)\pi$ vel $2k\pi$, denotante k numerum integrum. Cafus
hos ideo diftinguo, quod eorum Cofinus fint differentes; priori
enim cafu erit $cof.(2k + 1)\pi = -1$ pofteriori cafu autem
$cof.2k\pi = +1$. Patet autem priorem formam $n\phi = (2k+1)\pi$
fumi debere, quippe quæ dat $cof.n\phi = -1$, unde fit $o = a^n$
$- r^n$, hincque porro $r = a = \frac{p}{q}$. Erit ergo $p = a, q = 1$,
& $\phi = \frac{(2k + 1)\pi}{n}$, unde Functionis $a^n + z^n$ Factor erit
$aa - 2az.cof.\frac{(2k + 1)\pi}{n} + zz$. Cum igitur pro k nume-
rum quemque integrum ponere liceat, prodeunt hoc modo plu-
res Factores, neque tamen infiniti, quoniam fi $2k + 1$, ultra n
augetur, Factores priores recurrunt, quod ex exemplis clarius
patebit, cum fit $cof.(2\pi \pm \phi) = cof.\phi$. Deinde fi n eft
numerus impar, pofito $2k + 1 = n$, erit Factor quadratus
$aa + 2az + zz$ neque vero hinc fequitur quadratum $(a + z)^2$
effe Factorem Functionis $a^n + z^n$, quoniam (in §. 148) uni-
ca æquatio refultat, qua tantum patet $a + z$ effe Diviforem
<div align="right">formulæ</div>

formulæ $a^n + z^n$; quæ regula femper eft tenenda quoties *cof.* ϕ fit vel $+ 1$ vel $- 1$.

E X E M P L U M.

Evolvamus aliquot cafus, quo ifti Factores clarius ob oculos ponantur, atque hos cafus in duas claffes diftribuamus, prout *n* fuerit numerus vel par vel impar.

Si $n = 1$	Si $n = 2$
Formulæ	Formulæ
$a + z$	$a^2 + z^2$
Factor eft	Factor eft
$a + z$	$a^2 + z^2$
Si $n = 3$	Si $n = 4$
Formulæ	Formulæ
$a^3 + z^3$	$a^4 + z^4$
Factores funt	Factores funt
$aa - 2az.cof.\dfrac{1}{3}\pi + zz$	$aa - 2az.cof.\dfrac{1}{4}\pi + zz$
$a + z$	$aa - 2az.cof.\dfrac{3}{4}\pi + zz$
Si $n = 5$	Si $n = 6$
Formulæ	Formulæ
$a^5 + z^5$	$a^6 + z^6$
Factores funt	Factores funt
$aa - 2az.cof.\dfrac{1}{5}\pi + zz$	$aa - 2az.cof.\dfrac{1}{6}\pi + zz$
$aa - 2az.cof.\dfrac{3}{5}\pi + zz$	$aa - 2az.cof.\dfrac{3}{6}\pi + zz$
$a + z$	$aa - 2az.cof.\dfrac{5}{6}\pi + zz$

Ex quibus exemplis patet omnes Factores obtineri, fi loco $2k + 1$ omnes numeri impares non majores, quam Exponens

n

n, fubftituantur, iis vero cafibus quibus Factor quadratus prodit, Cap. IX. tantum ejus radicem Factoribus annumerari debere.

151. Si propofita fit hæc Functio $a^n - z^n$, ejus Factor trinomialis erit $pp - 2pqz. cof. \varphi + qqzz$, fi pofito $r = \frac{p}{q}$, fuerit $0 = a^n - r^n. cof. n\varphi$ & $0 = r^n. fin. n\varphi$. Erit ergo iterum *fin.* $n\varphi = 0$, ideoque $n\varphi = (2k+1)\pi$ vel $n\varphi = 2k\pi$. Hoc autem cafu valor pofterior fumi debet, ut fit *cof.* $n\varphi = +1$, qui dat $0 = a^n - r^n$ & $r = \frac{p}{q} = a$. Habebitur itaque $p = a$; $q = 1$; & $\varphi = \frac{2k\pi}{n}$; unde Factor trinomialis formulæ propofitæ erit $= aa - 2az. cof. \frac{2k}{n}\pi + zz$; quæ forma, fi loco $2k$ omnes numeri pares non majores quam *n* ponantur, fimul dabit omnes Factores; ubi de Factoribus quadratis idem eft tenendum quod ante monuimus. Ac primo quidem, pofito $k = 0$, prodit Factor $aa - 2az + zz$, pro quo vero radix $a - z$ capi debet. Similiter, fi *n* fuerit numerus par & ponatur $2k = n$, prodit $aa + 2az + zz$, unde patet $a + z$ effe diviforem formæ $a^n - z^n$

E X E M P L U M.

Cafus Exponentis *n* ut ante tractati ita fe habebunt, prout *n* fuerit numerus vel impar vel par.

LIB. I.

Si $n = 1$

Formulæ

$$a - z$$

ipsa erit Factor

$$a - z$$

Si $n = 2$

Formulæ

$$a^2 - z^2$$

Factores erunt

$$a - z$$
$$a + z$$

Si $n = 3$

Formulæ

$$a^3 - z^3$$

Factores erunt

$$a - z$$

$$aa - 2az.cof.\frac{2}{3}\pi + zz$$

Si $n = 4$

Formulæ

$$a^4 - z^4$$

Factores erunt

$$a - z$$

$$aa - 2az.cof.\frac{2}{4}\pi + zz$$

$$a + z$$

Si $n = 5$

Formulæ

$$a^5 - z^5$$

Factores erunt

$$a - z$$

$$aa - 2az.cof.\frac{2}{5}\pi + zz$$

$$aa - 2az.cof.\frac{4}{5}\pi + zz$$

Si $n = 6$

Formulæ

$$a^6 - z^6$$

Factores erunt

$$a - z$$

$$aa - 2az.cof.\frac{2}{6}\pi + zz$$

$$aa - 2az.cof.\frac{4}{6}\pi + zz$$

$$a + z$$

152. His igitur confirmatur id, quod supra jam innuimus, omnem Functionem integram, si non in Factores simplices reales, tamen in Factores duplices reales resolvi posse. Vidimus enim hanc Functionem indefinitæ dimensionis $a^n + z^n$ semper in Factores duplices reales, præter simplices reales, resolvi posse. Progrediamur ergo ad Functiones magis compositas, uti: $\alpha + \mathfrak{6}z^n + \gamma z^{2n}$, cujus quidem, si duos habeat Factores formæ $\eta + \theta z^n$, resolutio ex præcedentibus abunde patet. Hoc ergo tantum erit efficiendum, ut formæ $\alpha + \mathfrak{6}z^n + \gamma z^{2n}$, eo casu, quo non habet duos Factores reales formæ $\eta + \theta z^n$,

reso-

refolutionem in Factores reales, vel fimplices vel duplices, do- C a p. IX. ceamus.

153. Confideremus ergo hanc Functionem: $a^{2n} - 2a^n z^n \times$ *cof.* $g + z^{2n}$, quæ in duos Factores formæ $\eta + \theta. z^n$ reales refolvi nequit. Quod fi ergo ponamus hujus Functionis Factorem duplicem realem effe $pp - 2pqz. cof. \varphi + qqzz$, pofito $r = \frac{p}{q}$, duæ fequentes æquationes erunt refolvendæ:

$$0 = a^{2n} - 2a^n r^n \cdot cof. g. cof. n\varphi + r^{2n}. cof. 2n\varphi \& 0 = -$$
$2a^n r^n \cdot cof. g. fin. n\varphi + r^{2n}. fin. 2n\varphi.$ Vel, loco prioris æquationis fumatur ex §. 149, (ponendo $m = 2\nu$), hæc $0 =$ $a^{2n}. fin. 2n\varphi - 2a^n r^n. cof. g. fin. n\varphi$, quæ cum pofteriori collata dat $r = a$; tum vero erit *fin.* $2n\varphi = 2 cof. g. fin. n\varphi$: At eft *fin.* $2n\varphi = 2 fin. n\varphi. cof. n\varphi.$ unde fit $cof. n\varphi = cof. g.$ At eft femper $cof. (2k\pi \pm g) = cof. g$, ex quo habetur $n\varphi = 2k\pi \pm g \& \varphi = \frac{2k\pi \pm g}{n}$. Hinc ergo Factor generalis duplex formæ propofitæ erit $= aa - 2az. cof.$ $\frac{2k\pi \pm g}{n} + zz$; atque omnes Factores prodibunt, fi pro $2k$ omnes numeri pares non majores quam n fucceffive fubftituantur, uti ex applicatione ad cafus videre licebit.

E X E M P L U M.

Confideremus ergo cafus quibus n eft 1, 2, 3, 4, &c., ut ratio Factorum appareat. Erit ergo

Formulæ
$$a^2 - 2az. cof. g + zz$$
Unicus Factor
$$aa - 2az. cof. g + zz$$
Formulæ
$$a^4 - 2a^2z^2. cof. g + z^4$$

P 2 Facto-

Factores duo

$$aa - 2\,a\,z.\,\cos.\,\frac{g}{2} + z^2$$

$$aa - 2az\cos.\left(\frac{2\pi \stackrel{\mp}{} g}{2} + zz \text{ feu } aa + 2az\,\cos.\,\frac{g}{2} + zz\right)$$

Formulæ

$$a^6 - 2\,a^3\,z^3.\,\cos.\,g + z^6$$

Factores tres

$$aa - 2\,a\,z.\,\cos.\,\frac{g}{3} + z^2$$

$$aa - 2\,az.\cos.\frac{2\pi - g}{3} + z^2$$

$$aa - 2\,az.\cos.\frac{2\pi + g}{3} + z^2$$

Formulæ

$$a^8 - 2\,a^4.\,z^4\,\cos.\,g + z^8$$

Factores quatuor

$$aa - 2\,a\,z.\,\cos.\,\frac{g}{4} + zz$$

$$aa - 2\,az.\cos.\frac{2\pi - g}{4} + zz$$

$$aa - 2\,az.\cos.\frac{2\pi + g}{4} + zz$$

$$aa - 2az.\cos.\frac{4\pi \mp g}{4} + zz \text{ feu } aa + 2az.\cos.\frac{g}{4} + zz$$

Formulæ

$$a^{10} - 2\,a^5\,z^5.\,\cos.\,g + z^{10}$$

Factores quinque

$$aa - 2\,a\,z.\,\cos.\,\frac{g}{5} + zz$$

$$aa - 2\,az.\cos.\frac{2\pi - g}{5} + zz$$

$$aa - 2\,az.\cos.\frac{2\pi + g}{5} + zz$$

$$aa - 2\,az.\cos.\frac{4\pi - g}{5} + zz$$

$$aa - 2\,az.\cos.\frac{4\pi + g}{5} + zz$$

Confirmatur ergo etiam his exemplis omnem Functionem integram in Factores reales, five fimplices five duplices, refolvi poffe.

154. Hinc ulterius progredi licebit ad Functionem hanc $\alpha + \theta z^n + \gamma z^{2n} + \delta z^{3n}$, quæ certo habebit unum Factorem realem formæ $\eta + \theta z^n$, cujus igitur Factores reales, vel fimplices vel duplices, exhiberi poffunt; alter vero multiplicator formæ $\iota + \varkappa z^n + \lambda z^{2n}$, utcumque fuerit comparatus, per §. præced. pari modo in Factores refolvi poterit. Deinde hæc Functio $\alpha + \theta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n}$, cum perpetuo habeat duos Factores reales formæ hujus $\eta + \theta z^n + \iota z^{2n}$, fimiliter in Factores, vel fimplices vel duplices, reales refolvitur. Quin etiam progredi licet ad formam $\alpha + \theta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \xi z^{5n}$ quæ, cum certo habeat unum Factorem formæ $\eta + \theta z^n$, alter Factor erit formæ præcedentis; unde etiam hæc Functio refolutionem in Factores reales, vel fimplices vel duplices, admittet. Quare fi ullum dubium manfiffet circa hujufmodi refolutionem omnium Functionum integrarum, hoc nunc fere penitus tolletur.

155. Traduci vero etiam poteft hæc in Factores refolutio ad Series infinitas; fcilicet, quia vidimus fupra effe $1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \&c. = e^x$; at vero effe $e^x = (1 + \frac{x}{i})^i$, denotante i numerum infinitum, perfpicuum eft Seriem $1 + \frac{1}{x} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ habere Factores infinitos fimplices inter fe æquales nempe $1 + \frac{x}{i}$. At fi ab eadem Serie primus terminus dematur, erit $\frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$

&c. $= e^x - 1 = \left(1 + \frac{x}{i}\right)^{i} - 1$, cujus formæ cum §.

151 comparatæ, quo fit $a = 1 + \frac{x}{i}$; $n = i$ & $z = 1$,

Factor quicunque erit $= \left(1 + \frac{x}{i}\right)^{2} - 2\left(1 + \frac{x}{i}\right)cof. \frac{2k}{i}\pi +$

1, unde, fubftituendo pro $2k$ omnes numeros pares, fimul omnes Factores prodibunt. Pofito autem $2k = 0$ prodit Factor quadratus $\frac{xx}{ii}$, pro quo autem tantum ob rationes allegatas radix $\frac{x}{i}$ fumi debet, erit ergo x Factor expreffionis $e^x - 1$ quod quidem fponte patet. Ad reliquos Factores inveniendos notari oportet effe, ob Arcum $\frac{2k}{i}\pi$ infinite parvum, $cof.$

$\frac{2k}{i}\pi = 1 - \frac{2kk}{ii}\pi\pi$ (134), terminis fequentibus, ob i numerum infinitum, in nihilum abeuntibus. Hinc erit Factor quilibet $= \frac{xx}{ii} + \frac{4kk}{ii}\pi\pi + \frac{4kk\pi\pi}{i^3}x$, atque adeo forma $e^x - 1$

erit divifibilis per $1 + \frac{x}{i} + \frac{xx}{4kk\pi\pi}$. Quare expreffio $e^x - 1$

$= x\left(1 + \frac{x}{1.2} + \frac{x^2}{1.2.3} + \frac{x^3}{1.2.3.4} + \&c.\right)$, præter Factorem x, habebit hos infinitos $\left(1 + \frac{x}{i} + \frac{xx}{4\pi\pi}\right)\left(1 + \frac{x}{i} + \right.$

$\frac{xx}{16\pi\pi}\left.\right)\left(1 + \frac{x}{i} + \frac{xx}{36\pi\pi}\right)\left(1 + \frac{x}{i} + \frac{xx}{64\pi\pi}\right)$ \&c.

156. Cum autem hi Factores contineant partem infinite parvam $\frac{x}{i}$, quæ, cum in fingulis infit, atque per multiplicationem omnium, quorum numerus eft $\frac{1}{2}i$, producat terminum $\frac{x}{2}$, omitti non poteft. Ad hoc ergo incommodum vitandum confideremus hanc expreffionem $e^x - e^{-x} =$

$\left(1 + \frac{x}{i}\right)^{i} - \left(1 - \frac{x}{i}\right)^{i} = 2\left(\frac{x}{1} + \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} + \&c.\right)$

eft

eft enim $e^{-x} = 1 - \frac{x}{1} + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \&c.$; quæ cum

§. 151. comparata dat $n = i$, $a = 1 + \frac{x}{i}$ & $z = 1 - \frac{x}{i}$

unde hujus expreffionis Factor erit $= aa - 2az . cof. \frac{2k}{n} \pi +$

$zz = 2 + \frac{2xx}{ii} - 2 (1 - \frac{xx}{ii}) cof. \frac{2k}{i} \pi = \frac{4xx}{ii} + \frac{4kk}{ii} \pi\pi -$

$\frac{4kk\pi\pi xx}{i^4}$, ob $cof. \frac{2k}{i} \pi = 1 - \frac{2kk\pi\pi}{ii}$. Functio ergo $e^x -$

e^{-x} divifibilis erit per $1 + \frac{xx}{kk\pi\pi} - \frac{xx}{ii}$, ubi autem termi-

nus $\frac{xx}{ii}$ tuto omittitur, quia etfi per i multiplicetur, tamen

manet infinite parvus. Præterea vero ut ante, fi $k = 0$, erit

primus Factor $= x$. Quocirca, his Factoribus in ordinem re-

dactis, erit $\frac{e^x - e^{-x}}{2} = x (1 + \frac{xx}{\pi\pi})(1 + \frac{xx}{4\pi\pi})(1 + \frac{xx}{9\pi\pi})$

$(1 + \frac{xx}{16\pi\pi})(1 + \frac{xx}{25\pi\pi}) \&c. = x (1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} +$

$\frac{x^6}{1.2...7} + \&c.)$. Singulis fcilicet Factoribus per multiplica-

tionem conftantis ejufmodi formam dedi, ut per actualem mul-

tiplicationem primus terminus x refultet.

157. Eodem modo cum fit $\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} +$

$\frac{x^4}{1.2.3.4} + \&c. = \frac{(1 + \frac{x}{i})^i + (1 - \frac{x}{i})^i}{2}$, hujus expreffio-

nis cum fuperiori $a^n + z^n$ comparatio dabit $a = 1 + \frac{x}{i}$;

$z = 1 - \frac{x}{i}$ & $n = i$: erit ergo Factor quicunque $= aa - 2az \times$

$cof. \frac{2k+1}{n} \pi + zz = 2 + \frac{2xx}{ii} - 2 (1 - \frac{xx}{ii}) cof. \frac{2k+1}{i} \pi$. Eft

autem

autem *cof.* $\frac{2k+1}{i}\pi = 1 - \frac{(2k+1)^2\pi\pi}{2ii}$, unde forma Facto-

ris erit $= \frac{4xx}{ii} + \frac{(2k+1)^2\pi^2}{ii}$, evanefcente termino cujus de-

nominator eft i^4. Quoniam ergo omnis Factor expreffionis

$1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + $ &c. hujufmodi formam habere debet

$1 + \alpha xx$, quo Factor inventus ad hanc formam reducatur,

dividi debet per $\frac{(2k+1)^2\pi^2}{ii}$: hinc Factor formæ propofitæ erit

$1 + \frac{4xx}{(2k+1)^2\pi\pi}$, ex eoque omnes Factores infiniti inve-

nientur, fi loco $2k+1$ fucceffive omnes numeri impares

fubftituantur. Hanc ob rem erit

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} + \&c. =$$

$$\left(1 + \frac{4xx}{\pi\pi}\right)\left(1 + \frac{4\alpha x}{9\pi\pi}\right)\left(1 + \frac{4xx}{25\pi\pi}\right)\left(1 + \frac{4xx}{49\pi\pi}\right) \&c.$$

158. Si x fiat quantitas imaginaria, formulæ hæ exponen-
tiales in Sinum & Cofinum cujufpiam Arcus realis abeunt.

Sit enim $x = z\sqrt{-1}$; erit $\dfrac{e^{z\sqrt{-1}} - e^{-z\sqrt{-1}}}{2\sqrt{-1}} =$

fin. $z = z - \frac{z^3}{1.2.3} + \frac{z^5}{1.2.3.4.5} - \frac{z^7}{1.2.3....7} + \&c.$,

quæ adeo expreffio hos habet Factores numero infinitos

$z\left(1 - \frac{zz}{\pi\pi}\right)\left(1 - \frac{zz}{4\pi\pi}\right)\left(1 - \frac{zz}{9\pi\pi}\right)\left(1 - \frac{zz}{16\pi\pi}\right)\left(1 - \frac{zz}{25\pi\pi}\right)$

&c., feu erit *fin.* $z = z\left(1 - \frac{z}{\pi}\right)\left(1 + \frac{z}{\pi}\right)\left(1 - \frac{z}{2\pi}\right)$

$\left(1 + \frac{z}{2\pi}\right)\left(1 - \frac{z}{3\pi}\right)\left(1 + \frac{z}{3\pi}\right)$ &c.. Quoties ergo Ar-

cus z ita eft comparatus, ut quifpiam Factor evanefcat, quod

fit fi $z = 0$, $z = \pm\pi$; $z = \pm 2\pi$, & generaliter fi $z =$

$\pm k\pi$, denotante k numerum quemcunque integrum, fimul Si-
nus

nus ejus Arcus debet effe $= 0$, quod quidem ita patet, ut
hinc iftos Factores a pofteriori eruere licuiffet.

Simili modo, cum fit $e \dfrac{e^{z\sqrt{-1}} + e^{-z\sqrt{-1}}}{2} = \text{cof. } z$,

erit quoque $\text{cof. } z = (1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})$

$(1 - \frac{4zz}{49\pi\pi})$ &c., feu, his Factoribus in binos refolvendis,

erit quoque $\text{cof. } z = (1 - \frac{2z}{\pi})(1 + \frac{2z}{\pi})(1 - \frac{2z}{3\pi})(1 + \frac{2z}{3\pi})$

$(1 - \frac{2z}{5\pi})(1 + \frac{2z}{5\pi})$ &c., ex qua pari modo patet, fi fuerit

$z = \pm \dfrac{(2k+1)}{2} \pi$, fore $\text{cof.} z = 0$, id quod etiam ex natura
Circuli liquet.

159. Ex §. 152. etiam inveniri poffunt Factores hujus ex-
preffionis $e^x - 2 \, \text{cof.} g + e^{-x} = 2 (1 - \text{cof.} g + \frac{xx}{1.2} +$

$\frac{x^4}{1.2.3.4} +$ &c.). Tranfit enim hæc expreffio in hanc

$(1 + \frac{x}{i})^i - 2 \, \text{cof.} g + (1 - \frac{x}{i})^i$, quæ cum illa forma

comparata dat $2n = i$; $a = 1 + \frac{x}{i}$, & $z = 1 - \frac{x}{i}$, unde
Factor quicunque hujus formulæ erit $= aa - 2az. \text{cof.}$
$\frac{2k\pi \pm g}{n} + zz = 2 + \frac{2xx}{ii} - 2 (1 - \frac{xx}{ii}). \text{cof.} \frac{2(2k\pi \pm \pi)}{i}$:

at eft $\text{cof.} \frac{2(2k\pi \pm g)}{i} = 1 - \frac{2(2k\pi \pm g)^2}{ii}$, unde Factor

erit $= \frac{4xx}{ii} + \frac{4(2k\pi \pm g)^2}{ii}$, feu hujus formæ $1 + \frac{xx}{(2k\pi \pm g)^2}$.
Si ergo expreffio per $2 (1 - \text{cof.} g)$ dividatur, ut in Serie infini-
ta terminus conftans fit $= 1$, erit, fumendis omnibus Factori-

bus, $\dfrac{e^x - 2 \, \text{cof.} g + e^{-x}}{2 (1 - \text{cof.} g)} = (1 + \frac{xx}{gg})(1 + \frac{xx}{(2\pi - g)^2})$

$$\left(1 + \frac{xx}{(2\pi+g)^2}\right)\left(1 + \frac{xx}{(4\pi-g)^2}\right)\left(1 + \frac{xx}{(4\pi+g)^2}\right)$$

$$\left(1 + \frac{xx}{(6\pi-g)^2}\right)\left(1 + \frac{xx}{(6\pi+g)^2}\right) \&c..$$ Atque, fi loco

x ponatur $z\sqrt{-1}$, erit $\frac{cof. z - cof. g}{1 - cof. g} = \left(1 - \frac{z}{g}\right)\left(1 + \frac{z}{g}\right)$

$$\left(1 - \frac{z}{2\pi-g}\right)\left(1 + \frac{z}{2\pi-g}\right)\left(1 - \frac{z}{2\pi+g}\right)\left(1 + \frac{z}{2\pi+g}\right)$$

$$\left(1 - \frac{z}{4\pi-g}\right)\left(1 + \frac{z}{4\pi-g}\right) \&c., = 1 - \frac{zz}{1.2(1 - cof. g)} +$$

$$\frac{z^4}{1.2.3.4(1 - cof\, g)} - \frac{z^6}{1.2\ldots6(1 - cof\, g)} + \&c..$$ Hu-
jus adeo Seriei in infinitum continuatæ Factores omnes cog-
nofcuntur.

160. Commode etiam hujufmodi Functionis $e^{b+x} \pm e^{c-x}$
Factores inveniri omnefque affignari poffunt. Tranfmutatur enim

in hanc formam $\left(1 + \frac{b+x}{i}\right)^i \pm \left(1 + \frac{c-x}{i}\right)^i$, quæ comparata

cum forma $a^i \pm z^i$, Factorem habebit $aa - 2az. cof. \frac{m\pi}{i} + zz$,

denotante m numerum imparem fi valeat fignum fuperius, con-
tra vero numerum parem. Cum autem, ob i numerum infinite

magnum, fit $cof. \frac{m\pi}{i} = 1 - \frac{mm\pi\pi}{2ii}$, erit Factor ille generalis

$= (a - z)^2 + \frac{mm\pi\pi}{ii} az$. At hoc cafu erit $a = 1 + \frac{b+x}{i}$

$\& z = 1 + \frac{c-x}{i}$, unde fit $(a - z)^2 = \frac{(b - c + 2x)^2}{ii}$

$\& az = 1 + \frac{b+c}{i} + \frac{bc + (c-b)x - xx}{ii}$; ideoque Fa-
ctor erit per ii multiplicatus $= (b - c)^2 + 4(b - c)x +$
$4xx + mm\pi\pi$, neglectis terminis per i vel ii divifis, quoniam
jam omnis generis termini adfunt, præ quibus hi evanefcerent.
Termino ergo conftante ad unitatem per divifionem reducto

erit Factor $= 1 + \frac{4(b-c)x + 4xx}{mm\pi\pi + (b-c)^2}$.

161. Nunc,

161. Nunc, quoniam in omnibus Factoribus terminus con-
ſtans eſt $= 1$, ipſa Functio $e^{b+x} \pm e^{c-x}$ per ejuſmodi
conſtantem dividi debet, ut terminus conſtans fiat $= 1$, ſeu
ut ejus valor, poſito $x = 0$, fiat $= 1$; talis Diviſor erit
$e^b \pm e^c$, & hanc ob rem expreſſio hæc $\dfrac{e^{b+x} \pm e^{c-x}}{e^b \pm e^c}$ per

Factores numero infinitos exponi poterit. Erit ergo, ſi va-
leat ſignum ſuperius atque m denotet numerum imparem,

$$\frac{e^{b+x} + e^{c-x}}{e^b + e^c} = (1 + \frac{4(b-c)x+4xx}{\pi\pi+(b-c)^2})(1 + \frac{4(b-c)x+4xx}{9\pi\pi+(b-c)^2})$$

$(1 + \dfrac{4(b-c)x+4xx}{25\pi\pi+(b-c)^2})$ &c., ſin autem ſignum inferius va-

leat, atque ideo m denotet numerum parem, caſuque $m = 0$

radix Factoris quadrati ponatur, erit $\dfrac{e^{b+x} - e^{c-x}}{e^b - e^c} =$

$(1 + \dfrac{2x}{b-c})(1 + \dfrac{4(b-c)x+4xx}{4\pi\pi+(b-c)^2})(1 + \dfrac{4(b-c)x+4xx}{16\pi\pi+(b-c)^2})$

$(1 + \dfrac{4(b-c)x+4xx}{36\pi\pi+(b-c)^2})$ &c.

162. Ponatur $b = 0$, quod ſine detrimento univerſalitatis

fieri poteſt, eritque $\dfrac{e^x + e.^c e^{-x}}{1+e^c} = (1 - \dfrac{4cx+4xx}{\pi\pi+cc})$

$(1 - \dfrac{4cx+4xx}{9\pi\pi+cc})(1 - \dfrac{4cx+4xx}{25\pi\pi+cc})$ &c.; $\dfrac{e^x - e.^c e^{-x}}{1-e^c}$

$= (1 - \dfrac{2x}{c})(1 - \dfrac{4cx+4xx}{4\pi\pi+cc})(1 - \dfrac{4cx+4xx}{16\pi\pi+cc})$

$(1 - \dfrac{4cx+4xx}{36\pi\pi+cc})$ &c.. Jam ponatur c negativum, atque

habebuntur hæ duæ æquationes: $\dfrac{e^x + e.^{-c} e^{-x}}{1+e^{-c}} =$

$(1+$

$$\left(1 + \frac{4cx + 4xx}{\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{9\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{25\pi\pi + cc}\right) \&c.,$$

$$\frac{e^{x} - e^{-c}\,e^{-x}}{1 - e^{-c}} = \left(1 + \frac{2x}{c}\right)\left(1 + \frac{4cx + 4xx}{4\pi\pi + cc}\right)$$

$\left(1 + \frac{4cx + 4xx}{16\pi\pi + cc}\right)\left(1 + \frac{4cx + 4xx}{36\pi\pi + cc}\right)$ &c.. Multiplicetur forma prima per

tertiam, ac prodibit $\dfrac{e^{2x} + e^{-2x} + e^{c} + e^{-c}}{2 + e^{c} + e^{-c}}$; ponatur ve-

ro y loco $2x$, eritque $\dfrac{e^{y} + e^{-y} + e^{c} + e^{-c}}{2 + e^{c} + e^{-c}} = \left(1 - \dfrac{2cy + yy}{\pi\pi + cc}\right)$

$\left(1 + \dfrac{2cy + yy}{\pi\pi + cc}\right)\left(1 - \dfrac{2cy + yy}{9\pi\pi + cc}\right)\left(1 + \dfrac{2cy + yy}{9\pi\pi + cc}\right)\left(1 - \dfrac{2cy + yy}{25\pi\pi + cc}\right)$

$\left(1 + \dfrac{2cy + yy}{25\pi\pi + cc}\right)$ &c.. Multiplicetur prima forma per quar-

tam, erit productum $= \dfrac{e^{2x} - e^{-2x} + e^{c} - e^{-c}}{e^{c} - e^{-c}}$; po-

natur y pro $2x$, eritque $\dfrac{e^{y} - e^{-y} + e^{c} - e^{-c}}{e^{c} - e^{-c}} =$

$\left(1 + \dfrac{y}{c}\right)\left(1 - \dfrac{2cy + yy}{\pi\pi + cc}\right)\left(1 + \dfrac{2cy + yy}{4\pi\pi + cc}\right)\left(1 - \dfrac{2cy + yy}{9\pi\pi + cc}\right)$

$\left(1 + \dfrac{2cy + yy}{16\pi\pi + cc}\right)\left(1 - \dfrac{2cy + yy}{25\pi\pi + cc}\right)$ &c.. Si secunda forma

per quartam multiplicetur, prodibit eadem æquatio nisi quod
c capiendum sit negativum, erit nempe

$$\frac{e^{c} - e^{-c} - e^{y} + e^{-y}}{e^{c} - e^{-c}} = \left(1 - \frac{y}{c}\right)\left(1 + \frac{2cy + yy}{\pi\pi + cc}\right)$$

$\left(1 - \dfrac{2cy + yy}{4\pi\pi + cc}\right)\left(1 + \dfrac{2cy + yy}{9\pi\pi + cc}\right)\left(1 - \dfrac{2cy + yy}{16\pi\pi + cc}\right)$

$\left(1 + \dfrac{2cy + yy}{25\pi\pi + cc}\right)\left(1 - \dfrac{2cy + yy}{36\pi\pi + cc}\right)$ &c.. Multiplicetur de-

nique

nique forma secunda per quartam eritque $\dfrac{e^{y}+e^{-y}-e^{c}-e^{-c}}{2-e^{c}-e^{-c}}$

$$=(1-\tfrac{yy}{cc})(1-\tfrac{2cy+yy}{4\pi\pi+cc})(1+\tfrac{2cy+yy}{4\pi\pi+cc})(1-\tfrac{2cy+yy}{16\pi\pi+cc})$$

$$(1+\tfrac{2cy+yy}{16\pi\pi+cc})(1-\tfrac{2cy+yy}{36\pi\pi+cc})(1+\tfrac{2cy+yy}{36\pi\pi+cc}) \ \&c.$$

163. Hæ quatuor combinationes nunc commode ad Circulum transferri possunt, ponendo $c = g\sqrt{-1}$ & $y = v\sqrt{-1}$: erit enim $e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} = 2 \, cof. \, v$; $e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} = 2\sqrt{-1}.fin. \, v.$ & $e^{g\sqrt{-1}} + e^{-g\sqrt{-1}} = 2 \, cof. \, g$; $e^{g\sqrt{-1}} - e^{-g\sqrt{-1}} = 2\sqrt{-1}.fin. \, g.$ Hinc prima combinatio dabit $\dfrac{cof. \, v + cof. \, g}{1 + cof. \, g} = 1 - \dfrac{vv}{1.2(1 + cof. \, g)} +$

$$\dfrac{v^4}{1.2.3.4(1 + cof. \, g)} - \dfrac{v^6}{1.2....6(1 + cof. g)} + \&c. = (1 + \tfrac{2gv - vv}{\pi\pi - gg})$$

$$(1 - \tfrac{2gv - vv}{\pi\pi - gg})(1 + \tfrac{2gv - vv}{9\pi\pi - gg})(1 - \tfrac{2gv - vv}{9\pi\pi - gg})$$

$$(1 + \tfrac{2gv - vv}{25\pi\pi - gg})(1 - \tfrac{2gv - vv}{25\varpi\varpi - gg}) \&c. = (1 + \tfrac{v}{\varpi - g})$$

$$(1 - \tfrac{v}{\varpi + g})(1 - \tfrac{v}{\varpi - g})(1 + \tfrac{v}{\varpi + g})(1 + \tfrac{v}{3\varpi - g})$$

$$(1 - \tfrac{v}{3\varpi + g})(1 - \tfrac{v}{3\varpi - g})(1 + \tfrac{v}{3\varpi + g}) \&c. =$$

$$(1 - \tfrac{vv}{(\varpi - g)^2})(1 - \tfrac{vv}{(\varpi + g)^2})(1 - \tfrac{vv}{(3\varpi - g)^2})$$

$$(1 - \tfrac{vv}{(3\varpi + g)^2})(1 - \tfrac{vv}{(5\varpi - g)^2}) \&c..$$ Quarta vero

combinatio dat $\dfrac{cof. \, v - cof. \, g}{1 - cof. \, g} = 1 - \dfrac{vv}{1.2(1 - cof. \, g)} +$

$$\dfrac{v^4}{1.2.3.4(1 - cof. g)} - \dfrac{v^6}{1.2.....6(1 - cof. g)} + \&c. = (1 - \tfrac{vv}{gg})$$

$$(1 + \tfrac{2gv - vv}{4\varpi\varpi - gg})(1 - \tfrac{2gv - vv}{4\varpi\varpi - gg})(1 + \tfrac{2gv - vv}{16\varpi\varpi - gg})$$

$$(1 -$$

$$\left(1 - \frac{2gv - vv}{16\varpi\varpi - gg}\right) \&c. = \left(1 - \frac{v}{g}\right)\left(1 + \frac{v}{g}\right)\left(1 + \frac{v}{2\varpi - g}\right)$$

$$\left(1 - \frac{v}{2\varpi + g}\right)\left(1 - \frac{v}{2\varpi - g}\right)\left(1 + \frac{v}{2\varpi + g}\right)\left(1 + \frac{v}{4\varpi - g}\right)$$

$$\left(1 - \frac{v}{4\varpi + g}\right) \&c. = \left(1 - \frac{vv}{gg}\right)\left(1 - \frac{vv}{(2\varpi - g)^2}\right)$$

$$\left(1 - \frac{vv}{(2\varpi + g)^2}\right)\left(1 - \frac{vv}{(4\varpi - g)^2}\right)\left(1 - \frac{vv}{(4\varpi + g)^2}\right) \&c..$$

Secunda combinatio dat $\dfrac{\sin. g + \sin. v}{\sin. g} = 1 + \dfrac{v}{\sin. g} - \dfrac{v^3}{1.2.3 \, \sin. g} +$

$$\frac{v^5}{1.2 \ldots 5 \, \sin. g} - \&c. = \left(1 + \frac{v}{g}\right)\left(1 + \frac{2gv - vv}{\varpi\varpi - gg}\right)$$

$$\left(1 - \frac{2gv - vv}{4\varpi\varpi - gg}\right)\left(1 + \frac{2gv - vv}{9\varpi\varpi - gg}\right)\left(1 - \frac{2gv - vv}{16\varpi\varpi - gg}\right) \&c.$$

$$= \left(1 + \frac{v}{g}\right)\left(1 + \frac{v}{\varpi - g}\right)\left(1 - \frac{v}{\varpi + g}\right)\left(1 - \frac{v}{2\varpi - g}\right)$$

$$\left(1 + \frac{v}{2\varpi + g}\right)\left(1 + \frac{v}{3\varpi - g}\right)\left(1 - \frac{v}{3\varpi + g}\right)\left(1 - \frac{v}{5\varpi - g}\right) \&c..$$

Ac fumto v negativo prodit tertia combinatio.

164. Ipfæ vero etiam expreffiones in §. 162. primum inventæ ad Arcus circulares traduci poffunt hoc modo: cum fit

$$\frac{e^x + e^c \cdot e^{-x}}{1 + e^c} = \frac{(1 + e^{-c})(e^x + e^c \, e^{-x})}{2 + e^c + e^{-c}} =$$

$$\frac{e^x + e^{-x} + e^{c-x} + e^{-c+x}}{2 + e^c + e^{-c}}, \text{ fi ponamus } c = g\sqrt{-1} \&$$

$x = z\sqrt{-1}$, hæc expreffio abit in hanc $\dfrac{\cos. z + \cos.(g - z)}{1 + \cos. g} =$

$\cos. z + \dfrac{\sin. g. \sin. z}{1 + \cos. g}$. Erit ergo (ob $\dfrac{\sin. g}{1 + \cos. g} = tang. \dfrac{1}{2} g$)

$\cos. z + tang. \dfrac{1}{2} g. \sin. z = 1 + \dfrac{z}{1} tang. \dfrac{1}{2} g - \dfrac{zz}{1.2} -$

$\dfrac{z^3}{1.2.3} tang. \dfrac{1}{2} g + \dfrac{z^4}{1.2.3.4} + \dfrac{z^5}{1.2.3.4.5} tang. \dfrac{1}{2} g - \&c.$

$= \left(1 + \dfrac{4gz - 4zz}{\varpi\varpi - gg}\right)\left(1 + \dfrac{4gz - 4zz}{9\varpi\varpi - gg}\right)\left(1 + \dfrac{4gz - 4zz}{25\varpi\varpi - gg}\right) \&c.$

$$= \left(1 + \right.$$

$$= (1 + \frac{2z}{\varpi - g})(1 - \frac{2z}{\varpi + g})(1 + \frac{2z}{3\varpi - g})(1 - \frac{2z}{3\varpi + g})$$

$(1 + \frac{2z}{5\varpi - g})(1 - \frac{2z}{5\varpi + g})$ &c. . Simili modo altera ex-

preffio, fi Numerator & Denominator per $1 - e^{-c}$ multi-

plicetur, abit in $\dfrac{e^x + e^{-x} - e^{c-x} - e^{-c+x}}{2 - e^c - e^{-c}}$; quæ,

facto $c = z\sqrt{-1}$ & $x = z\sqrt{-1}$, dat $\dfrac{cof.\ z - cof.\ (g-z)}{1 - cof.\ g} =$

$cof.\ z - \dfrac{fin.\ g.\ fin.\ z.}{1 - cof.\ g} = cof.\ z - \dfrac{fin.\ z}{tang.\ \frac{1}{2}\ g}$. Erit ergo $cof.\ z -$

$cot.\ \frac{1}{2}\ g.\ fin.\ z = 1 - \frac{z}{1}\ cot.\ \frac{1}{2}\ g - \frac{zz}{1.2} + \frac{z^3}{1.2.3}\ cot.\ \frac{1}{2}\ g +$

$\frac{z^4}{1.2.3.4} - \frac{z^5}{1.2..5}\ cot.\ \frac{1}{2}\ g + $ &c. $= (1 - \frac{2z}{g})(1 + \frac{4gz - 4zz}{4\varpi\varpi - gg})$

$(1 + \frac{4gz - 4zz}{16\varpi\varpi - gg})(1 + \frac{4gz - 4zz}{36\pi\pi - gg})$ &c. $= (1 - \frac{2z}{g})$

$(1 + \frac{2z}{2\pi - g})(1 - \frac{2z}{2\pi + g})(1 + \frac{2z}{4\pi - g})(1 - \frac{2z}{4\pi + g})$ &c.

Quod fi ergo ponatur $v = 2z$ feu $z = \frac{1}{2}\ v$; habebitur

$\dfrac{cof.\ \frac{1}{2}\ (g - v)}{cof.\ \frac{1}{2}\ g} = cof.\ \frac{1}{2}\ v + tang.\ \frac{1}{2}\ g.\ fin.\ \frac{1}{2}\ v =$

$(1 + \frac{v}{\varpi - g})(1 - \frac{v}{\varpi + g})(1 + \frac{v}{3\varpi - g})(1 - \frac{v}{3\varpi + g})$&c.;

$\dfrac{cof.\ \frac{1}{2}\ (g + v)}{cof.\ \frac{1}{2}\ g} = cof.\ \frac{1}{2}\ v - tang.\ \frac{1}{2}\ g.\ fin.\ \frac{1}{2}\ v =$

$(1 - \frac{v}{\varpi - g})(1 + \frac{v}{\varpi + g})(1 - \frac{v}{3\varpi - g})(1 + \frac{v}{3\varpi + g})$&c.;

$\dfrac{fin.\ \frac{1}{2}\ (g - v)}{fin.\ \frac{1}{2}\ g} = cof.\ \frac{1}{2}\ v - cot.\ \frac{1}{2}\ g.\ fin.\ \frac{1}{2}\ v =$

$(1 - \frac{v}{g})(1 + \frac{v}{2\varpi - g})(1 - \frac{v}{2\varpi + g})(1 + \frac{z}{4\varpi - g})$&c.

$\dfrac{fin.\ \frac{1}{2}\ (g + v)}{fin.\ \frac{1}{2}\ g}$

LIB. I. $\dfrac{\textit{fin. } \frac{1}{2}\,(g + v)}{\textit{fin. } \frac{1}{2}\,g} = \textit{cof. } \dfrac{1}{2}\,v + \textit{cot. } \dfrac{1}{2}\,g . \textit{fin. } \dfrac{1}{2}\,v =$

$$(1 + \frac{v}{g})(1 - \frac{v}{2\varpi - g})(1 + \frac{v}{2\varpi + g})(1 - \frac{v}{4\varpi - g}) \ \&c.$$

Quorum Factorum lex progreffionis fatis eft fimplex & uniformis; atque ex his expreffionibus per multiplicationem oriuntur eæ ipfæ, quæ §. præcedente funt inventæ.

C A P U T X.

De ufu Factorum inventorum in definiendis fummis Serierum infinitarum.

165. SI fuerit $1 + Az + Bz^2 + Cz^3 + Dz^4 + \&c. = (1 + \alpha z)(1 + 6z)(1 + \gamma z)(1 + \delta z) \&c.$, hi Factores, five fint numero finiti five infiniti, fi in fe actu multiplicentur, illam expreffionem $1 + A + Bz^2 + Cz^3 + Dz^4 + \&c.$, producere debent. Æquabitur ergo coëfficiens A fummæ omnium quantitatum $\alpha + 6 + \gamma + \delta + \varepsilon + \&c.$. Coëfficiens vero B æqualis erit fummæ productorum ex binis, eritque $B = \alpha 6 + \alpha \gamma + \alpha \delta + 6\gamma + 6\delta + \gamma\delta + \&c.$. Tum vero coëfficiens C æquabitur fummæ productorum ex ternis, nempe erit $C = \alpha 6 \gamma + \alpha 6 \delta + 6\gamma\delta + \alpha\gamma\delta + \&c.$. Atque ita porro erit $D =$ fummæ productorum ex quaternis, $E =$ fummæ productorum ex quinis, &c., id quod ex Algebra communi conftat.

166. Quia fumma quantitatum $\alpha + 6 + \gamma + \delta + \&c.$, datur una cum fumma productorum ex binis, hinc fumma Quadratorum $\alpha^2 + 6^2 + \gamma^2 + \delta^2 + \&c.$, inveniri poterit, quippe quæ æqualis eft Quadrato fummæ demtis duplicibus productis ex binis. Simili modo fumma Cuborum, Biquadratorum & altiorum Poteftatum definiri poteft: fi enim ponamus

$$P =$$

$$P = \alpha + 6 + \gamma + \delta + \epsilon + \&c.$$
$$Q = \alpha^2 + 6^2 + \gamma^2 + \delta^2 + \epsilon^2 + \&c.$$
$$R = \alpha^3 + 6^3 + \gamma^3 + \delta^3 + \epsilon^3 + \&c.$$
$$S = \alpha^4 + 6^4 + \gamma^4 + \delta^4 + \epsilon^4 + \&c.$$
$$T = \alpha^5 + 6^5 + \gamma^5 + \delta^5 + \epsilon^5 + \&c.$$
$$V = \alpha^6 + 6^6 + \gamma^6 + \delta^6 + \epsilon^6 + \&c.$$
$$\&c.$$

Valores P, Q, R, S, T, V &c. fequenti modo ex cognitis A, B, C, D, &c., determinabuntur.

$$P = A$$
$$Q = AP - 2B$$
$$R = AQ - BP + 3C$$
$$S = AR - BQ + CP - 4D$$
$$T = AS - BR + CQ - DP + 5E$$
$$V = AT - BS + CR - DQ + EP - 6F$$
$$\&c.$$

quarum formularum veritas examine inftituto facile agnofcitur: interim tamen in calculo differentiali fummo cum rigore demonftrabitur.

167. Cum igitur fupra (§. 156.) invenerimus effe :

$$\frac{e^x - e^{-x}}{2} = x \left(1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.....7} + \&c. \right) =$$

$$x \left(1 + \frac{xx}{\pi\pi} \right) \left(1 + \frac{xx}{4\pi\pi} \right) \left(1 + \frac{xx}{9\pi\pi} \right) \left(1 + \frac{xx}{16\pi\pi} \right)$$

$$\left(1 + \frac{xx}{25\pi\pi} \right) \&c., \text{ erit } 1 + \frac{xx}{1.2.3} + \frac{x^4}{1.2.3.4.5} + \frac{x^6}{1.2.3.....7} +$$

$$\&c. = \left(1 + \frac{xx}{\pi\pi} \right) \left(1 + \frac{xx}{4\pi\pi} \right) \left(1 + \frac{xx}{9\pi\pi} \right) \left(1 + \frac{xx}{16\pi\pi} \right) \&c.$$

Ponatur $xx = \pi\pi z$, eritque $1 + \frac{\pi\pi}{1.2.3} z + \frac{\pi^4}{1.2.3.4.5} z^2 +$

$$\frac{\pi^6}{1.2.3.....7} z^3 + \&c. = (1 + z)(1 + \frac{1}{4}z)(1 + \frac{1}{9}z)(1 + \frac{1}{16}z)$$

Euleri *Introduct. in Anal. infin. parv.*   R $(1 +$

LIB. I. $(1 + \frac{1}{25}z)$ &c.. Facta ergo applicatione superioris regulæ ad hunc casum, erit $A = \frac{\pi\pi}{6}$; $B = \frac{\pi^4}{120}$; $C = \frac{\pi^6}{5040}$; $D = \frac{\pi^8}{362880}$ &c.. Quod si ergo ponatur

$$P = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \&c.$$

$$Q = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} + \frac{1}{25^2} + \frac{1}{36^2} + \&c.$$

$$R = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} + \frac{1}{25^3} + \frac{1}{36^3} + \&c.$$

$$S = 1 + \frac{1}{4^4} + \frac{1}{9^4} + \frac{1}{16^4} + \frac{1}{25^4} + \frac{1}{36^4} + \&c.$$

$$T = 1 + \frac{1}{4^5} + \frac{1}{9^5} + \frac{1}{16^5} + \frac{1}{25^5} + \frac{1}{36^5} + \&c.$$

$$\&c.$$

atque harum litterarum valores ex A, B, C, D, &c. determinentur, prodibit.

$$P = \frac{\pi\pi}{6}$$

$$Q = \frac{\pi^4}{90}$$

$$R = \frac{\pi^6}{945}$$

$$S = \frac{\pi^8}{9450}$$

$$T = \frac{\pi^{10}}{93555}$$

$$\&c.$$

168. Patet ergo omnium Serierum infinitarum in hac forma generali $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \&c.$, contentarum, quoties n fuerit numerus par, ope Peripheriæ Circuli π exhiberi posse; habebit enim semper summa Seriei ad π^n rationem rationalem.

lem. Quo autem valor harum summarum clarius perspicia- CAP. X.
tur, plures hujusmodi Serierum summas commodiori modo
expressas hic adjiciam.

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. = \frac{2^0 \cdot}{1.2.3} \cdot \frac{1}{1} \pi^2$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c. = \frac{2^2}{1.2.3.4.5} \cdot \frac{1}{3} \pi^4$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c. = \frac{2^4}{1.2.3....7} \cdot \frac{1}{3} \pi^6$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \&c. = \frac{2^6}{1.2.3....9} \cdot \frac{3}{5} \pi^8$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{4^{10}} + \frac{1}{5^{10}} + \&c. = \frac{2^8}{1.2.3.....11} \cdot \frac{5}{3} \pi^{10}$$

$$1 + \frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{4^{12}} + \frac{1}{5^{12}} + \&c. = \frac{2^{10} \cdot}{1.2.3.....13} \cdot \frac{691}{105} \pi^{12}$$

$$1 + \frac{1}{2^{14}} + \frac{1}{3^{14}} + \frac{1}{4^{14}} + \frac{1}{5^{14}} + \&c. = \frac{2^{12}}{1.2.3.....15} \cdot \frac{35}{1} \pi^{14}$$

$$1 + \frac{1}{2^{16}} + \frac{1}{3^{16}} + \frac{1}{4^{16}} + \frac{1}{5^{16}} + \&c. = \frac{2^{14}}{1.2.3.....17} \cdot \frac{3617}{15} \pi^{16}$$

$$1 + \frac{1}{2^{18}} + \frac{1}{3^{18}} + \frac{1}{4^{18}} + \frac{1}{5^{18}} + \&c. = \frac{2^{16}}{1.2.3.....19} \cdot \frac{43867}{21} \pi^{18}$$

$$1 + \frac{1}{2^{20}} + \frac{1}{3^{20}} + \frac{1}{4^{20}} + \frac{1}{5^{20}} + \&c. = \frac{2^{18}}{1.2.3....21} \cdot \frac{1222277}{55} \pi^{20}$$

$$1 + \frac{1}{2^{22}} + \frac{1}{3^{22}} + \frac{1}{4^{22}} + \frac{1}{5^{22}} + \&c. = \frac{2^{20}}{1.2.3...23} \cdot \frac{854513}{3} \pi^{22}$$

$$1 + \frac{1}{2^{24}} + \frac{1}{3^{24}} + \frac{1}{4^{24}} + \frac{1}{5^{24}} + \&c. = \frac{2^{22}}{1.2.3........25} \cdot \frac{1181820455}{273} \pi^{24}$$

$$1 + \frac{1}{2^{26}} + \frac{1}{3^{26}} + \frac{1}{4^{26}} + \frac{1}{5^{26}} + \&c. = \frac{2^{24}}{1.2.3........27} \cdot \frac{76977927}{1} \pi^{26}.$$

Hucusque istos Potestatum ipsius π Exponentes artificio alibi
exponendo continuare licuit, quod ideo hic adjunxi, quod

Seriei

Seriei fractionum primo intuitu perquam irregularis 1 , $\frac{1}{3}$, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{3}$, $\frac{691}{105}$, $\frac{35}{1}$, &c. in plurimis occafionibus eximius eft ufus.

169. Tractemus eodem modo æquationem §. 157. inventam, ubi erat

$$\frac{e^{x}+e^{-x}}{2} = 1 + \frac{xx}{1.2} + \frac{x^4}{1.2.3.4} + \frac{x^6}{1.2.3.4.5.6} +$$

&c., $= (1+\frac{4xx}{\pi\pi})(1+\frac{4xx}{9\pi\pi})(1+\frac{4xx}{25\pi\pi})(1+\frac{4xx}{49\pi\pi})$ &c. .

Pofito ergo $xx = \frac{\pi\pi z}{4}$ erit $1 + \frac{\pi\pi}{1.2.4} z + \frac{\pi^4}{1.2.3.4.4^2} zz +$

$\frac{\pi^6}{1.2....6.4^3} z^3 +$ &c. , $= (1+z)(1+\frac{1}{9}z)(1+\frac{1}{25}z)$

$(1+\frac{1}{49}z)$ &c. . Unde, facta applicatione, erit $A = \frac{\pi\pi}{1.2.4}$;

$B = \frac{\pi^4}{1.2.3.4.4^2}$; $C = \frac{\pi^6}{1.2.3....6.4^3}$; &c.. Quod fi ergo ponamus

$$P = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.$$

$$Q = 1 + \frac{1}{9^2} + \frac{1}{25^2} + \frac{1}{49^2} + \frac{1}{81^2} + \&c.$$

$$R = 1 + \frac{1}{9^3} + \frac{1}{25^3} + \frac{1}{49^3} + \frac{1}{81^3} + \&c.$$

$$S = 1 + \frac{1}{9^4} + \frac{1}{25^4} + \frac{1}{49^4} + \frac{1}{81^4} + \&c.$$

&c.

reperientur fequentes pro P, Q, R, S, &c., valores :

$$P = \frac{1}{1} \cdot \frac{\pi^2}{2^3} ; \qquad Q = \frac{2}{1.2.3} \cdot \frac{\pi^4}{2^5}$$

$$R = \frac{16}{1.2.3.4.5} \cdot \frac{\pi^6}{2^7} ; \qquad S = \frac{272}{1.2.3....7} \cdot \frac{\pi^8}{2^9}$$

$$T =$$

$$T = \frac{7936}{1.2.3\ldots\ldots9}\cdot\frac{\pi^{10}}{2^{11}}; \quad V = \frac{353792}{1.2.3\ldots\ldots11}\cdot\frac{\pi^{12}}{2^{13}}$$

$$W = \frac{22368256}{1.2.3\ldots\ldots\ldots13}\cdot\frac{\pi^{14}}{2^{15}}.$$

170. Eædem fummæ Poteftatum numerorum imparium inveniri poffunt ex fummis præcedentibus, in quibus omnes numeri occurrunt; fi enim fuerit $M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$

$\frac{1}{5^n} +$ &c., erit ubique, per $\frac{1}{2^n}$ multiplicando, $\frac{M}{2^n} = \frac{1}{2^n} +$

$\frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$ &c., quæ Series numeros tantum pares continens, fi a priori fubtrahatur, relinquet numeros impares, eritque ideo $M - \frac{M}{2^n} = \frac{2^n - 1}{2^n} M = 1 + \frac{1}{3^n} +$

$\frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} +$ &c.. Quod fi autem Series $\frac{M}{2^n}$ bis fumta fubtrahatur ab M figna prodibunt alternantia, eritque $M -$

$\frac{2M}{2^n} = \frac{2^{n-1} - 1}{2^{n-1}} M = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} -$

$\frac{1}{6^n} +$ &c.. Per tradita ergo præcepta fummari poterunt hæ Series

$$1 \pm \frac{1}{2^n} + \frac{1}{3^n} \pm \frac{1}{4^n} + \frac{1}{5^n} \pm \frac{1}{6^n} + \frac{1}{7^n} \pm \text{ &c.}$$

$$1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \text{ &c.}$$

Si quidem n fit numerus par, atque fumma erit $= A\pi^n$ exiftente A numero rationali.

