THE METHOD OF FLUXIONS

BOTH

DIRECT *and* INVERSE.

The FORMER being

A Translation from the Celebrated Marquis De L'HOSPITAL'S *Analyse des Infinements Petits:*

And the LATTER

Supply'd by the TRANSLATOR,

E. STONE, *F.R.S.*

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PREFACE.

THE Analyſis explain'd in the following Work ſuppoſes the common Analyſis, but is of a very different Nature from it; the latter being confin'd to Finite Quantities, whereas the former extends to Infinity itſelf. By means of this Analyſis we compare the infinitely ſmall (Differences or) Parts of finite Magnitudes, and find their Ratio's to each other; and hereby likewiſe learn the Ratio's of finite Magnitudes, thoſe being in reality ſo many infinitely great Magnitudes, in reſpect of the other infinitely ſmall ones. This Analyſis may ever be ſaid to go beyond the Bounds of Infinity itſelf; as not being confined to infinitely ſmall (Differences or) Parts, but diſcovering the Ratio's of Differences of Differences, or of infinitely ſmall Parts of infinitely ſmall Parts, and even the Ratio's of infinitely ſmall Parts of theſe again, without End. So that it not only contains the Doctrine of Infinites, but that of an Infinity of Infinites. It is an Analyſis of this kind that can alone lead us to the Knowledge of the true Nature and Principles of Curves: For Curves being no other than Polygons, having an infinite Number of Sides, and their Differences arising altogether from the different Angles which their infinitely small Sides make with each other, it is the Doctrine of Infinites alone that must enable us to determine the Position of these Sides, in order to get the Curvature formed by them; and thence the Tangents, Perpendiculars, Points of Inflexion and Retrogression, reflected and refracted Rays, *&c.* of the Curves.

Polygons circumscribed about or inscribed in Curves, whose Number of Sides infinitely augmented till at last they coincide with the Curves, have always been taken for the Curves themselves. But the Notion rested here, without farther Improvement, for many Ages: And it was the Discovery of the Analysis of Infinites that first pointed out the vast Extent and Fecundity of this Principle.

What the Ancients, and particularly *Archimedes*, have done herein, may claim our Wonder; but then they have only consider'd a few Curves, and those too, slightly enough: They have left us little other than a Course of particular Propositions, which give no Indication of any uniform and consistent Method. Yet ought they by no means to be reproached on this score, considering the great Force of Genius * they shew'd in penetrating such Obscurities, and setting foot in a Land utterly unknown before: If they did not tra-

* *Though I have twice or thrice read over* Archimedes's *Treatise of Spirals with the utmost Attention, to comprehend the Art employ'd in his subtle Demonstrations relating to the Tangents of Spirals, yet cou'd I never rise from him without some suspicion that I had not here taken the whole Force of the Demonstration,* &c. BULLIALDUS Præf. de Lineis Spiralibus.

vel far, nor took the most direct Way, yet whatever *Vieta* * may imagine, they did not deviate far; and the more difficult and thorny the Routs they took, the more surprizing it is they were not lost. In a word, the Ancients appear to have gone as far as their Time could admit of, and done the same that our later Genius's would, if under the same Circumstances: And had the Ancients lived in ours, it is reasonable to believe, they would have had our Views; this being a Consequence of the natural Equality we find in Genius's, and the necessary Time required for the Succession of Discoveries. 'Tis therefore no Surprize the Antients have not gone farther in this Affair; but very strange indeed that great Men, undoubtedly as great as any of the Ancients, shou'd so long have sat down here, and with a kind of superstitious Veneration contented themselves to read and comment the Works of Antiquity, without allowing themselves any other Use of their Faculties, than barely serv'd them as Followers, that durst not think for themselves, or carry their Views beyond the Discoveries of their Predecessors.

In this manner many employ'd themselves; they wrote; the Number of Books increased, yet no farther Progress made; the Labours of many Ages serving only to fill the World with obsequious Comments and multiplied Translations of Originals, that in themselves were often worthless. Such was the State of Mathematicks, and principally of Philosophy, till the Time of *Descartes*; who, at the Instigation of his

* *Si verè* Archimedes, *fallaciter conclusit* Euclides, *&c.* Suppl. Geom.

great and commanding Genius, left the Ancients, to follow the ſame Guide, *Reaſon*, that had conducted them: And this happy Bravery, tho' treated as a Revolt, was follow'd by an infinite Number of new and uſeful Views, both in Philoſophy and Geometry. People now begun to open their Eyes and think for themſelves.