171. Præterea vero expreffiones §. 164 exhibitæ fimili mo-

do

do Series notatu dignas fuppeditabunt. Cum enim fit $cof. \frac{1}{2} v +$

$tang. \frac{1}{2} g . fin. \frac{1}{2} v = (1 + \frac{v}{\varpi - g}) (1 - \frac{v}{\varpi + g})$

$(1 + \frac{v}{3\varpi - g})$ &c., fi ponamus $v = \frac{x}{n} \varpi$ & $g = \frac{m}{n} \pi$ erit

$(1 + \frac{x}{n - m}) (1 - \frac{x}{n + m}) (1 + \frac{x}{3n - m}) (1 - \frac{x}{3n + m})$

$(1 + \frac{x}{5n - m}) (1 - \frac{x}{5n + m})$ &c. $= cof. \frac{x \varpi}{2 n} + tang.$

$\frac{m\varpi}{2n} . fin. \frac{x \varpi}{2 n} = 1 + \frac{\varpi x}{2 n} tang. \frac{m\varpi}{2n} - \frac{\varpi \varpi x x}{2. 4 nn} - \frac{\varpi^3 x^3}{2. 4. 6 n^3}$

$tang. \frac{m\varpi}{2n} + \frac{\varpi^4 x^4}{2. 4. 6. 8 n^4} +$ &c.. Hæc expreffio infinita cum

§. 165 collata dabit hos valores $A = \frac{\varpi}{2n} tang. \frac{m\varpi}{2n}$; $B =$

$\frac{- \varpi \varpi}{2. 4 n n}$; $C = \frac{- \varpi^3}{2. 4. 6 n^3} . tang. \frac{m\varpi}{2n}$, $D = \frac{\varpi^4}{2. 4. 6. 8n^4}$; $E =$

$\frac{\varpi^5}{2. 4. 6. 8. 10 n^5} . tang. \frac{m\varpi}{2n}$ &c. Tum vero erit $\alpha = \frac{1}{n - m}$;

$\varsigma = - \frac{1}{n + m}$; $\gamma = \frac{1}{3n - m}$; $\delta = - \frac{1}{3 n + m}$; $\varepsilon =$

$\frac{1}{5n - m}$; $\xi = - \frac{1}{5 n + m}$ &c.

172. Hinc ergo ad normam §. 166 fequentes Series exorientur.

$P = \frac{1}{n - m} - \frac{1}{n + m} + \frac{1}{3n - m} - \frac{1}{3n + m} + \frac{1}{5n - m} - \frac{1}{5n + m} +$&c.

$Q = \frac{1}{(n - m)^2} + \frac{1}{(n + m)^2} + \frac{1}{(3n - m)^2} + \frac{1}{(3 n + m)^2} +$

$\frac{1}{(5 n - m)^2} +$ &c.

$R = \frac{1}{(n - m)^3} - \frac{1}{(n + m)^3} + \frac{1}{(3n - m)^3} - \frac{1}{(3 n + m)^3} +$

$\frac{1}{(5 n - m)^3} -$ &c.

$S =$

C A P. X.

$$S = \frac{1}{(n-m)^4} + \frac{1}{(n+m)^4} + \frac{1}{(3n-m)^4} + \frac{1}{(3n+m)^4} +$$
$$\frac{1}{(5n-m)^4} + \&c.$$

$$T = \frac{1}{(n-m)^5} - \frac{1}{(n+m)^5} + \frac{1}{(3n-m)^5} - \frac{1}{(3n+m)^5} +$$
$$\frac{1}{(5n-m)^5} - \&c.$$

$$V = \frac{1}{(n-m)^6} + \frac{1}{(n+m)^6} + \frac{1}{(3n-m)^6} + \frac{1}{(3n+m)^6} +$$
$$\frac{1}{(5n-m)^6} + \&c.$$

$$\&c.$$

Pofito autem *tang.* $\frac{m\varpi}{2n} = k$ erit, uti oftendimus,

$$P = A = \frac{k\varpi}{2n} = \frac{k\pi}{2n}$$

$$Q = \frac{(kk+1)\pi\pi}{4nn} = \frac{(2kk+2)\pi^2}{2.\,4\,nn}$$

$$R = \frac{(k^3+k)\pi^3}{8n^3} = \frac{(6k^3+6k)\pi^3}{2.\,4.\,6.\,n^3}$$

$$S = \frac{(3k^4+4kk+1)\pi^4}{48n^4} = \frac{(24k^4+32k^2+8)\pi^4}{2.\,4.\,6.\,8\,n^4}$$

$$T = \frac{(3k^5+5k^3+2k)\pi^5}{96n^5} = \frac{(120k^5+200k^3+80k)\pi^5}{2.\,4.\,6.\,8.\,10\,n^5}$$

173. Pari modo ultima forma §. 164; $cof. \frac{1}{2}v + cot. \frac{1}{2}g \times$

$fin. \frac{1}{2}v = (1+\frac{v}{g})(1-\frac{v}{2\pi-g})(1+\frac{v}{2\pi+g})(1-\frac{v}{4\pi-g})$

$(1+\frac{v}{4\pi+g})$ &c. Si ponamus $v = \frac{x}{n}\pi$, $g = \frac{m}{n}\pi$, &

tang. $\frac{m\pi}{2n} = k$, ut fit $cot. \frac{1}{2}g = \frac{1}{k}$, dabit $cof. \frac{\pi x}{2n} + \frac{1}{k} \times$

$fin. \frac{\pi x}{2n} = 1 + \frac{\pi x}{2nk} - \frac{\pi\pi xx}{2.\,4\,nn} - \frac{\pi^3 x^3}{2.\,4.\,6\,n^3 k} + \frac{\pi^4 x^4}{2.\,4.\,6.\,8\,n^4} +$

$\frac{\pi^5 x^5}{2.\,4.\,6.\,8.\,10\,n^5 k} - \&c. = (1+\frac{x}{m})(1-\frac{x}{2n-m})(1+\frac{x}{2n+m})$

$$(1-$$

$(1 - \frac{x}{4n - m})(1 + \frac{x}{4n+m},)$ &c.. Comparatione ergo cum

forma generali (§. 165) inſtituta erit $A = \frac{\pi}{2nk}$; $B =$

$\frac{-\pi\pi}{2.4n^2}$; $C = \frac{-\pi^3}{2.4.6n^3k}$; $D = \frac{\pi^4}{2.4.6.8n^4}$; $E = \frac{\pi^5}{2.4.6.8.10n^5k}$;

&c.; ex Factoribus vero habebitur $\alpha = \frac{1}{m}$; $\mathfrak{b} = \frac{-1}{2n - m}$;

$\gamma = \frac{1}{2n+m}$; $\delta = \frac{-1}{4n - m}$; $\epsilon = \frac{1}{4n+m}$ &c.

174. Hinc ergo ad normam §. 166. ſequentes Series forma-
buntur, earumque ſummæ aſſignabuntur

$$P = \frac{1}{m} - \frac{1}{2n - m} + \frac{1}{2n+m} - \frac{1}{4n - m} + \frac{1}{4n+m} - \text{&c.}$$

$$Q = \frac{1}{m^2} + \frac{1}{(2n - m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n - m)^2} +$$

$$\frac{1}{(4n+m)^2} + \text{&c.}$$

$$R = \frac{1}{m^3} - \frac{1}{(2n - m)^3} + \frac{1}{(2n+m)^3} - \frac{1}{(4n - m)^3} +$$

$$\frac{1}{(4n+m)^3} - \text{&c.}$$

$$S = \frac{1}{m^4} + \frac{1}{(2n - m)^4} + \frac{1}{(2n+m)^4} + \frac{1}{(4n - m)^4} +$$

$$\frac{1}{(4n+m)^4} + \text{&c.}$$

$$T = \frac{1}{m^5} - \frac{1}{(2n - m)^5} + \frac{1}{(2n+m)^5} - \frac{1}{(4n - m)^5} +$$

$$\frac{1}{(4n+m)^5} - \text{&c.}$$

&c.

Hæ

Hæ autem summæ P, Q, R, S, &c, ita se habebunt

$$P = A = \frac{\pi}{2nk} \qquad\qquad = \frac{1\,\pi}{2\,n\,k}$$

$$Q = \frac{(kk+1)\,\pi\pi}{4\,n\,n\,kk} \qquad = \frac{(2+2kk)\,\pi^2}{2.\,4\,n^2\,k^2}$$

$$R = \frac{(kk+1)\,\pi^3}{8\,n^3\,k^3} \qquad = \frac{(6+6kk)\,\pi^3}{2.\,4.\,6\,n^3\,k^3}$$

$$S = \frac{(k^4+4kk+3)\,\pi^4}{48\,n^4\,k^4} \qquad = \frac{(24+32kk+3k^4)\,\pi^4}{2.\,4.\,6.\,8\,n^4\,k^4}$$

$$T = \frac{(2k^4+5kk+3)\,\pi^5}{96\,n^5\,k^5} \qquad = \frac{(120+200kk+80k^4)\,\pi^5}{2.\,4.\,6.\,8.\,10\,n^5\,k^5}$$

$$V = \frac{(2k^6+17k^4+30k^2+15)\pi^6}{960\,n^6\,k^6} = \frac{(720+1440kk+816k^4+96k^6)\pi^6}{2.\,4.\,6.\,8.\,10.\,12\,n^6\,k^6}$$

&c.

175. Series istæ generales merentur ut casus quosdam particulares inde derivemus, qui prodibunt si rationem *m* ad *n* in numeris determinemus. Sit igitur primum $m = 1$ & $n = 2$, fiet $k = tang. \frac{\varpi}{4} = tang.\ 45° = 1$, atque ambæ Serierum classes inter se congruent.   Erit ergo

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -\ \&c.$$

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} +\ \&c.$$

$$\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} -\ \&c.$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} +\ \&c.$$

$$\frac{5\,\pi^5}{1536} = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \frac{1}{9^5} -\ \&c.$$

$$\frac{\pi^6}{960} = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} +\ \&c.$$

&c.

Harum Serierum primam jam supra (§. 140) elicuimus, reliquarum illæ, quæ pares habent Dignitates, modo ante (§. 169)

Euleri Introduct. in Anal. infin. parv.   S   sunt

LIB. I. funt erutæ; ceteræ, in quibus Exponentes funt numeri impa-
res, hic primum occurrunt. Conftat ergo omnium quoque ifta-
rum Serierum:

$$1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \frac{1}{9^{2n+1}} - \&c.$$

fummas per valorem ipfius π affignari poffe.

176. Sit nunc $m = 1$, $n = 3$; erit $k = tang. \frac{\pi}{6} =$

$tang. \; 30° = \frac{1}{\sqrt{3}}$; atque Series §. 172 abibunt in has

$$\frac{\pi}{6\sqrt{3}} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{10} + \frac{1}{14} - \frac{1}{16} + \&c$$

$$\frac{\pi\pi}{27} = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{14^2} + \frac{1}{16^2} + \&c.$$

$$\frac{\pi^3}{162\sqrt{3}} = \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{8^3} - \frac{1}{10^3} + \frac{1}{14^3} - \frac{1}{16^3} + \&c.$$

$$\&c. , \quad five$$

$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c.$$

$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \&c.$$

$$\frac{4\pi^3}{81\sqrt{3}} = 1 - \frac{1}{2^3} + \frac{1}{4^3} - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{8^3} + \&c.$$

$$\&c.$$

in his Seriebus defunt omnes numeri per ternarium divifibi-
les: hinc pares dimenfiones ex jam inventis deducentur hoc
modo. Cum fit

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. , \; erit$$

$$\frac{\pi\pi}{6.9} = \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{9^2} + \frac{1}{12^2} + \&c. = \frac{\pi\pi}{54},$$

quæ pofterior Series continens omnes numeros per ternarium
divifi-

divisibiles , si subtrahatur a priore , remanebunt omnes numeri

non divisibiles per 3 : sicque erit $\frac{8\pi\pi}{54} = \frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} +$

$\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + $ &c., uti invenimus.

177. Eadem hypothesis $m = 1$, $n = 3$, & $k = \frac{i}{\sqrt{3}}$, ad

§. 174. accommodata has præbebit summationes

$$\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \frac{1}{19} - \text{\&c.}$$

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{\&c.}$$

$$\frac{\pi^3}{18\sqrt{3}} = 1 - \frac{1}{5^3} + \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} - \frac{1}{17^3} + \frac{1}{19^3} - \text{\&c.}$$

$$\text{\&c.}$$

in quarum denominatoribus numeri tantum impares occurrunt exceptis iis, qui per ternarium sunt divisibiles. Ceterum pares dimensiones ex jam cognitis deduci possunt, cum enim sit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \text{\&c.}, \quad \text{erit}$$

$$\frac{\pi\pi}{8.\ 9} = \frac{1}{3^2} + \frac{1}{9^2} + \frac{1}{15^2} + \frac{1}{21^2} + \text{\&c.} = \frac{\pi\pi}{72}$$

quæ Series , omnes numeros impares per 3 divisibiles continens, si subtrahatur a superiore, relinquet Seriem quadratorum numerorum imparium per 3 non divisibilium, eritque

$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{\&c.}$$

178. Si Series in §. §. 172. & 174 inventæ vel addantur vel subtrahantur, obtinebuntur aliæ Series notatu dignæ. Erit scilicet $\frac{k\pi}{2n} + \frac{\pi}{2nk} = \frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} +$

$$\frac{1}{2n+m}$$

Lib. I. $\dfrac{1}{2n+m} + $ &c. $= \dfrac{(kk+1)\pi}{2nk}$: at est $k = tang. \dfrac{m\varpi}{2n} = $

$\dfrac{\sin. \frac{m\pi}{2n}}{\cos. \frac{m\pi}{2n}}$, & $1 + kk = \dfrac{1}{(\cos. \frac{m\pi}{2n})^2}$, unde $\dfrac{2k}{1+kk} = 2\sin. \dfrac{m\pi}{2n} \times$

$\cos. \dfrac{m\pi}{2n} = \sin. \dfrac{m\pi}{n}$, quo valore substituto habebimus

$$\dfrac{\pi}{n \sin. \frac{m\varpi}{n}} = \dfrac{1}{m} + \dfrac{1}{n-m} - \dfrac{1}{n+m} - \dfrac{1}{2n-m} + \dfrac{1}{2n+m} +$$

$$\dfrac{1}{3n-m} - \dfrac{1}{3n+m} - \&c..$$ Simili modo per subtractionem

erit $\dfrac{\varpi}{2nk} - \dfrac{k\varpi}{2n} = \dfrac{(1-kk)\varpi}{2nk} = \dfrac{1}{m} - \dfrac{1}{n-m} + \dfrac{1}{n+m} -$

$$\dfrac{1}{2n-m} + \dfrac{1}{2n+m} - \dfrac{1}{3n-m} + \dfrac{1}{3n+m} - \&c.,$$ at est

$$\dfrac{2k}{1-kk} = tang. 2. \dfrac{m\varpi}{2n} = tang. \dfrac{m\varpi}{n} = \dfrac{\sin. \frac{m\varpi}{n}}{\cos. \frac{m\varpi}{n}},$$ hinc erit

$$\dfrac{\varpi \cos. \frac{m\varpi}{n}}{n \sin. \frac{m\varpi}{n}} = \dfrac{1}{m} - \dfrac{1}{n-m} + \dfrac{1}{n+m} - \dfrac{1}{2n-m} + \dfrac{1}{2n+m} -$$

$$\dfrac{1}{3n-m} + \&c..$$ Series Quadratorum & altiorum Potestatum

hinc ortæ facilius per differentiationem hinc deducentur infra.

179. Quoniam casus, quibus $m = 1$ & $n = 2$ vel 3, jam

evolvimus, ponamus $m = 1$ & $n = 4$; erit $\sin. \dfrac{m\pi}{n} = $

$\sin. \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$ & $\cos. \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$. Hinc itaque habebitur

$$\dfrac{\varpi}{2\sqrt{2}} = 1 + \dfrac{1}{3} - \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9} + \dfrac{1}{11} - \dfrac{1}{13} - \dfrac{1}{15} +$$

&c.

&

& $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} +$

&c.. Sit $m = 1$ & $n = 8$, erit $\frac{m\varpi}{n} = \frac{\pi}{8}$ & $\sin. \frac{\varpi}{8} =$

$\sqrt{(\frac{1}{2} - \frac{1}{2\sqrt{2}})}$ & $\cos. \frac{\pi}{8} = \sqrt{(\frac{1}{2} + \frac{1}{2\sqrt{2}})}$ & $\frac{\cos. \frac{\pi}{8}}{\sin. \frac{\pi}{8}} =$

$1 + \sqrt{2}$. Hinc itaque erit

$$\frac{\varpi}{4\sqrt{(2-\sqrt{2})}} = 1 + \frac{1}{7} - \frac{1}{9} - \frac{1}{15} + \frac{1}{17} + \frac{1}{23} - \text{&c.}$$

$$\frac{\varpi}{8(\sqrt{2}-1)} = 1 - \frac{1}{7} + \frac{1}{9} - \frac{1}{15} + \frac{1}{17} - \frac{1}{23} + \text{&c.}$$

Sit nunc $m = 3$ & $n = 8$, erit $\frac{m\varpi}{n} = \frac{3\varpi}{8}$ & $\sin. \frac{3\varpi}{8} =$

$\sqrt{(\frac{1}{2} + \frac{1}{2\sqrt{2}})}$, & $\cos. \frac{3\pi}{8} = \sqrt{(\frac{1}{2} - \frac{1}{2\sqrt{2}})}$, unde $\frac{\cos. \frac{3}{8}\varpi}{\sin. \frac{3}{8}\varpi} =$

$\frac{1}{\sqrt{2}+1}$; ac prodibunt hæ Series

$$\frac{\varpi}{4\sqrt{(2+\sqrt{2})}} = \frac{1}{3} + \frac{1}{5} - \frac{1}{11} - \frac{1}{13} + \frac{1}{19} + \frac{1}{21} - \text{&c.}$$

$$\frac{\varpi}{8(\sqrt{2}+1)} = \frac{1}{3} - \frac{1}{5} + \frac{1}{11} - \frac{1}{13} + \frac{1}{19} - \frac{1}{21} + \text{&c.}$$

180. Ex his Seriebus per combinationem nascuntur:

$$\frac{\pi\sqrt{(2+\sqrt{2})}}{4} = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} - \frac{1}{11} -$$

$$\frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \text{&c.}$$

$$\frac{\pi\sqrt{(2-\sqrt{2})}}{4} = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} +$$

$$\frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \text{&c.}$$

$$\frac{\pi(\sqrt{(4+2\sqrt{2})}+\sqrt{2}-1)}{8} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} +$$

$$\frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \text{&c.}$$

$\pi($

$$\frac{\pi\left(\sqrt{(4+2\sqrt{2})}-\sqrt{2}+1\right)}{8} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{9} -$$

$$\frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} + \&c.$$

$$\frac{\pi\left(\sqrt{2}+1+\sqrt{(4-2\sqrt{2})}\right)}{8} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} -$$

$$\frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \&c.$$

$$\frac{\pi\left(\sqrt{2}+1-\sqrt{(4-2\sqrt{2})}\right)}{8} = 1 - \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} +$$

$$\frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19} - \&c.$$

Simili modo, ponendo $n = 16$ & m vel 1 vel 3 vel 5 vel 7, ulterius progredi licet, hocque modo summæ reperientur Serierum 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{1}{9}$, &c., in quibus fignorum $+$ & $-$ viciffitudines alias leges fequantur.

181. Si in Seriebus §. 178. inventis bini termini in unam fummam colligantur, erit

$$\frac{\pi}{n\,\mathit{fin}.\,\frac{m\pi}{n}} = \frac{1}{m} + \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} + \frac{2m}{9nn-mm} -$$

$$\frac{2m}{16nn-mm} + \&c.$$

ideoque

$$\frac{1}{nn-mm} - \frac{1}{4nn-mm} + \frac{1}{9nn-mm} - \&c. =$$

$$\frac{\pi}{2mn.\,\mathit{fin}.\,\frac{m\pi}{n}} - \frac{1}{2mm}.$$

Altera vero Series dabit

$$\frac{\pi}{n\,tang.\,\frac{m}{n}\,\pi} = \frac{1}{m} - \frac{2m}{nn-mm} - \frac{2m}{4nn-mm} - \frac{2m}{9nn-mm} -$$

$$\&c.$$

<div align="center">hincque</div>

$$\frac{1}{nn-mm} + \frac{1}{4nn-mm} + \frac{1}{9nn-mm} + \&c. = \frac{1}{2\,m\,m} -$$

$$\frac{\pi}{2\,m\,n\,tang.\frac{m}{n}\pi}.$$

Ex his autem conjunctis nafcitur hæc

$$\frac{1}{nn-mm} + \frac{1}{9nn-mm} + \frac{1}{25nn-mm} + \&c. = \frac{\pi\,tang.\frac{m}{2\,n}\pi}{4\,m\,n}.$$

Si in hac Serie fit $n = 1$ & m numerus par quicunque $= 2\,k$, ob *tang.* $k\pi = 0$, erit femper, nifi fit $k = 0$,

$$\frac{1}{1-4kk} + \frac{1}{9-4kk} + \frac{1}{25-4kk} + \frac{1}{49-4kk} + \&c. = 0,$$

fin autem in illa Serie fiat $n = 2$ & m fuerit numerus quicunque impar $= 2k+1$, ob $\frac{1}{tang.\frac{m\pi}{n}} = 0$, erit $\frac{1}{4-(2k+1)^2} +$

$$\frac{1}{16-(2k+1)^2} + \frac{1}{36-(2k+1)^2} + \&c. = \frac{1}{2(2k+1)^2}.$$

182. Multiplicentur Series inventæ per nn fitque $\frac{m}{n} = p$, habebuntur iftæ formæ

$$\frac{1}{1-pp} - \frac{1}{4-pp} + \frac{1}{9-pp} - \frac{1}{16-pp} + \&c. =$$

$$\frac{\pi}{2p\,fin.\,p\pi} - \frac{1}{2pp}.$$

$$\frac{1}{1-pp} + \frac{1}{4-pp} + \frac{1}{9-pp} + \frac{1}{16-pp} + \&c. =$$

$$\frac{1}{2pp} - \frac{\pi}{2p\,fin.\,p\pi}. \text{ Sit } pp = a, \text{ atque nafcentur hæ Series}$$

$$\frac{1}{1-a} - \frac{1}{4-a} + \frac{1}{9-a} - \frac{1}{16-a} + \&c. = \frac{\pi\sqrt{a}}{2a\,fin.\,\pi\sqrt{a}} - \frac{1}{2a}$$

$$\frac{1}{1-a} + \frac{1}{4-a} + \frac{1}{9-a} + \frac{1}{16-a} + \&c. = \frac{1}{2a} - \frac{\pi\sqrt{a}}{2a\,tang.\pi\sqrt{a}}$$

Dummodo ergo a non fuerit numerus negativus nec quadratus integer, fumma harum Serierum per Circulum exhiberi poterit.

<div align="center">183</div>

183. Per reductionem autem exponentialium imaginariorum ad Sinus & Cosinus Arcuum circularium supra traditam poterimus quoque summas harum Serierum assignare si a sit numerus negativus. Cum enim sit $e^{x\sqrt{-1}} = cos. x + \sqrt{-1} \times sin. x$ & $e^{-x\sqrt{-1}} = cos. x - \sqrt{-1}. sin. x$, erit viciffim, posito $y\sqrt{-1}$ loco x; $cos. y \sqrt{-1} = \dfrac{e^{-y} + e^{y}}{2}$ & $sin.$ $y \sqrt{-1} = \dfrac{e^{-y} - e^{y}}{2\sqrt{-1}}$. Quod si ergo $a = -b$ & $y = \varpi \sqrt{b}$, erit $cos. \varpi \sqrt{-b} = \dfrac{e^{-\varpi\sqrt{b}} + e^{\varpi\sqrt{b}}}{2}$ & $sin. \varpi \sqrt{-b} = \dfrac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{2\sqrt{-1}}$; ideoque $tang. \pi \sqrt{-b} = \dfrac{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}{(e^{-\pi\sqrt{b}} + e^{\pi\sqrt{b}})\sqrt{-1}}$. Hinc erit $\dfrac{\pi \sqrt{-b}}{sin. \varpi \sqrt{-b}} = \dfrac{-2\varpi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$; & $\dfrac{\varpi \sqrt{-b}}{tang. \pi \sqrt{-b}} = \dfrac{(e^{-\varpi\sqrt{b}} + e^{\varpi\sqrt{b}})\pi\sqrt{b}}{e^{-\pi\sqrt{b}} - e^{\pi\sqrt{b}}}$. His ergo notatis, erit

$$\frac{1}{1+b} - \frac{1}{4+b} + \frac{1}{9+b} - \frac{1}{16+b} + \&c. = \frac{1}{2b} - \frac{\pi\sqrt{b}}{(e^{\pi\sqrt{b}} - e^{-\pi\sqrt{b}})b}$$

$$\frac{1}{1+b} + \frac{1}{4+b} + \frac{1}{9+b} + \frac{1}{16+b} + \&c. = \frac{(e^{\pi\sqrt{b}} + e^{-\pi\sqrt{b}})\varpi\sqrt{b}}{2b(e^{\pi\sqrt{b}} - e^{-\varpi\sqrt{b}})} - \frac{1}{2b}.$$

Eædem hæ Series deduci possunt ex §. 162. adhibendo eandem methodum, quæ in

in hoc capite fum ufus. Quoniam vero hoc pacto reductio Cap. X.
Sinuum & Cofinuum Arcuum imaginariorum ad quantitates
exponentiales reales, non mediocriter illuftratur, hanc expli-
cationem alteri præferendam duxi.

C A P U T   XI.

De aliis Arcuum atque Sinuum expreffionibus infinitis.

184. QUoniam fupra (158.), denotante z Arcum Cir-
culi quemcunque, vidimus effe *fin.* $z = z\,(1 -$
$\frac{zz}{\pi\pi})(1 - \frac{zz}{4\pi\pi})(1 - \frac{zz}{9\pi\pi})(1 - \frac{zz}{16\pi\pi})$ &c., & *cof.* $z =$
$(1 - \frac{4zz}{\pi\pi})(1 - \frac{4zz}{9\pi\pi})(1 - \frac{4zz}{25\pi\pi})(1 - \frac{4zz}{49\pi\pi})$ &c., po-
namus effe Arcum $z = \frac{m\pi}{n}$, erit *fin.* $\frac{m\pi}{n} = \frac{m\pi}{n}(1 - \frac{mm}{nn})$
$(1 - \frac{mm}{4nn})(1 - \frac{mm}{9nn})(1 - \frac{mm}{16nn})$ &c., & *cof.* $\frac{m}{n}\pi =$
$(1 - \frac{4mm}{nn})(1 - \frac{4mm}{9nn})(1 - \frac{4mm}{25nn})(1 - \frac{4mm}{49nn})$ &c..
Vel ponatur $2n$ loco n, ut prodeant hæ expreffiones

fin. $\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot(\frac{4nn-mm}{4nn}\cdot)(\frac{16nn-mm}{16nn}\cdot)(\frac{36nn-mm}{36nn})$ &c.

cof. $\frac{m\pi}{2n} = (\frac{nn-mm}{nn}\cdot)(\frac{9nn-mm}{9nn}\cdot)(\frac{25nn-mm}{25nn}\cdot)(\frac{49nn-mm}{49nn})$&c.,

quæ, in Factores fimplices refolutæ, dant

fin. $\frac{m\pi}{2n} = \frac{m\pi}{2n}\cdot(\frac{2n-m}{2n}\cdot)(\frac{2n+m}{2n}\cdot)(\frac{4n-m}{4n}\cdot)(\frac{4n+m}{4n}\cdot)$
$(\frac{6n-m}{6n})$ &c.

$$\cos \frac{m\pi}{2n} = \left(\frac{n-m}{n}\cdot\right)\left(\frac{n+m}{n}\right)\left(\frac{3n-m}{3n}\right)\left(\frac{3n+m}{3n}\right)\left(\frac{5n-m}{5n}\right)$$

$$\left(\frac{5n+m}{5n}\right) \&c.$$

Ponatur $n-m$ loco m, quia est $\sin \frac{(n-m)\pi}{2n} = \cos \frac{m\pi}{2n}$ & $\cos \frac{(n-m)\pi}{2n} = \sin \frac{m\pi}{2n}$, provenient hæ expreßiones.

$$\cos \frac{m\pi}{2n} = \left(\frac{(n-m)\pi}{2n}\right)\left(\frac{n+m}{2n}\right)\left(\frac{3n-m}{2n}\right)\left(\frac{3n+m}{4n}\right)$$

$$\left(\frac{5n-m}{4n}\right)\left(\frac{5n+m}{6n}\right) \&c.$$

$$\sin \frac{m\pi}{2n} = \frac{m}{n}\left(\frac{2n-m}{n}\right)\left(\frac{2n+m}{3n}\right)\left(\frac{4n-m}{3n}\right)\left(\frac{4n+m}{5n}\right)$$

$$\left(\frac{6n-m}{5n}\right) \&c.$$

185. Cum igitur pro Sinu & Cosinu Anguli $\frac{m\pi}{2n}$ binæ habeantur expreßiones, si eæ inter se comparentur dividendo, erit $1 = \frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot\frac{7}{8}\cdot\frac{9}{8}$. &c., ideoque $\frac{\pi}{2} = \frac{2.2.4.4.6.6.8.8.10.10.12.12.}{1.3.3.5.5.7.7.9.9.11.11.13.}$ &c., quæ est expreßio pro Peripheria Circuli, quam WALLISIUS invenit in *Arithmetica infinitorum*. Similes autem huic innumeras expreßiones exhibere licet ope primæ expreßionis pro Sinu; ex ea enim deducitur fore:

$$\frac{\pi}{2} = \frac{n}{m}\cdot\sin \frac{m\pi}{2n}\left(\frac{2n}{2n-m}\right)\left(\frac{2n}{2n+m}\right)\left(\frac{4n}{4n-m}\right)\left(\frac{4n}{4n+m}\right)\left(\frac{6n}{6n-m}\right) \&c.,$$

quæ, posito $\frac{m}{n} = 1$, præbet illam ipsam WALLISII formulam.

Sit

Sit ergo $\frac{m}{n} = \frac{1}{2}$, ob $\sin.\frac{1}{4}\pi = \frac{1}{\sqrt{2}}$, erit

$$\frac{\pi}{2} = \frac{\sqrt{2}}{1}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{16}{15}\cdot\frac{16}{17}. \text{ \&c.}$$

Sit $\frac{m}{n} = \frac{1}{3}$, ob $\sin.\frac{1}{6}\pi = \frac{1}{2}$, erit

$$\frac{\pi}{2} = \frac{3}{2}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{12}{11}\cdot\frac{12}{13}\cdot\frac{18}{17}\cdot\frac{18}{19}\cdot\frac{24}{23} \text{ \&c.}$$

Quod si Expreffio *Wallifiana* dividatur per illam ubi $\frac{m}{n} = \frac{1}{2}$,

erit $\sqrt{2} = \dfrac{2.2.6.6.10.10.14.14.18.18.}{1.3.5.7. \ 9.11.13.15.17.19.} \text{ \&c.}$

186. Quoniam Tangens cujufque Anguli æquatur Sinui per Cofinum divifo, Tangens quoque per hujufmodi Factores infinitos exprimi poterit. Quod fi autem prima Sinus expreffio dividatur per alteram Cofinus expreffionem, erit

$$tang. \frac{m\pi}{2n} = \frac{m}{n-m}\left(\frac{2n-m}{n+m}\right)\left(\frac{2n+m}{3n-m}\right)\left(\frac{4n-m}{3n+m}\right)\left(\frac{4n+m}{5n-m}\right)$$
\&c., \qquad \&

$$cot. \frac{m\pi}{2n} = \frac{n-m}{m}\left(\frac{n+m}{2n-m}\right)\left(\frac{3n-m}{2n+m}\right)\left(\frac{3n+m}{4n-m}\right)\left(\frac{5n-m}{4n+m}\right)$$
\&c..

Simili modo autem Secantes \& Cofecantes exprimentur

$$fec. \frac{m\pi}{2n} = \left(\frac{n}{n-m}\right)\left(\frac{n}{n+m}\right)\left(\frac{3n}{3n-m}\right)\left(\frac{3n}{3n+m}\right)\left(\frac{5n}{5n-m}\right)$$
$$\left(\frac{5n}{5n+m}\right) \text{ \&c.}$$

$$cofec. \frac{m\pi}{2n} = \frac{n}{m}\left(\frac{n}{2n-m}\right)\left(\frac{3n}{2n+m}\right)\left(\frac{3n}{4n-m}\right)\left(\frac{5n}{4n+m}\right)$$
$$\left(\frac{5n}{6n-m}\right) \text{ \&c.}$$

Sin autem alteræ Sinuum \& Cofinuum formulæ combinentur, erit

<center>T 2</center>

<div align="right">*tang.*</div>

$$\tan. \frac{m\pi}{2n} = \frac{\pi}{2} \cdot \frac{m}{n-m} \cdot \frac{1(2n-m)}{2(n+m)} \cdot \frac{3(2n+m)}{2(3n-m)} \cdot \frac{3(4n-m)}{4(3n+m)} \cdot \&c.$$

$$\cot. \frac{m\pi}{2n} = \frac{\pi}{2} \cdot \frac{n-m}{m} \cdot \frac{1(n+m)}{2(2n-m)} \cdot \frac{3(3n-m)}{2(2n+m)} \cdot \frac{3(3n+m)}{4(4n-m)} \cdot \&c.$$

$$\sec. \frac{m\pi}{2n} = \frac{2}{\pi} \cdot \frac{n}{n-m} \cdot \frac{2n}{n+m} \cdot \frac{2n}{3n-m} \cdot \frac{4n}{3n+m} \cdot \frac{4n}{5n-m} \cdot \&c.$$

$$\operatorname{cosec.} \frac{m\pi}{2n} = \frac{2}{\pi} \cdot \frac{n}{m} \cdot \frac{2n}{2n-m} \cdot \frac{2n}{2n+m} \cdot \frac{4n}{4n-m} \cdot \frac{4n}{4n+m} \cdot \&c.$$

187. Si loco *m* fcribatur *k*, fimilique modo Anguli $\frac{k\pi}{2n}$ Sinus & Cofinus definiantur, ac per has expreffiones illæ priores dividantur, prodibunt iftæ formulæ

$$\frac{\sin. \frac{m\pi}{2n}}{\sin. \frac{k\pi}{2n}} = \frac{m}{k} \cdot \frac{2n-m}{2n-k} \cdot \frac{2n+m}{2n+k} \cdot \frac{4n-m}{4n-k} \cdot \frac{4n+m}{4n+k} \cdot \&c.$$

$$\frac{\sin. \frac{m\pi}{2n}}{\cos. \frac{k\pi}{2n}} = \frac{m}{n-k} \cdot \frac{2n-m}{n+k} \cdot \frac{2n+m}{3n-k} \cdot \frac{4n-m}{3n+k} \cdot \frac{4n+m}{5n-k} \cdot \&c.$$

$$\frac{\cos. \frac{m\pi}{2n}}{\cos. \frac{k\pi}{2n}} = \left(\frac{n-m}{n-k}\right)\left(\frac{n+m}{n+k}\right)\left(\frac{3n-m}{3n-k}\right)\left(\frac{3n+m}{3n+k}\right)\left(\frac{5n-m}{5n-k}\right). \&c.$$

$$\frac{\cos. \frac{m\pi}{2n}}{\sin. \frac{k\pi}{2n}} = \left(\frac{n-m}{k}\right)\left(\frac{n+m}{2n-k}\right)\left(\frac{3n-m}{2n+k}\right)\left(\frac{3n+m}{4n-k}\right)\left(\frac{5n-m}{4n+k}\right). \&c.$$

Sumto ergo pro $\frac{k\pi}{2n}$ ejufmodi Angulo cujus Sinus & Cofinus dentur, per hos licebit alius cujufcunque Anguli $\frac{m\pi}{2n}$ Sinum & Cofinum determinare.

188. Viciffim igitur hujufmodi expreffionum, quæ ex Fa-
ctoribus

ctoribus infinitis conftant , valores veri vel per Circuli Peri-
pheriam , vel per Sinus & Cofinus Angulorum datorum affig-
nari poffunt , quod ipfum non parvi eft momenti , cum etiam-
nunc aliæ methodi non conftent , quarum ope hujufmodi pro-
ductiorum infinitorum valores exhiberi queant. Ceterum vero
hujufmodi expreffiones parum utilitatis afferunt , ad valores cum
ipfius π tum Sinuum Cofinuumve Angulorum $\frac{m\pi}{2n}$ per approxi-
mationem eruendos. Quanquam enim ifti Factores $\frac{\pi}{2} =$
$2(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., in fractionibus deci-
malibus non difficulter in fe multiplicantur , tamen nimis multi
termini in computum duci deberent , fi valorem ipfius π ad
decem tantum figuras juftum invenire vellemus.

189. Præcipuus autem ufus hujufmodi expreffionum , etfi in-
finitarum , in inventione Logarithmorum verfatur , in quo ne-
gotio Factorum utilitas tanta eft , ut fine illis Logarithmorum
fupputatio effet difficillima. Ac primo quidem , cum fit $\pi =$
$4(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})$ &c., erit , fumendis Loga-
rithmis , $l\pi = l4 + l(1 - \frac{1}{9}) + l(1 - \frac{1}{25}) + l(1 - \frac{1}{49}) +$
&c., vel $l\pi = l2 - l(1 - \frac{1}{4}) - l(1 - \frac{1}{16}) - l(1 - \frac{1}{36})$
— &c. , five Logarithmi communes five hyperbolici fu-
mantur. Quoniam vero ex Logarithmis hyperbolicis vulga-
res facile reperiuntur , infigne compendium adhiberi poterit
ad Logarithmum hyperbolicum ipfius π inveniendum.

190. Cum igitur , Logarithmis hyperbolicis fumendis , fit
$l(1 - x) = - x - \frac{xx}{2} - \frac{x^3}{3} - \frac{x^4}{4}$ — &c. , fi hoc modo
finguli termini evolvantur , erit

T 3

$$l\pi = l4 \begin{cases} - \dfrac{1}{9} - \dfrac{1}{2.\ 9^2} - \dfrac{1}{3.\ 9^3} - \dfrac{1}{4.\ 9^4} - \&c. \\[2mm] - \dfrac{1}{25} - \dfrac{1}{2.\ 25^2} - \dfrac{1}{3.\ 25^3} - \dfrac{1}{4.\ 25^4} - \&c. \\[2mm] - \dfrac{1}{49} - \dfrac{1}{2.\ 49^2} - \dfrac{1}{3.\ 49^3} - \dfrac{1}{4.\ 49^4} - \&c. \end{cases}$$

&c.

In his Seriebus numero infinitis verticaliter defcendendo ejus-
modi prodeunt Series, quarum fummas fupra jam invenimus,
quare fi brevitatis gratia ponamus

$$A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c.$$

$$B = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c.$$

$$C = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c.$$

$$D = 1 + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c.$$

&c.

erit $l\pi = l4 - (A - 1) - \dfrac{1}{2}(B - 1) - \dfrac{1}{3}(C - 1) - \dfrac{1}{4}(D - 1) - \&c.$

Eft vero, fummis fupra inventis proxime exprimendis,

$$A = 1, 23370055013616982735431$$
$$B = 1, 01467803160419205454625$$
$$C = 1, 00144707664094212190647$$
$$D = 1, 00015517902529611930298$$
$$E = 1, 00001704136304482550816$$
$$F = 1, 00000188584858311957590$$
$$G = 1, 00000020924051921150010$$
$$H = 1, 00000002323715737915670$$
$$I =$$

CAP. XI.

$$I = 1, 00000000258143755665977$$
$$K = 1, 0000000002868076974555 8$$
$$L = 1, 0000000000318667751 4044$$
$$M = 1, 00000000000354072294392$$
$$N = 1, 0000000000003934124669\text{г}$$
$$O = 1, 00000000000004371244859$$
$$P = 1, 000000000000000485693682$$
$$Q = 1, 000000000000000053965957$$
$$R = 1, 00000000000000005996217$$
$$S = 1, 0000000000000000666246$$
$$T = 1, 000000000000000000074027$$
$$V = 1, 0000000000000000008225$$
$$W = 1, 000000000000000000000913$$
$$X = 1, 0000000000000000000000101$$

Hinc fine tædiofo calculo reperitur Logarithmus hyperbolicus ipfius $\pi = 1$, 144729885849400174143412, qui fi multiplicetur per 0, 43429 &c., prodit Logarithmus vulgaris ipfius $\pi = 0$, 49714987269413385435126.

191. Quia porro tam Sinum quam Cofinum Anguli $\frac{m\varpi}{2n}$ expreffum habemus per Factores numero infinitos, utriufque Logarithmum commode exprimere poterimus. Erit autem ex formulis primo inventis

$$l\, fin.\, \frac{m\pi}{2n} = l\,\pi + l\,\frac{m}{2n} + l\left(1 - \frac{mm}{4nn}\right) + l\left(1 - \frac{mm}{16nn}\right) +$$
$$l\left(1 - \frac{mm}{36nn}\right) \,\&c.$$

$$l\, cof.\, \frac{m\pi}{2n} = l\left(1 - \frac{mm}{nn}\right) + l\left(1 - \frac{vm}{9nn}\right) + l\left(1 - \frac{mm}{25nn}\right) +$$
$$l\left(1 - \frac{mm}{49nn}\right) + \&c.$$

Hinc primum Logarithmi hyperbolici, ut ante, per Series maxime convergentes facile exprimuntur. Ne autem præter

necessi-

LIB. I. neceſſitatem Series infinitas multiplicemus, rerminos priores
——— actu in Logarithmis involutos relinquamus, eritque

$$l\,\textit{fin.}\,\frac{m\pi}{2n} = l\pi + lm + l(2n - m) + l(2n + m) - l8 - 3\,ln$$

$$- \frac{mm}{16mm} - \frac{m^4}{2.16^2n^4} - \frac{m^6}{3.16^3n^6} - \frac{m^8}{4.16^4n^8} - \&c.$$

$$- \frac{mm}{36mm} - \frac{m^4}{2.36^2n^4} - \frac{m^6}{3.36^3n^6} - \frac{m^8}{4.36^4n^8} - \&c.$$

$$- \frac{mm}{64mm} - \frac{m^4}{2.64^2n^4} - \frac{m^6}{3.64^3n^6} - \frac{m^8}{4.64^4n^8} - \&c.$$

$$\&c.$$

$$l\,\textit{cof.}\,\frac{m\pi}{2n} = l(n - m) + l(n + m) - 2\,ln$$

$$- \frac{mm}{9nn} - \frac{m^4}{2.\,9^2n^4} - \frac{m^6}{3.\,9^3n^6} - \frac{m^8}{4.\,9^4n^8} - \&c.$$

$$- \frac{mm}{25nn} - \frac{m^4}{2.25^2n^4} - \frac{m^6}{3.25^3n^6} - \frac{m^8}{4.25^4n^8} - \&c.$$

$$- \frac{mm}{49mm} - \frac{m^4}{2.49^3n^4} - \frac{m^6}{3.49^3n^6} - \frac{m^8}{4.49^4n^8} - \&c.$$

$$\&c.$$

192. Occurrunt ergo in his Seriebus ſingulæ Poteſtates pa-
res ipſius $\frac{m}{n}$, quæ ſunt multiplicatæ per Series, quarum ſum-
mas jam ſupra aſſignavimus.   Erit nempe

$$l\,\textit{fin.}\,\frac{m\varpi}{2n} = lm + l(2n - m) + l(2n + m) - 3\,ln + l\pi - l8$$

$$- \frac{mm}{nn}\left(\frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \frac{1}{10^2} + \frac{1}{12^2} + \&c.\right)$$

$$- \frac{m^4}{2n^4}\left(\frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \frac{1}{10^4} + \frac{1}{12^4} + \&c.\right)$$

$$- \frac{m^6}{3n^6}\left(\frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \frac{1}{10^6} + \frac{1}{12^6} + \&c.\right)$$

$$- \frac{m^8}{4n^8}\left(\frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \frac{1}{10^8} + \frac{1}{12^8} + \&c.\right)$$

$$\&c.$$

l cof.

$$l \cos \frac{m\pi}{2n} = l(n-m) + l(n+m) - 2ln$$

$$- \frac{mm}{nn} \left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \&c. \right)$$

$$- \frac{m^4}{2n^4} \left(\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{9^4} + \&c. \right)$$

$$- \frac{m^6}{3n^6} \left(\frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{9^6} + \&c. \right)$$

$$- \frac{m^8}{4n^8} \left(\frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{9^8} + \&c. \right)$$

$$\&c.$$

Serierum posteriorum modo ante (§. 190) summæ sunt exhibi-
tæ; priores Series quidem ex his derivari possent, at, quo fa-
cilius ad usum transferri queant, earum summas pariter hic
adjiciam.

193.   Quod si ergo, brevitatis gratia, ponamus

$$\alpha = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \&c.$$

$$\varsigma = \frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \frac{1}{8^4} + \&c.$$

$$\gamma = \frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \frac{1}{8^6} + \&c.$$

$$\delta = \frac{1}{2^8} + \frac{1}{4^8} + \frac{1}{6^8} + \frac{1}{8^8} + \&c.$$

$$\&c.$$

erunt summæ in numeris proxime expressæ hæ :

$$\alpha = 0,41123351671205660911810$$
$$\varsigma = 0,06764520210694613696975$$
$$\gamma = 0,01589598534350701780804$$
$$\delta = 0,00392217717264822007570$$
$$\varepsilon = 0,00097753376477325984898$$
$$\xi = 0,00024420070472492872274$$

$\eta = 0, 00006103889453949332915$

$\theta = 0, 00001525902225127269977$

$\iota = 0, 0000038147118274431318008$

$\varkappa = 0, 00000095367522617534053$

$\lambda = 0, 00000023841863595259154$

$\mu = 0, 00000005960464832831555$

$\nu = 0, 00000001490116141589813$

$\zeta = 0, 00000000372529031233986$

$\odot = 0, 00000000093132257548284$

$\pi = 0, 00000000023283064370807$

$\rho = 0, 00000000005820766091685$

$\sigma = 0, 00000000001455191522858$

$\tau = 0, 0000000000363797880710$

$\upsilon = 0, 0000000000090949470177$

$\varphi = 0, 0000000000022737367544$

$\chi = 0, 0000000000005684341886$

$\psi = 0, 0000000000001421085471$

$\omega = 0, 0000000000000355271367$

reiiquæ summæ in ratione quadrupla defcrefcunt.

194. His ergo in fubfidium vocatis, erit

$$l \sin. \frac{m\pi}{2n} = lm + l(2n-m) + l(2n+m) - 3ln + l\varpi - l8$$

$$- \frac{mm}{nn}\left(\alpha - \frac{1}{2^2}\right) - \frac{m^4}{2n^4}\left(6 - \frac{1}{2^4}\right) - \frac{m^6}{3n^6}\left(\gamma - \frac{1}{2^6}\right)$$

$$- \&c.$$

$$l \cos. \frac{m\varpi}{2n} = l(n-m) + l(n+m) - 2ln$$

$$- \frac{mm}{nn}(A-1) - \frac{m^4}{2n^4}(B-1) - \frac{m^6}{3n^6}(C-1) - \&c.$$

quoniam igitur Logarithmi $l\varpi$ & $l8$ dantur, erit

Logarith-

Logarithmus hyperbolicus Sinus Anguli $\frac{m}{n}$ 90° $=$
$lm + l(2n - m) + l(2n + m) - 3\,ln$

—		0, 93471165583043575410
—	$\frac{m^2}{n^2}.$	0, 16123351671205660911
—	$\frac{m^4}{n^4}.$	0, 00257260105347306848
—	$\frac{m^6}{n^6}.$	0, 00009032844783567260
—	$\frac{m^8}{n^8}.$	0, 00000398179316205501
—	$\frac{m^{10}}{n^{10}}.$	0, 00000019425295465196
—	$\frac{m^{12}}{n^{12}}.$	0, 00000001001328748812
—	$\frac{m^{14}}{n^{14}}.$	0, 00000000053404135618
—	$\frac{m^{16}}{n^{16}}.$	0, 00000000002914859658
—	$\frac{m^{18}}{n^{18}}.$	0, 00000000000161797979
—	$\frac{m^{20}}{n^{20}}.$	0, 00000000000009097690
—	$\frac{m^{22}}{n^{22}}.$	0, 00000000000000516827
—	$\frac{m^{24}}{n^{24}}.$	0, 00000000000000029607
—	$\frac{m^{26}}{n^{26}}.$	0, 00000000000000001708
—	$\frac{m^{28}}{n^{28}}.$	0, 00000000000000000099
—	$\frac{m^{30}}{n^{30}}.$	0, 00000000000000000005

V 2

At

At *Logarithmus hyperbolicus Cosinus Ang.* $\frac{m}{n}$ $90°$ $=$

$$l(n-m) + l(n+m) - 2\,ln$$

$$-\quad \frac{m^2}{n^2}.\quad 0,\ 2337005501361698 2735$$

$$-\quad \frac{m^4}{n^4}.\quad 0,\ 0073390158020960 2727$$

$$-\quad \frac{m^6}{n^6}.\quad 0,\ 0004823588031404063$$

$$-\quad \frac{m^8}{n^8}.\quad 0,\ 0000387947563240 2982$$

$$-\quad \frac{m^{10}}{n^{10}}.\quad 0,\ 0000034082726089 6510$$

$$-\quad \frac{m^{12}}{n^{12}}.\quad 0,\ 0000003143080971 8659$$

$$-\quad \frac{m^{14}}{n^{14}}.\quad 0,\ 0000000298915027 4450$$

$$-\quad \frac{m^{16}}{n^{16}}.\quad 0,\ 0000000029046446 7239$$

$$-\quad \frac{m^{18}}{n^{18}}.\quad 0,\ 0000000002868263 9518$$

$$-\quad \frac{m^{20}}{n^{20}}.\quad 0,\ 0000000000286807 6974$$

$$-\quad \frac{m^{22}}{n^{22}}.\quad 0,\ 0000000000028969 7956$$

$$-\quad \frac{m^{24}}{n^{24}}.\quad 0,\ 0000000000002950 6024$$

$$-\quad \frac{m^{26}}{n^{26}}.\quad 0,\ 0000000000000302 6249$$

$$-\quad \frac{m^{28}}{n^{28}}.\quad 0;\ 0000000000000031 2232$$

$$-\quad \frac{m^{30}}{n^{30}}.\quad 0,\ 0000000000000003 2379$$

$$-\quad \frac{m^{32}}{n^{32}}.\quad 0,\ 0000000000000000 3373$$

$$-\quad \frac{m^{34}}{n^{34}}.\quad 0,\ 0000000000000000 0352$$

$$-\;\frac{m^{36}}{n^{36}}\cdot\;0\;,\;0000000000000000000037$$

$$-\;\frac{m^{38}}{n^{38}}\cdot\;0\;,\;00000000000000000000004$$

195. Si isti Sinuum & Cosinuum Logarithmi hyperbolici multiplicentur per 0, 4342944819 &c., prodibunt eorundem Logarithmi vulgares ad Radium $=$ 1 relati. Quoniam vero in Tabulis Logarithmus Sinus totius statui solet $=$ 10, quo Logarithmi tabulares Sinuum & Cosinuum obtineantur, post multiplicationem addi debet 10. Hinc erit

Logarithmus tabularis Sinus Anguli $\frac{m}{n}$ $90° =$

$$l\,m + l\,(2\,n - m) + l\,(2\,n + m) - 3\,l\,n$$

$$+\qquad\quad 9\,,\;594059885702190$$

$$-\;\frac{m^{2}}{n^{2}}\cdot\quad 0\,,\;070022826605901$$

$$-\;\frac{m^{4}}{n^{4}}\cdot\quad 0\,,\;001117266441661$$

$$-\;\frac{m^{6}}{n^{6}}\cdot\quad 0\,,\;000039229146453$$

$$-\;\frac{m^{8}}{n^{8}}\cdot\quad 0\,,\;000001729270798$$

$$-\;\frac{m^{10}}{n^{10}}\cdot\quad 0\,,\;000000084362986$$

$$-\;\frac{m^{12}}{n^{12}}\cdot\quad 0\,,\;000000004348715$$

$$-\;\frac{m^{14}}{n^{14}}\cdot\quad 0\,,\;000000000231931$$

$$-\;\frac{m^{16}}{n^{16}}\cdot\quad 0\,,\;000000000012659$$

$$-\;\frac{m^{18}}{n^{18}}\cdot\quad 0\,,\;000000000000702$$

$$-\;\frac{m^{20}}{n^{20}}\cdot\quad 0\,,\;000000000000039$$

V 3

Loga-

Logarithmus tabularis Cosinus Anguli $\frac{m}{n}\,90° =$

$$l(n - m) + l(n + m) - 2\,l\,n$$

$$+ \qquad\qquad 10,000000000000000$$

$$- \quad \frac{m^2}{n^2}. \qquad 0,101494859341892$$

$$- \quad \frac{m^4}{n^4}. \qquad 0,003187294065451$$

$$- \quad \frac{m^6}{n^6}. \qquad 0,000209485800017$$

$$- \quad \frac{m^8}{n^8}. \qquad 0,000016848348597$$

$$- \quad \frac{m^{10}}{n^{10}}. \qquad 0,000001480193986$$

$$- \quad \frac{m^{12}}{n^{12}}. \qquad 0,000000136502272$$

$$- \quad \frac{m^{14}}{n^{14}}. \qquad 0,000000012981715$$

$$- \quad \frac{m^{16}}{n^{16}}. \qquad 0,000000001261471$$

$$- \quad \frac{m^{18}}{n^{18}}. \qquad 0,000000000124567$$

$$- \quad \frac{m^{20}}{n^{20}}. \qquad 0,000000000012456$$

$$- \quad \frac{m^{22}}{n^{22}}. \qquad 0,000000000001258$$

$$- \quad \frac{m^{24}}{n^{24}}. \qquad 0,000000000000128$$

$$- \quad \frac{m^{26}}{n^{26}}. \qquad 0,000000000000013$$

196. Harum ergo formularum ope inveniri poſſunt Logarithmi Sinuum & Coſinuum quorumvis Angulorum tam hyperbolici quam vulgares, etiam ignoratis ipſis Sinibus & Coſinibus. Ex Logarithmis autem Sinuum & Coſinuum per ſolam ſubtractionem inveniuntur Logarithmi Tangentium, Cotangen-

tangentium , & Secantium, Cosecantiumque, quamobrem pro Cap. XI.
his peculiaribus formulis non erit opus. Ceterum notandum
est numerorum *m* , *n*, *n* — *m*, *n* + *m* , &c. Logarithmos
hyperbolicos accipi oportere , cum Logarithmi hyperbolici Si-
nuum Cosinuumque quæruntur, vulgares autem, cum tales
ope posteriorum formularum sunt indagandi. Præterea *m* : *n*
denotat rationem, quam Angulus propositus habet ad Angu-
lum rectum ; sicque, cum Sinus Angulorum semirecto majorum
æquentur Cosinibus Angulorum semirecto minorum ac vi-
cissim, fractio $\frac{m}{n}$ nunquam major accipienda erit quam $\frac{1}{2}$,
hancque ob rem termini illi multo magis convergent, ut semissis
instituto sufficere possit.