To keep to Mathematicks: *Deſcartes* here began where the Antients left off, *viz.* with the Solution of a Problem, at which, according to *Pappus*, [in *Collect. Mathem.* Lib. 7. at the Beginning] all of them ſtuck. 'Tis well known to what a Pitch he carried Algebra and Geometry, and how eaſy, by the Introduction of the former into the latter, he has render'd Solutions of innumerable Problems, which no one before him could maſter. But as he principally applied himſelf to the Reſolutions of Equations, Curves were by him conſider'd no farther than as they might aſſiſt him in finding their Roots: So that common Algebra being ſufficient for this, he did not endeavour to find any other, except in the Buſineſs of drawing Tangents to Curves, where he has happily applied it; and the Way he diſcover'd for that purpoſe appear'd to him ſo excellent, that he did not ſcruple to ſay, "this "Problem * was the moſt uſeful and general, "not only that he then knew, but even that "he ever had a Deſire to know in Geometry."

* Geom. *Lib.* 2.

The Geometry of *Deſcartes* having brought the Conſtruction of Problems by the Reſolutions of Equations into great vogue, and given a conſiderable Inſight into the Affair, the major part of Geometricians now apply themſelves to ſtudy and improve it with their own

Diſco-

Discoveries, which thus daily advance it towards Perfection.

Monsieur *Paschal* indeed directed his Views quite another way: He examined Curves in themselves; and under the Form of Polygons, found out the Lengths of some, the Spaces contained under them, the Solids described by those Spaces, their Centres of Gravity, *&c.* And from a bare Consideration of their Elements, that is, their infinitely small Parts, he discovered some general Methods relating thereto: Which are the more surprizing, as he seems to have come at them without Algebra, by the sole Force of Imagination.

Soon after the Publication of *Descartes*'s Method of Tangents, Monsieur *De Fermat* discovered another, which *Descartes* himself at length allows, (*Sect.* 71. *Tom.* 3.) to be more simple than his own on several Occasions. Yet this itself is not so simple as Dr. *Barrow* afterwards made it, from a close Consideration of the Nature of Polygons, which naturally represent to the Mind a little Triangle consisting of a Particle of a Curve, (contained between two infinitely near Ordinates,) the Difference of the correspondent Abscis's; and this Triangle is similar to that formed by the Ordinate, Tangent, and Subtangent: So that by one simple Analogy, this Method of Dr. *Barrow*'s performs the Business, without the Calculus required in the Method of *Descartes* and *De Fermat*.

Dr. *Barrow* rested not here: He also invented a kind of Calculus suitable to the Method, (*Lect. Geom. page* 80.) tho' deficient as well as that of *Descartes*, in clearing Equations of Fractions and Surd Quantities.

The Defect of this Method was supplied by that of Mr. *Leibnitz*'s, * [*or rather the great Sir* Isaac Newton. †] He began where Dr. *Barrow* and others left off: His *Calculus* has carried him into Countries hitherto unknown; and he has made Discoveries by it astonishing to the greatest Mathematicians of *Europe*. The Messieurs *Bernouli* were the first who perceived the Beauty of the Method; and have carried it such a length, as by its means to surmount Difficulties that were before thought insuperable.

* Acta Erudit. Lipf. *Ann.* 1684. *p.* 467.

† *See* Commercium Epistolicum.

This Calculus is of vast Extent; as being suited to mechanical Curves, as well as geometrical ones. Radical Signs are no Incumbrance at all in it, but sometimes a Conveniency. It extends to any number of indeterminate Quantities; and the Comparison of infinitely small Quantities of all kinds is performed by it with equal Facility. Whence arise an Infinity of surprizing Discoveries with regard to Tangents, as well Curves as right Lines, to Problems *de maximis & minimis*, Points of Inflection and Regression, Evolutes, Causticks by Reflection and Refraction, *&c.* as will appear in the Work itself.

I have divided it into ten Sections; the first whereof contains the Principles of the Method; the second shews the use thereof, in finding the Tangents of all kinds of Curves, let what will be the number of indeterminate Quantities in the Equation expressing their Natures. Tho' Mr. *Craigs*, indeed, (in *Lib. de Quadr. Figurar. Curvilin. part* 2.) thinks it only applicable to geometrical Curves. The third shews the use of the Method in solving all Problems *de maximis & minimis*. The fourth determines the Points of Inflexion and Retrogression of Curves. The fifth shews how to find the Evolutes of Monsieur *Hugens* in all kinds of Curves. The sixth and seventh shew the Method of finding Causticks by Reflection and Refraction, whereof M. *Tschirnhausen* is the Inventor. The eighth contains the farther use of the Method in finding Points in Curves that touch an infinite number of Right Lines, or Curves of a given Position. The ninth contains a Solution of some Problems that depend upon the foregoing Discoveries: And the tenth exhibits a new way of using the *Calculus Differentialis*, (or Method of Fluxions) in geometrical Curves: From whence is deduced the Method of *Descartes* and *Hudde*, which is only applicable to such kind of Curves.