197. Antequam hoc argumentum relinquamus, aptiorem
aperiamus modum Tangentes & Secantes quorumvis Angulorum
inveniendi , quam Caput præcedens suppeditat. Quanquam
enim Tangentes & Secantes per Sinus & Cosinus determi-
nantur ; tamen hoc fit per divisionem , quæ operatio in tan-
tis numeris nimis est operosa. Ac Tangentes quidem & Co-
tangentes jam supra (§. 136.) exhibuimus, verum illo loco
rationem formularum reddere non licuit, quam huic Capiti
reservavimus.

198. Ex §. 181. ergo primum expressionem pro Tangente
Anguli $\frac{m}{2n}$ π elicimus. Cum enim sit $\frac{1}{mn - mm} + \frac{1}{9nn - mm} +$

$\frac{1}{25nn - mm} + $ &c. $= \frac{\pi}{4mn} . tang. \frac{m}{2n} \varpi$ erit $tang. \frac{m}{2n} \varpi =$

$\frac{4mn}{\varpi} (\frac{1}{mn - mm} + \frac{1}{9nn - mm} + \frac{1}{25nn - mm} + $ &c.). Cum

deinde sit $\frac{1}{mn - mm} + \frac{1}{4nn - mm} + \frac{1}{9nn - mm} + $ &c. $=$

$\frac{1}{2mm} - \frac{\varpi}{2mn} . cot. \frac{m}{n} \varpi$, si pro *n* scribamus 2 *n* erit

$cot. \frac{m}{2n} \pi = \frac{2n}{m\varpi} - \frac{4nn}{\varpi} (\frac{1}{4nn - mm} + \frac{1}{16nn - mm} +$

$\frac{1}{36nn - mm}$

Lɪʙ. I. $\dfrac{1}{\underline{}\ 36nn\ -\ mm}$ + &c.). Convertantur hæ fractiones, præter primas, quippe quæ facile in computum ducuntur, in Series infinitas, erit

$$tang.\ \frac{m}{2\,n}\ \pi = \frac{m\,n}{nn\ -\ mm}\cdot\frac{4}{\pi}$$

$$+\ \frac{4}{\pi}\left(\frac{m}{3^2 n} + \frac{m^3}{3^4 n^3} + \frac{m^5}{3^6 n^5} + \&c.\right)$$

$$+\ \frac{4}{\pi}\left(\frac{m}{5^2 n} + \frac{m^3}{5^4 n^3} + \frac{m^5}{5^6 n^5} + \&c.\right)$$

$$+\ \frac{4}{\pi}\left(\frac{m}{7^2 n} + \frac{m^3}{7^4 n^3} + \frac{m^5}{7^6 n^5} + \&c.\right.$$

&c.

$$cot.\ \frac{m}{2\,n}\ \varpi = \frac{n}{m}\cdot\frac{2}{\varpi} - \frac{m\,n}{4nn\ -\ mm}\cdot\frac{4}{\varpi}$$

$$-\ \frac{4}{\pi}\left(\frac{m}{4^2 n} + \frac{m^3}{4^4 n^3} + \frac{m^5}{4^6 n^5} + \&c.\right)$$

$$-\ \frac{4}{\pi}\left(\frac{m}{6^2 n} + \frac{m^3}{6^4 n^3} + \frac{m^5}{6^6 n^5} + \&c.\right)$$

$$-\ \frac{4}{\pi}\left(\frac{m}{8^2 n} + \frac{m^3}{8^4 n^3} + \frac{m^5}{8^6 n^5} + \&c.\right)$$

&c.

198. At ex valore ipsius ϖ cognito reperitur

$\dfrac{1}{\pi}$ = 0, 3183098861837906715377679267450287 24,

deinde hic eædem Series occurrunt, quas supra litteris A, B, C, D, &c., & α, ζ, γ, δ, &c., indicavimus. His ergo notatis, erit

$$tang.\frac{m}{2\,n}\ \pi = \frac{m\,n}{mm\ -\ m\,m}\cdot\frac{4}{\pi} + \frac{m}{n}\cdot\frac{4}{\pi}\left(A - 1\right) + \frac{m^3}{n^3}\times$$

$$\frac{4}{\pi}\left(B - 1\right) + \frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(C - 1\right) + \frac{m^7}{n^7}\cdot\frac{4}{\pi}\left(D - 1\right)\ \&c.$$

Deinde erit pro Cotangente

cot.

$$\cot. \frac{m}{2n}\varpi = \frac{n}{m}\cdot\frac{2}{\varpi} - \frac{4mn}{4mm-nnn}\cdot\frac{1}{\varpi} - \frac{m}{n}\cdot\frac{4}{\pi}\left(a - \frac{1}{2^2}\right)$$

$$- \frac{m^3}{n^3}\cdot\frac{4}{\varpi}\left(6 - \frac{1}{2^4}\right) - \frac{m^5}{n^5}\cdot\frac{4}{\pi}\left(\gamma - \frac{1}{2^6}\right) - \&c.;$$

atque ex his formulis natæ funt expreſſiones, quas fupra (§. 135.) pro Tangente & Cotangente dedimus; fimul vero (§. 137.) oſtendimus, quomodo ex Tangentibus & Cotangentibus inventis per folam additionem & fubtractionem Secantes & Cofecantes reperiantur. Harum ergo regularum ope univerfus Canon Sinuum, Tangentium & Secantium, eorumque Logarithmorum multo facilius fupputari poſſet, quam quidem hoc a primis conditoribus eſt factum.

CAPUT XII.

De reali Functionum fractarum evolutione.

199. J Am fupra, in Capite fecundo, methodus eſt tradita Functionem quamcunque fractam in tot partes refolvendi quot ejus denominator habeat Factores fimplices; hi enim præbent denominatores fractionum illarum partialium. Ex quo manifeſtum eſt, fi denominator quos habeat Factores fimplices imaginarios, fractiones quoque inde ortas fore imaginarias: his ergo cafibus parum juvabit fractionem realem in imaginarias refolviſſe. Cum igitur oſtendiſſem omnem Functionem integram, qualis eſt denominator cujufvis fractionis, quantumvis Factoribus fimplicibus imaginariis fcateat, tamen in Factores duplices, feu fecundæ dimenfionis, reales femper refolvi poſſe; hoc modo in refolutione fractionum quantitates imaginariæ evitari poterunt, fi pro denominatoribus fractionum partialium non Factores denominatoris principalis fimplices, fed duplices reales aſſumamus.

200. Sit igitur propofita hæc Functio fracta $\frac{M}{N}$, ex qua tot fractiones fimplices fecundum methodum fupra expofitam eliciantur, quot denominator N habuerit Factores fimplices reales. Sit autem, loco imaginariorum, hæc expreffio pp — $2pqz\,cof.\,\varphi + qqzz$ Factor ipfius N; &, quoniam in hoc negotio numeratorem & denominatorem in forma evoluta contemplari oportet, fit hæc fractio propofita

$$\frac{A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.}{(pp - 2pqz.\,cof.\,\varphi + qqzz)(\alpha + 6z + \gamma zz + \delta z^3 + \&c.),}$$

ac ponatur fractio partialis ex denominatoris Factore pp — $2pqz.\,cof.\,\varphi + qqzz$ oriunda hæc : $\dfrac{A + Az}{pp - 2pqz.\,cof.\,\varphi + qqzz}$,

quoniam enim variabilis z in denominatore duas habet dimenfiones, in numeratore unam habere poterit, non vero plures; alias enim integra Functio contineretur, quam fcorfim elici oportet.

201. Sit, brevitatis gratia, numerator $A + Bz + Cz^2 + \&c.$ $= M$ & alter denominatoris Factor. $\alpha + 6z + \gamma z^2 + \&c.$ $= Z$; ponatur altera pars ex denominatoris Factore Z oriunda $= \dfrac{\Upsilon}{Z}$, eritque $\Upsilon = \dfrac{M - AZ - AZz}{pp - 2pqz.cof.\,\varphi + qqzz}$, quæ expreffio Functio integra ipfius z effe debet, ideoque neceffe eft ut $M - AZ - AzZ$ divifibile fit per $pp - 2pqz.cof.\,\varphi + qqzz$. Evanefcet ergo $M - AZ - AzZ$, fi ponatur $pp - 2pqz \times cof.\,\varphi + qqzz = 0$, hoc eft fi ponatur tam $z = \dfrac{p}{q}$ ($cof.\,\varphi + \sqrt{-1}.\,fin.\,\varphi$) quam $z = \dfrac{p}{q}$ ($cof.\,\varphi - \sqrt{-1}.\,fin.\,\varphi$) ; fit $\dfrac{p}{q} = f$, eritque $z^n = f^n$ ($cof.\,n\varphi + \sqrt{-1}.\,fin.n\,\varphi$). Duplex ergo hic valor pro z fubftitutus duplicem dabit æquationem, unde ambas incognitas conftantes A & A definire licet.

202. Facta ergo hac fubftitutione, æquatio $M = AZ + AZz$ evoluta hanc duplicem dabit æquationem

$$A +$$

$A + Bf.cof. \varphi + Cff.cof. 2\varphi + Df^3.cof. 3\varphi + \&c.$
$\pm (Bf.fin. \varphi + Cff.fin. 2\varphi + Df^3.fin.3\varphi + \&c.) \sqrt{-1}$ $\Big\} =$

$\begin{cases} A(\alpha + \mathcal{E}f.cof. \varphi + \gamma ff.cof. 2\varphi + \delta f^3.cof. 3\varphi + \&c.) \\ \pm A(\mathcal{E}f.fin. \varphi + \gamma ff.fin. 2\varphi + \delta f^3.fin. 3\varphi + \&c.) \sqrt{-1} \\ + A(\alpha f.cof. \varphi + \mathcal{E}ff.cof. 2\varphi + \gamma f^3.cof. 3\varphi + \&c.) \\ \pm A(\alpha f.fin. \varphi + \mathcal{E}ff.fin. 2\varphi + \gamma f^3.fin. 3\varphi + \&c.) \sqrt{-1} \end{cases}$

Sit, ad calculum abbreviandum,

$A + Bf.cof. \varphi + Cff.cof. 2\varphi + Df^3.cof. 3\varphi + \&c. = P$

$Bf.fin. \varphi + Cff.fin. 2\varphi + Df^3.fin. 3\varphi + \&c. = \text{p}$

$\alpha + \mathcal{E}f.cof. \varphi + \gamma ff.cof. 2\varphi + \delta f^3.cof. 3\varphi + \&c. = Q$

$\mathcal{E}f.fin. \varphi + \gamma ff.fin. 2\varphi + \delta f^3.fin. 3\varphi + \&c. = \text{q}$

$\alpha f.cof. \varphi + \mathcal{E}ff.cof. 2\varphi + \gamma f^3.cof. 3\varphi + \&c. = R$

$\alpha f.fin. \varphi + \mathcal{E}ff.fin. 2\varphi + \gamma f^3.fin. 3\varphi + \&c. = \text{r}$

eritque, his pofitis,

$P \pm \text{p}\sqrt{-1} = AQ \pm A\text{q} \sqrt{-1} + AR \pm A\text{r}\sqrt{-1}.$

203. Ob fignorum ambiguitatem hæ duæ oriuntur æquationes,

$$P = AQ + \text{a}R$$
$$\text{p} = A\text{q} + \text{a}\text{r}$$

ex quibus incognitæ A & a ita definiuntur, ut fit

$$A = \frac{\text{P}\text{r} - \text{p}\text{R}}{\text{Q}\text{r} - \text{q}\text{R}} \quad \& \quad \text{a} = \frac{\text{P}\text{q} - \text{p}\text{Q}}{\text{q}\text{R} - \text{Q}\text{r}}.$$

Propofita ergo fractione $\dfrac{M}{(pp - 2pqz.cof.\varphi + qqzz)Z}$

per fequentem regulam fractio partialis ex ea oriunda

$\dfrac{A + \text{a}z}{pp + 2pqz.cof.\varphi + qqzz}$ definietur. Pofito $f = \dfrac{p}{q}$, & evolutis fingulis terminis, fiat ut fequitur,

X 2

pofito

pofito $z^n = f^n . cof. n\, \phi$, fit $M = P$

..... $z^n = f^n . fin. n\, \phi$, fit $M = P$

..... $z^n = f^n . cof. n\, \phi$, fit $Z = Q$

..... $z^n = f^n . fin. n\, \phi$, fit $Z = Q$

..... $z^n = f^n . cof. n\, \phi$, fit $z\,Z = R$

..... $z^n = f^n . fin. n\, \phi$, fit $z\,Z = R$

Inventis hoc modo valoribus P, Q, R, P, Q, R erit

$$A = \frac{P\,R - P\,R}{Q\,R - Q\,R}, \quad \& \; A = \frac{P\,Q - P\,Q}{Q\,R - Q\,R}.$$

E x e m p l u m I.

Si fuerit propofita hæc Functio fracta $\dfrac{z\,z}{(1 - z + z z)(1 + z^4)}$

ex qua partem a denominatoris Factore $1 - z + z z$ oriundam definire oporteat, quæ fit $\dfrac{A + A z}{1 - z + z z}$. Ac primo quidem hic Factor, cum forma generali $p p - 2 p q z . cof. \phi + q q z z$ comparatus, dat $p = 1$, $q = 1$ & $cof. \phi = \frac{1}{2}$, unde fit

$\phi = 60° = \dfrac{\pi}{3}$. Quia itaque eft $M = z z$; $Z = 1 + z^4$ & $f = 1$ erit

$$P = \quad cof. \frac{2}{3} \pi = -\frac{1}{2}; \quad P = \quad \frac{\sqrt{3}}{2}$$

$$Q = 1 + cof. \frac{4}{3} \pi = \frac{1}{2}; \quad Q = -\frac{\sqrt{3}}{2}$$

$$R = cof. \frac{\pi}{3} + cof. \frac{5\pi}{3} = 1 \quad ; \quad R = 0.$$

Ex his invenitur $A = -1$; & $A = 0$, ideoque fractio quæfita eft $\dfrac{-1}{1 - z + z z}$, hujufque complementum erit

$$1 + z$$

$\frac{1 + z + zz}{1 + z^4}$, cujus denominator $1 + z^4$ cum habeat Factores $1 + z \sqrt{2 + zz}$ & $1 - z \sqrt{2 + zz}$, refolutio denuo fufcipi poteft; fit autem $\phi = \frac{\pi}{4}$ & priori cafu $f = - 1$ pofteriori $f = + 1$.

E X E M P L U M. II.

Sit igitur propofita hæc fractio refolvenda

$$\frac{1 + z + zz}{(1 + z \sqrt{2 + zz})(1 - z \sqrt{2 + zz})}$$

& erit $M = 1 + z + zz$; & pro priore Factore habebitur $f = - 1$; $\phi = \frac{\pi}{4}$, & $Z = 1 - z \sqrt{2 + zz}$, unde erit

$$P = 1 - \quad cof. \frac{\pi}{4} + cof. \frac{2\pi}{4} = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

$$P = \quad - \quad fin. \frac{\pi}{4} + fin. \frac{2\pi}{4} = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

$$Q = 1 + \sqrt{2}. cof. \frac{\pi}{4} + cof. \frac{2\pi}{4} = 2$$

$$Q = \quad + \sqrt{2}. fin. \frac{\pi}{4} + fin. \frac{2\pi}{4} = 2$$

$$R = - cof. \frac{\pi}{4} - \sqrt{2}. cof. \frac{2\pi}{4} - cof. \frac{3\pi}{4} = 0$$

$$R = - fin. \frac{\pi}{4} - \sqrt{2}. fin. \frac{2\pi}{4} - fin. \frac{3\pi}{4} = - 2\sqrt{2}$$

Ex his reperitur $Q_R - QR = - 4\sqrt{2} :$ &

$$A = \frac{\sqrt{2} - 1}{2\sqrt{2}}, \quad \& \quad A = 0$$ unde ex denominatoris Factore

$1 + z \sqrt{2 + zz}$ hæc orietur fractio partialis $\frac{(\sqrt{2} - 1) : 2\sqrt{2}}{1 + z \sqrt{2 + zz}}$,

alter autem Factor dabit fimili modo hanc $\frac{(\sqrt{2} + 1) : 2\sqrt{2}}{1 - z \sqrt{2 + zz}}$.

Hinc Functio primum propofita $\frac{zz}{(1 - z + zz)(1 + z^4)}$ refol-

vitur

vitur in has $\dfrac{-1}{1-z+zz} + \dfrac{(\sqrt{2}-1):2\sqrt{2}}{1+z\sqrt{2}+zz} + \dfrac{(\sqrt{2}+1):2\sqrt{2}}{1-z\sqrt{2}+zz}$

E x e m p l u m III.

Sit propofita hæc fractio refolvenda

$$\frac{1+2z+zz}{(1-\tfrac{8}{5}z+zz)(1+2z+3zz)}$$

Pro Factore denominatoris $1-\dfrac{8}{5}z+zz$ oriatur ifta fractio

$\dfrac{A+Az}{1-\tfrac{8}{5}z+zz}$; eritque $p=1$; $q=1$; *cof.* $\phi=\dfrac{4}{5}$, unde

$f=1$; $M=1+2z+zz$; $Z=1+2z+3zz$. Quia vero hic ratio Anguli ϕ ad rectum non conftat, Sinus & Cofinus ejus multiplorum feorfim debent inveftigari. Cum fit

$$cof. \ \phi = \frac{4}{5} ; \ erit \ \textit{fin.} \ \phi = \frac{3}{5}$$

$$cof. \ 2\phi = \frac{7}{25} ; \qquad \textit{fin.} \ 2\phi = \frac{24}{25}$$

$$cof. \ 3\phi = \frac{-44}{125} ; \quad \textit{fin.} \ 3\phi = \frac{117}{125} ;$$

hinc fit

$$P = 1 + 2.\frac{4}{5} + \frac{7}{25} = \frac{72}{25}$$

$$P = \qquad 2.\frac{3}{5} + \frac{24}{25} = \frac{54}{25}$$

$$Q = 1 + 2.\frac{4}{5} + 3.\frac{7}{25} = \frac{86}{25}$$

$$Q = \qquad 2.\frac{3}{5} + 3.\frac{24}{25} = \frac{102}{25}$$

$$R = \frac{4}{5} + 2.\frac{7}{25} - 3.\frac{44}{125} = \frac{38}{125}$$

$$R = \frac{3}{5} + 2.\frac{24}{25} + 3.\frac{117}{125} = \frac{666}{125}$$

ideoque $QR - QR = \dfrac{53400}{25.125} = \dfrac{2136}{125}$. Ergo

$$A =$$

$$A = \frac{1836}{2136} = \frac{153}{178}; \; A = - \frac{540}{2136} = - \frac{45}{178}.$$

Quare fractio ex Factore $1 - \frac{8}{5} z + zz$ oriunda erit

$$\frac{9(17 - 5z) : 178}{1 - \frac{8}{5} z + zz}.$$

Quæramus fimili modo fractionem alteri Factori refpondentem; erit $p = 1$, $q = - \sqrt{3}$ & $cof. \phi = \frac{1}{\sqrt{3}}$, ergo $f = - \frac{1}{\sqrt{3}}$, $M = 1 + 2z + zz$ & $Z = 1 - \frac{8}{5} z + zz$. Fiet autem, ob $cof. \phi = \frac{1}{\sqrt{3}}$, $fin. \phi = \frac{\sqrt{2}}{\sqrt{3}}$

$$cof. 2\phi = - \frac{1}{3}, \quad fin. 2\phi = \frac{2\sqrt{2}}{3}$$

$$cof. 3\phi = - \frac{5}{3\sqrt{3}}, \quad fin. 3\phi = \frac{\sqrt{3}}{3\sqrt{3}}$$

confequenter

$$P = 1 - \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot - \frac{1}{3} = \frac{2}{9}$$

$$P = - \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = - \frac{4\sqrt{2}}{9}$$

$$Q = 1 + \frac{8}{5\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{3} \cdot - \frac{1}{3} = \frac{64}{45}$$

$$Q = + \frac{8}{5\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} + \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = \frac{34\sqrt{2}}{45}$$

$$R = - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} - \frac{8}{5 \cdot 3} \cdot - \frac{1}{3} - \frac{1}{3\sqrt{3}} \cdot - \frac{5}{3\sqrt{3}} = \frac{4}{135}$$

$$R = - \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} - \frac{8}{5 \cdot 3} \cdot \frac{2\sqrt{2}}{3} - \frac{1}{3\sqrt{3}} \cdot \frac{\sqrt{2}}{3\sqrt{3}} = - \frac{98\sqrt{2}}{135}$$

ideoque $Q R - Q R = - \frac{712\sqrt{2}}{675}$; fiet ergo

$$A = \frac{100}{712} = \frac{25}{178}; \; A = \frac{540}{712} = \frac{135}{178}.$$

Fractio

Fractio ergo propofita $\dfrac{1 + 2z + zz}{\left(1 - \frac{8}{5}z + zz\right)\left(1 + 2z + 3zz\right)}$ re-

folvitur in $\dfrac{9(17 - 5z):178}{1 - \frac{8}{5}z + zz} + \dfrac{5(5 + 27z):178}{1 + 2z + 3zz}$.

204. Poffunt autem valores litterarum R & ʀ ex litteris Q & ǫ definiri, cum enim fit

$$Q = \alpha + \mathfrak{S}f.\,cof.\,\phi + \gamma f^2.\,cof.\,2\phi + \delta f^3.\,cof.\,3\phi\;\&c.$$
$$\mathfrak{Q} = \qquad \mathfrak{S}f.\,fin.\,\phi + \gamma f^2.\,fin.\,2\phi + \delta f^3.\,fin.\,3\phi\;\&c.$$

erit

$$Q.\,cof.\phi - \mathfrak{Q}.\,fin.\phi = \alpha.\,cof.\phi + \mathfrak{S}f.\,cof.2\phi + \gamma f^2.\,cof.3\phi + \&c.$$

ideoque $R = f(Q.\,cof.\phi - \mathfrak{Q}.\,fin.\,\phi)$

deinde erit

$$Q.\,fin.\phi + \mathfrak{Q}.\,cof.\phi = \alpha.\,fin.\,\phi + \mathfrak{S}f.\,fin.2\phi + \gamma ff.\,fin.3\phi + \&c.$$

ergo $\mathfrak{R} = f(Q.\,fin.\,\phi + \mathfrak{Q}.\,cof.\,\phi)$

Ex his porro fit

$$Q\mathfrak{R} - \mathfrak{Q}R = (QQ + \mathfrak{Q}\mathfrak{Q})f.\,fin.\,\phi$$
$$PR - \mathfrak{P}R = (PQ + \mathfrak{P}\mathfrak{Q})f.fin.\,\phi + (P\mathfrak{Q} - \mathfrak{P}Q)f.cof.\,\phi$$

eritque confequenter

$$A = \frac{PQ + \mathfrak{P}\mathfrak{Q}}{QQ + \mathfrak{Q}\mathfrak{Q}} + \frac{P\mathfrak{Q} - \mathfrak{P}Q}{QQ + \mathfrak{Q}\mathfrak{Q}} \cdot \frac{cof.\,\phi}{fin.\,\phi}$$
$$\mathfrak{A} = - \frac{P\mathfrak{Q} + \mathfrak{P}Q}{(QQ + \mathfrak{Q}\mathfrak{Q})f.fin.\,\phi}.$$

Quare ex denominatoris Factore $pp - 2pqz.cof.\,\phi + qqzz$ nafcitur ifta fractio partialis

$$\frac{(PQ + \mathfrak{P}\mathfrak{Q})f.\,fin.\,\phi + (P\mathfrak{Q} - \mathfrak{P}Q)(f.\,cof.\,\phi - z)}{(pp - 2pqz.cof.\,\phi + qqzz)(QQ + \mathfrak{Q}\mathfrak{Q})f.\,fin.\,\phi}$$

feu, ob $f = \dfrac{p}{q}$, hæc

$$\frac{(PQ + \mathfrak{P}\mathfrak{Q})p.\,fin.\,\phi + (P\mathfrak{Q} - \mathfrak{P}Q)(p.cof.\,\phi - qz)}{(pp - 2pqz.cof.\,\phi + qqzz)(QQ + \mathfrak{Q}\mathfrak{Q})p.fin.\,\phi}.$$

205. Oritur ergo hæc fractio partialis ex Functionis propo-
fitæ $\dfrac{M}{(pp - 2pqz.cof.\,\phi + qqzz)Z}$ Factore denominatoris
$pp - 2pqz.cof.\,\phi + qqzz$, atque litteræ P, ᴘ, Q & ǫ
fequenti modo ex Functionibus M & Z inveniuntur:

posito

posito $z^n = \dfrac{p^n}{q^n} \cdot \cos n\phi$, fit $M = \mathrm{P}$,

$$\& \quad Z = \mathrm{Q};$$

$\&$ posito $z^n = \dfrac{p^n}{q^n} \cdot \sin n\phi$, fit $M = \mathrm{p}$,

$$\& \quad Z = \mathrm{Q}:$$

ubi notandum eft Functiones M & Z, antequam hæc fubfti-
tutio fiat, omnino evolvi debere, ut hujufmodi habeant for-
mas

$$M = A + Bz + Cz^2 + Dz^3 + Ez^4 + \&c.,$$
$$\& \quad Z = \alpha + \mathfrak{6}z + \gamma z^2 + \delta z^3 + \varepsilon z^4 + \&c.;$$

eritque ideo

$$\mathrm{P} = A + B \frac{p}{q} . \cos \phi + C \frac{p^2}{q^2} . \cos 2\phi + D \frac{p^3}{q^3} . \cos 3\phi + \&c.$$

$$\mathrm{P} = \qquad B \frac{p}{q} . \sin \phi + C \frac{p^2}{q^2} . \sin 2\phi + D \frac{p^3}{q^3} . \sin 3\phi + \&c.$$

$$\mathrm{Q} = \alpha + \mathfrak{6} \frac{p}{q} . \cos \phi + \gamma \frac{p^2}{q^2} . \cos 2\phi + \delta \frac{p^3}{q^3} . \cos 3\phi + \&c.$$

$$\mathrm{Q} = \qquad \mathfrak{6} \frac{p}{q} . \sin \phi + \gamma \frac{p^2}{q^2} . \sin 2\phi + \delta \frac{p^3}{q^3} . \sin 3\phi + \&c..$$

206. Ex præcedentibus autem intelligitur hanc refolutio-
nem locum habere non poffe, fi Functio Z eundem Factorem
$pp - 2pqz . \cos \phi + qqzz$ adhuc in fe complectatur; hoc
enim cafu in æquatione $M = \mathrm{A} Z + \mathrm{a} Zz$ facta fubftitutione
$z^n = f^n (\cos n\phi \pm \sqrt{-1} . \sin \phi)$, ipfa quantitas Z eva-
nefceret, nihilque propterea colligi poffet. Quamobrem, fi
Functionis fractæ $\dfrac{M}{N}$ denominator habeat Factorem $(pp - 2pqz . \cos \phi + qqzz)^2$ vel altiorem Poteftatem, peculiari opus
erit refolutione. Sit igitur $N = (pp - 2pqz . \cos \phi + qqzz)^2 Z$;
atque ex denominatoris Factore $(pp - 2pqz . \cos \phi + qqzz)^2$
orientur hujufmodi duæ fractiones partiales

$$\frac{A + Az}{(pp - 2pqz.\cos.\varphi + qqzz)^2} + \frac{B + Bz}{pp - 2pqz.\cos.\varphi + qqzz},$$

ubi litteras conftantes A, A, B, B determinari oportet.

207. His pofitis, debebit ifta expreffio

$$\frac{M - (A + Az)Z - (B + Bz)Z(pp - 2pqz.\cos.\varphi + qqzz)}{(pp - 2pqz.\cos.\varphi + qqzz)^2}$$

effe Functio integra, & hanc ob rem numerator divifibilis erit per denominatorem. Primum ergo hæc expreffio $M -$ $Az - AzZ$ divifibilis effe debet per $pp - 2pqz.\cos.\varphi +$ $qqzz$; qui cum fit cafus præcedens, eodem quoque modo litteræ A & A determinabuntur.

Quare, pofito $z^n = \frac{p^n}{q^n}.\cos. n\varphi$, fit $M = P$,

$$\& \quad z = N:$$

&, pofito $z^n = \frac{p^n}{q^n}.\sin. n\varphi$, fit $M = P$,

$$\& \quad z = N.$$

Hifque factis fecundum regulam fupra datam, erit

$$A = \frac{PN + PN}{N^2 + N^2} + \frac{PN - PN}{N^2 + N^2} \cdot \frac{\cos.\varphi}{\sin.\varphi}$$

$$A = - \frac{PN + PN}{N^2 + N^2} \cdot \frac{q}{p\sin.\varphi}.$$

208. Inventis ergo hoc modo A & A, fiet

$$\frac{M - (A + Az)Z}{pp - 2pqz.\cos.\varphi + qqzz}$$ Functio integra, quæ fit $= P$; atque fupereft ut $P - BZ - BzZ$ divifibile evadat per $pp - 2pqz.\cos.\varphi + qqzz$, quæ expreffio cum fimilis fit præcedenti, fi

pofito $z^n = \frac{p^n}{q^n}.\cos. n\varphi$, vocetur $P = R$,

&, pofito $z^n = \frac{p^n}{q^n}.\sin. n\varphi$, vocetur $P = R$; erit

$$B =$$

$$B = \frac{RN + rN}{N^2 + n^2} + \frac{RN - rN}{N^2 + n^2} \cdot \frac{cof.\,\phi}{fin.\,\phi}$$

$$B = -\frac{RN + rN}{N^2 + n^2} \cdot \frac{q}{p\,fin.\phi}.$$

209. Hinc jam generaliter concludere licet quomodo refolutio inftitui debeat, fi denominator Functionis propofitæ $\frac{M}{N}$, Factorem habeat $(pp - 2pqz.\,cof.\,\phi + qqzz)^k$: fit enim

$$N = (pp - 2pqz.\,cof.\phi + qqzz)^k\, z \,, \text{ ita ut hæc refolvenda fit}$$

Functio fracta

$$\frac{M}{(pp - 2\,p\,q\,z.cof.\phi + q\,q\,z\,z)^k\, z}.$$

Præbeat ergo Factor denominatoris $(pp - 2pqz.\,cof.\,\phi + qqzz)^k$ has partes:

$$\frac{A + Az}{(pp - 2pqz.\,cof.\,\phi + qqzz)^k} + \frac{B + Bz}{(pp - 2pqz.\,cof.\,\phi + qqzz)^{k-1}} +$$

$$\frac{C + cz}{(pp - 2pqz.cof.\phi + qqzz)^{k-2}} + \frac{D + Dz}{(pp - 2pqz.cof.\phi + qqzz)^{k-3}} + \&c.$$

Jam, pofito $z^n = \frac{p^n}{q^n} \cdot cof.\,n\,\phi$, fit $M = M$,

$$\& \quad z = N \,;$$

$\&$, pofito $z^n = \frac{p^n}{q^n} \cdot fin.\,n\,\phi$, fit $M = M \,;$

$$\& \quad z = N \,;$$

erit

$$A = \frac{MN + MN}{N^2 + n^2} + \frac{MN - MN}{N^2 + n^2} \cdot \frac{cof.\,\phi}{fin.\,\phi}$$

$$A = -\frac{MN + MN}{N^2 + n^2} \cdot \frac{q}{p\,fin.\phi}.$$

Deinde vocetur $\dfrac{M - (A + Az)\,Z}{pp - 2ppz.\,cof.\phi + qqzz} = P$; atque,

Y 2   pofito

posito $z^n = \dfrac{p^n}{q^n} \cdot cof. \, n\,\Phi$, sit $P = P$,

&, posito $z^n = \dfrac{p^n}{q^n} \cdot fin. \, n\,\Phi$, sit $P = P$;

erit

$$B = \frac{PN + PN}{N^2 + N^2} + \frac{PN - PN}{N^2 + N^2} \cdot \frac{cof. \, \Phi}{fin. \, \Phi}$$

$$B = - \frac{PN + PN}{N^2 + N^2} \cdot \frac{q}{p \, fin. \, \Phi}.$$

Tum vocetur $\dfrac{P - (B + Bz)\,Z}{pp - 2pqz. \, cof. \, \Phi + qqzz} = Q$, atque

posito $z^n = \dfrac{p^n}{q^n} \cdot cof. \, n\,\Phi$, sit $Q = Q$,

&, posito $z^n = \dfrac{p^n}{q^n} \cdot fin. \, n\,\Phi$, sit $Q = Q$;

erit

$$C = \frac{QN + QN}{N^2 + N^2} + \frac{QN - QN}{N^2 + N^2} \cdot \frac{cof. \, \Phi}{fin. \, \Phi}$$

$$C = - \frac{QN + QN}{N^2 + N^2} \cdot \frac{q}{p \, fin. \, \Phi}.$$

Porro vocetur $\dfrac{Q - (C + Cz)\,Z}{pp - 2pqz. \, cof. \, \Phi + qqzz} = R$, atque

posito $z^n = \dfrac{p^n}{q^n} \cdot cof. \, n\Phi$, sit $R = R$,

&, posito $z^n = \dfrac{p^n}{q^n} \cdot fin. \, n\Phi$, sit $R = R$;

erit

$$D = \frac{RN + RN}{N^2 + N^2} + \frac{RN - RN}{N^2 + N^2} \cdot \frac{cof. \, \Phi}{fin. \, \Phi}$$

$$D = - \frac{RN + RN}{N^2 + N^2} \cdot \frac{q}{p \, fin. \, \Phi}.$$

hocque

hocque modo progrediendum eſt donec ultimæ fractionis, C A P.
cujus denominator eſt $pp - 2pq\, z.\, cof.\, \phi + qqzz$, nume- X I I.
rator fuerit determinatus.

E X E M P L U M.

Sit iſta propoſita Functio fracta

$$\frac{z - z^3}{(1 + zz)^4\,(1 + z^4)}$$

ex cujus denominatoris Factore $(1 + zz)^4$ oriantur hæ fra-
ctiones partiales,

$$\frac{A + Az}{(1 + zz)^4} + \frac{B + Bz}{(1 + zz)^3} + \frac{C + Cz}{(1 + zz)^2} + \frac{D + Dz}{1 + zz}.$$

Comparatione ergo inſtituta, erit $p = 1$, $q = 1$, $cof.\phi = 0$;
ideoque $\phi = \frac{1}{2}\,\pi$, · porroque $M = z - z^3$ & $Z = 1 + z^4$.
Hinc erit $M = 0$; $M = 2$; $N = 2$; $N = 0$, & $fin.\,\phi = 1$.
Hinc itaque invenitur

$$A = -\frac{4}{4}.\,0 = 0, \,\&\, A = 1.$$

ergo $A + Az = z$; hincque $P = \frac{z - z^3 - z - z^5}{1 + zz} = -z^3$,

& $P = 0$, $P = 1$: unde reperitur

$$B = 0, \,\&\, B = \frac{1}{2}.$$

Ergo $B + Bz = \frac{1}{2}z$, & $Q = \dfrac{-z^3 - \frac{1}{2}z - \frac{1}{2}z^5}{1 + zz} = -$

$$\frac{1}{2}z - \frac{1}{2}z^3;$$

unde $Q = 0$ & $Q = 0$, ergo
$C = 0$ & $C = 0$, hincque $R = -$

$$\frac{\frac{1}{2}z - \frac{1}{2}z^3}{1 + zz} = -\frac{1}{2}z,$$

·ergo $R = 0$; $R = -\frac{1}{2}$; unde fit

$$D = 0 \,\&\, D = -\frac{1}{4}.$$

Y $_3$

Quam-

Lib. I. Quamobrem fractiones quæſitæ ſunt hæ

$$\frac{z}{(1+zz)^4} + \frac{z}{2(1+zz)^3} - \frac{z}{4(1+zz)}.$$ Reliquæ vero fra-

ctionis numerator eſt $= S = \dfrac{R - (D + Dz)Z}{1+zz} = -\dfrac{1}{4}z+$

$\dfrac{1}{4}z^3$, quæ ergo erit $= \dfrac{-z+z^3}{4(1+z^4)}.$

210. Hac ergo methodo ſimul innoteſcit fractio complementi, quæ cum inventis conjuncta producat fractionem propoſitam ipſam. Scilicet ſi fractionis

$$\frac{M}{(pp - 2pqz. \; coſ. \, \Phi + qqzz)^k \, Z}$$

inventæ fuerint omnes fractiones partiales ex Factore $(pp - 2pqz. \; coſ. \, \Phi + qqzz)^k$ oriundæ, pro quibus formati ſunt valores Functionum P, Q, R, S, T, ſi harum litterarum Series ulterius continuetur, erit ea, quæ ultimam, qua opus eſt ad numeratores inveniendos, ſequitur, numerator reliquæ fractionis denominatorem Z habentis; nempe, ſi $k = 1$, erit reliqua fractio $\dfrac{P}{Z}$; ſi $k = 2$, erit reliqua fractio $\dfrac{Q}{Z}$; ſi $k = 3$, erit ea $\dfrac{R}{Z}$, & ita porro. Inventa autem hac reliqua fractione denominatorem Z habente, ea per has regulas ulterius reſolvi poterit.

CAPUT

C A P U T X I I I.

De Seriebus recurrentibus.

211. AD hoc Serierum genus, quas MOIVRÆUS *re-currentes* vocare folet, hic refero omnes Series quæ ex evolutione Functionis cujufque fractæ per divifionem actualem instituta nafcuntur. Supra enim jam oftendimus has Series ita effe comparatas, ut quivis terminus ex aliquot præcedentibus fecundum legem quandam conftantem determinetur, quæ lex a denominatore Functionis fractæ pendet. Cum autem nunc Functionem quamcunque fractam in alias fimpliciores refolvere docuerim, hinc Series quoque recurrens in alias fimpliciores refolvetur. In hoc igitur Capite propofitum eft Serierum recurrentium cujufvis gradus refolutionem in fimpliciores exponere.

212. Sit propofita ifta Functio fracta genuina

$$\frac{a + bz + czz + dz^3 + \&c.}{1 - \alpha z - \mathcal{6}zz - \gamma z^3 - \delta z^4 - \&c.}$$

quæ per divifionem evolvatur in hanc Seriem recurrentem

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes quemadmodum progrediantur, fupra eft oftenfum. Quod fi jam Functio illa fracta refolvatur in fractiones fuas fimplices, & unaquæque in Seriem recurrentem evolvatur, manifeftum eft fummam omnium harum Serierum ex fractionibus partialibus ortarum æqualem effe debere Seriei recurrenti.

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.$$

Fractiones ergo partiales, quas fupra invenire docuimus, da-
bunt

L I B. I. bunt Series partiales, quarum indoles ob fimplicitatem facile
perfpicitur; omnes autem Series partiales junctim fumtæ pro-
ducent Seriem recurrentem propofitam; unde & hujus natura
penitius cognofcetur.

213. Sint Series recurrentes ex fingulis fractionibus partia-
libus ortæ hæ.

$$a \ + \ bz \ + \ czz \ + \ dz^3 \ + \ ez^4 \ + \ \&c.$$
$$a' \ + \ b'z \ + \ c'zz \ + \ d'z^3 \ + \ e'z^4 \ + \ \&c.$$
$$a'' \ + \ b''z \ + \ c''zz \ + \ d''z^3 \ + \ e''z^4 \ + \ \&c.$$
$$a''' \ + \ b'''z \ + \ c'''zz \ + \ d'''z^3 \ + \ e'''z^4 \ + \ \&c.$$
$$\&c.$$

Quoniam hæ Series junctim fumtæ æquales effe debent huic

$$A + Bz + Czz + Dz^3 + Ez^4 + \&c.,$$

neceffe eft ut fit

$$A = a \ + \ a' \ + \ a'' \ + \ a''' \ + \ \&c.$$
$$B = b \ + \ b' \ + \ b'' \ + \ b''' \ + \ \&c.$$
$$C = c \ + \ c' \ + \ c'' \ + \ c''' \ + \ \&c.$$
$$D = d \ + \ d' \ + \ d'' \ + \ d''' \ + \ \&c.$$
$$\&c.$$

Hinc, fi fingularum Serierum ex fractionibus partialibus orta-
rum definiri queant coëfficientes Poteftatis z^n, horum fumma
dabit coëfficientem Poteftatis z^n in Serie recurrente $A + Bz +$
$Cz^2 + D^3 + \&c.$

214. Dubium hic fuboriri poffet, an, fi duæ hujufmodi
Series fuerint inter fe æquales

$$A + Bz + Cz^2 + Dz^3 + \&c. = A + Bz + Cz^2 + Dz^3 + \&c.,$$

neceffario inde fequatur, coëfficientes fimilium Poteftatum ip-
fius z inter fe effe æquales; feu an fit $A = A$; $B = B$;
$$C = C;$$

$C = $ C; $D = $ D; &c. . Hoc autem dubium facile tolletur, fi perpendamus hanc æqualitatem fubfiftere debere quemcunque valorem obtineat variabilis z. Sit igitur $z = 0$, atque manifeftum eft fore $A = $ A. His ergo terminis æqualibus utrinque fublatis, ac reliqua æquatione per z divifa, habebitur

$$B + Cz + Dz^2 + \&c. = B + Cz + Dz^2 + \&c.;$$

unde fequitur fore $B = $ B : fimili autem modo oftendetur effe $C = $ C; $D = $ D, & ita porro in infinitum.

215. Contemplemur ergo Series, quæ ex fractionibus partialibus, in quas fractio quæpiam propofita refolvitur, oriuntur. Ac primo quidem patet fractionem $\dfrac{A}{1 - pz}$ dare Seriem $A + Apz + Ap^2z^2 + Ap^3z^3 + \&c.$, cujus terminus generalis eft $Ap^n z^n$; hæc enim expreffio vocari folet *terminus generalis*, quoniam ex ea, loco n numeros omnes fucceffive fubftituendo, omnes Seriei termini nafcuntur. Deinde ex fractione $\dfrac{A}{(1 - pz)^2}$ oritur Series $A + 2Apz + 3Ap^2z^2 + 4Ap^3z^3 + \&c.$, cujus terminus generalis eft $(n + 1) Ap^n z^n$.

Tum ex fractione $\dfrac{A}{(1 - pz)^3}$, oritur Series $A + 3Apz + 6Ap^2z^2 + 10Ap^3z^3 + \&c.$, cujus terminus generalis eft $\dfrac{(n + 1)(n + 2)}{1. \quad 2} Ap^n z^n$. Generatim autem fractio $\dfrac{A}{(1 - pz)^k}$ præbet Seriem hanc $A + kApz + \dfrac{k(k + 1)}{1. \quad 2.} Ap^2z^2 + \dfrac{k(k + 1)(k + 2)}{1. \quad 2. \quad 3} Ap^3z^3 + \&c.$, cujus terminus generalis eft $\dfrac{(n + 1)(n + 2)(n + 3) \ldots \ldots (n + k - 1)}{1. \quad 2. \quad 3 \ldots \ldots \ldots (k - 1)} Ap^n z^n$. Ex ipfa autem Seriei progreffione colligitur hic idem terminus $= \dfrac{k(k + 1)(k + 2) \ldots \ldots \ldots (k + n - 1)}{1. \quad 2. \quad 3 \ldots \ldots \ldots n} Ap^n z^n$: hæc vero

Euleri *Introduct. in Anal. infin. parv.*   Z   expreffio

LIB. I. expreffio illi eft æqualis, id quod multiplicatione per crucem inftituta patebit, fiet enim,

$$1.2.3.....n(n+1).....(n+k-1) = 1.2.3.....(k-1)k.....(k+n-1)$$

quæ eft æquatio identica.

216. Quoties ergo in refolutione Functionum fractarum ad hujufmodi fractiones partiales $\dfrac{A}{(1-pz)^k}$ pervenitur, toties Seriei recurrentis ex illa Functione fracta ortæ $A + Bz + Cz^2 + Dz^3 + \&c.$, terminus generalis affignari poterit, quippe qui erit fumma terminorum generalium Serierum, quæ ex fractionibus partialibus nafcuntur.

EXEMPLUM I.

Invenire terminum generalem Seriei recurrentis, quæ ex hac fractione $\dfrac{1-z}{1-z-2zz}$ *nafcitur.*

Series hinc nata eft $1 + 0z + 2zz + 2z^3 + 6z^4 + 10z^5 + 22z^6 + 42z^7 + 86z^8 + \&c.$. Ad coëfficientem poteftatis generalis z^n inveniendum, fractio $\dfrac{1-z}{1-z-2zz}$ refolvatur in

$$\frac{\frac{2}{3}}{1+z} + \frac{\frac{1}{3}}{1-2z},$$ unde oritur terminus generalis quæfitus

$$\left(\frac{2}{3}(-1)^n + \frac{1}{3}\cdot 2^n\right)z^n = \frac{2^n \pm 2}{3}z^n,$$ ubi fignum $+$ valet fi n fit numerus par, fignum $-$ fi n fit impar.

EXEMPLUM II.

Invenire terminum generalem Seriei recurrentis quæ oritur ex fractione $\dfrac{1-z}{1-5z+6zz}$, *feu Seriei hujus* $1 + 4z + 14z^2 + 46z^3 + 146z^4 + 454z^5 + \&c.$

Ob

Ob denominatorem $= (1 - 2z)(1 - 3z)$ refolvitur fractio in has $\frac{-1}{1-2z} + \frac{2}{1-3z}$, ex quibus fit terminus generalis $2 . 3^n z^n - 2^n z^n = (2 . 3^n - 2^n) z^n$.

E X E M P L U M   I I I.

Invenire terminum generalem Seriei hujus $1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + 29z^6 + 47z^7 +$ &c., *quæ oritur ex evolutione fractionis* $\frac{1+2z}{1-z-zz}$.

Ob denominatoris Factores $1 - (\frac{1 + \sqrt{5}}{2}) z$ & $1 - (\frac{1 - \sqrt{5}}{2}) z$, per refolutionem prodeunt $\frac{\frac{\sqrt{5}+1}{2}}{1-(\frac{1+\sqrt{5}}{2})z} +$

$\frac{\frac{1-\sqrt{5}}{2}}{1-(\frac{1-\sqrt{5}}{2})z}$, unde erit terminus generalis $=$

$(\frac{1 + \sqrt{5}}{2})^{n+1} z^n + (\frac{1 - \sqrt{5}}{2})^{n+1} z^n$.

E X E M P L U M   IV.

Invenire terminum generalem Seriei hujus

$z + (\alpha a + b)z + (\alpha^2 a + \alpha b + \beta a)z^2 + (\alpha^3 a + \alpha^2 b + 2\alpha\beta a + \beta b)z^3 +$ &c,

quæ oritur ex evolutione fractionis $\frac{a+bz}{1-\alpha z - \beta zz}$.

Per refolutionem oriuntur hæ duæ fractiones:

Z 2

(a

$$\frac{\left(a\,(\alpha + \sqrt{(\alpha\alpha + 4b)}) + 2b\right) : 2\sqrt{(\alpha\alpha + 4b)}}{1 - \left(\dfrac{\alpha + \sqrt{(\alpha\alpha + 4\beta)}}{2}\right) z} +$$

$$\frac{\left(a\,(\sqrt{(\alpha\alpha + 4b)} - \alpha) - 2b\right) : 2\sqrt{(\alpha\alpha + 4b)}}{1 - \left(\dfrac{\alpha - \sqrt{\alpha\alpha + 4\beta}}{2}\right) z} , \quad \text{hinc}$$

terminus generalis erit $\dfrac{a\,(\sqrt{(\alpha\alpha + 4\beta)} + \alpha) + 2b}{2\sqrt{(\alpha\alpha + 4\beta)}}$

$\left(\dfrac{\alpha + \sqrt{(\alpha\alpha + 4\beta)}}{2}\right)^n z^n + \dfrac{a\,(\sqrt{(\alpha\alpha + 4\beta)} - \alpha) - 2b}{2\sqrt{(\alpha\alpha + 4\beta)}}$

$\left(\dfrac{\alpha - \sqrt{(\alpha\alpha + 4\beta)}}{2}\right)^n z^n$; ex quo omnium Serierum re-
currentium, quarum quifque terminus per duos præcedentes
determinatur, termini generales expedite definiri poterunt.

E x e m p l u m　V.

Invenire terminum generalem hujus Seriei $1 + z + 2z^2 + 2z^3 +$
$3 z^4 + 3 z^5 + 4 z^6 + 4 z^7 +$ &c., *quæ oritur ex fractione*

$$\frac{1}{1 - z - zz + z^3} = \frac{1}{(1 - z)^2\,(1 + z)}.$$

Quanquam lex progreffionis primo intuitu ita eft manifefta
ut explicatione non indigeat, tamen fractiones per refolutio-
nem ortæ $\dfrac{\frac{1}{2}}{(1 - z)^2} + \dfrac{\frac{1}{4}}{1 - z} + \dfrac{\frac{1}{4}}{1 + z}$ dant hunc termi-

num generalem $\dfrac{1}{2}\,(n + 1)\, z^n + \dfrac{1}{4}\, z^n + \dfrac{1}{4}\,(-1)^n z^n =$

$\dfrac{2n + 3 \pm 1}{4}\, z^n$, ubi fignum fuperius valet fi n fuerit numerus
par, inferius fi n fuerit impar.

217. Hoc pacto omnium Serierum recurrentium termini
generales exhiberi poffunt, quoniam omnes fractiones in hu-
jufmodi fractiones partiales fimplices refolvere licet. Quod fi
autem expreffiones imaginarias vitare velimus, fæpenumero
ad hujufmodi fractiones partiales pervenietur

$$A +$$

$$\frac{A + Bpz}{1 - 2pz.\, cof.\, \varphi + ppzz} \; ; \quad \frac{A + Bpz}{(1 - 2pz.cof.\, \varphi + ppzz)^2} \; ; \; \&$$

$$\frac{A + Bpz}{(1 - 2pz.\, cof.\, \varphi + ppzz)^k}, \text{ ex quarum evolutione cujufmodi}$$

Series nafcantur videndum eft. Ac primo quidem, ob $cof.\, n\, \varphi = 2\, cof.\, \varphi.\, cof.\, (n - 1)\, \varphi - cof.\, (n - 2)\, \varphi$, fractio

$$\frac{A}{1 - 2pz.\, cof.\, \varphi + ppzz} \text{ evoluta dabit}$$

$$A + 2Apz.cof.\varphi + 2Appzz.cof.2\varphi + 2Ap^3z^3.cof.3\varphi + 2Ap^4z^4.cof.4\varphi$$
$$+ \; Appzz., \quad + 2Ap^3z^3.\, cof.\, \varphi + 2Ap^4z^4.\, cof.2\varphi$$
$$+ Ap^4z^4. \qquad \&c.$$

cujus Seriei terminus generalis non tam facile apparet.