In the *2d*, *3d*, *4th*, *5th*, *6th*, *7th*, and *8th* Sections, there are indeed only a few Propositions: But then they are all General. And, as it were, so many Methods, easily applicable to any number of particular Propositions. But I have only applied them in some select Examples, being persuaded, that in Mathematicks general Methods are best; and that the Books containing Details, or particular Propositions, misemploy the Time both of the Reader and Author. Whence I should not have added the Problems of the *9th* Section, if they were not curious and exceedingly general. Thus the *10th* Section likewise contains nothing but Methods which the *Calculus Differentialis* gives to the manner of *Descartes* and *Hudde*, for drawing of Tangents. And if these are less general, all the preceeding Methods shew, that the Fault is not in our Calculus

culus, but in the Manner of *Descartes*, whereto it is confined.

On the other hand, there can be no better Proof of the vast Use of our Calculus, than this great Variety of Methods, as comprehending the Whole of what *Descartes* and *Hudde* have done in the Affair of Tangents. And the universal Proof it gives us of the use of arithmetical Progressions therein, leaves no room to doubt of the Certainty of this last Method.

I intended to have added another Section, to shew the surprizing use of this *Calculus* in Physicks, and to what degree of Exactness it may bring the same; as likewise the use thereof in Mechanicks; But Sickness has prevented me herein. However, I hope to effect it hereafter, and present it the Publick with Interest. And indeed the Whole of the present Treatise is only the First Part of the Calculus of M. *Leibnitz*, or the *Direct Method*, wherein we descend from Whole Magnitudes to their infinitely small Parts, of what kind soever. comparing them with each other, which is called the *Calculus Differentialis*: But the other Part, called the *Calculus Integralis*, (or *Inverse Method of Fluxions*,) consists in ascending from these infinitely small Parts to the Magnitudes, or Wholes, whereof they are the Parts. This Inverse Method also I designed to publish; but Mr. *Leibnitz*'s having wrote to me, that he was at work upon this Subject, in order for a Treatise *de Scientia Infiniti*, I was unwilling to deprive the Publick of so fine a Piece, which must needs contain whatever is curious in the Inverse Method of Tangents, Rectifications of Curves, Quadratures, Investigation of Superficies

perficies of Solids, and their Solidities, Centres of Gravity, *&c.* Neither would I ever have published the present Treatise, had he not intreated me to it by Letter; as likewise because I believed it might prove a necessary Introduction to whatever shall hereafter be discovered on the Subject.

I must own my self very much obliged to the Labours of Messieurs *Bernoulli*, but particularly to those of the present Professor at *Groenengen*, as having made free with their Discoveries as well as those of Mr. *Leibnitz*: So that whatever they please to claim as their own, I frankly return them.

I must here in justice own, (as Mr. *Leibnitz* himself has done, in *Journal des Sçavans* for *August*, 1694.) that the learned Sir *Isaac Newton* likewise discover'd something like the *Calculus Differentialis*, as appears by his excellent *Principia*, published first in the Year 1687. which almost wholly depends upon the Use of the said *Calculus*. But the Method of Mr. *Leibnitz*'s is much more easy and expeditious, on account of the Notation he uses, not to mention the wonderful Assistance it affords on many Occasions.

When this Treatise was nearly printed off, Mr. *Nieuwentiit*'s Performance happened to come to hand; the Title whereof being *Analysis Infinitorum*, gave me the Curiosity of running it over; upon which I found it very different from mine: For the Author not only uses a Notation different from Mr. *Leibnitz*'s, but absolutely rejects second, third, *&c.* Differences (or Fluxions;) and as the greater Part of my Book is built upon that Foundation, I should have thought my self obliged to answer

his

his Objections, and ſhew their Inſufficiency, if Mr. *Leibnitz*'s had not already fully done it to my hands, in the *Acta Eruditorum* An. 1695. p. 310 and 369.

To conclude: The two *Poſtulata* or Suppoſitions laid down at the Entrance of this Work as the ſole Foundation thereof, appear to me ſo ſelf-evident, as not to leave the leaſt Scruple about their Truth and Certainty on the Mind of an attentive Reader. They might however have been demonſtrated after the Manner of the Antients, if I had not intended to be ſhort in things already known, and apply my ſelf principally to ſuch as are new.

THE TRANSLATOR TO THE READER.

IT is needless for me to say any thing in Commendation of the Author's Piece, the Character whereof is so well establish'd: And had he wrote likewise a second Part, or the Inverse Method of Fluxions, or Calculus Integralis, (as Foreigners call it) the Whole would, no doubt, have been an excellent Introduction to this admirable Doctrine.