218. Quo igitur ad fcopum perveniamus, confideremus has duas Series

$$Ppz.\, fin.\, \varphi + Pp^2z^2.\, fin.\, 2\varphi + Pp^3z^3.\, fin.\, 3\varphi + Pp^4z^4.\, fin.\, 4\varphi + \&c.$$
$$Q + Qpz.\, cof.\, \varphi + Qp^2z^2.\, cof.\, 2\varphi + Qp^3z^3.\, cof.3\varphi + Qp^4z^4.cof.4\varphi + \&c.$$

quæ duæ Series utique nafcuntur ex evolutione fractionis, cujus denominator eft $1 - 2pz.cof.\, \varphi + ppzz$. Ac prior quidem oritur ex hac fractione $\dfrac{Ppz.\, fin.\, \varphi}{1 - 2pz.\, cof.\, \varphi + ppzz}$, pofterior vero ex hac $\dfrac{Q - Qpz.\, cof.\, \varphi}{1 - 2pz.\, cof.\, \varphi + ppzz}$. Addantur hæ duæ fractiones, atque fumma $\dfrac{Q + Ppz.\, fin.\, \varphi - Qpz.\, cof.\, \varphi}{1 - 2pz.\, cof.\, \varphi + ppzz}$ dabit Seriem cujus terminus generalis erit $= (P\, fin.n\, \varphi + Q\, cof.n\varphi)$ $p^n\, z^n$. Fiat autem hæc fractio propofitæ $\dfrac{A + Bpz}{1 - 2pz.\, cof.\, \varphi + ppzz}$ æqualis, erit $Q = A$, & $P = A\, cot.\varphi + B\, cofec.\varphi$. Seriei ergo ex hac fractione $\dfrac{A + Bpz}{1 - 2pz.\, cof.\, \varphi + ppzz}$ ortæ terminus generalis erit $= \dfrac{A\, cof.\, \varphi\, fin.\, n\varphi + B\, fin.\, n\varphi + A\, fin.\, \varphi.\, cof.\, n\varphi}{fin.\, \varphi}\, p^n\, z^n =$

$$\frac{A\, fin.\, (n + 1)\, \varphi + B\, fin.\, n\varphi}{fin.\, \varphi}\, p^n\, z^n.$$

219. Ad terminum generalem inveniendum, fi denomina-tor fractionis fuerit Poteftas, ut $(1 - 2pz.cof.\phi + ppzz)^k$, con-veniet hanc fractionem refolvi in duas etfi imaginarias

$$\frac{a}{(1 - (cof.\phi + \sqrt{-1}.fin.\phi) pz)^k} + \frac{b}{(1 - (cof.\phi - \sqrt{-1}.fin.\phi)pz)^k}$$

quarum fimul fumtarum terminus generalis Seriei ex ipfis ortæ erit

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1. \quad 2. \quad 3 \ldots\ldots (k-1)} (cof.n\phi + \sqrt{-1}.fin.n\phi) ap^n z^n +$$

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1. \quad 2. \quad 3 \ldots\ldots (k-1)} (cof.n\phi - \sqrt{-1}.fin.n\phi) bp^n z^n.$$

Sit $a + b = f$; $a - b = \frac{g}{\sqrt{-1}}$. ut fit $a = \frac{f\sqrt{-1} + g}{2\sqrt{-1}}$ &

$b = \frac{f\sqrt{-1} - g}{2\sqrt{-1}}$, eritque hæc expreffio

$$\frac{(n+1)(n+2)(n+3)\ldots\ldots(n+k-1)}{1. \quad 2. \quad 3 \ldots\ldots (k-1)} (f.cof.n\phi + g.fin.n\phi)p^n z^n$$

terminus generalis Seriei, quæ oritur ex his fractionibus

$$\frac{\frac{1}{2}f + \frac{1}{2\sqrt{-1}}g}{(1 - (cof.\phi + \sqrt{-1}.fin.\phi)pz)^k} + \frac{\frac{1}{2}f - \frac{1}{2\sqrt{-1}}g}{(1 - (cof.\phi - \sqrt{-1}.fin.\phi)pz)^k},$$

feu quæ oritur ex hac fractione una

$$\frac{f - kfpz.cof.\phi + \frac{k(k-1)}{1. 2}fp^2z^2.cof.2\phi - \frac{k(k-1)(k-2)}{1. 2. 3}fp^3z^3.cof.3\phi}{(1 - 2pz.cof.\phi + ppzz)^k}$$
$$+ kgpz.fin.\phi - \frac{k(k-1)}{1. 2}gp^2z^2.fin.2\phi + \frac{k(k-1)(k-2)}{1. 2. 3}gp^3z^3.fin.3\phi \quad \&c.$$

220. Pofito ergo $k = 2$, erit Seriei ex hac fractione

$$\frac{f - 2pz.(f.cof.\phi - g.fin.\phi) + ppzz.(f.cof.2\phi - g.fin.2\phi)}{(1 - 2pz.cof.\phi + ppzz)^2}$$

ortæ terminus generalis $= (n+1)(f.cof.n\phi + g.fin.n\phi)p^n z^n$.

At Seriei ex hac fractione $\frac{a}{1 - 2pz.cof.\phi + ppzz}$, feu hac

$$\frac{a - 2apz.\cos.\phi + appzz}{(1 - 2pz.\cos.\phi + ppzz)^2}$$ ortæ terminus generalis est $=$

$\frac{a\sin.(n+1)\phi}{\sin.\phi} p^n z^n$. Addantur hæ fractiones invicem, ac

ponatur $a + f = A$; $2a.\cos.\phi + 2f.\cos.\phi - 2g.\sin.\phi = -$ B

$\&$ $a + f.\cos.2\phi - g.\sin.2\phi = 0$, hinc erit $g =$

$\frac{B + 2A\cos.\phi}{2\sin.\phi}$, $a = \frac{A + B\cos.\phi}{1 - \cos.2\phi} = \frac{A + B\cos.\phi}{2(\sin.\phi)^2}$ $\&$ $f =$ $-$

$\frac{A\cos.2\phi - B\cos.\phi}{2(\sin.\phi)^2}$, $\&$ $g = \frac{B\sin.\phi + A\sin.2\phi.}{2(\sin.\phi)^2}$. Hanc ob

rem Seriei ex hac fractione $\frac{A + Bpz}{(1 - 2pz.\cos.\phi + ppzz)^2}$, ortæ ter-

minus generalis est $\frac{A + B\cos.\phi}{2(\sin.\phi)^3} \sin.(n+1)\phi.p^n z^n + (n+1)$

$\frac{(B\sin.\phi.\sin.n\phi + A\sin.2\phi.\sin.n\phi - B\cos.\phi.\cos.n\phi - A\cos.2\phi.\cos.n\phi)}{2(\sin.\phi)^2}$

$p^n z^n = -\frac{(n+1)(A\cos.(n+2)\phi + B\cos.(n+1)\phi)}{2(\sin.\phi)^2} p^n z^n +$

$\frac{(A + B\cos.\phi)\sin.(n+1)\phi}{2(\sin.\phi)^3} p^n z^n =$

$\frac{(\frac{1}{2}(n+3)\sin.(n+1)\phi - \frac{1}{2}(n+1)\sin.(n+3)\phi)}{2(\sin.\phi)^3} Ap^n z^n +$

$\frac{(\frac{1}{2}(n+2)\sin.n\phi - \frac{1}{2}n\sin.(n+2)\phi)}{2(\sin.\phi)^3} Bp^n z^n$. Est ergo

iste terminus generalis quæsitus $=$

$\frac{(n+3)\sin.(n+1)\phi - (n+1)\sin.(n+3)\phi}{4(\sin.\phi)^3} Ap^n z^n +$

$\frac{(n+2)\sin.n\phi - n.\sin.(n+2)\phi}{4(\sin.\phi)^3} Bp^n z^n$: Seriei quæ oritur ex

fractione $\frac{A + Bpz}{(1 - 2pz.\cos.\phi + ppzz)^2}$.

221. Sit $k = 3$, eritque Seriei ex hac fractione ortæ

$$\frac{f - 3pz.(f.\cos.\phi - g.\sin.\phi) + 3ppzz(f.\cos.2\phi - g.\sin.2\phi) - p^3 z^3(f.\cos.3\phi - g.\sin.3\phi)}{(1 - 2pz.\cos.\phi + ppzz)^3}$$

terminus generalis $= \frac{(n+1)(n+2)}{1. \quad 2} (f.\cos.n\phi + g.\sin.n\phi)p^n z^n$.

Deinde

Deinde Seriei ex fractione $\dfrac{a + bpz}{(1 - 2pz . cof. \Phi + ppzz)}$, feu ex hac

$$\dfrac{a - 2apz . cof. \Phi + appzz + bpz - 2bppzz . cof. \Phi + b p^3 z^3}{(1 - 2pz . cof. \Phi + ppzz)^3}$$ ortæ terminus ge-

neralis eft $\dfrac{(n+3) fin. (n+1) \Phi - (n+1) fin. (n+3) \Phi}{4 (fin. \Phi)^3} ap^n z^n +$

$\dfrac{(n+2) fin. n\Phi - n . fin. (n+2) \Phi}{4 (fin. \Phi)^3} bp^n z^n$. Addantur hæ fra-

ctiones ac ponatur numerator $= A$, erit $a + f = A$

$3f . cof. \Phi - 3g . fin . \Phi + 2a . cof . \Phi - b = 0$, $3f . cof . 2\Phi - 3g . fin . 2\Phi + a$

$- 2b . cof . \Phi = 0$; $\& \; b = f . cof . 3\Phi - g . fin . 3\Phi$, hinc erit $a =$

$\dfrac{f . cof . 3\Phi - g . fin . 3\Phi - 3f . cof . \Phi + 3g . fin . \Phi}{2 cof . \Phi} = 2g . (fin . \Phi)^2 tang . \Phi -$

$f - 2f . (fin . \Phi)^2$. Deinde reperitur $\dfrac{f}{g} = \dfrac{fin . 5\Phi - 2 fin 3\Phi + fin \Phi}{cof . 5\Phi - 2 cof . 3\Phi + cof . \Phi}$

$\& \; a + f = A = 2g . (fin . \Phi)^2 tang . \Phi - 2f (fin . \Phi)^2$; ergo

$\dfrac{A}{2 (fin . \Phi)^2} = \dfrac{g fin . \Phi - f . cof . \Phi}{cof . \Phi}$; ex quibus tandem oritur

$f = \dfrac{A (fin . \Phi - 2 fin 3\Phi + fin . 5\Phi)}{16 (fin . \Phi)^5}$, $g = \dfrac{A (cof . \Phi - 2 cof . 3\Phi + cof . 5\Phi)}{16 (fin \Phi)^5}$,

ob $16 (fin . \Phi)^5 = fin . 5\Phi - 5 fin . 3\Phi + 10 fin . \Phi$, erit $a =$

$\dfrac{A (9 fin . \Phi - 3 fin . 3\Phi)}{16 (fin . \Phi)^5} \& \; b = \dfrac{A (- fin . 2\Phi + fin . 2\Phi)}{16 (fin . \Phi)^5} = 0$. Eft

autem $3 fin . \Phi - fin . 3\Phi = 4 (fin . \Phi)^3$; ergo $a = \dfrac{3 A}{4 (fin . \Phi)^2}$. Quo-

circa erit terminus generalis $\dfrac{(n+1)(n+2)}{1 . \quad 2} p^n z^n$

$A \dfrac{(fin . (n+1) \Phi - 2 fin (n+3) \Phi + fin (n+5) \Phi)}{16 (fin . \Phi)^5} +$

$3 A p^n z^n . \dfrac{((n+3) fin . (n+1) \Phi - (n+1) fin . (n+3) \Phi)}{16 (fin . \Phi)^5} =$

$\dfrac{A p^n z^n}{16 (fin . \Phi)^5} \left(\dfrac{(n+4)(n+5)}{1 . \quad 2} fin . (n+1) \Phi - \dfrac{2(n+1)(n+5)}{1 . \quad 2} \right.$

$\left. fin . (n+3) \Phi + \dfrac{(n+1)(n+2)}{1 . \quad 2} fin . (n+5) \Phi \right)$.

222. Se-

222. Seriei ergo quæ oritur ex hac fractione

$$\frac{A + B p z}{(1 - 2 p z. \cos. \phi + p p z z)^3}$$

terminus generalis erit hic

$$\frac{A p^n z^n}{16 (\sin. \phi)^5} \left(\frac{(n + 5)(n + 4)}{1. \quad 2} \sin.(n+1)\phi - \frac{2(n + 1)(n + 5)}{1. \quad 2} \times \right.$$

$$\sin.(n+3)\phi + \frac{(n + 1)(n + 2)}{1. \quad 2} \sin. (n + 5) \phi)$$

$$+ \frac{B p^n z^n}{16 (\sin. \phi)^5} \left(\frac{(n+4)(n+3)}{1. \quad 2} \sin.n\phi - \frac{2n(n + 4)}{1. \quad 2} \sin.(n+2)\phi + \right.$$

$$\frac{n(n + 1)}{1. \quad 2} \sin. (n + 4) \phi).$$

Atque, ulterius progrediendo, Seriei, quæ oritur ex hac fractione

$$\frac{A + B p z}{(1 - 2 p z. \cos. \phi + p p z z)^4}$$

terminus generalis erit hic

$$+ \frac{A p^n z^n}{64 (\sin. \phi)^7} \left(\frac{(n + 7)(n + 6)(n + 5)}{1. \quad 2. \quad 3} \sin. (n + 1) \phi - \right.$$

$$\frac{3(n + 1)(n + 7)(n + 6)}{1. \quad 2. \quad 3} \sin.(n+3)\phi + \frac{3(n+1)(n + 2)(n + 7)}{1. \quad 2. \quad 3} \times$$

$$\sin. (n + 5) \phi - \frac{(n + 1)(n + 2)(n + 3)}{1. \quad 2. \quad 3} \sin. (n + 7)\phi)$$

$$+ \frac{B p^n z^n}{64 (\sin. \phi)^7} \left(\frac{(n + 6)(n + 5)(n + 4)}{1. \quad 2. \quad 3} \sin. n \phi - \right.$$

$$\frac{3 n(n+6)(n+5)}{1. \quad 2. \quad 3} \sin.(n + 2)\phi + \frac{3 n(n + 1)(n + 6)}{1. \quad 2. \quad 3}$$

$$\sin. (n + 4) \phi - \frac{n(n + 1)(n + 2)}{1. \quad 2. \quad 3} \sin. (n + 6) \phi).$$

Ex his autem expreſſionibus facile intelligitur, quemadmodum formæ terminorum generalium pro altioribus dignitatibus progrediantur. Ad naturam vero harum expreſſionum penitius inſpiciendam notari convenit eſſe

$$\sin.\phi = \sin.\phi$$
$$4\,(\sin.\phi)^3 = 3\sin.\phi - \sin.3\phi$$
$$16\,(\sin.\phi)^5 = 10\sin.\phi - 5\sin.3\phi + \sin.5\phi$$
$$64\,(\sin.\phi)^7 = 35\sin.\phi - 21\sin.3\phi + 7\sin.5\phi - \sin.7\phi$$
$$256(\sin.\phi)^9 = 126\sin.\phi - 84\sin.3\phi + 36\sin.5\phi - 9\sin.7\phi + \sin.9\phi$$

&c.

223. Cum igitur hoc pacto omnes functiones fractæ in fractiones partiales reales resolvi queant, simul omnium Serierum recurrentium termini generales per expressiones reales exhiberi poterunt. Quod quo clarius appareat, exempla sequentia adjuncta sunt.

E X E M P L U M   I.

Ex fractione $\dfrac{1}{(1-z)(1-zz)(1-z^3)} =$

$\dfrac{1}{1-z-zz+z^4+z^5-z^6}$, oritur ista Series recurrens

$1+z+2z^2+3z^3+4z^4+5z^5+7z^6+8z^7+10z^8+12z^9+\&c.$; cujus terminus generalis desideratur. Fractio proposita secundum Factores ordinata fit $= \dfrac{1}{(1-z)^3(1+z)(1+z+zz)}$, quæ

resolvitur in has fractiones $\dfrac{1}{6(1-z)^3} + \dfrac{1}{4(1-z)^2} +$

$\dfrac{17}{72(1-z)} + \dfrac{1}{8(1+z)} + \dfrac{(2+z)}{9(1+z+zz)}$. Harum prima

$\dfrac{1}{6(1-z)^3}$, dat terminum generalem $\dfrac{(n+1)(n+2)}{1.\ \ 2.} \cdot \dfrac{1}{6} z^n =$

$\dfrac{nn+3n+2}{12} z^n$: secunda $\dfrac{1}{4(1-z)^2}$ dat $\dfrac{n+1}{4} z^n$: tertia

$\dfrac{17}{72(1-z)}$ dat $\dfrac{17}{72} z^n$: quarta $\dfrac{1}{8(1+z)}$ dat $\dfrac{1}{8}(-1)^n z^n$.

Quinta

Quinta vero $\dfrac{2+z}{9\,(1+z+zz)}$ comparata cum forma

$\dfrac{A+B\,p\,z}{1-2pz.co\int.\Phi+ppzz}$ (218) dat $p=1$, $\varphi=\dfrac{\varpi}{3}=60°$;

$A=+\dfrac{2}{9}$; & $B=-\dfrac{1}{9}$, unde oritur terminus generalis

$+\dfrac{2\int in.(n+1)\Phi-\int in.n\Phi}{9\int in.\,\Phi}(-1)^n\,z^n=+\dfrac{4\int in.(n+1)\Phi-2\int in.n\Phi}{9\sqrt{3}}$

$(-1)^n\,z^n=+\dfrac{4\int in.(n+1)\dfrac{\varpi}{3}-2\int in.\,n\dfrac{\varpi}{3}}{9\sqrt{3}}(-1)^n z^n.$ Colligantur hæ expressiones omnes in unam summam, ac prodibit Seriei propositæ terminus generalis quæsitus $=(\dfrac{nn}{12}+\dfrac{n}{2}+$

$\dfrac{47}{72})z^n\pm\dfrac{1}{8}\,z^n\pm\dfrac{4\int in.(n+1)\dfrac{\varpi}{3}-2\int in.\,n\dfrac{\varpi}{3}}{9\sqrt{3}}\,z^n$, ubi signa superiora valent si n numerus par, inferiora sin impar. Ubi notandum est si fuerit n numerus formæ $3\,m$ fore

$\dfrac{4\int in.\dfrac{1}{3}(n+1)\varpi-2\int in.\dfrac{1}{3}\,n\varpi}{9\sqrt{3}}=\pm\dfrac{2}{9}$; si fuerit $n=$

$3\,m+1$ erit hæc expressio $=\mp\dfrac{1}{9}$; at si $n=3\,m+2$ erit ista expressio $=\mp\dfrac{1}{9}$, prout n fuerit numerus vel par vel impar. Ex his natura Seriei ita explicari potest, ut

si

si fuerit	terminus generalis futurus sit
$n = 6\,m + 0$	$\left(\dfrac{nn}{12} + \dfrac{n}{2} + 1\right) z^n$
$n = 6\,\dot{m} + 1$	$\left(\dfrac{nn}{12} + \dfrac{n}{2} + \dfrac{5}{12}\right) z^n$
$n = 6\,m + 2$	$\left(\dfrac{nn}{12} + \dfrac{n}{2} + \dfrac{2}{3}\right) z^n$
$n = 6\,m + 3$	$\left(\dfrac{nn}{12} + \dfrac{n}{2} + \dfrac{3}{4}\right) z^n$
$n = 6\,m + 4$	$\left(\dfrac{nn}{12} + \dfrac{n}{2} + \dfrac{2}{3}\right) z^n$
$n = 6\,m + 5$	$\left(\dfrac{nn}{12} + \dfrac{n}{2} + \dfrac{5}{12}\right) z^n$

Sic, si fuerit $n = 50$, valet forma $n = 6\,m + 2$, eritque termi-
nus Seriei $= 234 z^{50}$.

E x e m p l u m   II.

Ex fractione $\dfrac{1 + z + zz}{1 - z - z^4 + z^5}$, oritur hæc Series recurrens

$1 + 2z + 3zz + 3z^3 + 4z^4 + 5z^5 + 6z^6 + 6z^7 + 7z^8 + \&c.$,

cujus terminum generalem invenire oportet. Fractio proposi-
ta ad hanc formam reducitur $\dfrac{1 + z + zz}{(1 - z)^2 (1 + z)(1 + zz)}$,
quæ propterea resolvitur in has fractiones partiales
$\dfrac{3}{4(1 - z)^2} + \dfrac{3}{8(1 - z)} + \dfrac{1}{8(1 + z)} - \dfrac{1 + z}{4(1 + zz)}$. Ha-
rum prima $\dfrac{3}{4(1 - z)^2}$ dat terminum generalem $\dfrac{3(n + 1)}{4} z^n$;
secunda $\dfrac{3}{8(1 - z)}$ dat $\dfrac{3}{8} z^n$; tertia dat $\dfrac{1}{8}(-1)^n z^n$; &
quarta $- \dfrac{1 + z}{4(1 + zz)}$ comparata cum forma $\dfrac{A + B p z}{1 - 2pz.cos.\Phi + ppzz}$,
dat $p = 1$; $cos.\,\Phi = 0$; & $\Phi = \dfrac{1}{2}\,\pi$; $A = -\dfrac{1}{4}$;
$B =$

$B = +\frac{1}{4}$, unde fit terminus generalis $= (- \frac{1}{4}$ *fin.* $\frac{1}{2}$ $(n+1) \varpi + \frac{1}{4}$ *fin.* $\frac{1}{2} n\varpi) z^n$. Quare colligendo erit terminus generalis quæfitus $= (\frac{3}{4} n + \frac{9}{8}) z^n \pm \frac{1}{8} z^n - \frac{1}{4} ($*fin.* $\frac{1}{2} (n+1) \varpi -$ *fin.* $\frac{1}{2} n \varpi) z^n$. Hinc

fi fuerit	erit terminus generalis
$n = 4m + 0$	$(\frac{3}{4} n + 1) z^n$
$n = 4m + 1$	$(\frac{3}{4} n + \frac{5}{4}) z^n$
$n = 4m + 2$	$(\frac{3}{4} n + \frac{3}{2}) z^n$
$n = 4m + 3$	$(\frac{3}{4} n + \frac{3}{4}) z^n$

Ita, fi $n = 50$, valebit $n = 4m + 2$, eritque terminus $= 39z^{50}$.

224. Propofita ergo Serie recurrente , quoniam illa fractio unde oritur , facile cognofcitur , ejus terminus generalis fecundum præcepta data reperietur. Ex lege autem Seriei recurrentis , qua quifque terminus ex præcedentibus definitur , ftatim innotefcit denominator fractionis, hujufque Factores præbebunt formam termini generalis , per numeratorem enim tantum coefficientes determinantur. Sit nempe propofita hæc Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c. ,$$

cujus lex progreffionis , qua unufquifque terminus ex aliquot præcedentibus determinatur , præbeat hunc fractionis denominatorem $1 - \alpha z - 6z^2 - \gamma z^3$. Ita ut fit $D = \alpha C + 6B + \gamma A$; $E = \alpha D + 6C + \gamma B$; $F = \alpha E + 6D + \gamma C$; &c.,

A a 3

&c., qui multiplicatores α, $+\,6$, $+\,\gamma$ a Moivræo *fcalam relationis* conftituere dicuntur. Lex ergo progreffionis pofita eft in fcala relationis, atque fcala relationis ftatim præbet denominatorem fractionis, ex cujus refolutione propofita Series recurrens oritur.

225. Ad terminum ergo generalem, feu coëfficientem Poteftatis indefinitæ z^n, inveniendum, quæri debent denominatoris $1 - \alpha z - 6 z^2 - \gamma z^3$ Factores vel fimplices vel duplices, fi imaginarios vitare velimus. Sint primo Factores fimplices omnes inter fe inæquales & reales hi $(1 - pz)(1 - qz)(1 - rz)$; atque fractio generans Seriem propofitam refolvetur in $\dfrac{A}{1 - pz} + \dfrac{B}{1 - qz} + \dfrac{C}{1 - rz}$; unde Seriei terminus generalis erit $(Ap^n + Bq^n + Cr^n)z^n$. Si duo Factores fuerint æquales nempe $q = p$, tum terminus generalis hujufmodi erit $((An + B)p^n + Cr^n)z^n$, &, fi infuper fuerit $r = q = p$, erit terminus generalis $(An^2 + Bn + C)p^n z^n$. Quod fi vero denominator $1 - \alpha z - 6 z^2 - \gamma z^3$ duplicem habeat Factorem, ut fit $= (1 - pz)(1 - 2qz.cof.\phi + qqzz)$ tum terminus generalis erit $= (Ap^n + \dfrac{B fin.(n+1)\phi + C fin. n\phi}{fin.\phi} q^n)z^n$. Cum igitur, pofitis pro n fucceffive numeris 0, 1, 2, prodire debeant termini A, Bz, Cz^2, hinc valores litterarum A, B, C determinabuntur.

226. Sit fcala relationis bimembris, feu determinetur quifque terminus per duos præcedentes, ita ut fit

$$C = \alpha B - 6A; \quad D = \alpha C - 6B; \quad E = \alpha D - 6C, \text{ &c.,}$$

atque manifeftum eft Seriem hanc recurrentem, quæ fit

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots\ldots + Pz^n + Qz^{n+1} + \text{&c.,}$$

oriri ex fractione cujus denominator fit $1 - \alpha z + 6 zz$. Sint hujus denominatoris Factores $(1 - pz)(1 - qz)$ erit $p +$
$$q =$$

$q = \alpha$ & $pq = 6$: atque Seriei terminus generalis erit $(Ap^n + Bq^n) z^n$. Hinc facto $n = 0$, erit $A = A + B$; & facto $n = 1$ erit $A = Ap + Bq$; unde fit $Aq - B =$. $A(q - p)$ & $A = \frac{Aq - B}{q - p}$; & $B = \frac{Ap - B}{p - q}$. Inventis autem valoribus A & B, erit $P = Ap^n + Bq^n$ & $Q = Ap^{n+1} + Bq^{n+1}$. Tum vero erit $AB = \frac{BB - \alpha AB + 6AA}{46 - \alpha\alpha}$.

227. Hinc deduci poteft modus quemvis terminum ex unico præcedente formandi, cum ad hoc per legem progreffionis duo requirantur. Cum enim fit

$$P = Ap^n + Bq^n \ \& \ Q = Ap \cdot p^n + Bq \cdot q^n$$

erit

$$Pq - Q = A(q - p)p^n \ \& \ Pp - Q = B(p - q)q^n :$$

multiplicentur hæ expreffiones in fe invicem ; eritque

$$P^2 pq - (p + q)PQ + QQ - AB(p - q)^2 p^n q^n = 0.$$

At eft

$$p + q = \alpha; \ pq = 6; \ (p - q)^2 = (p + q)^2 - 4pq = \alpha\alpha - 46 \ \& \ p^n q^n = 6^n.$$ Quibus fubftitutis habebitur

$$6P^2 - \alpha PQ + QQ = (6AA - \alpha AB + BB)6^n, \text{ feu}$$

$$\frac{QQ - \alpha PQ + 6PP}{BB - \alpha AB + 6AA} = 6^n;$$ quæ eft infignis proprietas Serierum recurrentium, quárum quifque terminus per duos præcedentes determinatur. At cognito quovis termino P, erit fequens $Q = \frac{1}{2}\alpha P + \sqrt{((\frac{1}{4}\alpha^2 - 6)P^2 + (B^2 - \alpha AB + 6AA)6^n)}$, quæ expreffio, etfi fpeciem irrationalitatis præ fe

fert,

fert , tamen femper eft rationalis , propterea quod termini ir-
rationales in Serie non occurrunt.

228. Ex datis porro duobus terminis contiguis quibufvis
Pz^n & Qz^{n+1} commode affignari poteft terminus multo ma-

gis remotus Xz^{2n}　Ponatur enim

$X = fP^2 + gPQ - hAB\mathfrak{C}^n$　　　Quoniam eft

$P = Ap^n + Bq^n$ & $Q = Ap.p^n + Bq.q^n$　atque

$X = Ap^{2n} + Bq^{2n}$; erit ut fequitur

$$fP^2 = fA^2 p^{2n} + fB^2 q^{2n} + 2fAB\mathfrak{C}^n$$

$$gPQ = gA^2 p.p^{2n} + gB^2 q.q^{2n} + gAB\alpha\mathfrak{C}^n$$

$$-hAB\mathfrak{C}^n = \qquad\qquad\qquad -hAB\mathfrak{C}^n$$

$$\overline{\quad X = \quad Ap^{2n} + Bq^{2n}\quad}$$

Fiet ergo $f + gp = \frac{1}{A}$; $f + gq = \frac{1}{B}$ & $h = 2f + g\alpha$,

unde $g = \frac{B-A}{AB(p-q)}$ & $f = \frac{Ap - Bq}{AB(p-q)}$. At eft $B - A =$

$\frac{\alpha A - 2B}{p-q}$; $Ap - Bq = \frac{\alpha B - 2A\mathfrak{C}}{p-q}$. Ergo $f = \frac{\alpha B - 2A\mathfrak{C}}{AB(\alpha\alpha - 4\mathfrak{C})}$

& $g = \frac{\alpha A - 2B}{AB(\alpha\alpha - 4\mathfrak{C})}$ feu $f = \frac{2A\mathfrak{C} - \alpha B}{BB - \alpha AB + \mathfrak{C}AA}$ &

$g = \frac{2B - \alpha A}{BB - \alpha AB + \mathfrak{C}AA}$; ideoque $h = \frac{(4\mathfrak{C} - \alpha\alpha)A}{BB - \alpha AB + \mathfrak{C}AA}$.

Eritque ergo

$$X = \frac{(2A\mathfrak{C} - \alpha B)P^2 + (2B - \alpha A)PQ}{BB - \alpha AB + \mathfrak{C}AA} - A\mathfrak{C}^n.$$

Simili vero modo reperitur

$$X = \frac{(\alpha\mathfrak{C}A - (\alpha\alpha - 2\mathfrak{C})B)P^2 + (2B - \alpha A)Q^2}{\alpha(BB - \alpha AB + \mathfrak{C}AA)} - \frac{2B\mathfrak{C}^n}{\alpha}.$$

His conjungendis per eliminationem termini $\mathfrak{C}^n$ reperitur

$$X = \frac{(\mathfrak{C}A - \alpha B)P^2 + 2BPQ - AQQ}{BB - \alpha AB + \mathfrak{C}AA}.$$

229. Si

229. Simili modo, fi ftatuantur termini fequentes

$$A + Bz + Cz^2 + \ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \ldots +$$

$$Xz^{2n} + Yz^{2n+1} + Zz^{2n+2},$$

erit

$$Z = \frac{(\mathfrak{C} A - \alpha B) Q^2 + 2BQR - ARR}{BB - \alpha AB + \mathfrak{C} AA}, \&, \text{ ob } R = \alpha Q - \mathfrak{C} P,$$

erit

$$Z = \frac{-\mathfrak{C} \mathfrak{C} A P^2 + 2 \mathfrak{C} (\alpha A - B) PQ + (\alpha B - (\alpha\alpha - \mathfrak{C})A)Q^2}{BB - \alpha AB + \mathfrak{C} AA}$$

At eft

$$Z = \alpha Y - \mathfrak{C} X, \text{ ergo } Y = \frac{Z + \mathfrak{C} X}{\alpha}; \text{ unde fit}$$

$$Y = \frac{-\mathfrak{C} B P^2 + 2 \mathfrak{C} A PQ + \alpha(B - \alpha A) QQ}{BB - \alpha AB + \mathfrak{C} AA}. \text{ Sic igitur}$$

porro ex X & Y definiri poterunt fimili modo coëfficientes poteftatum z^{4n}, & z^{4n+1}; hincque ipfarum z^{8n}, z^{8n+1}, & ita porro.

E X E M P L U M.

Sit propofita ifta Series recurrens

$$1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \ldots + Pz^n + Qz^{n+1} + \&c.,$$

cujus cum quilibet coëfficiens fit fumma duorum præcedentium, erit denominator fractionis hanc Seriem producentis $1 - z - zz$; ideoque $\alpha = 1$; $\mathfrak{C} = -1$; & $A = 1$; $B = 3$; unde fit $BB - \alpha AB + \mathfrak{C} AA = 5$, ex quo

orietur primum $Q = \frac{P + \sqrt{(5 PP + 20 (-1)^n)}}{2} =$

$\frac{P + \sqrt{(5 PP \pm 20)}}{2}$, ubi fignum fuperius valet, fi n fit numerus par, inferius fi impar. Sic, fi $n = 4$, ob $P = 11$, erit

$Q = \dfrac{11 \doteq \sqrt{(5.121 + 20)}}{2} = \dfrac{11 \doteq 25}{2} = 18$. Si porro coëf-

ficiens termini z^{2n} fit X, erit $X = \dfrac{- 4PP + 6PQ - QQ}{5}$; ergo

Poteftatis z^8 coëfficiens erit $= \dfrac{-4.121 + 6.198 - 324}{5} = 76$.

Cum autem fit $Q = \dfrac{P + \sqrt{(5PP \doteq 20)}}{2}$ erit $QQ =$

$\dfrac{3PP \doteq 10 + P\sqrt{(5PP \doteq 20)}}{2}$; ideoque $X = \dfrac{- PP \mp 2 + P\sqrt{(5PP \doteq 20)}}{2}$.

Ex termino ergo Seriei quocunque $P z^n$, obtinentur hi

$\dfrac{P \doteq \sqrt{(5PP \doteq 20)}}{2} z^{n+1}$, & $\dfrac{- PP \mp 2 + P\sqrt{(5PP \doteq 20)}}{2} z^{2n}$.

230. Simili modo in Seriebus recurrentibus, quarum quilibet terminus ex tribus antecedentibus determinatur, quivis terminus ex duobus antecedentibus definiri poteft. Sit enim Series hujufmodi recurrens

$A + Bz + Cz^2 + Dz^3 + \ldots \ldots + Pz^n + Qz^{n+1} + Rz^{n+2} + \&c.$,

cujus fcala relationis fit α, $- 6$, $+ \gamma$, feu quæ oriatur ex fractione cujus denominator $= 1 - \alpha z + 6 z^2 - \gamma z^3$. Quod fi jam termini P, Q, R eodem modo per Factores hujus denominatoris, qui fint $(1 - pz)(1 - qz)(1 - rz)$ exprimantur, ut fit $P = Ap^n + Bq^n + Cr^n$; $Q = Ap \cdot p^n + Bq \cdot q^n + Cr \cdot r^n$; & $R = Ap^2 \cdot p^n + Bq^2 \cdot q^n + Cr^2 \cdot r^n$; ob $p + q + r = \alpha$; $pq + pr + qr = 6$ & $pqr = \gamma$, reperietur hæc proportio

$$R^3 \left.\begin{array}{l} - 2\alpha Q \\ + 6 P \end{array}\right\} \quad R^2 \left.\begin{array}{l} + (\alpha\alpha + 6) \, Q^2 \\ - (\alpha 6 + 3\gamma) PQ \\ + \alpha\gamma \quad\quad P^2 \end{array}\right\} \quad R \begin{array}{l} - (\alpha 6 - \gamma) \, Q^3 \\ + (\alpha\gamma + 66)PQ^2 \\ - \quad 2 6 \gamma \quad P^2 Q \\ + \quad \gamma\gamma \quad\quad P^3 \end{array} : c^n =$$

$$C^3 \begin{array}{l} -2\alpha B \\ +\mathcal{6}A \end{array} \Bigg\} \quad C^2 \begin{array}{l} +(\alpha^2+\mathcal{6})B^2 \\ -(\alpha\mathcal{6}+3\gamma)AB \\ +\alpha\gamma \quad A^2 \end{array} \Bigg] \quad C \begin{array}{l} -(\alpha\mathcal{6}-\gamma)B^3 \\ +(\alpha\gamma+\mathcal{66})AB^2 \\ - \quad 2\mathcal{6}\gamma \, A^2 B \\ + \quad \gamma\gamma \, A^3 \end{array} : \text{ I.}$$

Pendet ergo inventio termini R ex duobus præcedentibus P & Q a resolutione æquationis cubicæ.

231. His de terminis generalibus Serierum recurrentium notatis , superest ut earumdem Serierum summas investigemus. Ac primo quidem manifestum est summam Seriei recurrentis in infinitum extensæ æqualem esse fractioni ex qua oritur: cujus fractionis cum denominator ex ipsa progressionis lege pateat , reliquum est ut numeratorem definiamus. Sit itaque proposita hæc Series

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + Gz^6 + \&c.,$$

cujus lex progressionis præbeat hunc denominatorem $1 - \alpha z + \mathcal{6}z^2 - \gamma z^3 + \delta z^4$. Sumamus fractionem summæ Seriei in infinitum æqualem esse $= \dfrac{a + bz + cz^2 + dz^3}{1 - \alpha z + \mathcal{6}z^2 - \gamma z^3 + \delta z^4}$, ex qua cum Series proposita oriri debeat , erit comparando

$$\begin{aligned} a &= A \\ b &= B - \alpha A \\ c &= C - \alpha B + \mathcal{6}A \\ d &= D - \alpha C + \mathcal{6}B - \gamma A : \end{aligned}$$

Hinc erit summa quæsita

$$\frac{A + (B - \alpha A)z + (C - \alpha B + \beta A)z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A)z^3}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}.$$

232. Hinc facile intelligitur quemadmodum Seriei recurrentis summa ad datum terminum usque inveniri debeat.

Quæ-

LIB I. Quæratur scilicet Seriei modo assumtæ summa ad terminum $P\,z^n$, atque ponatur

$$s = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots + P z^n,$$

quoniam hujus Seriei summa in infinitum constat, quæratur summa terminorum ultimum $P z^n$ in infinitum sequentium, qui sint

$$t = Q z^{n+1} + R z^{n+2} + S z^{n+3} + T z^{n+4} + \&c.,$$

hæc Series per z^{n+1} divisa dat Seriem recurrentem propofitæ æqualem, cujus propterea summa erit $t =$

$$\frac{Q z^{n+1} + (R - \alpha Q) z^{n+2} + (S - \alpha R + \beta Q) z^{n+3} +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q) z^{n+4}.}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

Unde orietur summa quæsita $s =$

$$\frac{+ A + (B - \alpha A) z + (C - \alpha B + \beta A) z^2 +}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(D - \alpha C + \beta B - \gamma A) z^3 - Q z^{n+1} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(R - \alpha Q) z^{n+2} - (S - \alpha R + \beta Q) z^{n+3} -}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

$$\frac{(T - \alpha S + \beta R - \gamma Q) z^{n+4}.}{1 - \alpha z + \beta z^2 - \gamma z^3 + \delta z^4}$$

233. Quod si ergo scala relationis fuerit bimembris
$\alpha, -$

α, $-\beta$; Seriei $A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n$,

quæ oritur ex fractione $\dfrac{A + (B - \alpha A)z}{1 - \alpha z + \beta zz}$, fumma erit

$$\frac{A + (B - \alpha A)z - Qz^{n+1} - (R - \alpha Q)z^{n+2}}{1 - \alpha z + \beta zz}$$

At eft, ex natura Seriei, $R = \alpha Q - \beta P$, unde prodibit fumma

$$\frac{A + (B - \alpha A)z - Qz^{n+1} + \beta Pz^{n+2}}{1 - \alpha z + \beta zz}$$

EXEMPLUM.

Sit propofita Series $1 + 3z + 4z^2 + 7z^3 + \ldots\ldots + Pz^n$
ubi eft $\alpha = 1$; $\beta = -1$; $A = 1$; $B = 3$; erit hujus fumma

$$\frac{1 + 2z - Qz^{n+1} - Pz^{n+2}}{1 - z - zz} \quad .$$ Pofito vero $z = 1$,

erit fumma Seriei $1 + 3 + 4 + 7 + 11 + \ldots\ldots + P = P + Q - 3$. Summa ergo termini ultimi & fequentis ternario excedit fummam Seriei. Quia vero eft $Q = \dfrac{P + \sqrt{(5PP \pm 20)}}{2}$ erit fumma Seriei $1+3+4+7+11+\ldots+$
$P = \dfrac{3P - 6 + \sqrt{(5PP \pm 20)}}{2}$. Ex folo ergo termino ultimo fumma poteft exhiberi.

C A P U T   XIV.

De multiplicatione ac divisione Angulorum.

234. SIt Angulus, vel Arcus, in Circulo cujus Radius $= 1$, quicunque $= z$; ejus Sinus $= x$; Cosinus $= y$, & Tangens $= t$; erit $xx + yy = 1$ & $t = \frac{x}{y}$. Cum igitur, uti supra vidimus; tam Sinus quam Cosinus Angulorum z; $2z$; $3z$; $4z$; $5z$; &c., constituant Seriem recurrentem cujus scala relationis est $2y$, — 1; primum Sinus horum Arcuum ita se habebunt :

$$\begin{aligned}
\text{sin.}\ 0z &= 0 \\
\text{sin.}\ 1z &= x \\
\text{sin.}\ 2z &= 2xy \\
\text{sin.}\ 3z &= 4xy^2 - x \\
\text{sin.}\ 4z &= 8xy^3 - 4xy \\
\text{sin.}\ 5z &= 16xy^4 - 12xy^2 + x \\
\text{sin.}\ 6z &= 32xy^5 - 32xy^3 + 6xy \\
\text{sin.}\ 7z &= 64xy^6 - 80xy^4 + 24xy^2 - x \\
\text{sin.}\ 8z &= 128xy^7 - 192xy^5 + 80xy^3 - 0xy
\end{aligned}$$

hinc concluditur fore

$$\begin{aligned}
\text{sin.}\ nz = x \Big(& 2^{n-1} y^{n-1} - (n-2)2^{n-3}y^{n-3} + \\
& \frac{(n-3)(n-4)}{1.\ 2} 2^{n-5} y^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.\ 2.\ 3} 2^{n-7} y^{n-7} + \\
& \frac{(n-5)(n-6)(n-7)(n-8)}{1.\ 2.\ 3.\ 4} 2^{n-9} y^{n-9} - \&c. \Big)
\end{aligned}$$

235. Si ponamus Arcum $nz = s$; erit $\text{sin.}\ nz = \text{sin.}\ s = \text{sin.}\ (\varpi - s) = \text{sin.}\ (2\varpi + s) = \text{sin.}\ (3\varpi - s)$ &c., hi enim

enim Sinus omnes funt inter fe æquales. Hinc obtinemus plures valores pro x, qui erunt

$$\textit{fin.}\ \frac{s}{n};\ \textit{fin.}\ \frac{\pi - s}{n};\ \textit{fin.}\ \frac{2\pi + s}{n};\ \textit{fin.}\ \frac{3\pi - s}{n};\ \textit{fin.}\ \frac{4\pi + s}{n};\ \&c.,$$

qui ergo omnes æquationi inventæ æque conveniunt. Tot autem prodibunt diverfi pro x valores, quot numerus n continet unitates, qui propterea erunt radices æquationis inventæ. Cavendum ergo eft, ne valores æquales pro iifdem habeantur, quod fiet dum alternæ tantum expreffiones affumantur. Cognitis igitur radicibus æquationis a pofteriori, earum comparatio cum terminis æquationis notatu dignas præbebit proprietates. Quoniam autem ad hoc æquatio, in qua tantum x tanquam incognita infit, requiritur, pro y fuus valor $\sqrt{(1 - xx)}$ fubftitui debet; unde duplex operatio inftituenda erit, prout n fuerit vel numerus par vel impar.

236. Sit n numerus impar, quia Arcuum $- z, + z, + 3z, + 5z$; &c., differentia eft $2z$, hujufque Cofinus $= 1 - 2xx$, erit progreffionis Sinuum fcala relationis hæc $2 - 4xx, - 1$. Hinc erit

$$\textit{fin.} - z = - x$$
$$\textit{fin.}\ z = x$$
$$\textit{fin.}\ 3z = 3x - 4x^3$$
$$\textit{fin.}\ 5z = 5x - 20x^3 + 16x^5$$
$$\textit{fin.}\ 7z = 7x - 56x^3 + 112x^5 - 64x^7$$
$$\textit{fin.}\ 9z = 9x - 120x^3 + 432x^5 - 576x^7 + 256x^9$$

ergo

$$\textit{fin.}\ nz = nx - \frac{n(nn-1)}{1.2.3}x^3 + \frac{n(nn-1)(nn-9)}{1.2.3.4.5}x^5 -$$
$$\frac{n(nn-1)(nn-9)(nn-25)}{1.2.3.4.5.6.7}x^7 + \&c.,$$

fi quidem n fuerit numerus impar. Hujufque æquationis radices
funt

funt *fin.z*; *fin.* ($\frac{2\varpi}{n}$ + *z*); *fin.* ($\frac{4\varpi}{\nu}$ + *z*); *fin.* ($\frac{6\varpi}{n}$ + *z*);

fin.($\frac{8\varpi}{n}$ + *z*); &c., quarum numerus eft *n*.

237. Hujus ergo æquationis

$$0 = 1 - \frac{nx}{fin.nz} + \frac{n(nn-1)x^3}{1.\ 2.\ 3\ fin.\ nz} - \frac{n(nn-1)(nn-9)x^5}{1.\ 2.\ 3.\ 4.\ 5\ fin.\ nz} \cdots \pm$$

$$\frac{2^{n-1} x^n}{fin.\ nz}, \text{(ubi fignum fuperius valet fi } n \text{ unitate deficiat a}$$

multiplo quaternarii, contra inferius,) Factores funt $\left(1 - \frac{x}{fin.z}\right)$

$\left(1 - \dfrac{x}{fin.\left(\frac{2\varpi}{n}+z\right)}\right)\left(1 - \dfrac{x}{fin.\left(\frac{4\pi}{n}+z\right)}\right)$ &c., ex quibus con-

cluditur fore

$$\frac{n}{fin.nz} = \frac{1}{fin.z} + \frac{1}{fin.\left(\frac{2\varpi}{n}+z\right)} + \frac{1}{fin.\left(\frac{4\varpi}{n}+z\right)} + \frac{1}{fin.\left(\frac{6\varpi}{n}+z\right)} +$$

&c., donec habeantur *n* termini. Tum vero productum omnium

erit $\mp \dfrac{2^{n-1}}{fin.nz} = \dfrac{1}{fin.z.fin.\left(\frac{2\varpi}{n}+z\right).fin.\left(\frac{4\varpi}{n}+z\right).fin.\left(\frac{6\varpi}{n}+z\right)\&c.}$

feu *fin. nz* $= \mp 2^{n-1}$ *fin. z. fin.* ($\frac{2\varpi}{n}$ + *z*). *fin.* ($\frac{4\varpi}{n}$ + *z*)×

fin.($\frac{6\varpi}{n}$ + *z*) &c.. Et, quia terminus penultimus deeft, erit

$$0 = fin.z + fin.\left(\frac{2\varpi}{n}+z\right) + fin.\left(\frac{4\pi}{n}+z\right) + fin.\left(\frac{6\pi}{n}+z\right)\&c.$$

E X E M P L U M   I.

Si ergo fuerit *n* = 3, prodibunt hæ æqualitates
$0 = fin.z + fin.(120° + z) + fin.(240° + z) = fin.z + fin.(60 - z) - fin.(60 + z)$.

$$\frac{3}{fin.\ 3z}$$

$$\frac{3}{fin.3z} = \frac{1}{fin.z} + \frac{1}{fin.\,(\,120 + z\,)} + \frac{1}{fin.\,(\,240 + z\,)} = \frac{1}{fin.z} +$$

$$\frac{1}{fin.\,(\,60 - z\,)} - \frac{1}{fin.\,(\,60 + z\,)}$$

$$fin.\,3\,z = - 4\,fin.\,z.\,fin.\,(\,120 + z\,)\,fin.\,(\,240 + z\,) =$$

$$4\,fin.\,z.\,fin.\,(\,60 - z\,).\,fin.\,(\,60 + z\,).$$

Erit ergo, uti jam supra notavimus,

$$fin.\,(\,60 + z\,) = fin.\,z + fin.\,(\,60 - z\,),\,\&$$

$$3\,cofec.\,3z = cofec.\,z + cofec.\,(\,60 - z\,) - cofec.\,(\,60 + z\,).$$

E X E M P L U M   II.

Ponamus esse $n = 5$, atque prodibunt hæ æquationes:

$$0 = fin.\,z + fin.\,(\,\frac{2}{5}\,\pi + z\,) + fin.\,(\,\frac{4}{5}\,\pi + z\,) +$$

$$fin.\,(\,\frac{6}{5}\,\pi + z\,) + fin.\,(\,\frac{8}{5}\,\pi + z\,)$$

seu $0 = fin.\,z + fin.\,(\,\frac{2}{5}\,\pi + z\,) + fin.\,(\,\frac{1}{5}\,\pi - z\,) -$

$$fin.\,(\,\frac{1}{5}\,\pi + z\,) - fin.\,(\,\frac{2}{5}\,\pi - z\,)$$

seu $0 = fin.\,z + fin.\,(\,\frac{1}{5}\,\pi - z\,) - fin.\,(\,\frac{1}{5}\,\pi + z\,) +$

$$fin.\,(\,\frac{2}{5}\,\pi + z\,) - fin.\,(\,\frac{2}{5}\,\pi - z\,)$$

deinde erit

$$\frac{5}{fin.\,5z} = \frac{1}{fin.\,z} + \frac{1}{fin.\,(\,\frac{1}{5}\,\pi - z\,)} - \frac{1}{fin.\,(\,\frac{1}{5}\,\pi + z\,)} -$$

$$\frac{1}{fin.\,(\,\frac{2}{5}\,\pi - z\,)} + \frac{1}{fin.\,(\,\frac{2}{5}\,\pi + z\,)}$$

$$fin.\,5\,z = 16\,fin.\,z.\,fin.\,(\,\frac{1}{5}\,\pi - z\,).\,fin.\,(\,\frac{1}{5}\,\pi + z\,) \times$$

$$fin.\,(\,\frac{2}{5}\,\pi - z\,).\,fin.\,(\,\frac{2}{5}\,\pi + z\,)$$

EXEMPLUM III.

Hoc modo, fi ponamus $n = 2m + 1$, erit

$$0 = \text{fin. } z + \text{fin. } \left(\frac{\varpi}{n} - z\right) - \text{fin. } \left(\frac{\varpi}{n} + z\right) - \text{fin. } \left(\frac{2\varpi}{n} - z\right) +$$

$$\text{fin. } \left(\frac{2\varpi}{n} + z\right) + \text{fin. } \left(\frac{3\varpi}{n} - z\right) - \text{fin. } \left(\frac{3\varpi}{n} + z\right) - \cdots \pm$$

$$\text{fin. } \left(\frac{m}{n}\pi - z\right) \mp \text{fin. } \left(\frac{m}{n}\pi + z\right)$$

ubi figna fuperiora valent fi *m* fit numerus impar, inferiora fi fit par. Altera æquatio erit hæc.

$$\frac{n}{\text{fin. } nz} = \frac{1}{\text{fin. } z} + \frac{1}{\text{fin.}\left(\frac{\pi}{n} - z\right)} - \frac{1}{\text{fin. } \left(\frac{\pi}{n} + z\right)} -$$

$$\frac{1}{\text{fin. } \left(\frac{2\varpi}{n} - z\right)} + \frac{1}{\text{fin. } \left(\frac{2\varpi}{n} + z\right)} + \frac{1}{\text{fin. } \left(\frac{3\varpi}{n} - z\right)} -$$

$$\frac{1}{\text{fin. } \left(\frac{3\varpi}{n} + z\right)} - \cdots \pm \frac{1}{\text{fin. } \left(\frac{m\pi}{n} - z\right)} \mp \frac{1}{\text{fin. } \left(\frac{m\varpi}{n} + z\right)}$$

quæ ad Cofecantes commode transfertur. Tertio habetur hoc productum:

$$\text{fin. } nz = 2^{2m} \text{fin. } z . \text{fin. } \left(\frac{\pi}{n} - z\right). \text{fin.}\left(\frac{\varpi}{n} + z\right). \text{fin.}\left(\frac{2\varpi}{n} - z\right) \times$$

$$\text{fin. } \left(\frac{2\varpi}{n} + z\right). \text{fin.}\left(\frac{3\varpi}{n} - z\right). \text{fin. } \left(\frac{3\varpi}{n} + z\right). \cdots \cdots \times$$

$$\text{fin. } \left(\frac{m\pi}{n} - z\right). \text{fin. } \left(\frac{m\pi}{n} + z\right).$$

238. Sit *n* nunc numerus par, & quoniam eft $y = \sqrt{(1 - xx)}$ & *cof.* $2z = 1 - 2xx$, ita ut Seriei Sinuum fit fcala relationis, ut ante, $2 - 4xx, - 1$, erit

<div align="right">fin.</div>

fin. 0 $z = 0$

fin. 2 $z = 2\ x \surd (1 - xx)$

fin. 4 $z = (4x - 8\ x^3)\surd(1 - xx)$

fin. 6 $z = (6x - 32x^3 + 32x^5)\ \surd(1 - xx)$

fin. 8 $z = (8x - 80x^3 + 192x^5 - 128x^7)\ \surd(1 - xx)$

<div align="center">& generaliter</div>

$$\text{fin. } nz = (\ nx - \frac{n(nn - 4)}{1.\ 2.\ 3}\ x^3 + \frac{n(nn - 4)(nn - 16)}{1.\ 2.\ 3.\ 4.\ 5}\ x^5 -$$

$$\frac{n(nn - 4)(nn - 16)(nn - 36)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}\ x^7 + \dots \pm 2^{n-1}\ x^{n-1}\ )\surd(1 - xx)$$

denotante *n* numerum quemcunque parem.

239. Ad æquationem hanc rationalem efficiendam fumantur utrinque quadrata, ac prodibit hujufmodi æquatio

$$(\text{fin. } nz)^2 = nn\,xx + P\,x^4 + Q\,x^6 + \dots - 2^{2n-2}\ x^{2n}$$

feu $x^{2n} \dots \dots - \frac{nn}{2^{2n-2}}\ xx + \frac{1}{2^{2n-2}}\ (\text{fin. } nz)^2 = 0$

cujus æquationis radices erunt tam affirmativæ quam negativæ; Scilicet $+$ *fin.* z; $\pm$ *fin.* $(\frac{\pi}{n} - z)$; $\pm$ *fin.* $(\frac{2\varpi}{n} + z)$; $\pm$ *fin.* $(\frac{3\varpi}{n} - z)$; $\pm$ *fin.* $(\frac{4\varpi}{n} + z)$ &c. Sumendo omnino *n* hujufmodi expreffiones. Cum igitur ultimus terminus fit productum omnium harum radicum, extrahendo utrinque radicem quadratam erit

$$\text{fin. } nz = \pm\ 2^{n-1}\ \text{fin. } z.\text{fin. } (\frac{\varpi}{n} - z).\ \text{fin.} (\frac{2\varpi}{n} + z\ ) \times$$

fin. $(\frac{3\varpi}{n} - z\ ) \dots\dots$; ubi, quibus cafibus utrumvis fignum valeat, ex cafibus particularibus erit difpiciendum.