My Design at first was to have publish'd the Translation alone; but considering that it would be imperfect without the Inverse Method, I therefore have supplied it, and shewn its Uses in the Quadratures of Curv'd-lin'd Spaces, the Rectifications of Curves, Cubations of Solids, Quantities of their Superficies, and the Investigations of Centres of Gravity and Percussion; the Whole concluding with a few miscellaneous Problems.

The first Section, being a general Introduction, handles the Doctrine of infinite Series, which is so necessary to what follows, that without it, a

Person

Person can have but a very slender Skill in the Inverse Method *of* Fluxions.

In the second Section, concerning the finding of Fluents, *you have not only the usual Ways of performing the Business, (as well when they cannot be had in finite Terms, as otherwise,) but likewise the Manner of using the Tables of the late learned Mr.* Cotes *for that end: By which, and the Tables of Logarithms and natural Sines, elegant Expressions of Fluents may be had, without the Labour of throwing Quantities into Series, which in many Cases is not a little embarassing.*

In the succeeding Sections you have a Number of select Examples, whose Solutions will more divert than trouble, and at the same time sufficiently instruct the industrious Learner. Among these you have the analytical Processes of all the Problems of Mr. Cotes, *about Quadratures, Rectifications, Cubations, and the Quantities of the Superficies of round Solids, (the bare Constructions whereof he has given us in his admirable Treatise, entituled,* Harmonia Mensurarum,) *from whence we gain the neat and elegant Constructions he has given in the aforesaid Treatise.*

In the Section concerning the Investigations of the Centres of Gravity, you will find a Mean observ'd in the Choice of Examples, both as to Number and Facility of Solution: You have here also the Determination of the Centres of Gravity of Hyperbolick and Elliptick Spaces in the Measures of Ratio's and Angles. In the next Section you have nothing new: Those that have a mind to try, may perhaps get not inelegant Constructions of Fluents, determining the Centres of Percussion of Hyperbolick and Elliptick Spaces, &c. *after Mr.* Cotes's *Way.*

The first four Problems at the End may serve to give the Learner a Taste of the Inverse Method of Tangents, about which I might have been more full, but these alone will teach the Manner of solving others of the like nature. And it is for this Reason I have given also some physical Problems, which may direct him to master those of a more difficult Kind, whenever they occur.

As the illustrious Author has omitted the Exponential Calculus, *or Manner of finding the Fluxions of Exponential Quantities, such as* $x^x = a$, $x^x = y^y$, *&c. where the Index's of the variable Quantities are also variable; thinking, as I suppose, this Branch of Doctrine to be of very little or no use, so I have been silent in this Matter also; which it is much better to be, than take up the Reader's Time in learning what is only mere Speculation. Thus you have a summary View of what is contained in the Appendix. Once for all; that I may not be thought a Plagiary, I freely own, many things are here taken from Sir* Isaac Newton, *&c.*

In a word, I am of opinion that the Translation, together with the Appendix or second Part, may serve very well for an Introduction to the Doctrine of Fluxions, and will so far qualify the Learner, as to render him capable of understanding with moderate Ease the more sublime Discoveries in Mechanicks, Physicks, &c. *that have been found by this Art. Such a Work becomes the more necessary, because there are but two* English *Treatises on the Subject, (that I know of) the one being* Hays's Introduction to Mathematical Philosophy, *and the other,* Ditton's Institution of Fluxions; *the former of which is by far too prolix for Learners, abounding in general Rules and Examples, many of which will rather* *confound than instruct him; and is at the same time deficient in things that would not a little forward him; not to mention the Book's being out of Print, and very likely for ever so to continue.*

On the other hand, Mr. Ditton's *Book is by much too sparing in Examples of the Uses of the Methods: Those few he has given us being by no means fit for Beginners. He is also too redundant at his first setting out, in the Explanation of the Definition of Fluxions, as Sir* Isaac Newton *has it: Which, tho' it be true and exact, it is next to impossible for one who has not been conversant about Infinites to apprehend it. That of our Author is much easier, tho less Geometrical, who calls a* Differential *(or* Fluxion) *the infinitely small Part of a Magnitude; not deterring his Readers at first from proceeding, by dwelling long on the Explication of an intricate Definition, but comes immediately to the Algorithm, or Arithmetick of the Art; and thence to plain Examples of Solutions by it.*

But I would not here be thought in any wise to lessen the Value of Sir Isaac Newton's *Definition: When the Learner has made some Progress, I would have him then make himself Master of it.*

Thus much for the Work; preparatory to the due Perusal whereof, it may not be amiss to give some Notion of the general Nature and Origin of Fluxions, according to the Sense of the great Author and Inventor thereof, Sir Isaac Newton.