<div align="center">E X E M P L U M.</div>

Subftituendo autem pro *n* fucceffive numeros 2, 4, 6, &c. & eligendo *n* Sinus diverfos erit.

<div align="center">C c 2</div>

<div align="right">*fin.*</div>

$$\sin. 2z = 2 \sin. z . \sin. \left(\frac{\pi}{2} - z \right)$$

$$\sin. 4z = 8 \sin. z . \sin. \left(\frac{\pi}{4} - z \right).\sin. \left(\frac{\pi}{4} + z \right).\sin.\left(\frac{\pi}{2} - z \right)$$

$$\sin. 6z = 32 \sin. z . \sin. \left(\frac{\pi}{6} - z \right).\sin.\left(\frac{\pi}{6} + z \right).\sin. \left(\frac{2\pi}{6} - z \right) \times$$

$$\sin. \left(\frac{2\pi}{6} + z \right) . \sin. \left(\frac{3\pi}{6} - z \right)$$

$$\sin. 8z = 128 \sin. z . \sin.\left(\frac{\pi}{8} - z \right).\sin.\left(\frac{\pi}{8} + z \right).\sin. \left(\frac{2\pi}{8} - z \right) \times$$

$$\sin. \left(\frac{2\pi}{8} + z \right).\sin.\left(\frac{3\pi}{8} - z \right).\sin.\left(\frac{3\pi}{8} + z \right).\sin. \left(\frac{4\varpi}{8} - z \right)$$

240. Patet ergo fore generatim

$$\sin. nz = 2^{n-1} \sin. z . \sin.\left(\frac{\varpi}{n} - z \right).\sin.\left(\frac{\pi}{n} + z \right).\sin.\left(\frac{2\pi}{n} - z \right) \times$$

$$\sin. \left(\frac{2\pi}{n} + z \right).\sin. \left(\frac{3\pi}{n} - z \right).\sin.\left(\frac{3\pi}{n} + z \right)\dots. \sin. \left(\frac{1}{2}\pi - z \right)$$

si *n* fuerit numerus par. Quod si autem hæc cum superiori, ubi *n* erat numerus impar, comparetur tanta similitudo adesse deprehenditur, ut utramque in unam redigere liceat. Erit ergo, sive *n* fuerit numerus par sive impar,

$$\sin. nz = 2^{n-1} \sin. z . \sin. \left(\frac{\pi}{n} - z \right).\sin. \left(\frac{\varpi}{n} + z \right).\sin.\left(\frac{2\pi}{n} - z \right) \times$$

$$\sin. \left(\frac{2\varpi}{n} + z \right).\sin. \left(\frac{3\pi}{n} - z \right).\sin.\left(\frac{3\pi}{n} + z \right) \text{ \&c.}$$

donec tot habeantur Factores, quot numerus *n* continet unitates.

241. Expressiones istæ, quibus Sinus Angulorum multiplorum per Factores exponuntur, non parum utilitatis afferre possunt ad Logarithmos Sinuum Angulorum multiplorum inveniendos, itemque ad plures expressiones Sinuum per Factores, quales supra (§. 184) dedimus, reperiendas. Erit autem

$sin. \quad z = 1 \; sin. \; z$

$sin. \; 2 \; z = 2 \; sin. \; z. \; sin. \left(\dfrac{\pi}{2} - z \right)$

$sin. \; 3 \; z = 4 \; sin. \; z. \; sin. \left(\dfrac{\pi}{3} - z \right). sin. \left(\dfrac{\pi}{3} + z \right)$

$sin. \; 4 \; z = 8 \; sin. \; z. \; sin. \left(\dfrac{\pi}{4} - z \right). sin. \left(\dfrac{\pi}{4} + z \right). sin. \left(\dfrac{2\pi}{4} - z \right)$

$sin. \; 5 \; z = 16 \; sin. \; z. \; sin. \left(\dfrac{\pi}{5} - z \right). sin. \left(\dfrac{\pi}{5} + z \right). sin. \left(\dfrac{2\pi}{5} - z \right) \times$
$$sin. \left(\dfrac{2\pi}{5} + z \right)$$

$sin. \; 6 \; z = 32 \; sin. \; z. \; sin. \left(\dfrac{\pi}{6} - z \right). sin. \left(\dfrac{\pi}{6} + z \right). sin. \left(\dfrac{2\pi}{6} - z \right) \times$
$$sin. \left(\dfrac{2\pi}{6} + z \right). sin. \left(\dfrac{3\pi}{6} - z \right)$$
$$\&c.$$

242. Cum deinde sit $\dfrac{sin. \; 2 \, n z}{sin. \; n z} = 2 cos. \; nz$, Cosinus Angulorum multiplorum simili modo per Factores exprimentur.

$cos. \; z = 1 \; sin. \left(\dfrac{\varpi}{2} - z \right)$

$cos. \; 2z = 2 \; sin. \left(\dfrac{\varpi}{4} - z \right). sin. \left(\dfrac{\pi}{4} + z \right)$

$cos. \; 3z = 4 \; sin. \left(\dfrac{\varpi}{6} - z \right). sin. \left(\dfrac{\pi}{6} + z \right). sin. \left(\dfrac{3\pi}{6} - z \right)$

$cos. \; 4z = 8 \; sin. \left(\dfrac{\varpi}{8} - z \right). sin. \left(\dfrac{\pi}{8} + z \right). sin. \left(\dfrac{3\pi}{8} - z \right) \times$
$$sin. \left(\dfrac{3\varpi}{8} + z \right)$$

$cos. \; 5z = 16 \; sin. \left(\dfrac{\varpi}{10} - z \right). sin. \left(\dfrac{\varpi}{10} + z \right). sin. \left(\dfrac{3\pi}{10} - z \right) \times$
$$sin. \left(\dfrac{3\varpi}{10} + z \right). sin. \left(\dfrac{5\pi}{10} - z \right)$$

& generaliter

C c 3

cos.

$$cof. \; nz = 2^{n-1} \sin.(\frac{\pi}{2n} - z).\sin.(\frac{\pi}{2n} + z).\sin.(\frac{3\pi}{2n} - z) \times$$

$$\sin.(\frac{3\pi}{2n} + z).\sin.(\frac{5\pi}{2n} - z) \; \&c.,$$

quoad tot habeantur Factores quot numerus *n* continet uni-
tates.

243. Eædem expressiones prodibunt ex consideratione Co-
sinuum Arcuum multiplorum, si enim fuerit *cof. z = y*, erit
ut sequitur

$$cof. \; 0z = 1$$
$$cof. \; 1z = y$$
$$cof. \; 2z = 2yy - 1$$
$$cof. \; 3z = 4y^3 - 3y$$
$$cof. \; 4z = 8y^4 - 8yy + 1$$
$$cof. \; 5z = 16y^5 - 20y^3 + 5y$$
$$cof. \; 6z = 32y^6 - 48y^4 + 18yy - 1$$
$$cof. \; 7z = 64y^7 - 112y^5 + 56y^3 - 7y$$

& generaliter.

$$cof. \; nz = 2^{n-1} y^n . - \frac{n}{1} 2^{n-3} y^{n-2} + \frac{n(n-3)}{1.\;2}$$

$$2^{n-5} y^{n-4} - \frac{n(n-4)(n-5)}{1.\;2.\;3} 2^{n-7} y^{n-6} +$$

$$\frac{n(n-5)(n-6)(n-7)}{1.\;2.\;3.\;4} 2^{n-9} y^{n-8} - \&c.,$$

cujus æquationis, cum sit *cof. nz = cof. (2π - nz) =
cof. (2π + nz) = cof. (4π ± nz) = cof. (6π ± nz) &c.,*
erunt radices ipsius *y* hæ: $cof. \; z$; $cof.(\frac{2\pi}{n} \pm z)$; $cof.(\frac{4\pi}{n} \pm z)$;

$cof.(\frac{6\pi}{n} \pm z)$ &c., quarum formularum tot diversæ sunt pro
y eligendæ quot dantur; dantur autem tot, quot *n* continet
unitates.

244. Primum igitur patet, ob terminum secundum deficientem excepto casu $n = 1$, fore summam harum radicum omnium $= 0$. Erit ergo

$$0 = cof.z + cof.(\frac{2\varpi}{n} - z) + cof.(\frac{2\varpi}{n} + z) + cof.(\frac{4\varpi}{n} - z) +$$
$$cof.(\frac{4\varpi}{n} + z) + \&c.,$$

sumendo tot terminos quot n continet unitates: Hæc autem æqualitas sponte se offert si n sit numerus par, cum quivis terminus ab alio sui negativo destruatur. Contemplemur ergo numeros impares, unitate exclusa, eritque, ob $cof.v = -cof.(\pi - v)$

$$0 = cof.z - cof.(\frac{\pi}{3} - z) - cof.(\frac{\pi}{3} + z)$$

$$0 = cof.z - cof.(\frac{\pi}{5} - z) - cof.(\frac{\pi}{5} + z) + cof.(\frac{2\pi}{5} - z) +$$
$$cof.(\frac{2\pi}{5} + z)$$

$$0 = cof.z - cof.(\frac{\pi}{7} - z) - cof.(\frac{\pi}{7} + z) + cof.(\frac{2\pi}{7} - z) +$$
$$cof.(\frac{2\pi}{7} + z) - cof.(\frac{3\pi}{7} - z) - cof.(\frac{3\pi}{7} + z)$$

& generaliter, si fuerit n numerus impar quicunque, erit

$$0 = cof.z - cof.(\frac{\pi}{n} - z) - cof.(\frac{\pi}{n} + z) + cof.(\frac{2\pi}{n} - z) +$$
$$cof.(\frac{2\pi}{n} + z) - cof.(\frac{3\pi}{n} - z) - cof.(\frac{3\pi}{n} + z) +$$
$$cof.(\frac{4\pi}{n} - z) + cof.(\frac{4\pi}{n} + z) - \&c.,$$

sumendo tot terminos, quot numerus n continet unitates: oportet autem n esse numerum imparem unitate majorem, uti jam monuimus.

245. Quod ad productum ex omnibus attinet, variæ quidem prodeunt expressiones, prout n fuerit numerus vel impar, vel impariter par, vel pariter par : omnes autem comprehenduntur in expressione generali (§. 242.) inventa, si singuli Sinus in Cosinus transmutentur : Erit scilicet

$$\cos z = 1 \cos z$$

$$\cos 2z = 2 \cos.\left(\frac{\pi}{4}+z\right).\cos.\left(\frac{\pi}{4}-z\right)$$

$$\cos 3z = 4 \cos.\left(\frac{2\pi}{6}+z\right).\cos.\left(\frac{2\pi}{6}-z\right).\cos z$$

$$\cos 4z = 8 \cos.\left(\frac{3\pi}{8}+z\right).\cos.\left(\frac{3\pi}{8}-z\right).\cos.\left(\frac{\pi}{8}+z\right)\times$$
$$\cos.\left(\frac{\pi}{8}-z\right)$$

$$\cos 5z = 16 \cos.\left(\frac{4\pi}{8}+z\right).\cos.\left(\frac{4\pi}{8}-z\right).\cos.\left(\frac{2\pi}{8}+z\right)\times$$
$$\cos.\left(\frac{2\pi}{8}-z\right)\cos z$$

$$\text{\& generaliter}$$

$$\cos nz = 2^{n-1} \cos.\left(\frac{n-1}{n}\pi + z\right).\cos.\left(\frac{n-1}{n}\varpi - z\right)\times$$
$$\cos.\left(\frac{n-3}{n}\pi + z\right).\cos.\left(\frac{n-3}{n}\varpi - z\right)\times$$
$$\cos.\left(\frac{n-5}{n}\varpi + z\right).\cos.\left(\frac{n-5}{n}\varpi - z\right)\times$$
$$\cos.\left(\frac{n-7}{n}\varpi + z\right) \&c.,$$

sumtis tot Factoribus, quot numerus n continet unitates.

246. Sit n numerus impar, atque æquatio incipiatur ab unitate, erit

$$0 = 1 \mp \frac{ny}{\cos nz} + \&c.,$$ ubi signum superius valet si n fuerit numerus impar formæ $4m+1$, inferius si $n = 4m-1$. Hinc erit

$+$

$$+ \frac{1}{cof.\,z} = \frac{1}{cof.\,z}$$

$$- \frac{3}{cof.\,3z} = \frac{1}{cof.\,z} - \frac{1}{cof.\left(\frac{\pi}{3} - z\right)} - \frac{1}{cof.\left(\frac{\pi}{3} + z\right)}$$

$$+ \frac{5}{cof.\,5z} = \frac{1}{cof.\,z} - \frac{1}{cof.\left(\frac{\pi}{5} - z\right)} - \frac{1}{cof.\left(\frac{\pi}{5} + z\right)} +$$

$$\frac{1}{cof.\left(\frac{2\pi}{5} - z\right)} + \frac{1}{cof.\left(\frac{2\pi}{5} + z\right)}$$

& generaliter, pofito $n = 2\,m + 1$, erit

$$\frac{n}{cof.\,nz} = \frac{2\,m + 1}{cof.(2m+1)z} = \frac{1}{cof.\left(\frac{m}{n}\pi + z\right)} + \frac{1}{cof.\left(\frac{m}{n}\pi - z\right)} -$$

$$\frac{1}{cof.\left(\frac{m - 1}{n}\pi + z\right)} - \frac{1}{cof.\left(\frac{m - 1}{n}\pi - z\right)} +$$

$$\frac{1}{cof.\left(\frac{m - 2}{n}\pi + z\right)} + \frac{1}{cof.\left(\frac{m - 2}{n}\pi - z\right)} -$$

$$\frac{1}{cof.\left(\frac{m - 3}{n}\pi + z\right)} - \&c.$$

fumendis tot terminis, quot n continet unitates.

247. Cum ergo fit $\frac{1}{cof.\,v} = fec.\,v.$, hinc pro Secantibus infignes proprietates deducuntur, erit nempe

$$fec.\ z = fec.\,z.$$

$$3fec.\,3z = fec.\left(\frac{\pi}{3} + z\right) + fec.\left(\frac{\pi}{3} - z\right) - fec.\left(\frac{0\pi}{3} + z\right)$$

$$5fec.\,5z = fec.\left(\frac{2\pi}{5} + z\right) + fec.\left(\frac{2\pi}{5} - z\right) - fec.\left(\frac{\pi}{5} + z\right) -$$

$$fec.\left(\frac{\pi}{5} - z\right) + fec.\left(\frac{0\pi}{5} + z\right)$$

Euleri *Introduct. in Anal. infin. parv.* D d 7 *fec.*

$$7 \sec. 7z = \sec.\left(\frac{3\pi}{7}+z\right) + \sec.\left(\frac{3\pi}{7}-z\right) - \sec.\left(\frac{2\pi}{7}+z\right) -$$

$$\sec.\left(\frac{2\pi}{7}-z\right) + \sec.\left(\frac{\pi}{7}+z\right) + \sec.\left(\frac{\pi}{7}-z\right) -$$

$$\sec.\left(\frac{0\pi}{7}+z\right)$$

& generaliter, posito $n = 2m+1$, erit

$$n.\sec. nz = \sec.\left(\frac{m}{n}\pi+z\right) + \sec.\left(-\frac{m}{n}\pi-z\right) -$$

$$\sec.\left(\frac{m-1}{n}\pi+z\right) - \sec.\left(\frac{m-1}{n}\pi-z\right) +$$

$$\sec.\left(\frac{m-2}{n}\pi+z\right) + \sec.\left(\frac{m-2}{n}\pi-z\right) -$$

$$\sec.\left(\frac{m-3}{n}\pi+z\right) - \sec.\left(\frac{m-3}{n}\pi-z\right) +$$

$$\sec.\left(\frac{m-4}{n}\pi+z\right) + \ldots\ldots\ldots \pm \sec. z.$$

248. Pro Cosecantibus autem erit ex §. 237.

$$\text{cosec. } z = \text{cosec.} z$$

$$3\,\text{cosec. } 3z = \text{cosec.} z + \text{cosec.}\left(\frac{\pi}{3}-z\right) - \text{cosec.}\left(\frac{\pi}{3}+z\right)$$

$$5\,\text{cosec. } 5z = \text{cosec.} z + \text{cosec.}\left(\frac{\pi}{5}-z\right) - \text{cosec.}\left(\frac{\pi}{5}+z\right) -$$

$$\text{cosec.}\left(\frac{2\pi}{5}-z\right) + \text{cosec.}\left(\frac{2\pi}{5}+z\right)$$

$$7\,\text{cosec. } 7z = \text{cosec.} z + \text{cosec.}\left(\frac{\pi}{7}-z\right) - \text{cosec.}\left(\frac{\pi}{7}+z\right) -$$

$$\text{cosec.}\left(\frac{2\pi}{7}-z\right) + \text{cosec.}\left(\frac{2\pi}{7}+z\right) +$$

$$\text{cosec.}\left(\frac{3\pi}{7}-z\right) - \text{cosec.}\left(\frac{3\pi}{7}+z\right)$$

& generaliter, ponendo $n = 2m+1$, erit

$$n.\text{cosec.}$$

$$n.\text{cosec}.nz = \text{cosec}.z + \text{cosec}.(\tfrac{\pi}{n} - z) - \text{cosec}.(\tfrac{\pi}{n} + z) -$$

$$\text{cosec}.(\tfrac{2\pi}{n} - z) + \text{cosec}.(\tfrac{2\pi}{n} + z) +$$

$$\text{cosec}.(\tfrac{3\pi}{n} - z) - \text{cosec}.(\tfrac{3\pi}{n} + z) -$$

$$\ldots\ldots\ldots \mp \text{cosec}.(\tfrac{m\pi}{n} - z) \pm \text{cosec}.(\tfrac{m\pi}{n} + z)$$

ubi figna fuperiora valent fi *m* fuerit numerus par, inferiora fi *m* fit impar.

249. Cum fit, uti fupra vidimus, $\cos.nz \pm \sqrt{-1}.\sin.nz = (\cos.z \pm \sqrt{-1}.\sin.z)^n$, erit $\cos.nz =$

$$\frac{(\cos.z + \sqrt{-1}.\sin.z)^n + (\cos.z - \sqrt{-1}.\sin.z)^n}{2}, \&\sin.nz =$$

$$\frac{(\cos.z + \sqrt{-1}.\sin.z)^n - (\cos.z - \sqrt{-1}.\sin.z)^n}{2\sqrt{-1}}, \text{ erit}$$

$$\text{tang}.nz = \frac{(\cos.z + \sqrt{-1}.\sin.z)^n - (\cos.z - \sqrt{-1}.\sin.z)^n}{(\cos.z + \sqrt{-1}.\sin.z)^n \sqrt{-1} + (\cos.z - \sqrt{-1}.\sin.z)^n \sqrt{-1}}.$$

Ponamus $\text{tang}.z = \dfrac{\sin.z}{\cos.z} = t$, erit $\text{tang}.nz =$

$$\frac{(1 + t\sqrt{-1})^n - (1 - t\sqrt{-1})^n}{(1 + t\sqrt{-1})^n \sqrt{-1} + (1 - t\sqrt{-1})^n \sqrt{-1}}, \text{ unde oriun-}$$

tur Tangentes Angulorum multiplorum fequentes

$$\text{tang}. \; z = t$$

$$\text{tang}. \, 2z = \frac{2t}{1 - tt}$$

$$\text{tang}. \, 3z = \frac{3t - t^3}{1 - 3tt}$$

$$\text{tang}. \, 4z = \frac{4t - 4t^3}{1 - 6tt + t^4}$$

$$\text{tang}. \, 5z = \frac{5t - 10t^3 + t^5}{1 - 10tt + 5t^4}$$

$$\text{tang.} nz = \frac{nt - \dfrac{n(n-1)(n-2)}{1.\ \ 2.\ \ 3} t^3 + \dfrac{n(n-1)(n-2)(n-3)(n-4)}{1.\ \ 2.\ \ 3\ \ 4\ \ 5} t^5 - \&c.}{1 - \dfrac{n(n-1)}{1.\ \ 2} tt + \dfrac{n(n-1)(n-2)(n-3)}{1.\ \ 2.\ \ 3.\ \ 4} t^4 - \&c.}.$$

Cum jam fit *tang. nz = tang.* ($\varpi + nz$) $=$ *tang.*$(2\pi + nz) =$ *tang.* ($3\varpi + nz$) &c., erunt valores ipfius *t*, feu radices æquationis, hæ, *tang.z* ; *tang.*$(\frac{\pi}{n} + z)$; *tang.*$(\frac{2\pi}{n} + z)$; *tang.*$(\frac{3\pi}{n} + z)$; &c., quarum numerus eft *n*.

250.　　Quod fi æquatio ab unitate incipiat, erit

$$0 = 1 - \frac{nt}{\text{tang. } nz} - \frac{n(n-1)tt}{1.\ \ 2} + \frac{n(n-1)(n-2)t^3}{1.\ \ 2.\ \ 3\ \ \text{tang. } nz} + \&c..$$

Ex comparatione ergo coëfficientium cum radicibus, erit

$$n. \cot. nz = \cot. z + \cot.(\frac{\pi}{n} + z) + \cot.(\frac{2\pi}{n} + z) + \cot.(\frac{3\pi}{n} + z) +$$
$$\cot.(\frac{4\varpi}{n} + z) + \ldots\ldots + \cot.(\frac{n-1}{n}\pi + z)$$

deinde erit fumma quadratorum harum Cotangentium omnium $= \dfrac{nn}{(\textit{fin. } nz)^2} - n$, fimilique modo ulteriores Poteftates poffunt definiri. Ponendo autem loco *n* numeros definitos, erit.

$$\cot.\ z = \cot. z$$
$$2\cot. 2z = \cot. z + \cot.(\frac{\pi}{2} + z)$$
$$3\cot. 3z = \cot. z + \cot.(\frac{\pi}{3} + z) + \cot.(\frac{2\pi}{3} + z)$$
$$4\cot. 4z = \cot. z + \cot.(\frac{\pi}{4} + z) + \cot.(\frac{2\pi}{4} + z) +$$
$$\cot.(\frac{3\pi}{4} + z)$$

$$5 cot. 5z = cot.z + cot.\left(\frac{\pi}{5}+z\right) + cot.\left(\frac{2\varpi}{5}+z\right) +$$
$$cot.\left(\frac{3\pi}{5}+z\right) + cot.\left(\frac{4\pi}{5}+z\right).$$

251. Quia vero eft *cot. v = — cot. (π — v)*, erit

$$cot.\ z = cot.z$$

$$2 cot.\ 2z = cot.z - cot.\left(\frac{\pi}{2}-z\right)$$

$$3 cot.\ 3z = cot.z - cot.\left(\frac{\pi}{3}-z\right) + cot.\left(\frac{\varpi}{3}+z\right)$$

$$4 cot.\ 4z = cot.z - cot.\left(\frac{\pi}{4}-z\right) + cot.\left(\frac{\pi}{4}+z\right) -$$
$$cot.\left(\frac{2\pi}{4}-z\right)$$

$$5 cot.\ 5z = cot.z - cot.\left(\frac{\pi}{5}-z\right) + cot.\left(\frac{\pi}{5}+z\right) -$$
$$cot.\left(\frac{2\pi}{5}-z\right) + cot.\left(\frac{2\pi}{5}+z\right)$$

& generaliter

$$n.cot.\ nz = cot.z - cot.\left(\frac{\pi}{n}-z\right) + cot.\left(\frac{\pi}{n}+z\right) -$$
$$cot.\left(\frac{2\varpi}{n}-z\right) + cot.\left(\frac{2\pi}{n}+z\right) -$$
$$cot.\left(\frac{3\varpi}{n}-z\right) + cot.\left(\frac{3\pi}{n}+z\right) -$$
&c.

donec tot habeantur termini , quot numerus *n* continet unitates.

252. Incipiamus æquationem inventam a Poteftate fumma , ubi primum diftingendi funt cafus , quibus *n* eft vel numerus par , vel impar. Sit *n* numerus impar , *n = 2 m + 1* erit

D d 3   *t —*

LIB. I.

$t \; - \; tang. \, z \; = \; 0$

$t^3 \; - \; 3tt . tang. \, 3z \; - \; 3t \; + \; tang. \, 3z \; = \; 0$

$t^5 \; - \; 5t^4 . tang. \, 5z \; - \; 10t^3 \; + \; 10tt . tang. \, 5z + 5t - tang. \, 5z = 0$

& generaliter

$t^n \; - \; nt^{n-1} \, tang. \, nz \; - \; \ldots\ldots\ldots\ldots \; \mp \, tang. \, nz \; = \; 0$

ubi fignum fuperius — valet, fi *m* fit numerus par, inferius + fi *m* fit numerus impar. Erit ergo ex coëfficiente fecundi termini

$tang. \; z \; = \; tang. \, z$

$3 tang. \, 3z \; = \; tang. \, z \; + \; tang. \, \left(\frac{\pi}{3} + z \right) + tang. \, \left(\frac{2\pi}{3} + z \right)$

$5 tang. \, 5z \; = \; tang. \, z \; + \; tang. \, \left(\frac{\varpi}{5} + z \right) + tang. \, \left(\frac{2\varpi}{5} + z \right) +$

$\qquad\qquad tang. \, \left(\frac{3\varpi}{5} + z \right) + tang. \, \left(\frac{4\varpi}{5} + z \right).$

&c.

253. Cum igitur fit *tang.* $v = - tang. \, (\pi - v)$, Anguli recto majores ad Angulos recto minores reducuntur, eritque

$tang. \; z \; = \; tang. \, z$

$3 tang. \, 3z \; = \; tang. \, z \; - \; tang. \, \left(\frac{\varpi}{3} - z \right) + tang. \, \left(\frac{\pi}{3} + z \right)$

$5 tang. \, 5z \; = \; tang. \, z \; - \; tang. \, \left(\frac{\varpi}{5} - z \right) + tang. \, \left(\frac{\varpi}{5} + z \right) -$

$\qquad\qquad tang. \, \left(\frac{2\pi}{5} - z \right) + tang. \, \left(\frac{2\pi}{5} + z \right)$

$7 tang. \, 7z \; = \; tang. \, z \; - \; tang. \, \left(\frac{\pi}{7} - z \right) + tang. \, \left(\frac{\pi}{7} + z \right) -$

$\qquad\qquad tang. \, \left(\frac{2\pi}{7} - z \right) + tang. \, \left(\frac{2\pi}{7} + z \right) -$

$\qquad\qquad tang. \, \left(\frac{3\pi}{7} - z \right) + tang. \, \left(\frac{3\varpi}{7} + z \right)$

& gene-

& generaliter, fi $n = 2m + 1$, erit

$$n.tang.nz = tang.z - tang.(\frac{\pi}{n} - z) + tang.(\frac{\pi}{n} + z) - \cdots$$
$$tang.(\frac{2\pi}{n} - z) + tang.(\frac{2\pi}{n} + z) - \cdots$$
$$tang.(\frac{3\pi}{n} - z) + \cdots\cdots\cdots\cdots - \cdots$$
$$tang.(\frac{m\pi}{n} - z) + tang.(\frac{m\pi}{n} + z).$$

254. Tum vero productum ex his Tangentibus omnibus erit $= tang.nz$, propterea quod per fignorum negativorum numerum alternatim parem & imparem, fuperior fignorum ambiguitas tollitur. Sic erit

$$tang.\ z = tang.z$$
$$tang.3z = tang.z.\,tang.(\frac{\pi}{3} - z).\,tang.(\frac{\pi}{3} + z)$$
$$tang.5z = tang.z.\,tang.(\frac{\pi}{5} - z).\,tang.(\frac{\pi}{5} + z).\,tang.(\frac{2\pi}{5} - z)\times$$
$$tang.(\frac{2\pi}{5} + z)$$

& generaliter, fi $n = 2m + 1$, erit

$$tang.nz = tang.z.\,tang.(\frac{\pi}{n} - z).\,tang.(\frac{\pi}{n} + z).\,tang.(\frac{2\pi}{n} - z)\times$$
$$tang.(\frac{2\pi}{n} + z).\,tang.(\frac{3\pi}{n} - z)\cdots\cdots\times$$
$$tang.(\frac{m\pi}{n} - z).\,tang.(\frac{m\pi}{n} + z).$$

255. Sit jam n numerus par, atque, incipiendo a Poteftate fumma, erit

$$tt + 2t.\,cot.2z - 1 = 0$$
$$t^4 + 4t^3.cot.4z - 6tt - 4t.cot.4z + 1 = 0$$

&

      & generaliter, fi $n = 2m$, erit

$$t^{n} + nt^{n-1} \, cot. \, nz - \ldots \ldots \ldots \ldots \ldots \ldots \mp 1 = 0$$

ubi fignum fuperius — valet fi m fit numerus impar, inferius + fi m fit par. Comparando ergo radices cum coëfficiente fecundi termini, erit

$$- 2 \, cot. \, 2z = tang. \, z + tang. \left(\frac{\pi}{2} + z \right)$$

$$- 4 \, cot. \, 4z = tang. \, z + tang. \left(\frac{\pi}{4} + z \right) + tang. \left(\frac{2\pi}{4} + z \right) +$$
$$tang. \left(\frac{3\pi}{4} + z \right)$$

$$- 6 \, cot. \, 6z = tang. \, z + tang. \left(\frac{\pi}{6} + z \right) + tang. \left(\frac{2\pi}{6} + z \right) +$$
$$tang. \left(\frac{3\pi}{6} + z \right) + tang. \left(\frac{4\pi}{6} + z \right) +$$
$$tang. \left(\frac{5\pi}{6} + z \right).$$
$$\&c.$$

256. Cum fit $tang. \, v = - tang. \, (\pi - v)$, fequentes formabuntur æquationes

$$2 \, cot. \, 2z = - tang. \, z + tang. \left(\frac{\pi}{2} - z \right)$$

$$4 \, cot. \, 4z = - tang. \, z + tang. \left(\frac{\pi}{4} - z \right) - tang. \left(\frac{\pi}{4} + z \right) +$$
$$tang. \left(\frac{2\pi}{4} - z \right)$$

$$6 \, cot. \, 6z = - tang. \, z + tang. \left(\frac{\pi}{6} - z \right) - tang. \left(\frac{\pi}{6} + z \right) +$$
$$tang. \left(\frac{2\pi}{6} - z \right) - tang. \left(\frac{2\pi}{6} + z \right) +$$
$$tang. \left(\frac{3\pi}{6} - z \right)$$

&

& generaliter, fi $n = 2m$, erit

$$n.cot.\, nz = -\, tang.z + tang.\left(\frac{\varpi}{n} - z\right) - tang.\left(\frac{\varpi}{n} + z\right) +$$

$$tang.\left(\frac{2\varpi}{n} - z\right) - tang.\left(\frac{2\varpi}{n} + z\right) +$$

$$tang.\left(\frac{3\varpi}{n} - z\right) - tang.\left(\frac{3\varpi}{n} + z\right) +$$

$$\cdots\cdots\cdots\cdots + tang.\left(\frac{m\varpi}{n} - z\right).$$

257. Per has formas iterum ambiguitas producti ex omnibus radicibus deftruitur; eritque idcirco

$$1 = tang.z.\, tang.\left(\frac{\varpi}{2} - z\right)$$

$$1 = tang.z.\, tang.\left(\frac{\varpi}{4} - z\right).tang.\left(\frac{\varpi}{4} + z\right).tang.\left(\frac{2\varpi}{4} - z\right)$$

$$1 = tang.z.\, tang.\left(\frac{\varpi}{6} - z\right).tang.\left(\frac{\varpi}{6} + z\right).tang.\left(\frac{2\varpi}{6} - z\right) \times$$

$$tang.\left(\frac{2\varpi}{6} + z\right).tang.\left(\frac{3\varpi}{6} - z\right).$$

&c.

Harum vero æquationum ratio ftatim fponte in oculos incurrit, cum perpetuo bini Anguli reperiantur, quorum alter eft alterius complementum ad rectum. Hujufmodi ergo binorum Angulorum Tangentes productum dant $= 1$; ideoque omnium productum unitati debet effe æquale.

258. Quoniam Sinus & Cofinus Angulorum progreffionem arithmeticam conftituentium Seriem recurrentem præbent, per Caput præcedens fumma hujufmodi Sinuum & Cofinuum quotcunque exhiberi poterit. Sint Anguli in arithmetica progreffione

$$a,\; a + b,\; a + 2b,\; a + 3b,\; a + 4b,\; a + 5b, \;\&c.$$

& quæratur primo fumma Sinuum horum Angulorum in infinitum progredientium; ponatur ergo

$s = \text{fin}.\, a + \text{fin}.\,(a+b) + \text{fin}.\,(a+2b) + \text{fin}.\,(a+3b) + \&c.$

& quia hæc Series est recurrens, cujus scala relationis est $2 \cos.b, -1$, orietur hæc Series ex evolutione fractionis, cujus denominator est $1 - 2z.\cos.b + zz$, posito $z = 1$. Ipsa vero fractio erit $= \dfrac{\text{fin}.\,a + z\,(\text{fin}.\,(a+b) - 2\,\text{fin}.\,a.\cos.b)}{1 - 2z.\cos.b + zz}$, quare,

facto $z = 1$, erit $s = \dfrac{\text{fin}.\,a + \text{fin}.\,(a+b) - 2\,\text{fin}.\,a.\cos.b}{2 - 2\cos.b} =$

$\dfrac{\text{fin}.a - \text{fin}.(a-b)}{2\,(1 - \cos.b)}$, ob $2\text{fin}.a.\cos.b = \text{fin}.(a+b) + \text{fin}.(a-b)$.

Cum autem sit $\text{fin}.f - \text{fin}.g = 2\cos.\dfrac{f+g}{2}.\,\text{fin}.\dfrac{f-g}{2}$, erit

$\text{fin}.a - \text{fin}.(a-b) = 2\cos.(a - \dfrac{1}{2}b).\text{fin}.\dfrac{1}{2}b$: & $1 - \cos.b =$

$2\,(\text{fin}.\dfrac{1}{2}b)^2$, unde erit $s = \dfrac{\cos.(a - \frac{1}{2}b)}{2\,\text{fin}.\frac{1}{2}b}$.

259. Hinc itaque summa quotcunque Sinuum, quorum Arcus in arithmetica progressione incedunt, assignari poterit; quæratur nempe summa hujus progressionis

$\text{fin}.\,a + \text{fin}.(a+b) + \text{fin}.\,(a+2b) + \text{fin}.(a+3b) + \ldots\ldots + \text{fin}.(a+nb)$.

Quia summa hujus progressionis in infinitum continuatæ est $\dfrac{\cos.(a - \frac{1}{2}b)}{2\,\text{fin}.\frac{1}{2}b}$, considerentur termini ultimum sequentes in infinitum hi

$\text{fin}.(a+(n+1)b) + \text{fin}.\,(a + (n+2)b) + \text{fin}.\,(a+(n+3)b) + \&c.$

quia horum Sinuum summa est $= \dfrac{\cos.(a+(n+\frac{1}{2})b)}{2\,\text{fin}.\frac{1}{2}b}$; si hæc a priori subtrahatur, remanebit summa quæsita. Scilicet, si fuerit $s = \text{fin}.a + \text{fin}.(a+b) + \text{fin}.\,(a+2b) + \ldots\ldots + \text{fin}.(a+nb)$,

erit $s = \dfrac{\cos.\,(a - \frac{1}{2}b) - \cos.\,(a + (n + \frac{1}{2})b)}{2\,\text{fin}.\frac{1}{2}b} =$

$\dfrac{\text{fin}.\,(a + \frac{1}{2}nb)\,\text{fin}.\frac{1}{2}(n+1)b}{\text{fin}.\frac{1}{2}b}$

260. Pari

C A P.
X I V.

260. Pari modo, si consideretur summa Cosinuum, atque ponatur

$$s = cos.\,a + cos.\,(a+b) + cos.\,(a+2b) + cos.\,(a+3b) + \&c.$$

in infinitum, erit $s = \dfrac{cos.\,a + z(\,cos.\,(a+b) - 2cos.\,a.cos.\,b)}{1 - 2z.\,cos.\,b + zz}$, posito

$z = 1$. Quare, ob $2cos.\,a.cos.\,b = cos.\,(a-b) + cos.\,(a+b)$, fiet

$s = \dfrac{cos.\,a - cos.\,(a-b)}{2(1 - cos.\,b)}$. At est $cos.\,f - cos.\,g = 2\,sin.\,\dfrac{f+g}{2} \times$

$sin.\,\dfrac{g-f}{2}$; unde erit $cos.\,a - cos.\,(a-b) = -2\,sin.\,(a - \frac{1}{2}b)\times$

$sin.\,\frac{1}{2}b$, & ob $1 - cos.\,b = 2\,(sin.\,\frac{1}{2}b)^2$, erit $s = -$

$\dfrac{sin.\,(a - \frac{1}{2}b)}{2\,sin.\,\frac{1}{2}b}$. Quare, cum simili modo sit hujus Seriei

$$cos.\,(a+(n+1)b) + cos.\,(a+(n+2)b) + cos.\,(a+(n+3)b) + \&c.$$

summa sit $= -\dfrac{sin.\,(a+(n+\frac{1}{2})b)}{2\,sin.\,\frac{1}{2}b}$, si hæc ab illa subtrahatur, relinquetur summa hujus Seriei

$$s = cos.\,a + cos.\,(a+b) + cos.\,(a+2b) + cos.\,(a+3b) + \ldots + cos.\,(a+nb):$$

eritque $s = -\dfrac{sin.\,(a - \frac{1}{2}b) + sin.\,(a+(n+\frac{1}{2})b)}{2\,sin.\,\frac{1}{2}b} =$

$\dfrac{cos.\,(a+\frac{1}{2}nb).sin.\,\frac{1}{2}(n+1)b}{sin.\,\frac{1}{2}b}$.

261. Plurimæ aliæ quæstiones circa Sinus & Tangentes ex principiis allatis resolvi possent ; cujusmodi sunt, si quadrata, altioresve Potestates Sinuum, Tangentiumve summari deberent, verum quia hæc ex reliquis æquationum superiorum coefficientibus similiter derivantur, iis hic diutius non immoror. Quod autem ad has postremas summationes attinet, notandum est quamcunque Sinuum Cosinuumque Potestatem per singulos Sinus Cosinusve explicari posse, quod, ut clarius perspiciatur, breviter exponamus.

E e 2   262. Ad

262. Ad hoc expediendum juvabit ex præcedentibus hæc Lemmata depromſiſſe

$$2 \sin.a.\sin.z = \cos.(a-z) - \cos.(a+z)$$
$$2\cos.a.\sin.z = \sin.(a+z) - \sin.(a-z)$$
$$2\sin.a\cos.z = \sin.(a+z) + \sin.(a-z)$$
$$2\cos.a.\cos.z = \cos.(a-z) + \cos.(a+z)$$

Hinc igitur primum Poteſtates Sinuum reperiuntur

$$\sin.z = \sin.z$$
$$2(\sin.z)^2 = 1 - \cos.2z$$
$$4(\sin.z)^3 = 3\sin.z - \sin.3z$$
$$8(\sin.z)^4 = 3 - 4\cos.2z + \cos.4z$$
$$16(\sin.z)^5 = 10\sin.z - 5\sin.3z + \sin.5z$$
$$32(\sin.z)^6 = 10 - 15\cos.2z + 6\cos.4z - \cos.6z$$
$$64(\sin.z)^7 = 35\sin.z - 21\sin.3z + 7\sin.5z - \sin.7z$$
$$128(\sin.z)^8 = 35 - 56\cos.2z + 28\cos.4z - 8\cos.6z + \cos.8z$$
$$256(\sin.z)^9 = 126\sin.z - 84\sin.3z + 36\sin.5z - 9\sin.7z + \sin.9z$$
$$\&c.$$

Lex, qua hi coëfficientes progrediuntur, ex unciis Binomii elevati intelligitur, niſi quod numerus abſolutus in Poteſtatibus paribus ſemiſſis tantum ſit ejus, quem unciæ præbent.

263. Pari modo Poteſtates Coſinuum definientur

$$\cos.z = \cos.z$$
$$2(\cos.z)^2 = 1 + \cos.2z$$
$$4(\cos.z)^3 = 3\cos.z + \cos.3z$$
$$8(\cos.z)^4 = 3 + 4\cos.2z + \cos.4z$$
$$16(\cos.z)^5 = 10\cos.z + 5\cos.3z + \cos.5z$$
$$32(\cos.z)^6 = 10 + 15\cos.2z + 6\cos.4z + \cos.6z$$
$$64(\cos.z)^7 = 35\cos.z + 21\cos.3z + 7\cos.5z + \cos.7z$$
$$\&c.$$

Hic ratione legis progreſſionis eadem ſunt monenda quæ circa Sinus notavimus.

CAPUT

C A P U T   X V.

De Seriebus ex evolutione Factorum ortis.

264. SIt propositum productum ex Factoribus, numero five finitis five infinitis, conftans hujufmodi

$$(1 + \alpha z)(1 + 6z)(1 + \gamma z)(1 + \delta z)(1 + \varepsilon z)(1 + \xi z) \&c.,$$

quod, fi per multiplicationem actualem evolvatur, det

$$1 + A z + B z^2 + C z^3 + D z^4 + E z^5 + F z^6 + \&c.,$$

atque manifeftum eft coëfficientes A, B, C, D, E, &c., ita formari ex numeris α, 6, γ, δ, ε, ξ, &c., ut fit

$A = \alpha + 6 + \gamma + \delta + \varepsilon + \xi + $ &c. $=$ fummæ fingulorum
$B = $ fummæ Factorum ex binis diverfis
$C = $ fummæ Factorum ex ternis diverfis
$D = $ fummæ Factorum ex quaternis diverfis
$E = $ fummæ Factorum ex quinis diverfis
&c.

donec perveniatur ad productum ex omnibus.

265. Quod fi ergo ponatur $z = 1$, productum hoc

$$(1 + \alpha)(1 + 6)(1 + \gamma)(1 + \delta)(1 + \varepsilon) \&c.$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, 6, γ, δ, ε, &c., vel fumendis fingulis, vel duobus pluri-bufve diverfis in fe multiplicandis, nafcuntur. Atque fi idem numerus duobus pluribufve modis refultare queat, etiam idem bis pluriefve in hac numerorum Serie occurret.

266. Si ponatur $z = - 1$, productum hoc

E e 3 $(1 -$

$(1 — \alpha)(1 — \zeta)(1 — \gamma)(1 — \delta)(1 — \varepsilon)$ &c.

æquabitur unitati cum Serie numerorum omnium, qui ex his α, ζ, γ, δ, ε, ξ, &c. vel fumendis fingulis, vel duobus pluribufve diverfis in fe multiplicandis, nafcuntur; ut ante quidem, verum hoc difcrimine, ut ii numeri, qui vel ex fingulis, vel ternis, vel quinis, vel numero imparibus nafcuntur, fint negativi, illi vero, qui vel ex binis, vel quaternis, vel fenis vel numero paribus refultant, fint affirmativi.

267. Scribantur pro α, ζ, γ, δ, &c., numeri primi omnes 2, 3, 5, 7, 11, 13, &c., atque hoc productum

$$(1 + 2)(1 + 3)(1 + 5)(1 + 7)(1 + 11)(1 + 13) \text{ \&c. } = P$$

æquabitur unitati, cum Serie omnium numerorum vel primorum ipforum, vel ex primis diverfis per multiplicationem ortorum.　Erit ergo

$$P = 1 + 2 + 3 + 5 + 6 + 7 + 10 + 11 + 13 + 14 + 15 + 17 + \text{ \&c. },$$

in qua Serie omnes occurrunt numeri naturales, exceptis Poteftatibus, iifque qui per quamvis Poteftatem funt divifibiles. Defunt fcilicet numeri 4, 8, 9, 12, 16, 18 &c., quoniam funt vel Poteftates, ut 4, 8, 9, 16, &c., vel per Poteftates divifibiles ut 12, 18, &c.

268. Simili modo res fe habebit, fi pro α, ζ, γ, δ, &c.. Poteftates quæcunque numerorum primorum fubftituantur. Scilicet fi ponamus

$$P = (1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \text{ \&c. },$$

erit enim multiplicatione inftituta :

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \text{ \&c.}$$

in

in quibus fractionibus omnes occurrunt numeri praeter illos qui vel ipſi ſunt Poteſtates, vel per Poteſtatem quampiam diviſibiles. Cum enim omnes numeri integri ſint vel primi vel ex primis per multiplicationem compoſiti, hic ii tantum numeri excludentur, in quorum formationem idem numerus primus bis vel pluries ingreditur.

269. Si numeri $\alpha, \beta, \gamma, \delta$, &c., negative capiantur, ut ante (266.) fecimus, atque ponatur

$$P = \left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right)\left(1 - \frac{1}{5^n}\right)\left(1 - \frac{1}{7^n}\right)\left(1 - \frac{1}{11^n}\right)$$

&c., erit

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} - \text{&c.},$$

ubi iterum, ut ante, omnes occurrunt numeri praeter Poteſtates ac diviſibiles per Poteſtates. Verum ipſi numeri primi, & qui ex ternis, quinis, numerove imparibus conſtant, ſignum habent praefixum —, qui autem ex binis, vel quaternis, vel ſenis, vel numero paribus formantur, ſignum habent +. Sic in hac Serie occurret terminus $\frac{1}{30^n}$, quia eſt $30 = 2 \cdot 3 \cdot 5$, neque adeo Poteſtatem complectitur, habebit vero hic terminus $\frac{1}{30^n}$ ſignum —, quia 30 eſt productum ex tribus numeris primis.

270. Conſideremus jam hanc expreſſionem

$$\frac{1}{(1 - \alpha z)(1 - \beta z)(1 - \gamma z)(1 - \delta z)(1 - \varepsilon z)} \text{ &c.}$$

quae per diviſionem actualem evoluta praebeat hanc Seriem:

1 +

$$1 + Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 + Fz^6 + \&c.$$

atque manifeftum eft coëfficientes A, B, C, D, E, &c., fequenti modo ex numeris α, ς, γ, δ, ε, &c., componi, ut fit

$A =$ fummæ fingulorum
$B =$ fummæ Factorum ex binis ⎫
$C =$ fummæ Factorum ex ternis ⎬ non exclufis Factori-
$D =$ fummæ Factorum ex quaternis ⎭ bus iifdem.
        &c.

271. Pofito ergo $z = 1$, ifta expreffio

$$\frac{1}{(1-\alpha)(1-\varsigma)(1-\gamma)(1-\delta)(1-\varepsilon) \&c.}$$

æquabitur unitati cum Serie numerorum omnium, qui ex his α, ς, γ, δ, ε, ξ, &c., vel fumendis fingulis, vel duobus pluribufve in fe multiplicandis, oriuntur, non exclufis æqualibus. Hoc ergo differt ifta numerorum Series ab illa, quæ (§.265.) prodiit, quod ibi Factores tantum diverfi fumi debebant, hic autem idem Factor. bis pluriefve occurrere poffit. Hic fcilicet omnes numeri occurrunt, qui per multiplicationem ex his α, ς, γ, δ, &c., provenire poffunt.

272. Hanc ob rem Series femper ex terminorum numero infinito conftat, five Factorum numerus fuerit infinitus, five finitus. Sic erit

$$\frac{1}{1 - \frac{1}{2}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \&c.,$$

ubi omnes numeri adfunt, qui ex binario folo per multiplicationem oriuntur; feu omnes binarii Poteftates. Deinde erit

$$\frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} +$$
$$\frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \frac{1}{16} + \frac{1}{18} + \&c.,$$

ubi alii numeri non occurrunt, nisi qui ex his duobus 2 & 3
per multiplicationem originem trahunt; seu qui alios Divisores
præter 2 & 3 non habent.

273. Si igitur pro α, ϵ, γ, δ, &c., unitas per singulos om-
nes numeros primos scribatur, ac ponatur

$$P = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 - \frac{1}{13})\ \&c.,}$$

fiet

$$P = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi omnes numeri tam primi, quam qui ex primis per multi-
plicationem nascuntur, occurrunt. Cum autem omnes numeri
vel sint ipsi primi, vel ex primis per multiplicationem oriundi,
manifestum est, hic omnes omnino numeros integros in deno-
minatoribus adesse debere.

274. Idem evenit, si numerorum primorum Potestates
quæcunque accipiantur : si enim ponatur

$$P = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n})\&c.,}$$

fiet

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

ubi omnes numeri naturales nullo excepto occurrunt. Quod
si autem in Factoribus ubique signum $+$ statuatur, ut sit

$$P = \frac{1}{(1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n})\ \&c.,}$$

erit

Euleri *Introduct. in Anal. infin. parv.*   F f $P =$

$$P = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c.,$$

ubi numeri primi habent fignum —; qui funt producti ex duobus primis, five iifdem five diverfis, fignum habent +; & generatim, quorum numerorum numerus Factorum primorum eft par, fignum habent +, qui autem ex Factoribus primis numero imparibus conftant, habent fignum —. Sic terminus $\frac{1}{240^n}$, ob 240 = 2. 2. 2. 2. 3. 5, habebit fignum +, cujus legis ratio percipitur ex §. 270, fi ponatur $z = -1$.

275. Si hæc cum fuperioribus conferantur, nafcentur binæ Series quarum productum unitati æquatur. Sit enim

$$P = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.,}$$

&

$$Q = (1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.$$

erit

$$P = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \&c.,$$

$$Q = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} \&c.$$

(269), atque manifeftum eft fore $PQ = 1$.

275.. Sin autem ponatur

$$P = \frac{1}{(1 + \frac{1}{2^n})(1 + \frac{1}{3^n})(1 + \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n}) \&c.}$$

&

$$Q =$$

$$Q = (1 + \tfrac{1}{2^n})(1 + \tfrac{1}{3^n})(1 + \tfrac{1}{5^n})(1 + \tfrac{1}{7^n})(1 + \tfrac{1}{11^n}) \ \&c.,$$

erit

$$P = 1 - \tfrac{1}{2^n} - \tfrac{1}{3^n} + \tfrac{1}{4^n} - \tfrac{1}{5^n} + \tfrac{1}{6^n} - \tfrac{1}{7^n} - \tfrac{1}{8^n} +$$

$$\tfrac{1}{9^n} + \&c.$$

$$Q = 1 + \tfrac{1}{2^n} + \tfrac{1}{3^n} + \tfrac{1}{5^n} + \tfrac{1}{6^n} + \tfrac{1}{7^n} + \tfrac{1}{10^n} + \tfrac{1}{11^n} +$$

$$\&c.$$

fimilique modo habebitur $PQ = 1$. Cognita ergo alterius Seriei fumma, fimul alterius innotefcet.

277. Viciffim porro ex cognitis fummis harum Serierum, affignari poterunt valores Factorum infinitorum. Sit nimirum

$$M = 1 + \tfrac{1}{2^n} + \tfrac{1}{3^n} + \tfrac{1}{4^n} + \tfrac{1}{5^n} + \tfrac{1}{6^n} + \tfrac{1}{7^n} + \&c.$$

$$N = 1 + \tfrac{1}{2^{2n}} + \tfrac{1}{3^{2n}} + \tfrac{1}{4^{2n}} + \tfrac{1}{5^{2n}} + \tfrac{1}{6^{2n}} + \tfrac{1}{7^{2n}} + \&c.,$$

eritque

$$M = \frac{1}{(1 - \tfrac{1}{2^n})(1 - \tfrac{1}{3^n})(1 - \tfrac{1}{5^n})(1 - \tfrac{1}{7^n})(1 - \tfrac{1}{11^n}) \ \&c.}$$

$$N = \frac{1}{(1 - \tfrac{1}{2^{2n}})(1 - \tfrac{1}{3^{2n}})(1 - \tfrac{1}{5^{2n}})(1 - \tfrac{1}{7^{2n}})(1 - \tfrac{1}{11^{2n}}) \ \&c.}$$

Hinc per divifionem nafcitur

$$\frac{M}{N} = (1 + \tfrac{1}{2^n})(1 + \tfrac{1}{3^n})(1 + \tfrac{1}{5^n})(1 + \tfrac{1}{7^n})(1 + \tfrac{1}{11^n}) \ \&c.$$

denique

denique vero erit

$$\frac{M M}{N} = \frac{2^n + 1}{2^n - 1} \cdot \frac{3^n + 1}{3^n - 1} \cdot \frac{5^n + 1}{5^n - 1} \cdot \frac{7^n + 1}{7^n - 1} \cdot \frac{11^n + 1}{11^n - 1} \cdot \&c.$$

Ex cognitis ergo M & N, præter valores horum productorum, summæ harum Serierum habebuntur

$$\frac{1}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} + \frac{1}{10^n} - \frac{1}{11^n} - \&c.$$

$$\frac{1}{N} = 1 - \frac{1}{2^{2n}} - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{6^{2n}} - \frac{1}{7^{2n}} + \frac{1}{10^{2n}} - \frac{1}{11^{2n}} - \&c.$$

$$\frac{M}{N} = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{10^n} + \frac{1}{11^n} + \&c.$$

$$\frac{N}{M} = 1 - \frac{1}{2^n} - \frac{1}{3^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{6^n} - \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{9^n} + \frac{1}{10^n} - \&c. ,$$

ex quarum combinatione multæ aliæ deduci possunt.

E X E M P L U M   I.

Sit $n = 1$, &, quoniam supra demonstravimus esse,

$l \, \frac{1}{1 - x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \&c.$, erit, posito $x = 1$, $l \, \frac{1}{1 - 1} = l \infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$ At Logarithmus numeri infinite magni ∞ ipse est infinite magnus, ex quo erit

$$M =$$

$$M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \&c. = \infty.$$

Hinc ob $\frac{1}{M} = \frac{1}{\infty} = 0$, fiet

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} +$$

$$\frac{1}{14} + \frac{1}{15} \&c..$$

Tum vero in productis habebitur

$$M = \infty = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11}) \&c.,}$$

unde fit

$$\infty = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{18} \&c.,$$

&

$$0 = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \&c..$$

Deinde per fummationem Serierum fupra traditam erit

$$N = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. =$$

$\frac{\pi\pi}{6}$, hinc obtinentur iftæ fummæ Serierum

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} -$$

&c.

$$\infty = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} +$$

&c.