In order to this, we are to consider Quantities not as made up of very small Parts, but as described by a continued Motion. For Example, a Line *is described, not by the Apposition of little Lines or Parts, but by the continual Motion of a Point. A* Surface *or plain* Superficies *is described*

bed

led by the Motion of a Line, (not according to its own Direction.) And a Solid, *by the continual Motion of a Superficies, (not according to its own Direction.)*

Now the Velocities of the Increases or Increments of Magnitudes thus moving in very small equal Particles of Time, at the first Instant of the Generation of those Increments are called Fluxions, *and those Magnitudes* Fluents, *or* Flowing Quantities.

These Fluxions *are nearly proportional to the Increments of the* Fluents *or* Flowing Quantities, *generated in very small equal Parts of Time; but accurately as the Velocities wherewith they arise and begin to be generated; that is, they are those very Velocities; as was said before.*

The reason why the Increments generated in small equal Parts of Time, are not exactly proportional to the Velocities wherewith they are generated, is, because those Velocities are not constantly equable, which, if they were, the little Spaces described in equal Particles of Time, would then be exactly as the Velocities, (which is an allowed Principle in Mechanicks;) but the Velocities are mutable, or accelerated continually, and so the said little Spaces described, cannot be constantly proportional to the Velocities wherewith they were first described, or proportional to any one of those mutable Velocities.

Yet if the Particles of Time be taken very small, and the Acceleration be so likewise, Fluxions *may be taken as proportional to those Increments, and so the said Increments, may represent* Fluxions *in all Operations; the Results of which will be as exact, as if they had been determined from the Velocities of the Motions wherewith the said Increments begin to be generated. But to scrupulous*

Persons the Processes will not be allowed so geometrical. Neither indeed is it so satisfactory to throw out Quantities from an Equation, on account of their being infinitely less than others, (as all who use an Increment for a Fluxion do,) as to reject them, because they really vanish and become equal to nothing, as Sir Isaac Newton *does.*

Upon this latter Foundation is built the Calculus Differentialis, *first published by Mr.* Leibnitz, *in the Year* 1684.; *having been since followed by almost all the Foreigners: Who represent the first Increment, or* Differential, *(as they call it,) by the Letter* d, *the second by* dd, *the third by* ddd, *&c.; the Fluents, or Flowing Quantities, being called* Integrals. *But since this Method in the Practice thereof, does not differ from that of Fluxions, and an Increment or Differential may be taken for a Fluxion; out of regard to Sir* Isaac Newton, *who invented* the same before the Year* 1669, *I have altered the Notation of our Author, and instead of* d, dd, d^3, *&c. put his Notation, viz.* $\dot{x}$, $\ddot{x}$, $\dddot{x}$, *&c. or some other of the final Letters of the Alphabet, pointed thus, and called the infinitely small Increment, or Differential of a Magnitude, the* Fluxion *of it.*

* *See* Commercium Epistolicum.

So far of the general Nature of Fluxions: I shall conclude with proposing the Solutions of the two following Problems to such as are capable.

I. The Latitude of a Place and the Day of the Year being given, to find the Time of that Day the Heat of the Sun shall be the greatest, admitting it be as any Power of the Sun's Duration above the Horizon, and as any Power of the Sine of his Altitude.

II. To

II. To find the Nature of one of the Superficies of a Wall of uniform Matter, of a given Height, Length, Breadth at Bottom (*viz.* the Ground) and Top, the other opposite upright Superficies being a right-angled Parallelogram, as also the Section at the Ground and the Top, that shall stand firmer, or be least disposed to fall, when the Wind blows directly against the upright plain Superficies of it, than when the Superficies (to be found) is of any other Nature; allowing the Particles of the Air to be equally distant from each other, of equal Magnitudes, and all to move with equal Velocities parallel to each other; and the Wall to break off at the Surface of the Ground, with a sufficient Force of Wind.

Note, *The three Sides of the infinitely small Triangle, mentioned in* Page vij. *aforegoing, are an infinitely small Part of the Curve, the Difference between two infinitely near Ordinates, and that of two infinitely near Abscis's.*

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TREATISE

OF

FLUXIONS.

PART I.

SECT. I.

Of finding the Fluxions of Quantities.

DEFINITION I.

V*Ariable Quantities* are those that continually increase or decrease; and *constant* or *standing* Quantities, are those that continue the same while others vary. As the *Ordinates* and *Abscisses* of a Parabola are variable Quantities, but the *Parameter* is a constant or standing Quantity.

DEFIN. II.

THE infinitely small Part whereby a variable Quantity is continually increased or decreas'd, is called the * *Fluxion* of that Quantity.

 For

* *See the Translator's Preface.*

FIG. 1. For Example: Let there be any Curve Line *AMB*, whose Axis or Diameter is the Line *AC*, and let the right Line *PM* be an Ordinate, and the right Line *pm* another infinitely near to the former.