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} +$$

$$\frac{1}{9} + \frac{1}{10} - \frac{1}{11} \&c.$$

Denique pro Factoribus orietur

Ff 3

$\frac{\pi\pi}{6}$

Lib. I.
$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \&c.,$$

seu

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{25}{24} \cdot \frac{49}{48} \cdot \frac{121}{120} \cdot \frac{169}{168} \cdot \&c.$$

&, ob $\frac{M}{N} = \infty$ seu $\frac{N}{M} = 0$, habebitur

$$\infty = \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{12}{11} \cdot \frac{14}{13} \cdot \frac{18}{17} \cdot \frac{20}{19} \cdot \&c.,$$

seu

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{20} \cdot \&c.,$$

atque

$$\infty = \frac{3}{1} \cdot \frac{4}{2} \cdot \frac{6}{4} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{14}{12} \cdot \frac{18}{16} \cdot \frac{20}{18} \cdot \&c.,$$

seu

$$0 = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \&c.,$$

quarum fractionum (excepta prima) numeratores unitate deficiunt a denominatoribus, summæ autem ex numeratoribus & denominatoribus cujusque fractionis constanter præbent numeros primos, 3, 5, 7, 11, 13, 17, 19, &c.

E x e m p l u m II.

Sit $n = 2$, eritque ex superioribus

$$M = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \&c. = \frac{\pi\pi}{6}$$

$$N = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \&c. = \frac{\pi^4}{90}$$

Hinc primo istæ Series summantur

$$\frac{6}{\pi\pi}$$

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \text{\&c.}$$

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4} - \text{\&c.}$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \text{\&c.}$$

$$\frac{\pi\pi}{15} = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} \text{\&c.},$$

Deinde valores fequentium productorum innotefcunt

$$\frac{\pi\pi}{6} = \frac{2^2}{2^2-1} \cdot \frac{3^2}{3^2-1} \cdot \frac{5^2}{5^2-1} \cdot \frac{7^2}{7^2-1} \cdot \frac{11^2}{11^2-1} \cdot \text{\&c.}$$

$$\frac{\varpi^4}{90} = \frac{2^4}{2^4-1} \cdot \frac{3^4}{3^4-1} \cdot \frac{5^4}{5^4-1} \cdot \frac{7^4}{7^4-1} \cdot \frac{11^4}{11^4-1} \cdot \text{\&c.}$$

$$\frac{15}{\pi\pi} = \frac{2^2+1}{2^2} \cdot \frac{3^2+1}{3^2} \cdot \frac{5^2+1}{5^2} \cdot \frac{7^2+1}{7^2} \cdot \frac{11^2+1}{11^2} \cdot \text{\&c.},$$

feu

$$\frac{\pi\pi}{15} = \frac{4}{5} \cdot \frac{9}{10} \cdot \frac{25}{26} \cdot \frac{49}{50} \cdot \frac{121}{122} \cdot \frac{169}{170} \cdot \text{\&c.},$$

&

$$\frac{5}{2} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \frac{11^2+1}{11^2-1} \cdot \text{\&c.},$$

five

$$\frac{5}{2} = \frac{5}{3} \cdot \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{\&c.},$$

vel

$$\frac{3}{2} = \frac{5}{4} \cdot \frac{13}{12} \cdot \frac{25}{24} \cdot \frac{61}{60} \cdot \frac{85}{84} \cdot \text{\&c.}.$$

In his fractionibus numeratores unitate fuperant denominatores, fimul vero fumti præbent quadrata numerorum primorum 3^2, 5^2, 7^2, 11^2, &c.

EXEM-

E X E M P L U M I I I.

Quia ex superioribus valores ipsius M tantum si n sit numerus par, assignare licet, ponamus $n = 4$, eritque

$$M = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \&c. = \frac{\pi^4}{90}$$

$$N = 1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} + \&c. = \frac{\pi^8}{9450}$$

Hinc primæ sequentes Series summantur

$$\frac{90}{\pi^4} = 1 - \frac{1}{2^4} - \frac{1}{3^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} + \frac{1}{10^4} - \frac{1}{11^4}$$
$$\&c.$$

$$\frac{9450}{\pi^8} = 1 - \frac{1}{2^8} - \frac{1}{3^8} - \frac{1}{5^8} + \frac{1}{6^8} - \frac{1}{7^8} + \frac{1}{10^8} - \frac{1}{11^8}$$
$$\&c.$$

$$\frac{105}{\pi^4} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{10^4} + \frac{1}{11^4}$$
$$\&c.$$

$$\frac{\pi^4}{105} = 1 - \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \frac{1}{6^4} - \frac{1}{7^4} - \frac{1}{8^4} +$$
$$\frac{1}{9^4} \&c..$$

Deinde etiam valores sequentium productorum obtinentur

$$\frac{\pi^4}{90} = \frac{2^4}{2^4 - 1} \cdot \frac{3^4}{3^4 - 1} \cdot \frac{5^4}{5^4 - 1} \cdot \frac{7^4}{7^4 - 1} \cdot \frac{11^4}{11^4 - 1} \cdot \&c.$$

$$\frac{\pi^8}{9450} = \frac{2^8}{2^8 - 1} \cdot \frac{3^8}{3^8 - 1} \cdot \frac{5^8}{5^8 - 1} \cdot \frac{7^8}{7^8 - 1} \cdot \frac{11^8}{11^8 - 1} \cdot \&c.$$

$$\frac{105}{\pi^4} = \frac{2^4 + 1}{2^4} \cdot \frac{3^4 + 1}{3^4} \cdot \frac{5^4 + 1}{5^4} \cdot \frac{7^4 + 1}{7^4} \cdot \frac{11^4 + 1}{11^4} \cdot \&c.$$
$$\&$$

$$\frac{7}{6} = \frac{2^4 + 1}{2^4 - 1} \cdot \frac{3^4 + 1}{3^4 - 1} \cdot \frac{5^4 + 1}{5^4 - 1} \cdot \frac{7^4 + 1}{7^4 - 1} \cdot \frac{11^4 + 1}{11^4 - 1} \cdot \&c.$$
$$\text{seu}$$

$$\frac{35}{34} = \frac{41}{40} \cdot \frac{313}{312} \cdot \frac{1201}{1200} \cdot \frac{7321}{7320} \cdot \&c.,$$

in

in his Factoribus numeratores unitate superant denominatores, simul vero sumti præbent bi-quadrata numerorum primorum imparium 3, 5, 7, 11, &c.

278. Quoniam hic summam Seriei

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \&c.$$

ad Factores reduximus, ad Logarithmos commode progredi licebit. Nam, cum sit

$$M = \frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n})(1 - \frac{1}{11^n}) \&c.,}$$

erit

$$lM = -l(1 - \frac{1}{2^n}) - l(1 - \frac{3}{3^n}) - l(1 - \frac{1}{5^n}) -$$

$$l(1 - \frac{1}{7^n}) - \&c..$$

Hinc, sumendis Logarithmis hyperbolicis, erit

$$lM = + 1(\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \&c.)$$

$$+ \frac{1}{2}(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c.)$$

$$+ \frac{1}{3}(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c.)$$

$$+ \frac{1}{4}(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c.)$$

$$\&c.$$

Quod si insuper ponamus

$$N = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \&c. \,,$$

ut fit

$$N = \frac{1}{(1 - \frac{1}{2^{2n}})(1 - \frac{1}{3^{2n}})(1 - \frac{1}{5^{2n}})(1 - \frac{1}{7^{2n}})(1 - \frac{1}{11^{2n}}) \, \&c. \,,}$$

fiet, Logarithmis hyperbolicis fumendis,

$$lN = + 1 \left(\frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \&c. \right)$$

$$+ \frac{1}{2} \left(\frac{1}{2^{4n}} + \frac{1}{3^{4n}} + \frac{1}{5^{4n}} + \frac{1}{7^{4n}} + \frac{1}{11^{4n}} + \&c. \right)$$

$$+ \frac{1}{3} \left(\frac{1}{2^{6n}} + \frac{1}{3^{6n}} + \frac{1}{5^{6n}} + \frac{1}{7^{6n}} + \frac{1}{11^{6n}} + \&c. \right)$$

$$+ \frac{1}{4} \left(\frac{1}{2^{8n}} + \frac{1}{3^{8n}} + \frac{1}{5^{8n}} + \frac{1}{7^{8n}} + \frac{1}{11^{8n}} + \&c. \right)$$

&c.

Ex his conjunctis fiet $\qquad lM - \frac{1}{2} lN =$

$$+ 1 \left(\frac{1}{2^{n}} + \frac{1}{3^{n}} + \frac{1}{5^{n}} + \frac{1}{7^{n}} + \frac{1}{11^{n}} + \&c. \right)$$

$$+ \frac{1}{3} \left(\frac{1}{2^{3n}} + \frac{1}{3^{3n}} + \frac{1}{5^{3n}} + \frac{1}{7^{3n}} + \frac{1}{11^{3n}} + \&c. \right)$$

$$+ \frac{1}{5} \left(\frac{1}{2^{5n}} + \frac{1}{3^{5n}} + \frac{1}{5^{5n}} + \frac{1}{7^{5n}} + \frac{1}{11^{5n}} + \&c. \right)$$

$$+ \frac{1}{7} \left(\frac{1}{2^{7n}} + \frac{1}{3^{7n}} + \frac{1}{5^{7n}} + \frac{1}{7^{7n}} + \frac{1}{11^{7n}} + \&c. \right)$$

&c.

279. Si $n = 1$ erit $M = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$ &c.

$= l\infty$, & $N = \frac{\pi\pi}{6}$; hincque erit $l.l\infty - \frac{1}{2} l\frac{\varpi\varpi}{6} =$

$$+ 1 \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \text{\&c.} \right)$$

$$+ \frac{1}{3} \left(\frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} + \text{\&c.} \right)$$

$$+ \frac{1}{5} \left(\frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} + \text{\&c.} \right)$$

$$+ \frac{1}{7} \left(\frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} + \text{\&c.} \right)$$

&c.

Verum hæ Series, præter primam, non folum fummas habent finitas, fed etiam cunctæ fimul fumtæ fummam efficiunt finitam, eamque fatis parvam: unde necefle eft ut Seriei primæ $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} +$ &c., fumma fit infinite magna, quantitate fcilicet fatis parva deficiet a Logarithmo hyperbolico Seriei $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$ &c.

280. Sit $n = 2$; erit $M = \frac{\pi\pi}{6}$ & $N = \frac{\varpi^4}{90}$: unde fit

$$2l\pi - l6 = 1 \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \text{\&c.} \right)$$

$$+ \frac{1}{2} \left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \text{\&c.} \right)$$

$$+ \frac{1}{2} \left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \text{\&c.} \right)$$

&c.

$4l\pi$

$$4l\pi - l90 = +1\left(\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \frac{1}{11^4} + \&c.\right)$$

$$+ \frac{1}{2}\left(\frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{5^8} + \frac{1}{7^8} + \frac{1}{11^8} + \&c.\right)$$

$$+ \frac{1}{3}\left(\frac{1}{2^{12}} + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \frac{1}{11^{12}} + \&c.\right)$$

$$\&c.$$

$$\frac{1}{2}l\frac{5}{2} = 1\left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \&c.\right)$$

$$+ \frac{1}{3}\left(\frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \&c.\right)$$

$$+ \frac{1}{5}\left(\frac{1}{2^{10}} + \frac{1}{3^{10}} + \frac{1}{5^{10}} + \frac{1}{7^{10}} + \frac{1}{11^{10}} + \&c.\right)$$

$$\&c.$$

281. Quanquam lex, qua numeri primi progrediuntur, non conftat, tamen harum Serierum altiorum Poteftatum fummæ non difficulter proxime affignari poterunt. Sit enim hæc Series

$$M = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \&c.,$$

&

$$S = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c.$$

erit

$$S = M - 1 - \frac{1}{4^n} - \frac{1}{6^n} - \frac{1}{8^n} - \frac{1}{9^n} - \frac{1}{10^n} - \&c.,$$

& ob

$$\frac{M}{2^n} = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \frac{1}{12^n} + \&c.,$$

erit

$$S = M - \frac{M}{2^n} - 1 + \frac{1}{2^n} - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \&c.;$$

feu

$$S =$$

$$S = (M-1)\left(1 - \frac{1}{2^n}\right) - \frac{1}{9^n} - \frac{1}{15^n} - \frac{1}{21^n} - \frac{1}{25^n} -$$

$$\frac{1}{27^n} - \&c.,$$

& ob

$$M\left(1 - \frac{1}{2^n}\right)\frac{1}{3^n} = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} + \&c.,$$

erit

$$S = (M-1)\left(1 - \frac{1}{2^n}\right)\left(1 - \frac{1}{3^n}\right) + \frac{1}{6^n} - \frac{1}{25^n} -$$

$$\frac{1}{35^n} - \frac{1}{45^n} - \&c..$$

Hinc, ob datam fummam M, valor ipfius S commode invenitur, fi quidem n fuerit numerus mediocriter magnus.

282. Inventis autem fummis altiorum Poteftatum, etiam fummæ Poteftatum minorum ex formulis inventis exhiberi poffunt. Atque hac methodo fequentes prodierunt fummæ Seriei

$$\frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} + \&c.,$$

fi fit   erit fumma Seriei

$n = 2$;	0, 45224742004I222
$n = 4$;	0, 076993139764252
$n = 6$;	0, 017070086850639
$n = 8$;	0, 004061405366515
$n = 10$;	0, 000993603573633
$n = 12$;	0, 000246026470033
$n = 14$;	0, 000061244396725
$n = 16$;	0, 000015282026219
$n = 18$;	0, 000003817278702

$n =$

$$n = 20; \qquad 0, 000000953961123$$
$$n = 22; \qquad 0, 000000238450446$$
$$n = 24; \qquad 0, 000000059608184$$
$$n = 26; \qquad 0, 000000014901555$$
$$n = 28; \qquad 0, 000000003725333$$
$$n = 30; \qquad 0, 000000000931323$$
$$n = 32; \qquad 0, 000000000232830$$
$$n = 34; \qquad 0, 000000000058207$$
$$n = 36; \qquad 0, 000000000014551$$

reliquæ fummæ parium Poteftatum in ratione quadrupla de-crefcunt.

283. Hæc autem Seriei $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} +$ &c.,

in productum infinitum converfio etiam directe inftitui poteft hoc modo : fit

$$A = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} +$$

&c., fubtrahe

$$\frac{1}{2^n} A = \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$$ &c.,

erit

$$\left(1 - \frac{1}{2^n}\right) A = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} + \frac{1}{11^n} + \&c.$$

$= B$: fic fublati funt omnes termini per 2 divifibiles,

fubtr. $\frac{1}{3^n} B = \frac{1}{3^n} + \frac{1}{9^n} + \frac{1}{15^n} + \frac{1}{21^n} +$ &c.,

erit

$$\left(1 - \frac{1}{3^n}\right) B = 1 + \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{11^n} + \frac{1}{13^n} + \&c. = C :$$

fic infuper fublati funt omnes termini per 3 divifibiles,

<div align="right">fubtr.</div>

subtr. $\dfrac{1}{5^n} C = \dfrac{1}{5^n} + \dfrac{1}{25^n} + \dfrac{1}{35^n} + \dfrac{1}{55^n} + \&c.$,

erit

$(1 - \dfrac{1}{5^n}) C = 1 + \dfrac{1}{7^n} + \dfrac{1}{11^n} + \dfrac{1}{13^n} + \dfrac{1}{17^n} + \&c.$,

sic sublati etiam sunt omnes termini per 5 divisibiles. Pari modo tolluntur termini divisibiles per 7, 11, reliquosque numeros primos; manifestum autem est sublatis omnibus terminis, qui per numeros primos divisibiles sint, solam unitatem relinqui. Quare pro B, C, D, E, &c., valoribus restitutis tandem orietur

$$A (1 - \dfrac{1}{2^n})(1 - \dfrac{1}{3^n})(1 - \dfrac{1}{5^n})(1 - \dfrac{1}{7^n})(1 - \dfrac{1}{11^n}) \&c. = 1,$$

unde Seriei propositæ summa erit $=$

$$A = \dfrac{1}{(1 - \dfrac{1}{2^n})(1 - \dfrac{1}{3^n})(1 - \dfrac{1}{5^n})(1 - \dfrac{1}{7^n})(1 - \dfrac{1}{11^n}) \&c.},$$

seu

$$A = \dfrac{2^n}{2^n - 1} \cdot \dfrac{3^n}{3^n - 1} \cdot \dfrac{5^n}{5^n - 1} \cdot \dfrac{7^n}{7^n - 1} \cdot \dfrac{11^n}{11^n - 1} \cdot \&c.$$

284. Hæc methodus jam commode adhiberi poterit ad alias Series, quarum summas supra invenimus, in producta infinita convertendas. Invenimus autem supra (175.) summas harum Serierum

$$1 - \dfrac{1}{3^n} + \dfrac{1}{5^n} - \dfrac{1}{7^n} + \dfrac{1}{9^n} - \dfrac{1}{11^n} + \dfrac{1}{13^n} - \&c.,$$

si n fuerit numerus impar, summa enim est $= N \varpi^n$ & valores ipsius N loco citato dedimus. Notandum autem est cum

Lib. I. cum hic tantum numeri impares occurrunt, eos qui sint formæ $4m+1$ habere signum $+$, reliquos formæ $4m-1$ signum $-$. Sit igitur

$$A = 1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{15^n} +$$

&c.

$$\frac{1}{3^n} A = \frac{1}{3^n} - \frac{1}{9^n} + \frac{1}{15^n} - \frac{1}{21^n} + \frac{1}{27^n} - \text{&c.}, \text{ addatur,}$$

erit

$$\left(1 + \frac{1}{3^n}\right) A = 1 + \frac{1}{5^n} - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} \text{ &c.},$$

$$= B$$

$$\frac{1}{5^n} B = \frac{1}{5^n} + \frac{1}{25^n} - \frac{1}{35^n} - \frac{1}{55^n} \text{ &c.}, \text{ subtrahatur,}$$

erit

$$\left(1 - \frac{1}{5^n}\right) B = 1 - \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{&c.} = C,$$

ubi jam numeri per 3 & 5 divisibiles desunt,

$$\frac{1}{7^n} C = \frac{1}{7^n} - \frac{1}{49^n} - \frac{1}{77^n} + \text{ &c.}, \text{ addatur,}$$

erit

$$\left(1 + \frac{1}{7^n}\right) C = 1 - \frac{1}{11^n} + \frac{1}{13^n} + \frac{1}{17^n} - \text{ &c.} = D,$$

sic etiam numeri per 7 divisibiles sunt sublati

$$\frac{1}{11^n} D = \frac{1}{11^n} - \frac{1}{121^n} + \text{ &c.}, \text{ addatur,}$$

erit

$$\left(1 + \frac{1}{11^n}\right) D = 1 + \frac{1}{13^n} + \frac{1}{17^n} - \text{ &c.} = E$$

sic numeri per 11 divisibiles quoque sunt sublati. Auferendis
autem

autem hoc modo reliquis numeris omnibus per reliquos numeros primos divifibilibus, tandem prodibit

$$A(1 + \frac{1}{3^n})(1 - \frac{1}{5^n})(1 + \frac{1}{7^n})(1 + \frac{1}{11^n})(1 - \frac{1}{13^n}) \ \&c. = 1,$$

feu

$$A = \frac{3^n}{3^n + 1} \cdot \frac{5^n}{5^n - 1} \cdot \frac{7^n}{7^n + 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \times$$

$$\frac{17^n}{17^n - 1} \ . \&c.,$$

ubi in numeratoribus occurrunt Poteftates omnium numerorum primorum, quæ in denominatoribus infunt unitate five auctæ five minutæ, prout numeri primi fuerint formæ $4m - 1$, vel $4m + 1$.

285. Pofito ergo $n = 1$, ob $A = \frac{\pi}{4}$, erit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \&c.,$$

fupra autem invenimus effe

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{3^2}{2.4} \cdot \frac{5^2}{4.6} \cdot \frac{7^2}{6.8} \cdot \frac{11^2}{10.12} \cdot \frac{13^2}{12.14} \cdot \frac{17^2}{16.18} \cdot \frac{19^2}{18.20} \cdot \&c..$$

Dividatur fecunda per primam & orietur

$$\frac{2\pi}{3} = \frac{4}{3} \cdot \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

feu

$$\frac{\pi}{2} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \&c.,$$

ubi numeri primi conftituunt numeratores, denominatores vero funt numeri impariter pares, unitate differentes a nume-

Euleri *Introduct. in Anal. infin. parv.*　　H h　　　ratori-

ratoribus. Quod ſi hæc denuo per primam $\frac{\varpi}{4}$ dividatur, erit

$$2 = \frac{4}{2} \cdot \frac{4}{6} \cdot \frac{8}{6} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{16}{18} \cdot \frac{20}{18} \cdot \frac{24}{22} \cdot \&c.,$$

seu

$$2 = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{12}{11} \cdot \&c.,$$

quæ fractiones oriuntur ex numeris primis imparibus 3, 5, 7, 11, 13, 17, &c., quemque in duas partes unitate differentes diſpeſcendo, & partes pares pro numeratoribus, impares pro denominatoribus ſumendo.

286. Si hæ expreſſiones cum *Walliſiana* comparentur

$$\frac{\varpi}{2} = \frac{2.\ 2.\ 4.\ 4.\ 6.\ 6.\ 8.\ 8.\ 10.\ 10.\ 12}{1.\ 3.\ 3.\ 5.\ 5.\ 7.\ 7.\ 9.\ 9.\ 11.\ 11} \ \&c.,$$

seu

$$\frac{4}{\pi} = \frac{3.\ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{9.\ 9}{8.\ 10} \cdot \frac{11.\ 11}{10.\ 12} \cdot \&c.,$$

cum fit

$$\frac{\pi\pi}{8} = \frac{3\ \ 3}{2.\ 4} \cdot \frac{5.\ 5}{4.\ 6} \cdot \frac{7.\ 7}{6.\ 8} \cdot \frac{11.\ 11}{10.\ 12} \cdot \frac{13.\ 13}{12.\ 14} \cdot \&c.,$$

illa per hanc diviſa dabit

$$\frac{32}{\varpi^3} = \frac{9.\ 9}{8.\ 10} \cdot \frac{15.\ 15}{14.\ 16} \cdot \frac{21.\ 21}{20.\ 22} \cdot \frac{25.\ 25}{24.\ 26} \cdot \&c.,$$

ubi in numeratoribus occurrunt omnes numeri impares non primi.

287. Sit jam $n = 3$ erit $A = \frac{\pi^3}{32}$, unde fit

$$\frac{\pi^3}{32} = \frac{3^3}{3^3 + 1} \cdot \frac{5^3}{5^3 - 1} \cdot \frac{7^3}{7^3 + 1} \cdot \frac{11^3}{11^3 + 1} \cdot \frac{13^3}{13^3 - 1} \cdot \frac{17^3}{17^3 - 1} \cdot$$

&c.. At ex Serie

$$\frac{\pi^6}{945} = 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c.,$$

fit

$$\frac{\pi^6}{945} = \frac{2^6}{2^6-1} \cdot \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1}$$

&c. , feu

$$\frac{\pi^6}{960} = \frac{3^6}{3^6-1} \cdot \frac{5^6}{5^6-1} \cdot \frac{7^6}{7^6-1} \cdot \frac{11^6}{11^6-1} \cdot \frac{13^6}{13^6-1} \cdot \&c. \, ,$$

quæ per primam divifa dabit

$$\frac{\pi^3}{30} = \frac{3^3}{3^3-1} \cdot \frac{5^3}{5^3+1} \cdot \frac{7^3}{7^3-1} \cdot \frac{11^3}{11^3-1} \cdot \frac{13^3}{13^3+1} \cdot \frac{17^3}{17^3+1} \cdot$$

&c. ,

hæc vero denuo per primam divifa dabit

$$\frac{16}{15} = \frac{3^3+1}{3^3-1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{7^3+1}{7^3-1} \cdot \frac{11^3+1}{11^3-1} \cdot \frac{13^3-1}{13^3+1} \cdot \frac{17^3-1}{17^3+1} \cdot$$

&c. , feu

$$\frac{16}{15} = \frac{14}{13} \cdot \frac{62}{63} \cdot \frac{172}{171} \cdot \frac{666}{665} \cdot \frac{1098}{1099} \cdot \&c. \, ,$$

quæ fractiones formantur ex cubis numerorum primorum imparium, quemque in duas partes unitate differentes difpefcendo, ac partes pares pro numeratoribus, impares pro denominatoribus fumendo.

288. Ex his expreffionibus denuo novæ Series formari poffunt, in quibus omnes numeri naturales denominatores conftituunt. Cum enim fit

$$\frac{\pi}{4} = \frac{3}{3+1} \cdot \frac{5}{5-1} \cdot \frac{7}{7+1} \cdot \frac{11}{11+1} \cdot \frac{13}{13-1} \cdot \&c. \, ,$$

erit

$$\frac{\pi}{6} = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13}) \&c.,}$$

unde per evolutionem hæc Series nafcetur

$$\frac{\pi}{6} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c,$$

ubi

L I B. I. ubi ratio fignorum ita eft comparata, ut binarius habeat —;
numeri primi formæ $4m - 1$ fignum —; & numeri primi
formæ $4m + 1$ fignum $+$; numeri autem compofiti ea ha-
bent figna, quæ ipfis ratione multiplicationis ex primis con-
veniunt. Sic patebit fignum fractionis $\dfrac{1}{60}$, ob $60 =$

$—$ $—$ $—$ $+$

$2 . 2 . 3 . 5$, quod erit —. Simili modo porro erit

$$\frac{\varpi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1-\frac{1}{13}) \&c.,}$$

unde orietur hæc Series

$$\frac{\pi}{2} = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} +$$
$$\frac{1}{9} + \frac{1}{10} \&c.,$$

ubi binarius habet fignum $+$; numeri primi formæ $4m - 1$
fignum —; numeri primi formæ $4m + 1$ fignum $+$; & nu-
merus quifque compofitus id habet fignum, quod ipfi ratio-
ne compofitionis ex primis convenit, fecundum regulas mul-
tiplicationis.

289. Cum deinde fit

$$\frac{\pi}{2} = \frac{1}{(1 - \frac{1}{3})(1 + \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13}) \&c.,}$$

erit per evolutionem

$$\frac{\pi}{2} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} \&c.,$$

ubi tantum numeri impares occurrunt, figna autem ita funt
comparata, ut numeri primi formæ $4m - 1$ fignum habe-
ant $+$; numeri primi formæ $4m + 1$ fignum —; unde fi-
mul numerorum compofitorum figna definiuntur. Binæ porro
Series hinc formari poffunt, ubi omnes numeri occurrunt,
erit fcilicet

$\pi =$

$$\pi = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.,}$$

unde per evolutionem oritur

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} +$$
$$\frac{1}{9} - \frac{1}{10}\ \&c.,$$

ubi binarius fignum habet $+$; numeri primi formæ $4m-1$ fignum $+$; numeri vero primi formæ $4m+1$ fignum $-$. Tum vero etiam erit

$$\frac{\pi}{3} = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{5})(1-\frac{1}{7})(1-\frac{1}{11})(1+\frac{1}{13})\ \&c.,}$$

unde per evolutionem oritur

$$\frac{\pi}{3} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} +$$
$$\frac{1}{9} + \frac{1}{10}\ \&c.,$$

ubi binarius habet fignum $-$, numeri primi formæ $4m-1$ fignum $+$, & numeri primi formæ $4m+1$ fignum $-$.

290. Poffunt hinc etiam innumerabiles aliæ fignorum conditiones exhiberi, ita ut Seriei

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{1}{6},\ \frac{1}{7},\ \frac{1}{8},\ \&c.,$$

fumma affignari queat. Cum fcilicet fit

$$\frac{\pi}{2} = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})\ \&c..}$$

Multi-

Multiplicetur hæc expressio per $\dfrac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = 2$, erit

$$\pi = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 + \frac{1}{7})(1 + \frac{1}{11}) \ \&c.,}$$

&

$$\pi = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} - \frac{1}{11} \ \&c.,$$

ubi binarius signum habet $+$; ternarius $+$; reliqui numeri primi omnes formæ $4m - 1$ signum $-$; at numeri primi formæ $4m + 1$ signum $+$; & unde pro numeris compositis ratio signorum intelligitur. Simili modo, cum sit

$$\pi = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11}) \ \&c.,}$$

multiplicetur per $\dfrac{1 + \frac{1}{5}}{1 - \frac{1}{5}} = \dfrac{3}{2}$, erit

$$\frac{3\pi}{2} = \frac{1}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{5})(1 - \frac{1}{7})(1 - \frac{1}{11})(1 + \frac{1}{13})(1 + \frac{1}{17}) \ \&c.,}$$

unde per evolutionem oritur

$$\frac{3\pi}{2} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \&c.,$$

ubi binarius habet signum $+$; numeri primi formæ $4m - 1$ signum $+$; & numeri primi formæ $4m + 1$, præter quinarium, signum $-$.

291. Possunt etiam innumerabiles hujusmodi Series exhiberi, quarum summa sit $= 0$. Cum enim sit

$$0 = \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \&c.,$$

erit

$$0 = \frac{1}{(1+\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13}) \&c.,}$$

unde, ut supra vidimus, oritur

$$0 = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \&c.,$$

ubi omnes numeri primi signum habent $-$; compositorumque numerorum signa regulam multiplicationis sequuntur. Multiplicemus autem illam expressionem per $\dfrac{1+\frac{1}{2}}{1-\frac{1}{2}} = 3$, erit

pariter

$$0 = \frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})(1+\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13}) \&c.}$$

unde per evolutionem nascitur

$$0 = 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \&c.,$$

ubi binarius habet signum $+$; reliqui numeri primi omnes signum $-$. Simili modo quoque erit

$$0 =$$

$$0 = \frac{1}{(1+\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11})(1+\frac{1}{13}) \,\&c.,}$$

unde oritur ista Series

$$0 = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \,\&c.,$$

ubi omnes numeri primi, præter 3 & 5, habent fignum —. In genere autem notandum eft, quoties omnes numeri primi, exceptis tantum aliquibus, habeant fignum —, fummam Seriei fore $= 0$. Contra autem quoties omnes numeri primi, exceptis tantum aliquibus, habeant fignum +, tum fummam Seriei fore infinite magnam.

292. Supra etiam (176.) fummam dedimus Seriei

$$A = 1 - \frac{1}{2^n} + \frac{1}{4^n} - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{8^n} + \frac{1}{10^n} - \frac{1}{11^n} + \frac{1}{13^n} \,\&c.,$$

fi fuerit n numerus impar : Erit ergo

$$\frac{1}{2^n} A = \frac{1}{2^n} - \frac{1}{4^n} + \frac{1}{8^n} - \frac{1}{10^n} + \frac{1}{14^n} - \,\&c.,$$

quæ addita dat

$$B = (1 + \frac{1}{2^n}) A = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \frac{1}{25^n} - \,\&c.$$

$$\frac{1}{5^n} B = \frac{1}{5^n} - \frac{1}{25^n} + \frac{1}{35^n} - \frac{1}{55^n} \,\&c. \,, \text{ addatur },$$

erit

$$C =$$

$$C = (1 + \frac{1}{5^n})B = 1 + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} -$$

$$\frac{1}{23^n} \text{ \&c.}$$

$$\frac{1}{7^n} C = \frac{1}{7^n} + \frac{1}{49^n} - \frac{1}{77^n} + \text{ \&c.}, \text{ fubtrahatur,}$$

erit

$$D = (1 - \frac{1}{7^n})C = 1 - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \text{ \&c..}$$

Ex his tandem fiet

$$A(1 + \frac{1}{2^n})(1 + \frac{1}{5^n})(1 - \frac{1}{7^n})(1 + \frac{1}{11^n})(1 - \frac{1}{13^n}) \text{ \&c.} = 1,$$

ubi numeri primi unitate excedentes multipla fenarii habent fignum —, deficientes autem fignum +. Eritque

$$A = \frac{2^n}{2^n + 1} \cdot \frac{5^n}{5^n + 1} \cdot \frac{7^n}{7^n - 1} \cdot \frac{11^n}{11^n + 1} \cdot \frac{13^n}{13^n - 1} \cdot \text{ \&c.,}$$

293. Confideremus cafum $n = 1$, quo $A = \frac{\pi}{3\sqrt{3}}$,

eritque

$$\frac{\pi}{3\sqrt{3}} = \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \text{ \&c.,}$$

ubi in numeratoribus poft 3 occurrunt omnes numeri primi, denominatores vero a numeratoribus unitate difcrepant, funt-que omnes per 6 divifibiles. Cum jam fit

$$\frac{\pi\pi}{6} = \frac{4}{3} \cdot \frac{9}{8} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \cdot \frac{11 \cdot 11}{10 \cdot 12} \cdot \frac{13 \cdot 13}{12 \cdot 14} \text{ \&c.,}$$

erit, hac expreffione per illam divifa,

$$\frac{\pi\sqrt{3}}{2} = \frac{9}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \text{ \&c.,}$$

LIB. I. ubi denominatores non funt per 6 divifibiles.   Vel erit

$$\frac{\pi}{2\sqrt{3}} = \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \&c.$$

$$\frac{2\pi}{3\sqrt{3}} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{22} \cdot \&c.,$$

quarum hæc per illam divifa dat

$$\frac{4}{3} = \frac{6}{4} \cdot \frac{6}{8} \cdot \frac{12}{10} \cdot \frac{12}{14} \cdot \frac{18}{16} \cdot \frac{18}{20} \cdot \frac{24}{22} \cdot \&c.,$$

feu

$$\frac{4}{3} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{9}{8} \cdot \frac{9}{10} \cdot \frac{12}{11} \cdot \&c.,$$

ubi fingulæ fractiones ex numeris primis 5, 7, 11, &c., formantur, fingulos numeros primos in duas partes unitate differentes difpefcendo, & partes per 3 divifibiles conftanter pro numeratoribus fumendo.

294.   Quoniam vero fupra vidimus effe

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \&c.,$$

feu

$$\frac{\pi}{3} = \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \&c.,$$

fi fuperiores $\frac{\pi}{2\sqrt{3}}$ & $\frac{2\pi}{3\sqrt{3}}$ per hanc dividantur, orietur

$$\frac{\sqrt{3}}{2} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdot \frac{14}{15} \cdot \frac{16}{15} \cdot \&c.$$

$$\frac{2}{\sqrt{3}} = \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{12}{11} \cdot \frac{18}{19} \cdot \frac{24}{23} \cdot \frac{30}{29} \cdot \&c.,$$

In priori expreffione fractiones formantur ex numeris primis formæ 12 m + 6 + 1, in pofteriore ex numeris primis formæ 12 m ± 1, fingulos in duas partes unitate difcrepantes difpefcendo, & partes pares pro numeratoribus, impares vero pro denominatoribus fumendo.

295.. Contemplemur adhuc Seriem fupra inventam (179), quæ ita progrediebatur

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \quad \begin{array}{c} \text{C A P.} \\ \text{X V.} \end{array}$$

&c. $= A$, erit

$$\frac{1}{3} A = \frac{1}{3} + \frac{1}{9} - \frac{1}{15} - \frac{1}{21} + \frac{1}{27} + \frac{1}{33} -$$

&c.: subtrahatur

$$\left(1 - \frac{1}{3}\right) A = 1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} + \frac{1}{19} -$$

&c. $= B$

$$\frac{1}{5} B = \frac{1}{5} - \frac{1}{25} - \frac{1}{35} + \frac{1}{55} - \text{\&c.:addatur, erit}$$

$$\left(1 + \frac{1}{5}\right) B = 1 - \frac{1}{7} + \frac{1}{11} - \frac{1}{13} + \frac{1}{17} \text{ \&c.} = C:$$

sicque pregrediendo tandem pervenietur ad

$$\frac{\pi}{2\sqrt{2}}\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 + \frac{1}{13}\right)$$

$$\left(1 - \frac{1}{17}\right)\left(1 - \frac{1}{19}\right) \text{ \&c.} = 1.$$

ubi signa ita se habent, ut numerorum primorum formæ $8m+1$, vel $8m+3$, signa sint $-$; numerorum primorum vero formæ $8m+5$, vel $8m+7$, signa sint $+$. Hinc itaque erit

$$\frac{\pi}{2\sqrt{2}} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{16} \cdot \frac{19}{18} \cdot \frac{23}{24} \cdot \text{\&c.,}$$

ubi omnes denominatores vel divisibiles sunt per 8, vel tantum sunt numeri impariter pares. Cum igitur sit

$$\frac{\pi}{4} = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \cdot \frac{13}{12} \cdot \frac{17}{16} \cdot \frac{19}{20} \cdot \frac{23}{24} \cdot \text{\&c.}$$

$$\frac{\pi}{2} = \frac{3}{2} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{11}{10} \cdot \frac{13}{14} \cdot \frac{17}{18} \cdot \frac{19}{18} \cdot \frac{23}{22} \cdot \text{\&c.,}$$

&

$$\frac{\pi\,\pi}{8}$$

$$\frac{\pi\pi}{8} = \frac{3\cdot3}{2\cdot4}\cdot\frac{5\cdot5}{4\cdot6}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{11\cdot11}{10\cdot12}\cdot\frac{13\cdot13}{12\cdot14}\cdot \&c.,$$

erit

$$\frac{\pi}{2\sqrt2} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{18}\cdot\frac{19}{20}\cdot\frac{23}{22}\cdot \&c.,$$

ubi nulli denominatores per 8 divifibiles occurrunt, pariter pares vero adfunt, quoties unitate differunt a numeratoribus. Prima vero per ultimam divifa dat

$$1 = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{9}{8}\cdot\frac{10}{9}\cdot\frac{11}{12}\cdot \&c.,$$

quæ fractiones formantur ex numeris primis, fingulos in duas partes unitate difcrepantes difpefcendo, & partes pares, (nifi fint pariter pares) pro numeratoribus fumendo.

296: Simili modo reliquæ Series, quas fupra pro expreffione arcuum circularium invenimus (179. *& feqq.*) in Factores transformari poffunt, qui ex numeris primis conftituantur. Sicque multæ aliæ infignes proprietates tam hujufmodi Factorum, quam Serierum infinitarum erui poterunt. Quoniam vero præcipuas hic jam commemoravi, pluribus evolvendis hic non immorabor. Sed ad aliud huic affine argumentum procedam. Quemadmodum fcilicet in hoc Capite numeri, quatenus per multiplicationem oriuntur, funt confiderati, ita in fequenti generatio numerorum per additionem perpendetur.

CAPUT

C A P U T   X V I.

De Partitione numerorum.

297. P Roposita sit ista expressio

$$(1 + x^{\alpha} z)(1 + x^{6} z)(1 + x^{\gamma} z)(1 + x^{\delta} z)(1 + x^{\varepsilon} z) \&c.,$$

quæ cujusmodi induat formam, si per multiplicationem evolvatur, inquiramus.   Ponamus prodire

$$1 + P z + Q z^{2} + R z^{3} + S z^{4} + \&c.,$$

atque manifestum est P fore summam Potestatum

$$x^{\alpha} + x^{6} + x^{\gamma} + x^{\delta} + x^{\varepsilon} + \&c. .$$ Deinde Q est summa Factorum ex binis Potestatibus diversis, seu Q erit aggregatum plurium Potestatum ipsius x, quarum Exponentes sunt summæ duorum terminorum diversorum hujus Seriei

$$\alpha, \quad 6, \quad \gamma, \quad \delta, \quad \varepsilon, \quad \xi, \quad \eta, \quad \&c.$$

Simili modo R erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ trium terminorum diversorum.   Atque S erit aggregatum Potestatum ipsius x, quarum Exponentes sunt summæ quatuor terminorum diversorum ejusdem Seriei, α, 6, γ, δ, ε, &c., & ita porro.

298.   Singulæ hæ Potestates ipsius x, quæ in valoribus literarum P, Q, R, S, &c., insunt, unitatem pro coëfficiente habebunt, si quidem earum Exponentes unico modo ex

α, 6,

α, $\mathcal{6}$, γ, $\mathcal{d}$, &c., formari queant : fin antem ejufdem Pote-
ſtatis Exponens pluribus modis poſſit eſſe ſumma duorum,
trium, pluriumve terminorum Seriei α, $\mathcal{6}$, γ, $\mathcal{d}$, ε, &c., tum
etiam Poteſtas illa coëfficientem habebit, qui unitatem toties
in ſe complectatur. Sic, ſi in valore ipſius Q reperiatur $N x^n$,
indicio hoc erit numerum n eſſe N diverſis modis ſummam
duorum terminorum diverforum Seriei α, $\mathcal{6}$, γ, &c.. Atque
ſi in evolutione Factorum propoſitorum occurrat terminus
$N x^n z^m$, ejus coëfficiens N indicabit quot variis modis nu-
merus n poſſit eſſe ſumma m terminorum diverforum Seriei
α, $\mathcal{6}$, γ, $\mathcal{d}$, ε, ξ, &c.

299. Quod ſi ergo productum propoſitum

$$(1 + x^\alpha z)(1 + x^\mathcal{6} z)(1 + x^\gamma z)(1 + x^\mathcal{d} z) \text{ &c. },$$

per multiplicationem veram evolvatur, ex expreſſione reſultan-
te ſtatim apparebit, quot variis modis datus numerus poſſit eſſe
ſumma tot terminorum diverforum Seriei α, $\mathcal{6}$, γ, $\mathcal{d}$, ε, ξ, &c.,
quot quis voluerit. Scilicet, ſi quæratur quot variis modis nume-
rus n poſſit eſſe ſumma m terminorum illius Seriei diverforum,
in expreſſione evoluta quæri debet terminus $x^n z^m$, ejuſque
coëfficiens indicabit numerum quæſitum.

300. Quo hæc fiant planiora, ſit propoſitum hoc produ-
ctum ex Factoribus conſtans infinitis

$$(1 + x z)(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \text{ &c., }$$

quod per multiplicationem actualem evolutum dat

$$1 + z$$

$$1 + z\,(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \&c.)$$
$$+ z^2\,(x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + \&c.)$$
$$+ z^3\,(x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + 10x^{14} + \&c.)$$
$$+ z^4\,(x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + 11x^{17} + 15x^{18} + \&c.)$$
$$+ z^5\,(x^{15} + x^{16} + 2x^{17} + 3x^{18} + 5x^{19} + 7x^{20} + 10x^{21} + 13x^{22} + 18x^{23} + \&c.)$$
$$+ z^6\,(x^{21} + x^{22} + 2x^{23} + 3x^{24} + 5x^{25} + 7x^{26} + 11x^{27} + 14x^{28} + 20x^{29} + \&c.)$$
$$+ z^7\,(x^{28} + x^{29} + 2x^{30} + 3x^{31} + 5x^{32} + 7x^{33} + 11x^{34} + 15x^{35} + 21x^{36} + \&c.)$$
$$+ z^8\,(x^{36} + x^{37} + 2x^{38} + 3x^{39} + 5x^{40} + 7x^{41} + 11x^{42} + 15x^{43} + 22x^{44} + \&c.)$$

&c.

Ex his ergo Seriebus ſtatim definire licet quot variis modis
propoſitus numerus ex dato terminorum diverſorum hujus Se-
rici 1, 2, 3, 4, 5, 6, 7, 8, &c., numero oriri queat. Sic,
ſi quæratur quot variis modis numerus 35 poſſit eſſe ſumma
ſeptem terminorum diverſorum Seriei 1, 2, 3, 4, 5, 6, 7, &c.,
quæratur in Serie z^7 multiplicante Poteſtas x^{35}, ejuſque coëf-
ficiens 15 indicabit numerum propoſitum 35 quindecim variis
modis eſſe ſummam ſeptem terminorum Seriei 1, 2, 3, 4, 5,
6, 7, 8, &c.

301. Quod ſi autem ponatur $z = 1$, & ſimiles Poteſtates
ipſius x in unam ſummam conjiciantur, ſeu, quod eodem re-
dit, ſi evolvatur hæc expreſſio infinita

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \&c.,$$

quo facto orietur hæc Series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + \&c.,$$

ubi quivis coëfficiens indicat, quot variis modis Exponens
Poteſtatis ipſius x conjunctæ ex terminis diverſis Seriei 1, 2,
3, 4, 5, 6, 7, &c., per additionem emergere poſſit. Sic ap-
paret numerum 8 ſex modis per additionem diverſorum nu-
merorum produci, qui ſunt

$$8 = 8 \qquad\qquad 8 = 5 + 3$$
$$8 = 7 + 1 \qquad\qquad 8 = 5 + 2 + 1$$
$$8 = 6 + 2 \qquad\qquad 8 = 4 + 3 + 1$$

ubi

LIB. I. ubi notandum eſt numerum propoſitum ipſum ſimul computari debere, quia numerus terminorum non definitur, ideoque unitas inde non excluditur.

302. Hinc igitur intelligitur, quomodo quiſque numerus per additionem diverſorum numerorum producatur. Conditio autem diverſitatis omittetur, ſi Factores illos in denominatorem tranſponamus. Sit igitur propoſita hæc expreſſio

$$\frac{1}{(1 - x^{\alpha}z)(1 - x^{6}z)(1 - x^{\gamma}z)(1 - x^{\delta}z)(1 - x^{\varepsilon}z) \&c.,}$$

quæ per diviſionem evoluta det

$$1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c..$$

Atque manifeſtum eſt fore P aggregatum. Poteſtatum ipſius x, quarum Exponentes contineantur in hac Serie

$$\alpha, \ 6, \ \gamma, \ \delta, \ \varepsilon, \ \xi, \ \eta, \ \&c.,$$

Deinde Q erit aggregatum Poteſtatum ipſius x, quarum Exponentes ſint ſummæ duorum terminorum hujus Seriei, ſive eorundem ſive diverſorum. Tum erit R ſumma Poteſtatum ipſius x, quarum Exponentes ex additione trium terminorum illius Seriei oriantur; & S ſumma Poteſtatum, quarum Exponentes ex additione quatuor terminorum in illa Serie contentorum formantur, & ita porro.

303. Si igitur tota expreſſio per ſingulos terminos explicetur, & termini ſimiles conjunctim exprimantur, intelligetur quot variis modis propoſitus numerus n per additionem m terminorum, ſive diverſorum ſive non diverſorum, Seriei α, 6, γ, δ, ε, ξ, &c., produci queat. Quæratur ſcilicet in expreſſione evoluta terminus $x^n z^m$, ejuſque coëfficiens, qui ſit N, ita ut totus terminus ſit $= N x^n z^m$, atque coëfficiens N indicabit quot variis modis numerus n per additionem m terminorum

minorum in Serie α, ϵ, γ, δ, ϵ, &c., contentorum produci queat. Hoc igitur pacto quæstio priori, quam ante sumus contemplati, similis resolvetur.

304. Accommodemus hæc ad casum inprimis notatu dignum, sitque proposita hæc expressio

$$\frac{1}{(1 - xz)(1 - x^2z)(1 - x^3z)(1 - x^4z)(1 - x^5z) \&c.,}$$

quæ per divisionem evoluta dabit

$$
\begin{aligned}
&1 + z\,(x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + \&c.)\\
&+ z^2(x^2 + x^3 + 2x^4 + 2x^5 + 3x^6 + 3x^7 + 4x^8 + 4x^9 + 5x^{10} + \&c\,)\\
&+ z^3(x^3 + x^4 + 2x^5 + 3x^6 + 4x^7 + 5x^8 + 7x^9 + 8x^{10} + 10x^{11} + \&c.)\\
&+ z^4(x^4 + x^5 + 2x^6 + 3x^7 + 5x^8 + 6x^9 + 9x^{10} + 11x^{11} + 15x^{12} + \&c.)\\
&+ z^5(x^5 + x^6 + 2x^7 + 3x^8 + 5x^9 + 7x^{10} + 10x^{11} + 13x^{12} + 18x^{13} + \&c.)\\
&+ z^6(x^6 + x^7 + 2x^8 + 3x^9 + 5x^{10} + 7x^{11} + 11x^{12} + 14x^{13} + 20x^{14} + \&c.)\\
&+ z^7(x^7 + x^8 + 2x^9 + 3x^{10} + 5x^{11} + 7x^{12} + 11x^{13} + 15x^{14} + 21x^{15} + \&c\,)\\
&+ z^8(x^8 + x^9 + 2x^{10} + 3x^{11} + 5x^{12} + 7x^{13} + 11x^{14} + 15x^{15} + 22x^{16} + \&c.)\\
&\&c.,
\end{aligned}
$$

Ex his ergo Seriebus statim definire licet quot variis modis propositus numerus per additionem ex dato terminorum hujus Seriei 1, 2, 3, 4, 5, 6, 7, &c., numero produci queat. Sic, si quæratur quot variis modis numerus 13 oriri possit per additionem quinque numerorum integrorum, spectari debebit terminus $x^{13}z^5$, cujus coëfficiens 18 indicat numerum propositum 13 ex quinque numerorum additione octodecim modis oriri posse.

305. Si ponatur $z = 1$, atque similes Potestates ipsius x conjunctim exprimantur, hæc expressio

$$\frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6) \&c.,}$$

evolvetur in hanc Seriem

$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8$ &c.,

in qua quilibet coëfficiens indicat, quot variis modis Exponens Poteſtatis adjunctæ per additionem produci queat ex numeris integris, ſive æqualibus ſive inæqualibus. Scilicet ex termino $11x^6$ cognoſcitur numerum 6 undecim modis par additionem numerorum integrorum produci poſſe, qui ſunt

$6 = 6$		$6 = 3 + 1 + 1 + 1$		
$6 = 5 + 1$		$6 = 2 + 2 + 2$		
$6 = 4 + 2$		$6 = 2 + 2 + 1 + 1$		
$6 = 4 + 1 + 1$		$6 = 2 + 1 + 1 + 1 + 1$		
$6 = 3 + 3$		$6 = 1 + 1 + 1 + 1 + 1 + 1$		
$6 = 3 + 2 + 1$				

ubi quoque notari debet, ipſum numerum propoſitum, cum in Serie numerorum 1, 2, 3, 4, 5, 6, &c., propoſita contineatur, unum modum præbere.

306. His in genere expoſitis, diligentius inquiramus in modum hanc compoſitionum multitudinem inveniendi. Ac primo quidem conſideremus eam ex numeris integris compoſitionem, in qua numeri tantum diverſi admittuntur, quam prius commemoravimus. Sit igitur in hunc finem propoſita hæc expreſſio

$$Z = (1 + xz)(1 + x^2z)(1 + x^3z)(1 + x^4z)(1 + x^5z) \ \&c.,$$

quæ evoluta & ſecundum Poteſtates ipſius z digeſta præbeat

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c.,$$

ubi methodus deſideratur has ipſius x Functiones P, Q, R, S, T, &c., expedite inveniendi, hoc enim pacto quæſtioni propoſitæ convenientiſſime ſatisfiet.

307. Patet autem, ſi loco z ponatur xz, prodire

$(1 +$

$$(1 + x^2 z)(1 + x^3 z)(1 + x^4 z)(1 + x^5 z) \ \&c. = \frac{Z}{1 + xz} \qquad \begin{matrix} \text{C A P.} \\ \text{X V I.} \end{matrix}$$

ergo, pofito xz loco z, valor producti, qui erat Z, abibit in

$\frac{Z}{1 + xz}$; ficque, cum fit

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + \&c.,$$

erit

$$\frac{Z}{1 + xz} = 1 + Pxz + Qx^2 z^2 + Rx^3 z^3 + Sx^4 z^4 + \&c.,$$

multiplicetur ergo actu per $1 + xz$, atque prodibit

$$Z = 1 + Pxz + Qx^2 z^2 + Rx^3 z^3 + Sx^4 z^4 + \&c.$$
$$\qquad + xz + Px^2 z^2 + Qx^3 z^3 + Rx^4 z^4 + \&c.,$$

qui valor ipfius Z cum fuperiori comparatus dabit

$$P = \frac{x}{1 - x}; \ Q = \frac{Px^2}{1 - x^2}; \ R = \frac{Qx^3}{1 - x^3}; \ S = \frac{Rx^4}{1 - x^4} \ \&c.,$$

Sequentes ergo pro P, Q, R, S, $\&c.$, obtinentur valores

$$P = \frac{x}{1 - x}$$

$$Q = \frac{x^3}{(1 - x)(1 - x^2)}$$

$$R = \frac{x^6}{(1 - x)(1 - x^2)(1 - x^3)}$$

$$S = \frac{x^{10}}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}$$

$$T = \frac{x^{15}}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)}$$

$$\&c.$$

308. Sic igitur feorfim unamquamque Seriem Poteftatum ipfius x exhibere poffumus, ex qua definire licet, quot variis modis propofitus numerus ex dato partium integrarum numero per additionem formari poffit. Manifeftum autem porro eft has fingulas Series effe recurrentes, quia ex evolutione Functionis fractæ ipfius x nafcuntur. Prima fcilicet expreffio

$P =$

LIB. I. $P = \dfrac{x}{1-x}$, dat Seriem geometricam

$$x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.,$$

ex qua quidem manifeſtum eſt quemvis numerum ſemel in Serie numerorum integrorum contineri.

309. Expreſſio ſecunda $\dfrac{x^3}{(1-x)(1-xx)}$, dat hanc Seriem

$$x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + \&c.,$$

in qua cujuſvis termini coëfficiens indicat quot modis Expo-nens ipſius x in duas partes inæquales diſpertiri poſſit. Sic terminus $4x^9$ indicat, numerum 9 quatuor modis in duas par-tes inæquales ſecari poſſe. Quod ſi hanc Seriem per x^3 divida-mus, prodibit Series, quam præbet iſta fractio $\dfrac{1}{(1-x)(1-x^2)}$, quæ erit

$$1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \&c.,$$

cujus terminus generalis ſit $= Nx^n$; atque ex geneſi hujus Seriei intelligitur coëfficientem N indicare, quot variis modis Exponens n ex numeris 1 & 2 per additionem naſci queat. Cum igitur prioris Seriei terminus generalis ſit $= Nx^{n+3}$, de-ducitur hinc iſtud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1 & 2 *produci poteſt, totidem variis modis numerus* n + 3 *in duas partes inæquales ſecari poterit.*

310. Expreſſio tertia $\dfrac{x^6}{(1-x)(1-x^2)(1-x^3)}$ in Se-riem evoluta dabit

$$x^6 + x^7 + 2x^8 + 3x^9 + 4x^{10} + 5x^{11} + 7x^{12} + 8x^{13} + \&c.,$$

in qua cujuſvis termini coëfficiens indicat quot variis modis Exponens Poteſtatis x adjunctæ in tres partes inæquales diſper-
tiri

tiri poſſit. Quod ſi autem hæc fractio $\dfrac{1}{(1-x)(1-x^2)(1-\lambda^3)}$

evolvatur, prodibit hæc Series

$$1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + \&c.,$$

cujus terminus generalis ſi ponatur $= Nx^n$, coëfficiens N indicabit quot variis modis numerus n ex numeris 1, 2, 3, per additionem produci poſſit. Cum igitur prioris Seriei terminus generalis ſit Nx^{n+6}, ſequetur hinc iſtud theorema.

Quot variis modis numerus n *per additionem ex numeris* 1, 2, 3, *produci poteſt, totidem variis modis numerus* n + 6 *in tres partes inæquales ſecari poterit.*

311. Expreſſio quarta $\dfrac{x^{10}}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$ in Seriem recurrentem evoluta dabit

$$x^{10} + x^{11} + 2x^{12} + 3x^{13} + 5x^{14} + 6x^{15} + 9x^{16} + \&c.,$$

in qua cujusvis termini coëfficiens indicabit quot variis modis Exponens Poteſtatis x adjunctæ in quatuor partes inæquales diſpertiri poſſit. Quod ſi autem hæc expreſſio

$$\dfrac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)}$$ evolvatur, prodibit

ſuperior Series per x^{10} diviſa, nempe

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + \&c.,$$

cujus terminum generalem ponamus $= Nx^n$; atque hinc patebit coëfficientem N indicare, quot variis modis numerus n per additionem oriri poſſit ex his quatuor numeris 1, 2, 3, 4. Cum igitur prioris Seriei terminus generalis futurus ſit $= Nx^{n+10}$, deducitur hoc theorema.

K k 3

Quot

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4, *totidem variis modis numerus* n + 10 *in quatuor partes inæquales ſecari poterit.*

312. Generaliter ergo, ſi hæc expreſſio

$$\frac{1}{(1-x)(1-x^2)(1-x^3) \ldots \ldots \ldots (1-x^m)}$$

in Seriem evolvatur, ejuſque terminus generalis fuerit =
Nx^n, coëfficiens N indicabit, quot variis modis numerus *n*
per additionem produci poſſit ex his numeris 1, 2, 3, 4
. . . . *m*. Quod ſi autem hæc expreſſio

$$\frac{x^{\frac{m(m+1)}{2}}}{(1-x)(1-x^2)(1-x^3) \ldots \ldots \ldots (1-x^m)}$$

in Seriem evolvatur, erit ejus terminus generalis =
$Nx^{n+\frac{m(m+1)}{2}}$: atque hic coëfficiens N indicat quot variis modis numerus $n+\frac{m(m+1)}{1.\ 2}$ in *m* partes inæquales ſecari poſſit, unde hoc habetur theorema.

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4 . . . . m, *totidem modis numerus* n + $\frac{m(m+1)}{1.\ 2}$ *in* m *partes inæquales ſecari poterit.*

313. Ex poſita partitione numerorum in partes inæquales, perpendamus quoque partitionem in partes, ubi æqualitas partium non excluditur; quæ partitio ex hac expreſſione originem habet

$$Z = \frac{1}{(1-xz)(1-x^2z)(1-x^3z)(1-x^4z)(1-x^5z)\ \&c.}$$

Ponamus evolutione per diviſionem inſtituta prodire

$$Z =$$

$$Z = 1 + Pz + Qz^2 + Rz^3 + Sz^4 + Tz^5 + \&c. .$$

Perſpicuum autem eſt, ſi loco z ponatur xz, prodire

$$\frac{1}{(1 - x^2z)(1 - x^3z)(1 - x^4z)(1 - x^5z) \&c.} = (1 - xz)\, Z.$$

Facta ergo in Serie evoluta eadem mutatione, fiet

$$(1 - xz)\, Z = 1 + Pxz + Qx^2z^2 + Rx^3z^3 + Sx^4z^4 + \&c. .$$

Multiplicetur ergo ſuperior Series pariter per $(1 - xz)$, eritque

$$(1 - xz)\, Z = 1 + Pz + Qz^2 + Rz^3 + S z^4 + \&c.$$
$$\qquad\qquad - xz - Pxz^2 - Qxz^3 - Rxz^4 - \&c. .$$

Comparatione ergo inſtituta orietur

$$P = \frac{x}{1 - x}; \; Q = \frac{Px}{1 - x^2}; \; R = \frac{Qx}{1 - x^3}; \; S = \frac{Rx}{1 - x^4}, \; \&c.,$$

unde pro P, Q, R, S, &c., ſequentes valores proveniunt.

$$P = \frac{x}{1 - x}$$

$$Q = \frac{x^2}{(1 - x)(1 - x^2)}$$

$$R = \frac{x^3}{(1 - x)(1 - x^2)(1 - x^3)}$$

$$S = \frac{x^4}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)}$$
$$\&c.$$

314. Expreſſiones iſtæ a ſuperioribus aliter non diſcrepant, niſi quod numeratores hic minores habeant Exponentes quam caſu præcedente. Atque hanc ob rem Series, quæ per evolutionem naſcuntur, ratione coëfficientium omnino convenient, quæ convenientia jam ex comparatione (§. §. 300. & 304.)

perſpi-

L I B. I. perſpicitur, nunc vero demum ejus ratio intelligitur.　Hinc ergo omnino ſimilia theoremata conſequentur, quæ ſunt.

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, *totidem modis numerus* n + 2 *in duas partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, *totidem modis numerus* n + 3 *in tres partes diſpertiri poterit.*

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, 4, *totidem modis numerus* n + 4 *in quatuor partes diſpertiri poterit.*

Atque generaliter habebitur hoc theorema:

Quot variis modis numerus n *per additionem produci poteſt ex numeris* 1, 2, 3, ... m, *totidem modis numerus* n + m *in* m *partes diſpertiri poterit.*

315.　Sive ergo quæratur quot modis datus numerus in *m* partes inæquales, ſive in *m* partes, æqualibus non excluſis, diſpertiri poſſit, utraque quæſtio reſolvetur ſi cognoſcatur quot modis quiſque numerus per additionem produci poſſit ex numeris 1, 2, 3, 4 .... *m*, quemadmodum hoc patebit ex ſequentibus theorematis, quæ ex ſuperioribus ſunt derivata.

Numerus n *tot modis in* m *partes inæquales diſpertiri poteſt, quot modis numerus* n — $\frac{m(m+1)}{2}$ *per additionem produci poteſt ex numeris* 1, 2, 3, 4, ... m.

Numerus n, *tot modis in* m *partes ſive æquales ſive inæquales diſpertiri poteſt quot modis numerus* n — m *per additionem produci poteſt ex numeris* 1, 2, 3, ... m.

Hinc porro ſequuntur hæc theoremata.

Numerus n *totidem modis in* m *partes inæquales ſecari poteſt, quot modis numerus* n — $\frac{m(m-1)}{2}$ *in* m *partes, ſive æquales ſive inæquales, diſpertitur.*

Numerus n *totidem modis in* m *partes, ſive inæquales ſive æquales,*
ſecari

secari poteft, quot modis numerus $n + \dfrac{m(m-1)}{2})$ *in* m *partes inæquales difpertiri poteft.*

316. Per formationem autem Serierum recurrentium inveniri poterit, quot variis modis datus numerus *n* per additionem produci poffit ex numeris 1, 2, 3, ... *m*. Ad hoc enim inveniendum evolvi debebit fractio

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

atque Series recurrens continuari debebit ufque ad terminum Nx^n, cujus coëfficiens N indicabit, quot modis numerus *n* per additionem produci poffit ex numeris 1, 2, 3, 4, ... *m*. At vero hic folvendi modus non parum habebit difficultatis, fi numeri *m* & *n* fint modice magni; fcala enim relationis, quam præbet denominator per multiplicationem evolutus, ex pluribus terminis conftat, unde operofum erit Seriem ad plures terminos continuare.

317. Hæc autem difquifitio minus erit molefta, fi cafus fimpliciores primum expediantur, ex his enim facile erit ad cafus magis compofitos progredi. Sit Seriei, quæ ex hac fractione oritur,

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

terminus generalis $= Nx^n$; at Seriei ex hac forma

$$\frac{x^m}{(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^m)}$$

ortæ terminus generalis fit Mx^n, ubi coëfficiens M indicabit quot variis modis numerus $n-m$ per additionem produci

Euleri Introduct. in Anal. infin. parv.　　　L l　　　poffit

poſſit ex numeris 1, 2, 3, ... *m*.   Subtrahatur poſterior expreſſio a priori, ac remanebit

$$\frac{1}{(1-x)(1-x^2)(1-x^3) \ . \ . \ . \ . \ . \ . \ . \ (1-x^{m-1})}$$

atque manifeſtum eſt Seriei hinc ortæ terminum generalem futurum eſſe $(N-M)x^n$; quare coëfficiens $N-M$ indicabit quot variis modis numerus *n* per additionem produci poſſit ex numeris 1, 2, 3, ... ($m-1$).

318. Hinc ergo ſequentem regulam nanciſcimur.

Sit L numerus modorum, quibus numerus *n* per additionem produci poteſt ex numeris 1, 2, 3, ... ($m-1$).

Sit M numerus modorum, quibus numerus $n-m$ per additionem produci poteſt ex numeris 1, 2, 3, ... *m*.

Sitqne N numerus modorum, quibus numerus *n* per additionem produci poteſt ex numeris 1, 2, 3, ... *m*.

His poſitis, erit, ut vidimus, $L=N-M$; ideoque $N=L+M$. Quod ſi ergo jam invenerimus quot variis modis numeri *n* & $n-m$ per additionem produci queant, ille ex numeris 1, 2, 3, ... ($m-1$) hic vero ex numeris 1, 2, 3, ... *m*; hinc addendo cognoſcemus, quot variis modis numerus *n* per additionem produci queat ex numeris 1, 2, 3, ... *m*.   Ope hujus theorematis a caſibus ſimplicioribus, qui nihil habent difficultatis, continuo ad magis compoſitos progredi licebit, hocque modo tabula hic annexa eſt computata, * cujus uſus ita ſe habet.

Si quæratur quot variis modis numerus 50 in 7 partes inæquales diſpertiri poſſit; ſumatur in prima columna verticali numerus $50 - \frac{7 \cdot 8}{2} = 22$, in horizontali autem ſuprema numerus romanus VII; atque numerus in angulo poſitus 522 indicabit modorum numerum quæſitum.

Sin autem quæratur, quot variis modis numerus 50 in 7 partes, ſive æquales ſive inæquales, diſpertiri poſſit, in prima

<div align="right">columna</div>

*   *Vide Tab. pag.* 275.

columna verticali fumatur numerus $50 - 7 = 43$, cui in co-
lumna 7ma refpondebit numerus quæfitus 8946.

319. Series hujus tabulæ verticales, etfi funt recurrentes,
tamen ingentem habent connexionem cum numeris naturalibus,
trigonalibus, pyramidalibus, & fequentibus, quam paucis ex-
ponere operæ pretium erit. Quoniam enim ex fractione

$\frac{1}{(1-x)(1-xx)}$ oritur Series $1 + x + 2x^2 + 2x^3 + 3x^4 +$

$3x^5 +$ &c., ac proinde ex fractione $\frac{x}{(1-x)(1-xx)}$ hæc

$x + x^2 + 2x^3 + 2x^4 + 3x^5 + 3x^6 +$ &c.. Si duæ hæ Se-
ries addantur, nafcitur ifta

$$1 + 2x + 3x^2 + 4x^3 \; 5x^4 + 6x^5 + 7x^6 +$$ &c.,

quæ per divifionem oritur ex fractione $\frac{1+x}{(1-x)(1-xx)} =$

$\frac{1}{(1-x)^2}$; unde patet Seriei poftremæ terminos numericos
Seriem numerorum naturalium conftituere. Hinc ex Serie ta-
bulæ fecunda addendo binos terminos proveniet Series nu-
merorum naturalium, pofito $x = 1$.

$$1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + 6 + 6 +$$ &c.
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 +$$ &c.

Viciffim ergo ex Serie numerorum naturalium fuperior inveni-
tur, fubtrahendo quemque terminum Seriei fuperioris a termi-
no inferioris fequente.

320. Series verticalis tertia oritur ex fractione

$\frac{1}{(1-x)(1-xx)(1-x^3)}$. Cum autem fit $\frac{1}{(1-x)^3} =$

$\frac{(1+x)(1+x+xx)}{(1-x)(1-xx)(1-x^3)}$, manifeftum eft, fi primo Se-
riei illius terni termini addantur, tum bini hujus novæ Seriei
denuo addantur, prodire debere numeros trigonales, id quod
ex fchemate fequente apparebit

$1 +$

LIB. I. $1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + 14 + 16 + 19$ &c.

$1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + 36 + 42 + 49$ &c.

$1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + 66 + 78 + 91$ &c..

Viciſſim autem apparet quomodo ex Serie trigonalium erui debeat Series ſuperior.

321. Simili modo, quia Series quarta oritur ex fractione

$$\frac{1}{(1-x)(1-xx)(1-x^3)(1-x^4)}, \text{ erit } \frac{(1+x)(1+x+xx)(1+x+xx+x^3)}{(1-x)(1-xx)(1-x^3)(1-x^4)}$$

$$= \frac{1}{(1-x)^4}.$$ Si in Serie quarta primum quaterni termini addantur, tum in Serie reſultante terni, denique in hac bini, prodibit Series numerorum pyramidalium uti ex ſequenti calculo videre licet.

$1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + 23 + 27 +$ &c.

$1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + 67 + 83 +$ &c.

$1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + 161 + 203 +$ &c.

$1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + 286 + 364 +$ &c.

Simili autem modo Series quinta deducet ad numeros pyramidales ſecundi ordinis, ſexta ad tertii ordinis, & ita porro.

322. Viciſſim igitur ex numeris figuratis illæ ipſæ Series, quæ in tabulis occurrunt, formari poterunt, per operationes, quæ ex inſpectione calculi ſequentis ſponte elucebunt.

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + $ &c.

$1 + 1 + 2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 + $ &c.   II

$1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 + 55 + $ &c.

$1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 + 25 + 30 + $ &c.

$1 + 1 + 2 + 3 + 4 + 5 + 7 + 8 + 10 + 12 + $ &c.   III

$1 + 4 + 10 + 20 + 35 + 56 + 84 + 120 + 165 + 220 + $ &c.

$1 + 3 + 7 + 13 + 22 + 34 + 50 + 70 + 95 + 125 + $ &c.

$1 + 2 + 4 + 7 + 11 + 16 + 23 + 31 + 41 + 53 + $ &c.

$1 + 1 + 2 + 3 + 5 + 6 + 9 + 11 + 15 + 18 + $ &c.   IV

$1 + 5 + 15 + 35 + 70 + 126 + 210 + 330 + 495 + 715 + $ &c.

$1 + 4 + 11 + 24 + 46 + 80 + 130 + 200 + 295 + 420 + $ &c.

$1 + 3 + 7 + 14 + 25 + 41 + 64 + 95 + 136 + 189 + $ &c.

$1 + 2 + 4 + 7 + 12 + 18 + 27 + 38 + 53 + 71 + $ &c.

$1 + 1 + 2 + 3 + 5 + 7 + 10 + 13 + 18 + 23 + $ &c.   V

&c.

In his ordinibus primæ Series funt numeri figurati, unde fubtrahendo quemvis terminum Seriei fecundæ a termino primæ fequente formatur Series fecunda. Tum Seriei tertiæ bini termini conjunctim fubtrahantur a termino fequente Seriei fecundæ, ficque oritur Series tertia; hocque modo fubtrahendo ulterius fummam trium, quatuor, & ita porro terminorum a termino fuperioris Seriei fequente, formabuntur reliquæ Series donec perveniatur ad Seriem, quæ incipit ab $1 + 1 + 2$ &c., hæcque erit Series in tabula exhibita.

323. Series verticales tabulæ omnes fimiliter incipiunt, continuoque plures habent terminos communes; ex quo intelligitur in infinitum has Series inter fe fore congruentes. Prodibit autem Series, quæ oritur ex hac fractione

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\&c.},$$

quæ cum fit recurrens, primum denominator fpectari debet, ut

     hinc

LIB. I. hinc ſcala relationis habeatur. Quod ſi autem Factores de-
nominatoris continuo in ſe multiplicentur, prodibit

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\&c.,$$

quæ Series ſi attentius conſideretur, aliæ Poteſtates ipſius x adeſſe non deprehenduntur, niſi quarum Exponentes conti-neantur in hac formula $\frac{3nn \pm n}{2}$; atque, ſi n ſit numerus im-par, Poteſtates erunt negativæ; affirmativæ autem ſi n fuerit numerus par.

324. Cum igitur ſcala relationis ſit

$$+1, +1, 0, 0, -1, 0, -1, 0, 0, 0, 0, +1, 0, 0, +1, 0, 0, \&c.,$$

Series recurrens ex evolutione fractionis

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)((1-x^7)\&c.,}$$

oriunda erit hæc

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 +$$
$$30x^9 + 42x^{10} + 56x^{11} + 77x^{12} + 101x^{13} + 135x^{14} +$$
$$176x^{15} + 231x^{16} + 297x^{17} + 385x^{18} + 490x^{19} + 627x^{20} +$$
$$792x^{21} + 1002x^{22} + 1250x^{23} + 1570x^{24} \&c. .$$

In hac ergo Serie coëfficiens quiſque indicat, quot variis mo-dis Exponens ipſius x per additionem ex numeris integris o-riri queat. Sic numerus 7 quindecim modis per additionem oriri poteſt.

$7 = 7$	$7 = 4+2+1$	$7 = 3+1+1+1+1$
$7 = 6+1$	$7 = 4+1+1+1$	$7 = 2+2+2+1$
$7 = 5+2$	$7 = 3+3+1$	$7 = 2+2+1+1+1$
$7 = 5+1+1$	$7 = 3+2+2$	$7 = 2+1+1+1+1+1$
$7 = 4+3$	$7 = 3+2+1+1$	$7 = 1+1+1+1+1+1+1$

325. Quod

325. Quod fi autem hoc productum

$$(1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \&c.,$$

evolvatur, fequens prodibit Series

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + 10x^{10} +$$
&c.,

in qua quifque coëfficiens indicat, quot variis modis Exponens ·
ipfius x per additionem numerorum inæqualium oriri poffit,
Sic numerus 9 octo variis modis per additionem ex numeris
inæqualibus formari poteft.

$9 = 9$	$9 = 6 + 2 + 1$	
$9 = 8 + 1$	$9 = 5 + 4$	
$9 = 7 + 2$	$9 = 5 + 3 + 1$	
$9 = 6 + 3$	$9 = 4 + 3 + 2$	

326. Ut comparationem inter has formas inftituamus, fit
$$P = (1 - x)(1 - x^2)(1 - x^3)(1 - x^4)(1 - x^5)(1 - x^6) \&c.,$$
&
$$Q = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)(1 + x^6) \&c.,$$
erit
$$PQ = (1 - x^2)(1 - x^4)(1 - x^6)(1 - x^8)(1 - x^{10})(1 - x^{12})$$
&c.,
qui Factores cum omnes in P contineantur, dividatur P per
PQ, erit $\frac{1}{Q} = (1 - x)(1 - x^3)(1 - x^5)(1 - x^7)(1 - x^9) \&c.,$

ideoque

$$Q = \frac{1}{(1 - x)(1 - x^3)(1 - x^5)(1 - x^7)(1 - x^9) \&c.,}$$

quæ fractio fi evolvatur, prodibit Series, in qua quifque coëf-
ficiens indicabit, quot variis modis Exponens ipfius x, per
additionem ex numeris imparibus produci poffit. Cum igitur
hæc expreffio æqualis fit illi, quam in §. præcedente contemplati
fumus, fequitur hinc iftud theorema.

Quot

Quot modis datus numerus per additionem formari poteſt ex omnibus numeris integris inter ſe inæqualibus ; totidem modis idem numerus formari poterit per additionem ex numeris tantum imparibus , ſive æqualibus ſive inæqualibus.

327. Cum igitur, ut ante vidimus, fit

$$P = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} +$$

&c., erit, ſcribendo xx loco x ,

$$PQ = 1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + x^{44} + x^{52} - \text{\&c.},$$

Quocirca erit hanc per illam dividendo

$$Q = \frac{1 - x^2 - x^4 + x^{10} + x^{14} - x^{24} - x^{30} + \text{\&c.}}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \text{\&c.}}.$$

Erit ergo Series Q pariter recurrens , atque ex Serie $\frac{1}{P}$ oritur, hane per $1 - x^2 - x^4 + x^{10} + x^{14} - x^{24}$ &c., multiplicando. Nempe, cum fit ex (324), $\frac{1}{P} = 1 + x + 2x^2 +$

$$3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{\&c.} ,$$

ſi is multiplicetur per

$$1 - x^2 - x^4 + x^{10} + x^{14} - \text{\&c.} ,$$

fiet

$$1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + 22x^8 + 30x^9 + \text{\&c.}$$
$$- x^2 - x^3 - 2x^4 - 3x^5 - 5x^6 - 7x^7 - 11x^8 - 15x^9 - \text{\&c.}$$
$$- x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - \text{\&c.}$$

aut

$$1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + 6x^8 + 8x^9 + \text{\&c.}$$

$= Q$ Hinc ergo, ſi formatio numerorum per additionem numerorum, ſive æqualium ſive inæqualium conſtet , deducetur formatio numerorum per additionem numerorum inæqualium, hincque porro formatio numerorum per additionem numerorum imparium tantum.

328. Reſtant in hoc genere caſus quidam memorabiles, quorum evolutio non omni utilitate carebit in numerorum natura cognoſcenda. Conſideretur nempe hæc expreſſio

$$(1 + x)$$

$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})(1 + x^{32})$ &c., C A P.

in qua Exponentes ipfius x in ratione dupla progrediuntur. ——— X V I.
Hæc expreffio fi evolvatur, reperietur quidem hæc Series

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{&c.};$$

quoniam vero dubium effe poteft, utrum hæc Series in infi-
nitum hac lege geometrica progrediatur, hanc ipfam Seriem
inveftigemus. Sit igitur

$$P = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16}) \text{ &c.,}$$

ac ponatur Series per evolutionem oriunda

$$P = 1 + \alpha x + \xi x^2 + \gamma x^3 + \delta x^4 + \varepsilon x^5 + \xi x^6 + \eta x^7 + \theta x^8 + \text{&c.,}$$

Patet autem fi loco x fcribatur $x x$, tum prodire productum
$(1 + xx)(1 + x^4)(1 + x^8)(1 + x^{16})(1 + x^{32})$ &c. $= \frac{P}{1 + x}$:
facta ergo in Serie eadem fubftitutione erit

$$\frac{P}{1 + x} = 1 + \alpha x^2 + \xi x^4 + \gamma x^6 + \delta x^8 + \varepsilon x^{10} + \xi x^{12} + \text{&c.,}$$

multiplicetur ergo per $1 + x$, eritque

$$P = 1 + x + \alpha x^2 + \alpha x^3 + \xi x^4 + \xi x^5 + \gamma x^6 + \gamma x^7 + \delta x^8 + \delta x^9 + \text{&c.,}$$

qui valor ipfius P fi cum fuperiori comparetur, habebitur

$$\alpha = 1; \ \xi = \alpha; \ \gamma = \alpha; \ \delta = \xi; \ \varepsilon = \xi; \ \xi = \gamma; \ \eta = \gamma; \ \text{&c.,}$$

erunt ergo omnes coëfficientes $= 1$, ideoque productum pro-
pofitum P evolutum dabit Seriem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \text{&c.,}$$

329. Cum igitur hic omnes ipfius x Poteftates, fingulæque
femel occurrant, ex forma producti $(1 + x)(1 + x^2)(1 + x^4)$ &c.,
fequitur, omnem numerum integrum ex terminis progreffionis

LIB. I. geometricæ duplæ 1, 2, 4, 8, 16, 32, &c., diverſis per additionem formari poſſe, hocque unico modo. Nota eſt hæc proprietas in praxi ponderandi, ſi enim habeantur pondera 1, 2, 4, 8, 16, 32, &c., librarum; his ſolis ponderibus omnia onera ponderari poterunt, niſi partes libræ requirant. Sic his decem ponderibus, nempe 1 ℔, 2 ℔, 4 ℔, 8 ℔, 16 ℔, 32 ℔, 64 ℔, 128 ℔, 256 ℔, 512 ℔, omnia pondera uſque ad 1024 ℔, librari poſſunt, & ſi unum pondus 1024 ℔, addatur omnibus oneribus uſque ad 2048 ℔, ponderandis ſufficient.

330. Oſtendi autem inſuper ſolet in praxi ponderandi paucioribus ponderibus, quæ ſcilicet in ratione geometrica tripla progrediantur, nempe 1, 3, 9, 27, 81, &c, librarum pariter omnia onera ponderari poſſe, niſi opus ſit fractionibus. In hac autem praxi pondera non ſolum uni lanci, ſed ambabus, uti neceſſitas exigit, imponi debent. Nititur ergo iſta praxis hoc fundamento, quod ex terminis progreſſionis geometricæ triplæ 1, 3, 9, 27, 81, &c., diverſis ſemper ſumendis per additionem ac ſubtractionem omnes omnino numeri produci queant; erit ſcilicet.

$$
\begin{array}{lll}
1 = 1 & \quad 5 = 9 - 3 - 1 & \quad 9 = 9 \\
2 = 3 - 1 & \quad 6 = 9 - 3 & \quad 10 = 9 + 1 \\
3 = 3 & \quad 7 = 9 - 3 + 1 & \quad 11 = 9 + 3 - 1 \\
4 = 3 + 1 & \quad 8 = 9 - 1 & \quad 12 = 9 + 3
\end{array}
$$

&c.

331. Ad hanc veritatem oſtendendam conſidero hoc productum infinitum

$$(x^{-1} + 1 + x^1)(x^{-3} + 1 + x^3)(x^{-9} + 1 + x^9)(x^{-27} + 1 + x^{27}) \, \&c.$$

$$= P,$$

quod evolutum alias non dabit Poteſtates ipſius x, niſi quarum Exponentes formari poſſint ex numeris 1, 3, 9, 27, 81, &c.,

ſive

TABULA

ad paginam 275 Tom. I.

	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2	2
3	1	2	3	3	3	3	3	3	3	3	3
4	1	3	4	5	5	5	5	5	5	5	5
5	1	3	5	6	7	7	7	7	7	7	7
6	1	4	7	9	10	11	11	11	11	11	11
7	1	4	8	11	13	14	15	15	15	15	15
8	1	5	10	15	18	20	21	22	22	22	22
9	1	5	12	18	23	26	28	29	30	30	30
10	1	6	14	23	30	35	38	40	41	42	42
11	1	6	16	27	37	44	49	52	54	55	56
12	1	7	19	34	47	58	65	70	73	75	76
13	1	7	21	39	57	71	82	89	94	97	99
14	1	8	24	47	70	90	105	116	123	128	131
15	1	8	27	54	84	110	131	146	157	164	169
16	1	9	30	64	101	136	164	186	201	212	219
17	1	9	33	72	119	163	201	230	252	267	278
18	1	10	37	84	141	199	248	288	318	340	355
19	1	10	40	94	164	235	300	352	393	423	445
20	1	11	44	108	192	282	364	434	488	530	560
21	1	11	48	120	221	331	436	525	598	653	695
22	1	12	52	136	255	391	522	638	732	807	863
23	1	12	56	150	291	454	618	764	887	984	1060
24	1	13	61	169	333	532	733	919	1076	1204	1303
25	1	13	65	185	377	612	860	1090	1291	1455	1586
26	1	14	70	206	427	709	1009	1297	1549	1761	1930
27	1	14	75	225	480	811	1175	1527	1845	2112	2331
28	1	15	80	249	540	931	1367	1801	2194	2534	2812
29	1	15	85	270	603	1057	1579	2104	2592	3015	3370
30	1	16	91	297	674	1206	1824	2462	3060	3590	4035
31	1	16	96	321	748	1360	2093	2857	3589	4242	4802
32	1	17	102	351	831	1540	2400	3319	4206	5013	5788
33	1	17	108	378	918	1729	2738	3828	4904	5888	6751
34	1	18	114	411	1014	1945	3120	4417	5708	6912	7972
35	1	18	120	441	1115	2172	3539	5066	6615	8070	9373
36	1	19	127	478	1226	2432	4011	5812	7657	9418	11004
37	1	19	133	511	1342	2702	4526	6630	8824	10936	12866
38	1	20	140	551	1469	3009	5102	7564	10156	12690	15021
39	1	20	147	588	1602	3331	5731	8588	11648	14663	17475
40	1	21	154	632	1747	3692	6430	9749	13338	16928	20298
41	1	21	161	672	1898	4070	7190	11018	15224	19466	23501
42	1	22	169	720	2062	4494	8033	12450	17354	22367	27169
43	1	22	176	764	2233	4935	8946	14012	19720	25608	31316
44	1	23	184	816	2418	5427	9953	15765	22380	29292	36043
45	1	23	192	864	2611	5942	11044	17674	25331	33401	41373
46	1	24	200	920	2818	6510	12241	19805	28629	38047	47420
47	1	24	208	972	3034	7104	13534	22122	32278	43214	54218
48	1	25	217	1033	3266	7760	14950	24699	36347	49037	61903
49	1	25	225	1089	3507	8442	16475	27493	40831	55494	70515
50	1	26	234	1154	3765	9192	18138	30588	45812	62740	80215
51	1	26	243	1215	4033	9975	19928	33940	51294	70760	91058
52	1	27	252	1285	4319	10829	21873	37638	57358	79725	103226
53	1	27	261	1350	4616	11720	23961	41635	64015	89623	116792
54	1	28	271	1425	4932	12692	26226	46031	71362	100654	131970
55	1	28	280	1495	5260	13702	28652	50774	79403	112804	148847
56	1	29	290	1575	5608	14800	31275	55974	88252	126299	167672
57	1	29	300	1650	5969	15944	34082	61575	97922	141136	188556
58	1	30	310	1735	6351	17180	37108	67696	108527	157564	211782
59	1	30	320	1815	6747	18467	40340	74280	120092	175586	237439
60	1	31	331	1906	7166	19858	43819	81457	132751	195491	266006
61	1	31	341	1991	7599	21301	47527	89162	146520	217280	297495
62	1	32	352	2087	8056	22856	51508	97539	161554	241279	332337
63	1	32	363	2178	8529	24473	55748	106522	177884	267507	370733
64	1	33	374	2280	9027	26207	60289	116263	195666	296320	413112
65	1	33	385	2376	9542	28009	65117	126692	214944	327748	459718
66	1	34	397	2484	10083	29941	70281	137977	235899	362169	511045
67	1	34	408	2586	10642	31943	75762	150042	258569	399705	567377
68	1	35	420	2700	11229	34085	81612	163069	283161	440725	629281
69	1	35	432	2808	11835	36308	87816	176978	309729	485315	697097

five addendo five fubtrahendo : num vero omnes Poteftates pro-
deant , fingulæque femel, fic exploro. Sit

$$P = \&c. + cx^{-3} + bx^{-2} + ax^{-1} + 1 + \alpha x^1 + \digamma x^2 + \gamma x^3 +$$
$$\delta x^4 + \varepsilon x^5 + \&c. ,$$

manifeftum vero eft, fi x^3 loco x fcribatur, tum prodire

$$\frac{P}{x^{-1} + 1 + x^1} = bx^{-6} + ax^{-3} + 1 + \alpha x^3 + \digamma x^6 + \gamma x^9 + \&c. .$$

Hinc igitur reperitur $P =$ &c.

$$+ ax^{-4} + ax^{-3} + ax^{-2} + x^{-1} + 1 + x + \alpha x^2 + \alpha x^3 + \alpha x^4 +$$
$$\digamma x^5 + \digamma x^6 + \digamma x^7 + \&c. ,$$

quæ expreffio cum affumta comparata dabit

$$\alpha = 1 ; \digamma = \alpha ; \gamma = \alpha ; \delta = \alpha ; \varepsilon = \digamma ; \xi = \digamma ; \&c. , \&$$
$$a = 1 , b = a , c = a , d = a , e = b , \&c. .$$

Hinc itaque erit

$$P = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + \&c.$$
$$+ x^{-1} + x^{-2} + x^{-3} + x^{-4} + x^{-5} + x^{-6} + x^{-7} + \&c. ,$$

unde patet omnes ipfius x Poteftates, tam affirmativas quam ne-
gativas, hic occurrere, atque adeo omnes numeros ex terminis
progreffionis geometricæ triplæ, vel addendo vel fubtrahendo ,
formari poffe ; & unumquemque numerum unico tantum modo.

M m 2 *a.*   CAPUT

CAPUT XVII.

De usu Serierum recurrentium in radicibus æquationum indagandis.

332. **I**Ndicavit *Vir Celeb. Daniel* BERNOULLI infignem ufum Serierum recurrentium in inveftigandis radicibus æquationum cujufvis gradus, in *Comment. Acad. Petropol. Tomo III.*, ubi oftendit, quemadmodum cujufque æquationis algebraicæ, quotcunque fuerit dimenfionum, valores radicum veris proximi ope Serierum recurrentium affignari queant. Quæ inventio, cum fæpenumero maximam afferat utilitatem, eam hic diligentius explicare conftitui, ut intelligatur, quibus cafibus adhiberi poffit. Interdum enim præter expeCtationem evenit, ut nulla æquationis radix ope hujus methodi cognofci queat. Quocirca, ut vis hujus methodi clarius perfpiciatur, ex proprietatibus Serierum recurrentium totum fundamentum, quo nititur, contemplemur.

333. Quoniam omnis Series recurrens ex evolutione cujufdam fraCtionis rationalis oritur, fit ifta fraCtio

$$= \frac{a + bz + cz^2 + dz^3 + ez^4 + \&c.}{1 - \alpha z - \beta z^2 - \gamma z^3 - \delta z^4 - \&c.},$$

unde oriatur fequens Series recurrens

$$A + Bz + Cz^2 + Dz^3 + Ez^4 + Fz^5 + \&c.,$$

cujus coëfficientes A, B, C, D, &c., ita determinantur ut fit

$$A = a$$

$$A = a$$
$$B = \alpha A + b$$
$$C = \alpha B + 6 A + c$$
$$D = \alpha C + 6 B + \gamma A + d$$
$$E = \alpha D + 6 C + \gamma B + \delta A + e$$
&c.

Terminus autem generalis, feu coëfficiens Poteftatis z^n, invenitur ex refolutione fractionis propofitæ in fractiones fimplices, quarum denominatores fint Factores denominatoris $1 - \alpha z - 6zz - \gamma z^3 - $ &c., uti (*Cap. XIII.*) eft oftenfum.

334. Forma autem termini generalis potiffimum pendet ab indole Factorum fimplicium denominatoris, utrum fint reales an imaginarii, & utrum fint inter fe inæquales & eorum bini plurefve æquales. Quos varios cafus ut ordine percurramus, ponamus primum omnes denominatoris Factores fimplices cum reales effe tum inter fe inæquales. Sint ergo Factores fimplices denominatoris omnes $(1-pz)(1-qz)(1-rz)(1-sz)$ &c., ex quibus fractio propofita in fequentes fractiones fimplices refolvatur $\frac{A}{1-pz} + \frac{B}{1-qz} + \frac{C}{1-rz} + \frac{D}{1-sz} + $ &c.. Quibus cognitis erit Seriei recurrentis terminus generalis $=$
$$z^n (A p^n + B q^n + C r^n + D s^n + \text{&c.}),$$ quem ftatuamus
$= P z^n$; fit fcilicet P coëfficiens Poteftatis z^n, fequentiumque Q, R, &c., ita ut Series recurrens fiat
$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots + Pz^n + Qz^{n+1} + Rz^{n+2} +$$
&c.

335. Ponamus jam n effe numerum maximum, feu Seriem recurrentem ad plurimos terminos effe continuatam; quoniam numerorum inæqualium Poteftates eo magis fiunt inæquales, quo fuerint altiores; tanta erit diverfitas in Poteftatibus

$A p^n$

$A p^n$, $B q^n$, $C r^n$, &c., ut ea, quæ oritur ex maximo nume-
rorum p, q, r, &c., reliquas magnitudine longe fuperet, præ
eaque reliquæ penitus evanefcant, fi n fuerit numerus plane
infinite magnus. Cum igitur numeri p, q, r, &c., fint inter
fe inæquales, ponamus inter eos p effe maximum; ac propte-
rea, fi n fit numerus infinitus, fiet $P = A p^n$; fin autem n fit

numerus vehementer magnus erit tantum proxime $P = A p^n$.

Simili vero modo erit $Q = A p^{n+1}$, ideoque $\frac{Q}{P} = p$. Un-
de patet, fi Series recurrens jam longe fuerit producta, coëffi-
cientem cujufque termini per præcedentem divifum proxime effe
exhibiturum valorem maximæ litteræ p.

336. Si igitur in fractione propofita

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \varepsilon z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

denominator habeat omnes Factores fimplices reales & inter
fe inæquales, ex Serie recurrente inde orta cognofci poterit
unus Factor fimplex, is fcilicet $1 - pz$, in quo littera p om-
nium maximum habet valorem. Neque in hoc negotio coëf-
ficientes numeratoris a, b, c, d. &c., in computum ingre-
diuntur, fed quicunque ii ftatuantur, tamen denique idem ve-
rus valor litteræ maximæ p invenitur. Verus quidem valor
ipfius p tum demum innotefcit, quando Series in infinitum
fuerit continuata; interim tamen fi jam plures ejus termini fue-
rint formati, eo propius valor ipfius p cognofcetur, quo ma-
jor fuerit terminorum numerus, & quo magis littera ifta p ex-
cedat reliquas q, r, s, &c.: perinde vero eft utrum hæc ma-
xima littera p fuerit figno $+$ an figno $-$ affecta, quoniam
ejus Poteftates æque increfcunt.

337. Quemadmodum nunc hæc inveftigatio ad inventio-
nem radicum æquationis cujufvis algebraicæ accommodari pof-

fit, fatis eft perfpicuum. Ex Factoribus enim denominatoris
$1 - \alpha z - \mathfrak{E} zz - \gamma z^3 - \delta z^4 - \&c.$, cognitis facile
affignantur radices æquationis hujus

$$1 - \alpha z - \mathfrak{E} z^2 - \gamma z^3 - \delta z^4 - \&c. = 0,$$

ita ut, fi Factor fuerit $1 - p z$, hujus æquationis radix una
futura fit $z = \frac{1}{p}$. Cum igitur ex Serie recurrente reperia-
tur maximus numerus p, indidem obtinebitur minima radix
æquationis $1 - \alpha z - \mathfrak{E} z^2 - \gamma z^3 - \&c. = 0$. Vel, fi
ponatur $z = \frac{1}{x}$ ut prodeat hæc æquatio

$$x^m - \alpha x^{m-1} - \mathfrak{E} x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

ejufdem methodi ope eruitur maxima hujus æquationis radix
$x = p$.

338. Si igitur proponatur æquatio hæc

$$x^m - \alpha x^{m-1} - \mathfrak{E} x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

quæ omnes radices habeat reales & inter fe inæquales, harum
radicum maxima fequenti modo reperietur. Formetur ex coëf-
ficientibus hujus æquationis fractio

$$\frac{a + b z + c z^2 + d z^3 + \&c.}{1 - \alpha z - \mathfrak{E} z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Hincque formetur Series recurrens, affumendo pro arbitrio
numeratorem, feu, quod eodem redit, affumendo pro libitu
terminos initiales; quæ fit

$$A + B z + C z^2 + D z^3 + \ldots\ldots\ldots + P z^n + Q z^{n+1}$$

dabitque fractio $\frac{Q}{P}$ valorem radicis maximæ x pro æquatione
propofita, eo propius, quo major fuerit numerus n.

EXEM-

Exemplum I.

Sit proposita ista æquatio xx — 3 x — 1 = o , *cujus maximam radicem inveniri oporteat.*

Formetur fractio $\frac{a + b z}{1 - 3 z - zz}$, unde positis duobus primis terminis 1 , 2 , orietur ista Series recurrens

1 , 2 , 7 , 23 , 76 , 251 , 829 , 2738 , &c. ,

erit ergo $\frac{2738}{829}$ proxime æqualis radici æquationis propositæ maximæ. Valor autem hujus fractionis in partibus decimalibus expressus est

3 , 3027744

æquationis vero radix maxima est $= \frac{3 + \sqrt{13}}{2} =$

3 , 3027756,

quæ inventam superat tantum una parte millionesima. Ceterum notandum est fractiones $\frac{Q}{P}$ alternatim vera radice esse majores & minores.

Exemplum II.

Proposita sit ista æquatio 3 x — 4 x³ $= \frac{1}{2}$ *cujus radices exhibent Sinus trium Arcuum , quorum triplorum Sinus est* $= \frac{1}{2}$.

Æquatione perducta ad hanc formam o = 1 — 6 x $*$ + 8x³ , quæratur hujus, ut in numeris integris maneamus, radix minima, ita ut non opus sit pro x ponere $\frac{1}{2}$. Formetur ergo hæc fractio

$$\frac{a + b x + c x x}{1 - 6x * + 8x^3}$$

ex qua sumendis pro lubitu tribus terminis initialibus o, o, 1,

quia

quia hoc modo calculus facillime expeditur, orietur hæc Se- C A P.
ries recurrens, omittendis poteſtatibus ipſius x quia tantum X V I I.
coëfficientibus opus eſt,

$$0; 0; 1; 6; 36; 208; 1200; 6912; 39808; 229248.$$

Erit ergo proxime æquationis radix minima $= \frac{39808}{229248} =$

$\frac{311}{1791} = 0, 1736515$, quæ propterea eſſe deberet Sinus anguli
$10°$; hic autem ex tabulis eſt $0, 1736482$, quem ſuperat radix
inventa parte $\frac{33}{10000000}$. Facilius autem hæc eadem radix in-
veniri poteſt ponendo $x = \frac{1}{2} y$, ut prodeat æquatio $1 —$
$3y \ast + y^3 = 0$, ex qua ſimili modo tractata oritur Series

$$0, 0, 1, 3, 9, 26, 75, 216, 622, 1791, 5157 \ \&c.,$$

erit ergo proxime æquationis radix minima $y = \frac{1791}{5157} =$

$\frac{199}{573} = 0, 3472949$, unde fit $x = \frac{1}{2} y = 0, 1736479$, qui
valor decies propius accedit quam præcedens.

E X E M P L U M   I I I.

Si deſideretur ejuſdem æquationis propoſitæ $0 = 1 — 6x \ast +$
$8x^3$, *radix maxima.*

Ponatur $x = \frac{y}{2}$, eritque $y^3 \ast — 3y + 1 = 0$. Cujus
æquationis radix maxima reperietur per Seriem recurrentem
cujus ſcala relationis eſt $0, 3, — 1$, unde ergo oritur, ſumtis
tribus terminis initialibus pro arbitrio,

$$1, 1, 1, 2, 2, 5, 4, 13, 7, 35, 8, 98, — 11, \&c.,$$

in qua Serie cum ad terminos negativos perveniatur, id indi-
cio eſt maximam radicem eſſe negativam, eſt enim $x = —$

Euleri *Indroduct. in Anal. infin. parv.*   N n   *ſin.*

Lıв. I. *fin.* 70° $= - 0,9396926$. Quare hujus ratio in terminis initialibus eft habenda, hoc modo

$$1-2+4-7+14-25+49-89+172-316+605- \&c.,$$

ex qua erit $y = \frac{-605}{316}$ & $x = \frac{-605}{632} = -0,957$, quæ a veritate vehementer abludit.

339. Ratio hujus diffenfus potiffimum eft, quod æquationis propofitæ radices fint *fin.* 10°, *fin.* 50°, & — *fin.* 70°, quarum binæ maximæ tam parum a fe invicem difcrepant, ut in Poteftatibus, ad quas Seriem continuavimus, fecunda radix *fin.* 50° adhuc notabilem teneat rationem ad radicem maximam, ideoque præ ea non evanefcant. Hincque etiam faltus pendet, quod alternatim valores inventi fiant nimis magni & nimis parvi : Sic, fumendo

$$y = \frac{-316}{172}, \text{fit } x = \frac{-158}{172} = \frac{-79}{86} = -0,918.$$

Nam, quoniam Poteftates radicis maximæ alternatim fiunt affirmativæ & negativæ, alternatim quoque Poteftates fecundæ radicis adduntur & tolluntur : quamobrem, quo hæc difcrepantia fiat infenfibilis, Series vehementer ulterius debet continuari.

340. Aliud vero remedium huic incommodo afferri poteft, tranfmutando æquationem ope idoneæ fubftitutionis in aliam formam, cujus radices fibi non amplius fint tam vicinæ. Sic, fi in æquatione $0 = 1 - 6x + 8x^3$ cujus radices funt — *fin.* 70°, + *fin.* 50°, + *fin.* 10°, ponatur $x = y - 1$, æquationis $0 = 8y^3 - 24yy + 18y - 1$ radices erunt $1 -$ *fin.* 70°; $1 +$ *fin.* 50°; $1 +$ *fin.* 10°; ideoque ejus radix minima erit $1 -$ *fin.* 70°, cum tamen hæc *fin.* 70° effet radix maxima æquationis præcedentis; atque $1 +$ *fin.* 50° nunc eft radix maxima, cum *fin.* 50° ante effet media. Atque hoc modo quævis radix per fubftitutionem in maximam minimamve radicem novæ æquationis tranfmutari, ideoque per methodum hic traditam inveniri poterit.

poterit. Quia præterea in hoc exemplo radix $1 - fin. 70°$ Cap. multo minor eft, quam binæ reliquæ, etiam facile per Seriem XVII. recurrentem proxime cognofcetur.

Exemplum IV.

Invenire radicem minimam æquationis $0 = 8y^3 - 24yy + 8y - 1$, *quæ ab unitate fubtracta relinquet Sinum anguli 70°.*

Ponatur $y = \frac{1}{2} z$, ut fit $0 = z^3 - 6zz + 9z - 1$, cujus radix minima invenietur per Seriem recurrentem, cujus fcala relationis eft $9, -6, +1$, pro radice autem maxima invenienda fcala relationis fumi deberet $6, -9, +1$. Pro minima ergo formetur hæc Series

$$1, 1, 1, 4, 31, 256, 2122, 17593; 145861; \&c.,$$

erit ergo proxime $z = \frac{17593}{145861} = 0, 12061483$ & $y = 0, 06030741$, atque *fin.* $70° = 1 - y = 0, 93969258$, quæ a veritate ne in ultima quidem figura difcrepat. Ex hoc ergo exemplo intelligitur quantam utilitatem idonea transformatio æquationis ope fubftitutionis ad inventionem radicum afferat, & quod hoc pacto methodus tradita non folum ad maximas minimafve radices adftringatur, fed etiam omnes radices exhibere queat.

341. Cognita ergo jam quacunque æquationis propofitæ radice proxime, ita ut, verbi gratia, numerus k quam minime a quapiam radice differat, ponatur $x - k = y$ feu $x = y + k$, hocque modo prodibit æquatio, cujus radix minima erit $= x - k$, quæ igitur fi per Series recurrentes indagetur, quod facillime fiet, quia hæc radix multo minor erit, quam ceteræ, fi ea ad k addatur habebitur radix vera ipfius x, pro æquatione propofita. Hoc vero artificium tam late patet, ut etiamfi æquatio contineat radices imaginarias, ufum fuum retineat.

342. Imprimis autem fine hoc artificio radix cognofci nequit,

L ɪ ʙ. I. quit, cui datur alia æqualis fed figno contrario affecta. Scili-
cet, fi æquatio cujus maxima radix p, eadem radicem habeat
$— p$, tum, etiamfi Series recurrens in infinitum continuetur,
tamen radix hæc p nunquam obtinebitur. Sit, ut hoc exemplo
illuftremus, propofita æquatio $x^3 — x^2 — 5x + 5 = 0$, cujus
maxima radix eft $\sqrt{5}$, præter quam vero ineft quoque $—\sqrt{5}$.
Si igitur modo ante præfcripto, pro radice maxima invenien-
da, utamur, atque Seriem recurrentem formemus ex fcala re-
lationis 1, $+5$, $— 5$, quæ erit

$$1, 2, 3, 8, 13, 38, 63, 188, 313, 938, 1563, \&c.,$$

ubi ad nullam rationem conftantem pervenitur. Termini vero
alterni rationem æquabilem induunt, quorum fi quifque per
præcedentem dividatur, reperietur quadratum maximæ radicis,
fic enim eft proxime $5 = \frac{1563}{313} = \frac{938}{188} = \frac{313}{63}$. Quoties ergo
termini tantum alterni fefe ad rationem conftantem componunt,
toties quadratum radicis quæfitæ proxime obtinetur. Ipfa au-
tem radix $x = \sqrt{5}$ invenitur ponendo $x = y + 2$ unde fit
$1 — 3y — 5yy — y^3 = 0$, cujus radix minima cognofcetur ex
Serie

$$1, 1, 1, 9, 33, 145, 609, 2585, 10945, \&c.,$$

erit enim proxime $= \frac{2585}{10945} = 0,2361$, at $2,2361$ eft pro-
xime $= \sqrt{5}$, quæ eft radix maxima æquationis.

343. Quanquam numerator fractionis, ex qua Series recur-
rens formatur, a noftro arbitrio pendet, tamen idonea ejus
conftitutio plurimum confert, ut valor radicis cito vero pro-
xime exhibeatur. Cum enim affumtis, ut fupra, Factoribus de-
nominatoris (334.), fit terminus generalis Seriei recurrentis $=$
$z^n (A p^n + B q^n + C r^n + \&c.)$, ifti coëfficientes A, B, C,
&c., per numeratorem fractionis determinantur; unde fieri po-
teft, ut A five magnum five parvum valorem obtineat: priori
cafu radix maxima p cito reperitur, pofteriore vero tarde.
Quin etiam numerator ita accipi poteft ut A prorfus evanefcat,
quo

quo cafu, etiamfi Series in infinitum continuetur, tamen nunquam radicem maximam *p* præbebit. Hoc autem evenit fi numerator ita accipiatur, ut ipfe eundem habeat Factorem $1 - pz$, fic enim ex computo penitus tolletur. Sic, fi proponatur æquatio $x^3 - 6xx + 10x - 3 = 0$, cujus maxima radix eft $= 3$, indeque formetur fractio

$$\frac{1 - 3z}{1 - 6z + 10z^2 - 3z^3}$$

ut Seriei recurrentis fit fcala relationis 6, — 10, + 3

$$1 + 3, 8, 21, 55, 144, 377, \&c.,$$

cujus termini prorfus non convergunt ad rationem, 1 : 3. Eadem enim Series oritur ex fractione $\dfrac{1}{1 - 3z + zz}$, ac propterea maximam radicem æquationis $x^2 - 3x + 1 = 0$ exhibet.

344. Quin etiam numerator ita affumi poteft, ut per Seriem recurrentem quævis radix æquationis reperiatur, quod fiet fi numerator fuerit productum ex omnibus Factoribus denominatoris præter eum, cui refpondet radix quam velimus. Sic, fi in priori exemplo fumatur numerator $1 - 3z + zz$, fractio $\dfrac{1 - 3z + zz}{1 - 6z + 10z^2 - 3z^3}$, dabit hanc Seriem recurrentem 1, 3, 9, 27, 81, 243, &c., quæ, cum fit geometrica, ftatim monftrat radicem $x = 3$. Fractio enim illa æqualis eft huic fimplici $\dfrac{1}{1 - 3z}$. Hinc apparet, fi termini initiales, quos pro lubitu affumere licet, ita accipiantur, ut progreffionem geometricam conftituant, cujus Exponens æquetur uni radici æquationis, tum totam Seriem recurrentem fore geometricam, ideoque eam ipfam radicem effe exhibituram, etiamfi neque fit maxima neque minima.

345. Ne igitur, dum quærimus radicem vel maximam vel minimam, præter expectationem nobis alia radix per Seriem recurrentem exhibeatur, ejufmodi numerator debet eligi, qui

cum

cum denominatore nullum Factorem habeat communem, quod fiet si pro numeratore unitas accipiatur, unde terminus primus Seriei erit $= 1$, ex quo solo secundum scalam relationis sequentes omnes definiantur. Hocque modo semper certe radix æquationis vel maxima vel minima, prout fuerit propositum, eruetur. Sic, proposita æquatione $y^3 \ast - 3y + 1 = 0$, cujus radix maxima desideratur, ex scala relationis $0, + 3, - 1$ incipiendo ab unitate sequens oritur Series recurrens

$$1 - 0 + 3 - 1 + 9 - 6 + 28 - 27 + 90 - 109 + 297$$
$$- 517 + 1000 - 1848 + 3517 - 6544 + \&c.,$$

quæ manifesto ad rationem constantem convergit, ostenditque

radicem maximam esse negativam, atque proxime $y = \dfrac{-6544}{3517}$

$= - 1,860676$, quæ esse debebat $= - 1,86793852$. Ratio autem supra est allata, cur tam lente ad verum valorem appropinquetur, propterea quod altera radix non multo sit minor maxima, simulque sit affirmativa.