Now if you draw the right Line *MR* parallel to *AC*, and the Chords *AM*, *Am*; and about the Centre *A* with the Distance *AM*, you describe the small circular Arch *MS*: then shall *Pp* be the Fluxion of *PA*; *Rm* the Fluxion of *Pm*; *Sm* the Fluxion of *AM*; and *Mm* the Fluxion of the Arch *AM*. In like manner, the little Triangle *MAm*, whose Base is the Arch *Mm*, shall be the Fluxion of the Segment *AM*; and the small Space *MPpm*, will be the Fluxion of the Space contained under the right Lines *AP*, *PM*, and the Arch *AM*.

COROLLARY.

1. It is manifest, that the Fluxion of a constant Quantity, (which is always one of the Initial Letters *a*, *b*, *c*, &c. of the Alphabet) is *o*: or (which is all one) that constant Quantities have no Fluxions.

SCHOLIUM.

THE Fluxion of a variable Quantity expressed by a single Letter, which is commonly one of the later Letters of the Alphabet, is represented by the same Letter with a Dot or Full-point over it: as the Fluxion of x *is* $\dot{x}$, *that of* y *is* $\dot{y}$, *that of* z *is* $\dot{z}$, *and that of* u *is* $\dot{u}$. *And if you call the variable Quantities* AP, x; PM, y; AM, z; *the Arch* AM, u; *the mixt-*

lin'd Space APM, s; *and the Segment* AM, t: *then will* $\dot{x}$ *express the Value of* Pp, $\dot{y}$ *the Value of* Rm, $\dot{z}$ *the Value of* Sm, $\dot{u}$ *the Value of the small Arch* Mm, $\dot{s}$ *the Value of the little Space* MPpm, *and* $\dot{t}$ *the Value of the small mixt-lin'd Triangle* MAm.

POSTULATE I.

2. GRANT that two Quantities, whose Difference is an infinitely small Quantity, may be taken (or used) indifferently for each other: or (which is the same thing) that a Quantity, which is increased or decreas'd only by an infinitely small Quantity, may be consider'd as remaining the same.

For Example: Grant that A*p* may be taken for *AP*; *pm* for *PM*; the Space *Apm* for *APM*; the small Space *MPpm* for the small Rectangle *MPpR*; the small Sector *AMS* for the small Triangle *AMm*; the Angle *pAm* for the Angle *PAM*, &c.

POSTULATE II.

3. GRANT that a Curve Line may be consider'd, as the Assemblage of an infinite Number of infinitely small right Lines: or (which is the same thing) as a Polygon of an infinite Number of Sides, each of an infinitely small Length, which determine the Curvature of the Line by the Angles they make with each other.

For Example: Grant that the Part *Mm* of the Curve, and the Circular Arch *MS*, may be consider'd as straight Lines, on account of their being infinitely small, so that the little

Triangle mSM may be looked upon as a right-lin'd Triangle.

Prop. I.

4. *To find the Fluxions of simple Quantities connected together with the Signs + and —.*

It is required to find the Fluxion of $a+x+y-z$. If you suppose x to increase by an infinitely small Part, *viz.* till it becomes $x+\dot{x}$; then will y become $y+\dot{y}$; and z, $z+\dot{z}$: and the constant Quantity a will * still be the same a. So that the given Quantity $a+x+y-z$ will become $a+x+\dot{x}+y+\dot{y}-z-\dot{z}$; and the Fluxion of it (which will be had in taking it from this last Expression) will be $\dot{x}+\dot{y}-\dot{z}$; and so of others. From whence we have the following

* Art. 1.

Rule I.

For finding the Fluxions of simple Quantities connected together with the Signs + and —.

Find the Fluxion of each Term of the Quantity proposed; which connected together by the same respective Signs will give another Quantity, which will be the Fluxion of that given.

Prop. II.

5. *To find the Fluxions of the Product of several Quantities multiplied, or drawn into each other.*

1. The Fluxion of xy is $y\dot{x}+x\dot{y}$: for y becomes $y+\dot{y}$, when x becomes $x+\dot{x}$; and therefore xy then becomes $xy+y\dot{x}+x\dot{y}+\dot{x}\dot{y}$. Which

Which is the Product of $x+\dot{x}$ into $y+\dot{y}$, and the Fluxion thereof will be $y\dot{x}+x\dot{y}+\dot{x}\dot{y}$, that is, * $y\dot{x}+x\dot{y}$: because $\dot{x}\dot{y}$ is a Quantity infinitely small, in respect of the other Terms $y\dot{x}$ and $x\dot{y}$: For if, for Example, you divide $y\dot{x}$ and $\dot{x}\dot{y}$ by $\dot{x}$, we shall have the Quotients y and $\dot{y}$, the latter of which is infinitely less than the former. * Art. 2.

Whence it follows, *that the Fluxion of the Product of two Quantities, is equal to the Product of the Fluxion of the first of those Quantities in the second* Plus *the Product of the Fluxion of the second into the first.*

2. The Fluxion of xyz is $yz\dot{x}+xz\dot{y}+xy\dot{z}$. For by considering the Product xy as one Quantity, we must (as has been before shewn) take the Product of the Fluxion $y\dot{x}+x\dot{y}$ into the second Quantity z, (which will be $yz\dot{x}+xz\dot{y}$) Plus the Product of the Fluxion $\dot{z}$ of the second Quantity z into the first Quantity xy (which is $xy\dot{z}$;) and therefore the Fluxion of xyz will be $yz\dot{x}+xz\dot{y}+xy\dot{z}$.

3. The Fluxion of $xyzu$ is $uyz\dot{x}+uxz\dot{y}+uxy\dot{z}+xyz\dot{u}$. Which is proved as in the Case aforegoing, by considering the Product xyz as one Quantity. Understand the same of others. Hence we have this

RULE II.

For the Fluxions of Quantities of several Dimensions.

The Fluxion of a Quantity of several Dimensions, or (which is the same) of the Product of several Quantities multiply'd into one another, is equal to the Sum of the Products of the Fluxion of each of those Quantities into the Product of the others.

So the Fluxion of ax is $xo + a\dot{x}$, that is, $a\dot{x}$; and that of $\overline{a+x}\times\overline{b-y}$ is $b\dot{x}-y\dot{x}-a\dot{y}-x\dot{y}$.

PROP. III.

6. *TO find the Fluxion of a Fraction.*

The Fluxion of $\frac{x}{y}$ is $\frac{y\dot{x}-x\dot{y}}{yy}$. For supposing $\frac{x}{y}=z$, then will x be $=yz$; and since these variable Quantities x and yz ought always to be equal to one another, whether they increase or decrease, it follows that their Fluxions, that is, their infinitely small Increments or Decrements shall likewise be equal.
* *Art.* 5. And therefore will * $\dot{x}=y\dot{z}+z\dot{y}$, and $\dot{z}=\frac{\dot{x}-z\dot{y}}{y}=\frac{y\dot{x}-x\dot{y}}{yy}$ by substituting z for its Value $\frac{x}{y}$. Which was to be done, &c. Hence we have this

RULE III.

For the Fluxions of Quantities divided by one another, or Fractions.

The Fluxion of a Fraction is equal to the Product of the Fluxion of the Numerator into the Denominator *Minus* the Product of the Fluxion of the Denominator into the Numerator: the whole being divided by the Square of the Denominator.

As

As the Fluxion of $\frac{a}{x}$ is $\frac{-a\dot{x}}{xx}$, and that of $\frac{x}{a+x}$ will be $\frac{a\dot{x}}{aa+2ax+xx}$.

PROP. IV.

7. *TO find the Fluxion of any whole or broken Powers of a variable Quantity.*

Before we lay down a general Rule for finding the Fluxions of all sorts of Powers, we must explain the Analogy that there is between their Exponents or Indices. In order to this, if there be a Geometrical Progression, having 1 for its first Term, and the second Term be any Quantity x; and if the Indices or Exponents be orderly set under them, it is plain that the said Exponents will form an Arithmetical Progression.

Prog. Geom. $1, x, xx, x^3, x^4, x^5, x^6, x^7$, &c.
Prog. Arithm. $0, 1, 2, 3, 4, 5, 6, 7$, &c.

And if the Geometrical Progression be continued downwards from Unity, and the Arithmetical Progression downwards from 0, the Terms of this last shall be the Exponents of those to which they answer in the other; as -1 is the Exponent of $\frac{1}{x}$, -2 of $\frac{1}{xx}$, &c.

Prog. Geom. $x, 1, \frac{1}{x}, \frac{1}{xx}, \frac{1}{x^3}, \frac{1}{x^4}$, &c.

Prog. Arith. $1, 0, -1, -2, -3, -4$, &c.

But if some new Term be brought into the Geometrical Progression, we must find an answerable Arithmetical one for the Exponent of it.

As if the Term be $\sqrt{x}$, its Exponent will be $\frac{1}{2}$: $\sqrt[3]{x}$, will have $\frac{1}{3}$ for its Exponent: $\sqrt[5]{x^4}$, $\frac{4}{5}$: $\frac{1}{\sqrt{x^3}}$, $-\frac{3}{2}$: $\frac{1}{\sqrt[3]{x^5}}$, $-\frac{5}{3}$: $\frac{1}{\sqrt{x^7}}$, $-\frac{7}{2}$, &c. So that these Expressions $\sqrt{x}$ and $x^{\frac{1}{2}}$, $\sqrt[3]{x}$ and $x^{\frac{1}{3}}$, $\sqrt[5]{x^4}$ and $x^{\frac{4}{5}}$, $\frac{1}{\sqrt{x^3}}$ and $x^{-\frac{3}{2}}$, &c. signify the same thing.