346. His probe perpensis, quæ cum in genere tum ad exempla allata monuimus, summa utilitas hujus methodi ad investigandas æquationum radices luculenter perspicietur; artificia vero, quibus operatio contrahi, eoque promtior reddi queat, satis quoque sunt indicata; ita ut nihil insuper addendum esset, nisi casus, quibus æquatio vel radices habet æquales vel imaginarias, evolvendi superessent. Ponamus ergo denominatorem fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \delta z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

habere Factorem $(1 - pz)^2$, reliquis Factoribus existentibus $1 - qz$, $1 - rz$, &c.. Seriei ergo recurrentis hinc natæ terminus generalis erit $= z^n ((n+1) A p^n + B p^n + C q^n + \&c.)$, quæ cujusmodi valorem sit adeptura, si n fuerit numerus vehemen-

vehementer magnus , duo cafus funt diftinguendi , alter quo p C A P.
eft numerus major reliquis q, r, &c., alter quo p non præbet X V I I.
radicem maximam. Cafu priori , quo p fimul eft radix maxima, ———
ob coëfficientem ($n + 1$) reliqui termini $B p^n + C q^n$ &c. ,
non tam cito præ eo evanefcent , quam ante : fin autem q fue-
rit $> p$, tum quoque tarde terminus ($n + 1$) $A p^n$ præ $B q^n$
evanefcet , ideoque inveftigatio radicis maximæ admodum eva-
det molefta.

E X E M P L U M   I.

Sit propofita æquatio $x - 3xx + 4 = 0$, *cujus maxima ra-
dix 2 bis occurrit.*
   Quæratur ergo maxima radix hæc modo ante expofito per
evolutionem fractionis

$$\frac{1}{1 - 3z \ * \ + 4z^3}$$

quæ dabit hanc Seriem recurrentem

$$1, 3, 9, 23, 57, 135, 313, 711, 1593, \&c.,$$

ubi quidem quivis terminus per præcedentem divifus dat quo-
tum binario majorem. Cujus ratio ex termino generali facil-
lime patet, rejectis enim in eo terminis $B p^n$, $C q^n$ &c. , erit
terminus poteftati z^n refpondens $= (n + 1) A p^n + B p^n$, fe-
quens $= (n + 2) A p^{n + 1} + B p^{n + 1}$, qui per illum divifus dat
$\frac{(n + 2) A + B}{(n + 1) A + B} p > p$, nifi n jam in infinitum excreverit.

E X E M-

EXEMPLUM II.

Sit jam proposita æquatio $x^3 - xx - 5x - 3 = 0$, *cujus maxima radix* $= 3$, *reliquæ duæ æquales* $= -1$, & quæratur maxima radix ope Seriei recurrentis, cujus scala relationis est $1, +5, +3$; unde oritur

$$1, 1, 6, 14, 47, 135, 412, 1228, \&c.,$$

quæ ideo satis cito valorem 3 exhibet, quod Potestates minoris radicis -1, etiamsi multiplicentur per $n + 1$, tamen mox præ Potestatibus ipsius 3 evanescant.

EXEMPLUM III.

Sin autem proponeretur æquatio $x^3 + xx - 8x - 12 = 0$, cujus radices sunt $3, -2, -2$, multo tardius maxima sese prodet. Orietur enim hæc Series

$$1, -1, 9, -5, 65, 3, 457, 347, 3345, 4915, \&c.,$$

quæ adhuc longissime continuari deberet, antequam pateret, radicem inde oriundam esse $= 3$.

347. Simili modo si tres Factores essent æquales, ita ut denominatoris Factor unus esset $(1 - pz)^3$, reliqui $1 - qz$, $1 - rz$, &c., Seriei recurrentis terminus generalis erit $=$
$$z^n \left(\frac{(n+1)(n+2)}{1 \cdot 2} A p^n + (n+1) B p^n + C p^n + D q^n + C r^n \&c. \right)$$
Si ergo p fuerit maxima radix, atque n fuerit numerus tantus, ut Potestates q^n, r^n &c. præ p^n evanescant, tum ex Serie recurrente orietur radix $=$

$$\frac{\frac{1}{2}(n+2)(n+3)A + (n+2)B + C}{\frac{1}{2}(n+1)(n+2)A + (n+1)B + C} p,$$

quæ, nisi sit n numerus maximus & quasi infinitus, verum ipsius

fius *p* valorem indicabit. Erit autem ifte radicis valor $= p +$

$$\frac{(n + 2)\ A + B}{\frac{1}{2}(n + 1)(n + 2)\ A + (n + 1)\ B + C} p.$$

Quod fi autem *p* non fuerit radix maxima, tum inventio ma-
xime multo magis adhuc impedietur; unde fequitur æquatio-
nes, quæ contineant radices æquales, hac methodo per Series
recurrentes multo difficilius refolvi, quam fi omnes radices ef-
fent inter fe inæquales.

348. Videamus nunc quomodo Series recurrens in infinitum
continuata debeat effe comparata, quando denominator frac-
tionis habet Factores imaginarios. Sint igitur fractionis

$$\frac{a + bz + cz^2 + dz^3 + \&c.}{1 - \alpha z - \varsigma z^2 - \gamma z^3 - \delta z^4 - \&c.}$$

Factores denominatoris reales $1 - qz$, $1 - rz$, &c., infu-
perque Factor trinomialis $1 - 2pz.\ cof.\ \phi + ppzz$ continens
duos Factores fimplices imaginarios. Quod fi ergo Series re-
currens ex illa fractione orta fuerit

$$A + Bz + Cz^2 + Dz^3 + \ldots\ldots\ldots + Pz^n + Qz^{n+1},$$

erit, per ea quæ fupra expofuimus, coëfficiens $P =$

$$\frac{A.\ fin.\ (n + 1)\ \phi + B.\ fin.n\phi}{fin.\ \phi}\ p^n + C q^n + D r^n + \&c..$$ Si igitur
numerus *p* minor fuerit, quam unus ceterorum *q*, *r*, &c.,
ita ut maxima radix æquationis

$$x^m - ax^{m-1} - \varsigma x^{m-2} - \gamma x^{m-3} - \&c. = 0,$$

fit realis, tum ea per Series recurrentes æque reperietur, ac fi
nullæ radices ineffent imaginariæ.

349. Inventio ergo maximæ radicis realis per radices imagi-
narias non perturbabitur, fi hæ ita fuerint comparatæ, ut bi-
narum, quæ Factorem realem componunt, productum non fit

LIB. I. majus quadrato radicis maximæ. Sin autem binæ ejufmodi
infint radices imaginariæ, ut earum productum adæquet vel
adeo fuperet quadratum maximæ radicis realis, tum inveftiga-
tio ante expofita nihil declarabit, propterea quod Poteftas
p^n, præ fimili Poteftate radicis maximæ nunquam evanefcit,
etiamfi Series in infinitum continuetur. Cujus exempla illuftra-
tionis caufa hic adjicere vifum eft.

E X E M P L U M I.

Sit propofita æquatio $x^3 - 2x - 4 = 0$, *cujus radicem ma-*
ximam inveftigari oporteat.

Refolvitur hæc æquatio in duos Factores $(x - 2)(xx +
2x + 2)$; unde unam habet radicem realem 2 & duas reliquas
imaginarias, quarum productum eft 2, minus quam quadratum
radicis realis. Quam ob rem ea per modum hactenus traditum
cognofci poterit. Formetur ergo Series recurrens ex fcala re-
lationis 0, $+ 2$, $+ 4$, quæ erit

1, 0, 2, 4, 4, 16, 24, 48, 112, 192, 416, 832, &c.,

unde fatis luculenter radix realis 2 cognofci poteft.

E X E M P L U M II.

Propofita fit æquatio $x^3 - 4xx + 8x - 8 = 0$, *cujus radix*
una realis eft 2, *binarum imaginariarum productum vero* $= 4$,
ideoque æquale quadrato radicis realis 2.

Quæramus ergo radicem per Seriem recurrentem, quod quo
facilius fieri queat, ponamus $x = 2y$, ut habeatur $y^3 -
2yy + 2y - 1 = 0$, unde formetur Series recurrens

1, 2, 2, 1, 0, 0, 1, 2, 2, 1, 0, 0, 1, 2, 2, 1, &c.,

in qua cum iidem termini perpetuo revertantur, nihil inde
aliud

aliud colligi poteſt , niſi radicem maximam vel non eſſe rea- CA P.
lem , vel dari imaginarias , quarum productum æquale ſit aut ſu- XVII.
peret quadratum radicis realis.

EXEMPLUM III.

Sit jam propoſita æquatio $x^3 - 3xx + 4x - 2 = 0$, *cujus*
radix realis eſt 1 , *imaginariarum vero productum* = 2.
Formetur ergo ex ſcala relationis 3 , — 4 , + 2 , Series

$$1, 3, 5, 5, 1, -7, -15, -15, —, +1, 33, 65, 65, 1, \&c.,$$

in qua cum termini modo fiant affirmativi , modo negativi ,
radix realis 1 inde nullo modo cognoſci poterit. Hujuſmodi
vero revolutiones ſemper oſtendunt radicem , quam Series præ-
bere debebat , eſſe imaginariam ; hic enim radices imaginariæ
poteſtate ſunt majores quam realis 1.

350. Sit igitur in fractione generali productum binarum ra-
dicum imaginariarum pp majus quam ullius radicis realis qua-
dratum , ita ut præ p^n reliquæ poteſtates q^n, r^n , &c. , eva-
neſcant ſi n ſit numerus infinitus. Hoc ergo caſu fiet $P =$
$\frac{A.ſin.(n+1)\phi + B.ſin.n\phi}{ſin. \phi} p^n$, & $Q = \frac{A.ſin(n+2)\phi + B.ſin.(n+1)\phi}{ſin. \phi} p^{n+1}$
ideoque $\frac{Q}{P} = \frac{A. ſin.(n+2) \phi + B. ſin.(n+1) \phi}{A. ſin.(n+1) \phi + B. ſin. n \phi} p$. Quæ ex-
expreſſio nunquam valorem conſtantem induet , etiamſi n ſit
numerus infinitus. Sinus enim Angulorum perpetuo maxime
manent mutabiles , ita ut mox ſint affirmativi mox negativi.

351. Interim tamen ſi fractiones ſequentes $\frac{R}{Q}$, $\frac{S}{R}$ ſimili
modo ſumantur , indeque litteræ A & B eliminentur , ſimul nu-
merus n ex calculo egredietur ; reperietur enim $Ppp + R =$
$2 Qp. \mathit{cof}. \phi$, unde fit $\mathit{cof}. \phi = \frac{Ppp + R}{2 Qp}$; ſimiliter vero erit

$\mathit{cof}. \phi =$

$cof. \phi = \dfrac{Qpp + S}{2Rp}$, ex quorum duorum valorum compa-

ratione fit $p = \sqrt{\dfrac{RR - QS}{QQ - PR}}$, atque $cof. \phi =$

$\dfrac{QR - PS}{2\sqrt{(Q^2 - PR)(R^2 - QS)}}$. Quam ob rem fi Series recurrens

jam eo ufque fuerit continuata, ut præ p^n reliquarum radicum

Poteftates evanefcant, tum hoc modo Factor trinomialis $1 -$

$2pz.cof.\phi + ppzz$ poterit inveniri.

352. Quoniam ifte calculus non fatis exercitatis moleftiam

creare poffet, eum totum hic apponam. Ex valore ipfius $\dfrac{Q}{P}$ in-

vento oritur $A P. p. fin. (n + 2) \phi + B Pp. fin. (n + 1) \phi =$

$A Q. fin. (n + 1) \phi + B Q. fin. n\phi$, unde fit $\dfrac{A}{B} =$

$\dfrac{Q. fin. n\phi - Pp. fin. (n + 1)\phi}{Pp. fin. (n + 2)\phi - Q. fin. (n + 1)\phi}$. Pari ratione erit $\dfrac{A}{B} =$

$\dfrac{R. fin. (n + 1)\phi - Qp. fin. (n + 2)\phi}{Qp. fin. (n + 3)\phi - R. fin. (n + 2)\phi}$; æquatis his duobus

valoribus fiet

$0 = QQp. fin. n\phi. fin. (n + 3)\phi - QR. fin. n\phi. fin. (n + 2)\phi -$
$PQpp. fin.(n+1)\phi. fin.(n+3)\phi - QQp. fin.(n+1)\phi. fin.(n+2)\phi +$
$QR. fin.(n+1)\phi. fin.(n+1)\phi + PQpp. fin.(n+1)\phi. fin.(n+2)\phi.$

Cum autem fit $fin.a \; fin.b = \dfrac{1}{2}. cof.(a - b) - \dfrac{1}{2}. cof. (a + b)$

fiet $0 = \dfrac{1}{2} QQp. (cof.3\phi - cof.\phi) + \dfrac{1}{2} QR.(1 - cof.2\phi) +$

$\dfrac{1}{2} PQpp. (1 - cof. 2\phi)$ quæ per $\dfrac{1}{2} Q$ divifa dat

$(Ppp. + R)(1 - cof.2\phi) = Qp. (cof.\phi - cof.3\phi)$. At eft

$cof.\phi = cof.2\phi.cof.\phi + fin. 2\phi.fin.\phi$ & $cof.3\phi = cof.2\phi.cof.\phi -$
$fin.2\phi \; fin.\phi$ unde $cof.\phi - cof.3\phi = 2 fin.2\phi.fin.\phi = 4 fin.\phi^2 \times$
$cof.\phi$ & $1 - cof.\phi = 2 fin.\phi^2$, ex quo erit $Ppp + R =$

$2 Qp.cof.\phi$, & $cof. \phi = \dfrac{Ppp + R}{2 Qp}$; atque $cof.\phi = \dfrac{Qpp + S}{2 Rp}$: unde

superiores

superiores valores prodeunt, scilicet $p = \sqrt{\dfrac{RR - QS}{QQ - PR}}$ & cof. $\varphi =$

$$\frac{QR - PS}{2\sqrt{(Q^2 - PR)(RR - QS)}}.$$

353. Si denominator fractionis, ex qua Series recurrens formatur, plures habeat Factores trinomiales inter se æquales, tum, spectata forma termini generalis supra data, patebit inventionem radicum multo magis fieri incertam. Interim tamen si una quæcunque radix realis jam proxime fuerit detecta, tum æquationis transformatione semper valor ejusdem radicis multo propior eruetur. Ponatur enim x æqualis valori illi jam detecto $+y$, atque novæ æquationis quæratur minima radix pro y, quæ addita ad illum valorem præbebit verum ipsius x valorem.

E X E M P L U M.

Sit proposita ista æquatio $x^3 - 3xx + 5x - 4 = 0$, *cujus unam radicem fere esse* $= 1$ *inde constat, quod, posito* $x = 1$, *prodit* $x^3 - 3xx + 5x - 4 = -1$.

Ponatur ergo $x = 1 + y$, fietque $1 - 2y - y^3 = 0$, unde pro radice minima invenienda formetur Series recurrens, cujus scala relationis $2, 0, +1$, quæ erit

$$1, 2, 4, 9, 20, 44, 97, 214, 472, 1041, 2296, \&c.,$$

unde radix minima ipsius y erit proxime $\dfrac{1041}{2296} = 0,453397$, ita ut sit $x = 1,453397$, qui valor tam prope vix alia methodo æque facile obtineri poterit.

354. Quod si autem Series quæcunque recurrens tandem tam prope ad progressionem geometricam convergat, tum ex ipsa lege progressionis statim facile cognosci poterit, cujusnam æquationis radix sit futura quotus qui ex divisione unius termini per præcedentem oritur. Sint

$P, Q,$

$$P, \ Q, \ R, \ S, \ T, \ \&c. \ ,$$

termini Seriei recurrentis a principio jam longiſſime remoti , ita ut cum progreſſione geometrica confundantur ; ſitque $T =$ $\alpha S + 6 R + \gamma Q + \delta P$, ſeu ſcala relationis $\alpha, +6, +\gamma + \delta$. Ponatur valor fractionis $\dfrac{Q}{P} = x$; erit $\dfrac{R}{P} = xx$; $\dfrac{S}{P} = x^3$ &

$\dfrac{T}{P} = x^4$, qui in ſuperiori æquatione ſubſtituti dabunt

$$x^4 = \alpha x^3 + 6x^2 + \gamma x + \delta.$$

unde patet quotum $\dfrac{Q}{P}$ tandem præbere radicem unam æquationis inventæ. Hoc vero & præcedens methodus indicat , præterea vero docet fractionem $\dfrac{Q}{P}$ dare maximam æquationis radicem.

355. Poteſt quoque hæc methodus inveſtigandarum radicum ſæpenumero utiliter adhiberi , ſi æquatio ſit infinita. Ad quod oſtendendum propoſita ſit æquatio $\dfrac{1}{2} = z - \dfrac{z^3}{6} +$ $\dfrac{z^5}{120} - \dfrac{z^7}{5040} + \&c.$, cujus radix minima z exhibet Arcum $30°$, ſeu Semiperipheriæ Circuli ſextantem. Perducatur ergo æquatio ad hanc formam

$$1 - 2z + \frac{z^3}{3} - \frac{z^5}{60} + \frac{z^7}{2520} - \&c. = 0.$$

Hinc ergo formetur Series recurrens , cujus ſcala relationis eſt infinita , ſcilicet

$$2, \ 0, \ - \frac{1}{3}, \ 0, \ + \frac{1}{60}, \ 0, \ - \frac{1}{2520}, \ 0 \ \&c.,$$

eritque Series recurrens

$$1, \ 2, \ 4, \ \frac{23}{3}, \ \frac{44}{3}, \ \frac{1681}{60}, \ \frac{2408}{45}, \ \&c.,$$

erit

erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1681.3}{2408.4} = \frac{5043}{9632} = 0,52356:$

At ex proportione Peripheriæ ad Diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero discrepet. Hoc autem in hac æquatione commode usu venit, quod ejus omnes radices sint reales, atque a minima reliquæ satis notabiliter discrepent. Quæ conditio cum rarissime in æquationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

C A P U T XVIII.

De fractionibus continuis.

356. QUoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim jam aliquoties ejusmodi specimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipsam Arithmeticam & Algebram communem non contemnenda subsidia affert ista speculatio, quæ hoc Capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco ejusmodi fractionem, cujus denominator constat ex numero integro cum fractione, cuius denominator denuo est aggregatum ex integro & fractione, quæ porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Hujusmodi ergo fractio continua erit sequens expressio

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f +}}}}}\&c. \qquad \text{vel } a + \cfrac{\alpha}{b + \cfrac{\mathfrak{G}}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f +}}}}}\&c.,$$

in quarum forma priori omnes fractionum numeratores funt unitates, quam potiffimum hic contemplabor, in altera vero forma funt numeratores numeri quicunque.

358. Expofita ergo fractionum harum continuarum forma, primum videndum eft, quemadmodum earum fignificatio confueto more expreffa inveniri queat. Quæ ut facilius inveniri poffit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in fecunda, poft in tertia & ita porro fractione; quo facto patebit fore

$$a \qquad\qquad = a$$

$$a + \frac{1}{b} \qquad\qquad = \frac{ab + 1}{b}$$

$$a + \cfrac{1}{b + \cfrac{1}{c}} \qquad\qquad = \frac{abc + a + c}{bc + 1}$$

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}} = \frac{abcd + ab + ad + cd + 1}{bcd + b + d}$$

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}} = \frac{abcde + abe + ade + cde + abc + a + c + e}{bcde + be + de + bc + 1}$$

&c.

359. Etfi in his fractionibus ordinariis non facile lex, fecundum quam numerator ac denominator ex litteris *a, b, c, d,* &c., componantur, perfpicitur, tamen attendenti ftatim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator eft aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex nu-

meratore

meratore penultimo fimplici: eademque lex in denominatori- C A P.
bus obfervatur. Scriptis ergo ordine litteris *a*, *b*, *c*, *d*, &c., X V I I I.
ex iis fractiones inventæ facile formabuntur hoc modo

$$\overset{a}{\frac{1}{0}}\;;\;\overset{b}{\frac{a}{1}}\;;\;\overset{c}{\frac{ab+1}{b}}\;;\;\overset{d}{\frac{abc+a+c}{bc+1}}\;;\;\overset{e}{\frac{abcd+ab+ad+cd+1}{bcd+b+d}}$$

ubi quilibet numerator invenitur, fi præcedentium ultimus per
indicem fupra fcriptum multiplicetur atque ad productum ante-
penultimus addatur; quæ eadem lex pro denominatoribus va-
let. Quo autem hac lege ab ipfo initio uti liceat, præfixi
fractionem $\frac{1}{0}$ quæ, etiamfi e fractione continua non oriatur,
tamen progreffionis legem clariorem efficit. Quælibet autem
fractio exhibet valorem fractionis continuæ ufque ad eam litte-
ram, quæ antecedenti imminet, inclufive continuata.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\mathfrak{b}}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\epsilon}{f+\&c.}}}}}\;,$$

dabit, prout aliis aliifque locis abrumpitur, fequentes valores

$$a \qquad\qquad = a$$
$$a+\frac{\alpha}{b} \qquad\qquad = \frac{ab+\alpha}{b}$$
$$a+\cfrac{\alpha}{b+\cfrac{\mathfrak{b}}{c}} \qquad = \frac{abc+\mathfrak{b}a+\alpha c}{bc+\mathfrak{b}}$$
$$a+\cfrac{\alpha}{b+\cfrac{\mathfrak{b}}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\mathfrak{b}ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\mathfrak{b}d+\gamma b}$$

&c.,

L ɪ ʙ. I. quarum fractionum quæque ex binis præcedentibus fequentem in modum invenietur

$$
\frac{\mathrm{I}}{\underset{\alpha}{0}} \;;\; \frac{\overset{b}{a}}{\underset{6}{\mathrm{I}}} \;;\; \frac{\overset{c}{ab+\alpha}}{\underset{\gamma}{bc}} \;;\; \frac{\overset{d}{abc+6a+\alpha c}}{\underset{\delta}{bc+6}} \;;\; \frac{\overset{e}{abcd+6ad+\alpha cd+\gamma ab+\alpha\gamma}}{\underset{\epsilon}{bcd+6d+\gamma b}}
$$

361. Fractionibus fcilicet formandis fupra infcribantur indices *a*, *b*, *c*, *d*, &c., infra autem fubfcribantur indices α, 6, γ, δ, &c.. Prima fractio iterum conftituatur $\frac{\mathrm{I}}{0}$, fecunda $\frac{a}{\mathrm{I}}$, tum fequentium quævis formabitur fi antecedentium ultimæ numerator per indicem fupra fcriptum, penultimæ vero numerator per indicem infra fcriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis fequentis: fimili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem fupra fcriptum, & ex penultimo denominatore per indicem infra fcriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum ufque denominatorem, qui fractioni antecedenti eft infcriptus, continuatæ inclufive.

362. Quod fi ergo hæ fractiones eoufque continuentur quoad fractio continua indices fuppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractiones vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem fuggerent. Ponamus enim verum valorem fractionis continuæ

$$
a + \cfrac{\alpha}{b + \cfrac{6}{c + \cfrac{\gamma}{d + \cfrac{e.}{e + \&c.,}}}} \qquad\qquad \text{effe} == x
$$

atque manifeftum eft fractionem primam $\frac{\mathrm{I}}{0}$ effe majorem

quam

quam x; fecunda vero $\frac{a}{1}$ minor erit quam x; tertia $a + \frac{a}{b}$
iterum vero valore erit major; quarta denuo minor, atque
ita porro hæ fractiones alternatim erunt majores & minores
quam x.   Porro autem perfpicuum eft quamlibet fractionem
propius accedere ad verum valorem x quam ulla præceden-
tium; unde hoc pacto citiffime & commodiffime valor ipfius x
proxime obtinetur; etiamfi fractio continua in infinitum pro-
grediatur, dummodo numeratores a, 6, γ, δ, &c., non nimis
crefcant; fin autem omnes ifti numeratores fuerint unitates,
tum appropinquatio nulli incommodo eft obnoxia.

363. Quo ratio hujus appropinquationis ad verum fractio-
nis continuæ valorem melius percipiatur, confideremus fractio-
num inventarum differentias. Ac, prima quidem $\frac{1}{0}$ præter-
miffa, differentia inter fecundam ac tertiam eft $= \frac{a}{b}$; quar-
ta a tertia fubtracta relinquit $\frac{a\,6}{b\,(b\,c\,+\,6)}$; quarta a quinta fub-
tracta relinquit $\frac{a\,6\,\gamma}{(b\,c\,+\,6)\,(b\,c\,d\,+\,6\,d\,+\,\gamma)}$, &c..   Hinc ex-
primetur valor fractionis continuæ per Seriem terminorum con-
fuetam hoc modo, ut fit

$$x = a + \frac{a}{b} - \frac{a\,6}{b\,(b\,c\,+\,6)} + \frac{a\,6\,\gamma}{(b\,c\,+\,6)(b\,c\,d\,+\,6\,d\,+\,\gamma\,b)} - \&c.,$$

quæ Series toties abrumpitur quoties fractio continua non in
infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quam-
cunque in Seriem terminorum, quorum figna alternantur,
convertendi, fi quidem prima littera a evanefcat. Si enim
fuerit

$x =$

$$x = \cfrac{\alpha}{b + \cfrac{\varsigma}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \&c.}}}}},$$

erit per ea, quæ modo invenimus,

$$x = \frac{\alpha}{b} - \frac{\alpha\varsigma}{b(bc + \varsigma)} + \frac{\alpha\varsigma\gamma}{(bc + \varsigma)(bcd + \varsigma d + \gamma b)} -$$

$$\frac{\alpha\varsigma\gamma\delta}{(bcd + \varsigma d + \gamma b)(bcde + \varsigma de + \gamma be + \delta bc + \varsigma\delta)} + \&c..$$

Unde, fi α, ς, γ, δ, &c. fuerint numeri non crefcentes, uti omnes unitates, denominatores vero a, b, c, d, &c. numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe confideratis, poterit viciffim Series quæcunque terminorum alternantium in fractionem continuam converti, feu fractio continua inveniri cujus valor æqualis fit fummæ Seriei propofitæ. Sit enim propofita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, fingulis terminis cum Serie ex fractione continua orta comparandis

$$A = \frac{\alpha}{b}; \qquad\qquad \text{hincque } \alpha = Ab,$$

$$\frac{B}{A} = \frac{\varsigma}{bc + \varsigma}; \qquad\qquad \text{unde fit } \varsigma = \frac{Bbc}{A - B}$$

$$\frac{C}{B} = \frac{\gamma b}{bcd + \varsigma d + \gamma b}; \qquad \gamma = \frac{Cd(bc + \varsigma)}{b(B - C)}$$

$$\frac{D}{C} = \frac{\delta(bc + \varsigma)}{bcde + \varsigma de + \gamma be + \delta bc + \varsigma\delta}; \quad \delta = \frac{De(bcd + \varsigma d + \gamma b)}{(bc + \varsigma)(C - D)}$$

$$\&c..$$

At, cum fit $\varsigma = \dfrac{Bbc}{A - B}$, erit $bc + \varsigma = \dfrac{Abc}{A - B}$; unde

$$\gamma =$$

$$\gamma = \frac{A C c d}{(A - B)(B - C)}.$$ Porro fit $b c d + 6 d + \gamma b = $

$$(b6 + 6) d + \gamma b = \frac{A b c d}{A - B} + \frac{A C b c d}{(A - B)(B - C)} = \frac{A B b c d}{(A - B)(B - C)},$$

unde erit $\dfrac{b c d + 6 d + \gamma b}{b c + 6} = \dfrac{B d}{B - C}$ & $\delta = \dfrac{B D d e}{(B - C)(C - D)}$;

fimili modo reperietur $\varepsilon = \dfrac{C E e f}{(C - D)(D - E)}$ & ita porro.

366. Quo ista lex .clarius appareat, ponamus esse

$P = b$
$Q = b c + 6$
$R = b c d + 6 d + \gamma b$
$S = b c d e + 6 d e + \gamma b e + \delta b c + 6 \delta$
$T = b 6 d e f + $ &c.
$V = b c d e f g + $ &c.,

erit ex lege harum expressionum

$Q = P c + 6$
$R = Q d + \gamma P$
$S = R e + \delta Q$
$T = S f + \varepsilon R$
$V = T g + \xi S$
&c..

Cum igitur his adhibendis litteris fit

$$x = \frac{a}{P} - \frac{a 6}{P Q} + \frac{a 6 \gamma}{Q R} - \frac{a 6 \gamma \delta}{R S} + \frac{a 6 \gamma \delta \varepsilon}{S T} - \text{&c.,}$$

367. Quoniam ergo ponimus esse

$$x = A - B + C - D + E - F + \text{&c.,}$$

erit

$$A = \frac{a}{P} ; \quad a = A P$$

P p 3

$$\frac{B}{A}$$

$$\frac{B}{A} = \frac{\mathcal{C}}{Q}; \quad \mathcal{C} = \frac{BQ}{A}$$

$$\frac{C}{B} = \frac{\gamma P}{R}; \quad \gamma = \frac{CR}{BP}$$

$$\frac{D}{C} = \frac{\delta Q}{S}; \quad \delta = \frac{DS}{CQ}$$

$$\frac{E}{D} = \frac{\varepsilon R}{T}; \quad \varepsilon = \frac{ET}{DR}$$

&c. &c.

Porro vero differentiis fumendis habebitur

$$A - B = \frac{\alpha(Q - \mathcal{C})}{PQ} \qquad = \frac{\alpha c}{Q} \qquad = \frac{APc}{Q}$$

$$B - C = \frac{\alpha\mathcal{C}(R - \gamma P)}{PQR} \qquad = \frac{\alpha\mathcal{C}d}{PR} \qquad = \frac{BQd_\circ}{R}$$

$$C - D = \frac{\alpha\mathcal{C}\gamma(S - \delta Q)}{QRS} \qquad = \frac{\alpha\mathcal{C}\gamma e}{QS} \qquad = \frac{CR\varepsilon}{S}$$

$$D - E = \frac{\alpha\mathcal{C}\gamma\delta(T - \varepsilon R)}{RST} = \frac{\alpha\mathcal{C}\gamma\delta f}{RT} = \frac{DSf}{T},$$

&c. &c. &c.

Si bini igitur in fe invicem ducantur, fiet

$$(A - B)(B - C) = ABcd.\frac{P}{R}; \quad \& \quad \frac{R}{P} = \frac{ABcd}{(A - B)(B - C)}$$

$$(B - C)(C - D) = BCde.\frac{Q}{S}; \quad \& \quad \frac{S}{Q} = \frac{BCed}{(B - C)(C - D)}$$

$$(C - D)(D - E) = CDef.\frac{R}{T}; \quad \& \quad \frac{T}{R} = \frac{CDef}{(C - D)(D - E)}$$

&c..

Unde, cum fit $P = b$; $Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\alpha = Ab$$

$$\mathcal{C} = \frac{Bbc}{A - B}$$

$$\gamma = \frac{ACcd}{(A - B)(B - C)}$$

$$\delta =$$

$$\delta = \frac{BDde}{(B-C)(C-D)}$$

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

&c..

368. Inventis ergo valoribus numeratorum α, ς, γ, δ, &c., denominatores b, c, d, e, &c., arbitrio noftro relinquuntur: ita autem eos aflumi convenit, ut, cum ipfi fint numeri integri, tum valores integros pro α, ς, γ, δ, &c., exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C, &c., utrum fint integri an fracti. Ponamus effe numeros integros, atque quæfito fatisfiet ftatuendo

$$b = 1 \qquad\qquad \alpha = A$$
$$c = A - B \qquad\qquad \varsigma = B$$
$$d = B - C \quad \text{unde fit} \quad \gamma = AC$$
$$e = C - D \qquad\qquad \delta = BD$$
$$f = D - E \qquad\qquad \varepsilon = CE$$
$$\text{\&c.} \qquad\qquad\qquad \text{\&c.}$$

Quocirca, fi fuerit,

$$x = A - B + C - D + E - F + \text{\&c.,}$$

idem ipfius x valor per fractionem continuam ita exprimi poterit, ut fit

$$x = \cfrac{A}{1 + \cfrac{B}{A-B+\cfrac{AC}{B-C+\cfrac{BD}{C-D+\cfrac{CE}{D-E+}}}}}\text{\&c.}$$

369. Sin autem omnes termini Seriei fint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{\&c.,}$$

habebuntur pro α, ς, γ, δ, &c., fequentes valores

$$\alpha =$$

$$\alpha = \frac{b}{A}; \quad \mathfrak{6} = \frac{A b c}{B - A}; \quad \gamma = \frac{B^2 c d}{(B - A)(C - B)};$$

$$\delta = \frac{C^2 d e}{(C - B)(D - C)}; \quad \varepsilon = \frac{D^2 e f}{(D - C)(E - D)}; \quad \&c..$$

Ponatur ergo ut fequitur

$b = A$;	$\alpha = 1$
$c = B - A$;	$\mathfrak{6} = A A$
$d = C - B$;	$\gamma = B B$
$e = D - C$;	$\delta = C C$

&c.,

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{A A}{B - A + \cfrac{B B}{C - B + \cfrac{C C}{D - C +}}}} \&c.,$$

E x e m p l u m I.

Transformetur hæc Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \&c.,$$

in fractionem continuam.

Erit ergo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c., atque, cum Seriei propofitæ valor fit $= l\,2$, erit

$$l\,2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 +}}}}}} \&c.$$

E x e m p l u m II.

Transformetur hæc Series infinita

$$\frac{\pi}{4}$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.,$$

ubi π denotat peripheriam circuli, cujus diameter $= 1$, in fractionem continuam.

Subſtitutis loco A, B, C, D, &c., numeris 1, 3, 5, 7, &c., orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}}},$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}},$$

quæ eſt expreſſio, quam BROUNCKERUS primum pro quadratura circuli protulit.

E X E M P L U M III.

Sit propoſita iſta Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.,$$

quæ, ob $A = m$; $B = m + n$; $C = m + 2n$; &c., in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

$$\frac{1}{x}$$

$$\frac{1}{x} - m = \cfrac{m\,m}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}$$

EXEMPLUM IV.

Quoniam, fupra §. 178., invenimus effe

$$\frac{\pi \, \cos. \frac{m\,\pi}{n}}{n \, \sin. \frac{m\,\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} -$$

&c.,

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; &c., unde fiet

$$\frac{\pi \, \cos. \frac{m\,\pi}{n}}{n \, \sin. \frac{m\,\pi}{n}} = \cfrac{1}{m + \cfrac{m\,m}{n-2m + \cfrac{(n-m)^2}{2\,m + \cfrac{(n+m)^2}{n-2m + \cfrac{(2n-m)^2}{2\,m + \cfrac{(2n+m)^2}{n-2m+}}}}}}$$

&c.

370. Si Series propofita per continuos Factores progrediatur, ut fit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.,$$

tum prodibunt fequentes determinationes

$$\alpha = \frac{b}{A}; \quad \beta = \frac{b\,c}{B-1}; \quad \gamma = \frac{B\,c\,d}{(B-1)(C-1)};$$

$$\delta = \frac{C\,d\,e}{(C-1)(D-1)}; \quad \varepsilon = \frac{D\,e\,f}{(D-1)(E-1)}; \quad \&c.,$$

fiat ergo ut fequitur,

$$b = A;$$

$$b = A \,;\qquad\qquad \alpha = 1$$
$$c = B - 1 \,;\qquad\quad \mathfrak{c} = A$$
$$d = C - 1 \,;\qquad\quad \gamma = B$$
$$e = D - 1 \,;\qquad\quad \delta = C$$
$$f = E - 1 \,;\qquad\quad \varepsilon = D$$

&c.,

unde confequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \&c.}}}}}$$

E X E M P L U M I.

Quoniam, pofito *e* numero cujus Logarithmus eft $= 1$, fupra invenimus effe

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.,$$

feu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.,$$

hæc Series in fractionem continuam convertetur ponendo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c. : quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.,}}}}}}$$

unde, afymmetria initio rejecta, erit

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \&c.}}}}}$$

EXEM-

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis fumitur, cofi-num effe $= 1 - \dfrac{1}{2} + \dfrac{1}{2.12} - \dfrac{1}{2.12.30} + \dfrac{1}{2.12.30.56} -$ &c.. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$, $E = 56$, &c., atque Cofinus arcus qui radio æquatur, po-natur $= x$; erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}},$$

feu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}.$$

371. Sit Series infuper cum geometrica conjuncta, fcilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \&c.,$$

erit

$$\alpha = Ab; \quad \epsilon = \frac{Bbcz}{A - Bz}; \quad \gamma = \frac{ACcdz}{(A - Bz)(B - Cz)};$$

$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)}; \quad \epsilon = \frac{CEefz}{(C - Dz)(D - Ez)}; \&c..$$

Ponatur nunc

$b = 1$;	$\alpha = A$
$\iota = A - Bz$;	$\epsilon = Bz$
$d = B - Cz$;	$\gamma = ACz$
$e = C - Dz$;	$\delta = BDz$,

unde fiet

$$x =$$

$$x = \frac{A}{1 +} \frac{Bz}{A - Bz +} \frac{ACz}{B - Cz +} \frac{BDz}{C - Dz +} \&c.$$

372. Quo autem hoc negotium generalius abfolvamus, ponamus effe

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione inftituta,

$$\alpha = \frac{Ab}{L}; \; \mathfrak{c} = \frac{BLbcy}{AMz - BLy}; \; \gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)};$$
$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)}; \&c.,$$

ftatuantur valores b, c, d, &c., fequenti modo

$b = L;$ erit $\alpha = A$
$c = AMz - BLy;$ $\mathfrak{c} = BLLy$
$d = BNz - CMy;$ $\gamma = ACM^2yz$
$e = COz - DNy;$ $\delta = BDN^2yz$
$f = DPz - EOy;$ $\varepsilon = CEO^2yz$
      &c.   &c.

unde Series propofita per fequentem fraßionem continuam exprimetur

$$x = \frac{A}{L +} \frac{BLLy}{AMz - BLy +} \frac{ACMMyz}{BNz - CMy +} \frac{BDNNyz}{COz - DNy +} \&c.$$

373. Habeat denique Series propofita hujufmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque fequentes valores prodibunt

     $\alpha =$

$$\alpha = \frac{Ab}{L}; \quad \mathfrak{b} = \frac{Bbcy}{Mz - By}; \quad \gamma = \frac{C Mcdyz}{(Mz - By)(Nz - Cy)};$$

$$\delta = \frac{D N deyz}{(Nz - Cy)(Oz - Dy)}; \quad \varepsilon = \frac{E Oefyz}{(Oz - Dy)(Pz - Ey)};$$

$$\&c.,$$

ad valores ergo integros inveniendos fiat

	erit	
$b = Lz;$		$\alpha = Az$
$c = Mz - By;$		$\mathfrak{b} = BLyz$
$d = Nz - Cy;$		$\gamma = CMyz$
$e = Oz - Dy;$		$\delta = DNyz$
$f = Pz - Ey;$		$\varepsilon = EOyz$
$\&c.$		$\&c.$

Unde valor Seriei propositæ ita exprimetur, ut sit

$$x = \frac{Az}{Lz +} \ \frac{BLyz}{Mz - By +} \ \frac{C Myz}{Nz - Cy +} \ \frac{D Nyz}{Oz - Dy +} \&c..$$

Vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \frac{BLyz}{Mz - By +} \ \frac{C Myz}{Nz - Cy +} \ \frac{D Nyz}{Oz - Dy +} \&c.$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex supra traditis, infinitæ Series, quarum summæ constent, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor summæ illius Seriei est æqualis. Exempla, quæ jam hic sunt allata, sufficiunt ad hunc usum ostendendum: verumtamen optandum esset, ut methodus detegeretur, cujus beneficio, si proposita fuerit fractio continua quæcunque, ejus valor immediate inveniri posset. Quanquam enim fractio con-

tinua tranfmutari poteft in Seriem infinitam, cujus fumma per
methodos cognitas inveftigari queat, tamen plerumque iftæ Se-
ries tantopere fiunt intricatæ, ut earum fumma, etiamfi fit fatis
fimplex, vix ac ne vix quidem obtineri poffit.

375. Quo autem clarius perfpiciatur, dari ejufmodi fractio-
nes continuas, quarum valor aliunde facile affignari queat,
etiamfi ex Seriebus infinitis, in quas convertuntur, nihil ad-
modum colligere liceat, confideremus hanc fractionem conti-
nuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

cujùs omnes denominatores funt inter fe æquales ; fi enim hinc
modo fupra expofito, fractiones formemus

$$\begin{array}{ccccccc} 0, & 2, & 2, & 2, & 2, & 2, & 2 \\ \dfrac{1}{0}, & \dfrac{0}{1}, & \dfrac{1}{2}, & \dfrac{2}{5}, & \dfrac{5}{12}, & \dfrac{12}{29}, & \dfrac{29}{70}, \&c. : \end{array}$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2.5} + \frac{1}{5.12} - \frac{1}{12.29} + \frac{1}{29.70} - \&c.,$$

vel, fi bini termini conjungantur, erit

$$x = \frac{2}{1.5} + \frac{2}{5.29} + \frac{2}{29.169} + \&c.,$$

$$\text{vel}$$

$$x = \frac{1}{2} - \frac{2}{2.12} - \frac{2}{12.70} - \&c. .$$

Quin etiam, cum fit

$$x = \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.$$

$$+ \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.,$$

erit

$$x = \frac{1}{4} + \frac{1}{1.5} - \frac{1}{2.12} + \frac{1}{5.29} - \frac{1}{12.70} \&c.,$$

quæ Series etiamsi vehementer convergant, tamen vera earum summa ex earum forma colligi nequit.

376. Pro hujusmodi autem fractionibus continuis, in quibus denominatores omnes vel sunt æquales, vel iidem revertuntur; ita ut ea fractio, si ab initio aliquot terminis truncetur, toti adhuc sit æqualis, facilis habetur modus earum summas explorandi. In exemplo enim proposito, cum sit

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.,}}}}$$

erit $x = \frac{1}{2+x}$, ideoque $xx + 2x = 1$ & $x + 1 = \sqrt{2}$; ita ut valor hujus fractionis continuæ sit $= \sqrt{2} - 1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Est enim $\sqrt{2} - 1$ tam prope $= \frac{29}{70}$, ut error sit insensibilis : namque, radicem extrahendo, erit

$$\sqrt{2} - 1 = 0, 41421356236,$$
atque
$$\frac{29}{70} = 0, 41428571428,$$

ita ut error tantum in partibus centesimis millesimis consistat.

377. Quemadmodum ergo fractiones continuæ commodissimum suppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum C a p.
proxime inveſtigandas. Ponamus hunc in finem X V I I I.

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \&c.}}}}} ,$$

erit $x = \dfrac{1}{a + x}$ & $xx + ax = 1$, unde fit $x = -\dfrac{1}{2} a +$

$\sqrt{(1 + \dfrac{1}{4} aa)} = \dfrac{\sqrt{(aa + 4)} - a}{2}$. Hæc ergo fractio con-
tinua inſerviet valori radicis quadratæ ex numero $aa + 4$ in-
veniendo. Hincque adeo ſubſtituendo loco a ſucceſſive nu-
meros 1, 2, 3, 4, &c., reperientur $\sqrt{5}$; $\sqrt{2}$; $\sqrt{13}$; $\sqrt{5}$;
$\sqrt{29}$; $\sqrt{10}$; $\sqrt{53}$; &c., perductis ſcilicet his radicibus ad for-
mam ſimpliciſſimam : erit ergo

$$
\begin{array}{cccccc}
1, & 1, & 1, & 1, & 1, & 1, \\
\dfrac{0}{1}, & \dfrac{1}{1}, & \dfrac{1}{2}, & \dfrac{2}{3}, & \dfrac{3}{5}, & \dfrac{5}{8}, \&c. = \dfrac{\sqrt{5} - 1}{2} \\
2, & 2, & 2, & 2, & 2, & 2 \\
\dfrac{0}{1}, & \dfrac{1}{2}, & \dfrac{2}{5}, & \dfrac{5}{12}, & \dfrac{12}{29}, & \dfrac{29}{70}, \&c. = \sqrt{2} - 1 \\
3, & 3, & 3, & 3, & 3, & 3 \\
\dfrac{0}{1}, & \dfrac{1}{3}, & \dfrac{3}{10}, & \dfrac{10}{33}, & \dfrac{33}{109}, & \dfrac{109}{360}, \&c. = \dfrac{\sqrt{13} - 3}{2} \\
4, & 4, & 4, & 4, & 4, & 4 \\
\dfrac{0}{1}, & \dfrac{1}{4}, & \dfrac{4}{17}, & \dfrac{17}{72}, & \dfrac{72}{305}, & \dfrac{305}{1292}, \&c. = \sqrt{5} - 2 \\
\end{array}
$$

&c.,

notandum autem eo promptiorem eſſe approximationem, quo
major fuerit numerus a : ſic in ultimo exemplo erit $\sqrt{5} = 2$
$\dfrac{305}{1292}$, ut error minor ſit quam $\dfrac{1}{1292.5473}$, ubi 5473 eſt deno-
minator ſequentis fractionis $\dfrac{1292}{5473}$.

378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \&c.}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}; \text{ ideoque } axx + abx = b.$$

&

$$x = -\frac{1}{2} b \pm \sqrt{\left(\frac{1}{4} bb + \frac{b}{a}\right)} = \frac{-ab + \sqrt{(aabb + 4ab)}}{2a}.$$

Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi g ratia, $a = 2$, $b = 7$; erit $x = \dfrac{-14 + \sqrt{14.18}}{4} = \dfrac{-7 + 3\sqrt{7}}{2}$;

at valorem ipsius x proxime exhibebunt sequentes fractiones

$$
\begin{array}{cccccc}
2, & 7, & 2, & 7, & 2, & 7 \\
\dfrac{0}{1}, & \dfrac{1}{2}, & \dfrac{7}{15}, & \dfrac{15}{32}, & \dfrac{112}{239}, & \dfrac{239}{510}, \quad \&c.,
\end{array}
$$

Erit ergo proxime $\dfrac{-7 + 3\sqrt{7}}{2} = \dfrac{239}{510}$ & $\sqrt{7} = \dfrac{2024}{765} = 2,6457516$; at revera est $\sqrt{7} = 2,6457513\,1$; ita ut error minor sit quam $\dfrac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \&c.}}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c};$$

unde $(ab + 1)xx + (abc + a - b + c)x = bc + 1$ atque

$$x =$$

$$x = \frac{-abc-a+b-c+\sqrt{((abc+a+b+c)^2+4)}}{2(ab+1)} ;$$

ubi quantitas poſt ſignum radicale poſita iterum eſt ſumma duo-
rum quadratorum, neque ergo hæc forma radicibus ex aliis
numeris extrahendis inſervit, niſi ad quos prima forma jam ſuf-
fecerat. Simili modo ſi quatuor litteræ *a, b, c, d*, continuo
repetitæ denominatores fractionis continuæ conſtituant, tum ea
plus non inſerviet quam ſecunda, quæ duas tantum litteras con-
tinebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extra-
ctionem radicis quadratæ adhiberi queant, ſimul inſervient æqua-
tionibus quadratis reſolvendis; quod quidem ex ipſo calculo eſt
manifeſtum, dum *x* per æquationem quadraticam affectam de-
terminatur. Poteſt autem viciſſim facile cujuſque æquationis
quadratæ radix per fractionem continuam hoc modo exprimi.
Sit propoſita iſta æquatio

$$xx = ax + b ,$$

ex qua, cum ſit $x = a + \dfrac{b}{x}$, ſubſtituatur in ultimo termino
loco *x* valor idem jam inventus, eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}} ,$$

ſimili ergo modo procedendo, erit per fractionem continuam
infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a +}}} \&c. ,$$

quæ autem, cum numeratores *b* non ſint unitates, non tam
commode adhiberi poteſt.

381. Ut autem uſus in arithmetica oſtendatur, primum no-
tandum eſt omnem fractionem ordinariam in fractionem con-

tinuam

LIB. I.

tinuam converti poffe. Sit enim propofita fractio $x = \frac{A}{B}$; in qua fit $A > B$; dividatur A per B, fitque quotus $= a$ & refiduum C; tum per hoc refiduum C dividatur præcedens divifor B, prodeatque quotus b & relinquatur refiduum D, per quod denuo præcedens divifor C dividatur; ficque hæc operatio, quæ vulgo ad maximum communem diviforem numerorum A & B inveftigandum ufurpari folet, continuetur, donec ipfa finiatur; fequenti modo

$$
\begin{array}{l}
B)\,A\,(a \\
\quad C)\,B\,(b \\
\qquad D)\,C\,(c \\
\qquad\quad E)\,D\,(d \\
\qquad\qquad F \;\&c.,
\end{array}
$$

eritque per naturam divifionis

$$A = aB + C; \text{ unde } \frac{A}{B} = a + \frac{C}{B};$$

$$B = bC + D; \qquad \frac{B}{C} = b + \frac{D}{C}; \quad \frac{C}{B} = \cfrac{1}{b + \cfrac{D}{C}}$$

$$C = cD + E; \qquad \frac{C}{D} = c + \frac{E}{D}; \quad \frac{D}{C} = \cfrac{1}{c + \cfrac{E}{D}}$$

$$D = dE + F; \qquad \frac{D}{E} = d + \frac{F}{E}; \quad \frac{E}{D} = \cfrac{1}{d + \cfrac{F}{E}}$$

&c. &c. &c.

hinc, fequentes valores in præcedentibus fubftituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \cfrac{1}{b + \cfrac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{E}{D}}},$$

unde tandem x per meros quotos inventos a, b, c, d, &c., fequentem in modum exprimetur, ut fit

$$x =$$

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \&c.}}}}}$$

E X E M P L U M   I.

Sit propofita ifta fractio $\frac{1461}{59}$, quæ fequenti modo in fractionem continuam tranfmutabitur, cujus omnes numeratores erunt unitates. Inftituatur fcilicet eadem operatio, qua maximus communis divifor numerorum 59 & 1461 quæri folet.

$$59 \,)\, 1461 \,(\, 24$$
$$\underline{118}$$
$$281$$
$$\underline{236}$$
$$45 \,)\, 59 \,(\, 1$$
$$\underline{45}$$
$$14 \,)\, 45 \,(\, 3$$
$$\underline{42}$$
$$3 \,)\, 14 \,(\, 4$$
$$\underline{12}$$
$$2 \,)\, 3 \,(\, 1$$
$$\underline{2}$$
$$1 \,)\, 2 \,(\, 2$$
$$\underline{2}$$
$$0$$

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}.$$

E X E M P L U M   II.

Fractiones quoque decimales eodem modo tranfmutari poterunt; fit enim propofita

$$\sqrt{2} = 1, 41421356 = \frac{141421356}{100000000},$$

unde hæc operatio inftituatur

100000000

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418728	1010160	2
&c.	209364	

Ex qua operatione jam patet omnes denominatores effe 2, atque

adeo effe $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\&c.}}}}}$,

cujus expreffionis ratio jam ex fuperioribus patet.

EXEMPLUM III.

Imprimis vero etiam hic attentione dignus eft numerus e, cujus logarithmus eft $= 1$, qui eft $e = 2,718281828459$, unde oritur $\dfrac{e-1}{2} = 0,8591409142295$, quæ fractio decimalis, fi fuperiori modo tractetur, dabit quotos fequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
&c.		

fi ifte calculus exactius adhuc, affumto valore ipfius e, ulterius continuetur, tum prodibunt ifti quoti

$$1, 6, 10, 14, 18, 22, 26, 30, 34, \&c.,$$

qui, demto primo, progreffionem arithmeticam conftituunt, unde patet fore $e-1$

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\&c.}}}}}}},$$

cujus fractionis ratio ex calculo infinitefimali dari poteft.

382. Cum igitur ex hujufmodi expreffionibus fractiones erui queant, quæ quam citiffime ad verum valorem expreffionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipfas proxime accedant, exprimendas. Quin etiam, fi fractio fuerit propofita cujus numerator & denominator fint numeri valde magni, fractiones ex minoribus numeris conftantes inveniri poterunt quæ, etiamfi propofitæ non fint penitus æquales, tamen ab ea quam minime difcrepent. Hincque problema a WALLISIO olim tractatum facile refolvi poteft, quo quæruntur fractiones minoribus numeris expreffæ, quæ tam prope exhauriant valorem fractionis cujufpiam in numeris majoribus propofitæ, quantum fieri poterit numeris non majoribus. Fractiones autem noftra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus conftantes dentur quæ propius accedant.

E X E M P L U M I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nifi numeri majores adhibeantur. Si fractio decimalis cognita

$$3, 1415926535 \ \&c.,$$

modo expofito per divifionem continuam evolvatur, reperientur fequentes quoti

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad \quad 1, \quad \&c.,$$

ex quibus fequentes fractiones formabuntur,

$$\frac{1}{0}, \ \frac{3}{1}, \ \frac{22}{7}, \ \frac{333}{106}, \ \frac{355}{113}, \ \frac{103993}{33102}, \ \&c.,$$

fecunda fractio jam oftendit effe diametrum ad peripheriam utt

$$1 : 3,$$

1 : 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7 : 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor sit parte $\dfrac{1}{113.33102}$. Ceterum hæ fractiones alternatim vero sunt majores minorefque.

E X E M P L U M   I I.

Exprimatur ratio diei ad annum folarem medium in numeris minimis proxime. Cum annus iste sit 365^d, 5^b, $48'$, $55''$, continebit in fractione annus unus $365 \, \dfrac{20935}{86400}$ dies. Tantum ergo opus est ut hæc fractio evolvatur, quæ dabit fequentes quotos

$$4, \quad 7, \quad 1, \quad 6, \quad 1, \quad 2, \quad 2, \quad 4$$

unde istæ eliciuntur fractiones

$$\frac{0}{1}, \quad \frac{1}{4}, \quad \frac{7}{29}, \quad \frac{8}{33}, \quad \frac{55}{227}, \quad \frac{63}{260}, \quad \frac{181}{747}, \quad \&c. .$$

Horæ ergo cum minutis primis & fecundis, quæ fupra 365 dies adfunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde fequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inferat 100 dies, *Gregorianus* quaternis feculis tres annos biffextiles in communes convertit.

FINIS TOMI PRIMI.