Progr. Geom. $1, \sqrt{x}, x.\ 1, \sqrt[3]{x}, \sqrt[3]{xx}, x.\ 1, \sqrt[5]{x}, \sqrt[5]{xx}, \sqrt[5]{x^3}, \sqrt[5]{x^4}, x.$

Progr. Arith. $0, \frac{1}{2}, 1.\ 0, \frac{1}{3}, \frac{2}{3}, 1.\ 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1.$

Prog. Geom. $\frac{1}{x}, \frac{1}{\sqrt{x^3}}, \frac{1}{xx}, \frac{1}{x}.\ \frac{1}{\sqrt[3]{x^4}}, \frac{1}{\sqrt[3]{x^5}}, \frac{1}{xx}.\ \frac{1}{x^3}, \frac{1}{\sqrt{x^7}}, \frac{1}{x^4}.$

Prog. Ar. $-1, -\frac{3}{2}, -2, -1. -\frac{4}{3}, -\frac{5}{3}, -2. -3, -\frac{7}{2}, -4.$

Where you may observe, that as $\sqrt{x}$ is a Geometrical Mean between 1 and x, $\frac{1}{2}$ is an Arithmetical Mean between their Exponents 0 and 1. In like manner, as $\sqrt[3]{x}$ is the first of two mean Proportionals between 1 and x, so $\frac{1}{3}$ is the first of two Arithmetical Means between their Exponents 0 and 1: Understand the same of others. Now from the Nature of these two Progressions it follows,

1. That the Sum of the Exponents of any two Terms of the Geometrical Progression, is the Exponent of the Product of these two Terms drawn into each other, as x^{4+3} or x^7 is the Product of x^3 into x^4, and $x^{\frac{1}{2}+\frac{1}{3}}$ or $x^{\frac{5}{6}}$ is the Product of $x^{\frac{1}{2}}$ into $x^{\frac{1}{3}}$, and $x^{-\frac{1}{3}+\frac{1}{5}}$ or $x^{-\frac{2}{15}}$ is the Product of $x^{-\frac{1}{3}}$ into $x^{\frac{1}{5}}$, &c. In like manner $x^{\frac{1}{3}+\frac{1}{3}}$ or $x^{\frac{2}{3}}$, is the Product of $x^{\frac{1}{3}}$ drawn into itself, that is, the Square of it,

it, and x^{2+2+2} or x^6 is the Product of x^2 into x^2 into x^2, that is, the Cube of it, and $x^{-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}-\frac{1}{3}}$, or $x^{-\frac{4}{3}}$, is the fourth Power of $x^{-\frac{1}{3}}$, and so of other Powers. Hence it is manifest, that the double, triple, &c. of the Exponent of any Term of the Geometrical Progression, is the Exponent of the Square, Cube, &c. of that Term; and consequently that the $\frac{1}{2}$, $\frac{1}{3}$, &c. of the Exponent of any Term of the Geometrical Progression, shall be the Exponent of the Square Root, Cube Root, &c. of that Term.

2. That the Difference of the Exponents of any two Terms of the Geometrical Progression, shall be the Exponent of the Quotient arising by the Division of one of those Terms by the other, as $x^{\frac{1}{2}-\frac{1}{3}} = x^{\frac{1}{6}}$ shall be the Exponent of the Quotient of $x^{\frac{1}{2}}$ divided by $x^{\frac{1}{3}}$, and $x^{-\frac{1}{3}-\frac{1}{4}} = x^{-\frac{7}{12}}$ shall be the Exponent of the Quotient of the Division of $x^{-\frac{1}{3}}$ divided by $x^{\frac{1}{4}}$; whence you see that it is the same thing to multiply $x^{-\frac{1}{3}}$ by $x^{-\frac{1}{4}}$, as to divide $x^{-\frac{1}{3}}$ by $x^{\frac{1}{4}}$: and so of others. This being well understood, there may happen two Cases in finding the Fluxions of Powers.

1. When the Power is a whole one, that is, when its Exponent is a whole Number. Now the Fluxion of xx is $2x\dot{x}$, of x^3 is $3xx\dot{x}$, of x^4 is $4x^3\dot{x}$, &c. for since the Square of x is only the Product of x into x, the Fluxion thereof shall be * $x\dot{x} + x\dot{x}$, that is, $2x\dot{x}$. In * Art. 5.
like manner, since the Cube of x is only the Product of x into x into x, the Fluxion of it will

* Art. 5. will be * $xx\dot{x}+xx\dot{x}+xx\dot{x}$, that is, $3xx\dot{x}$; and since it is the same of any other of these Powers whatsoever, it follows, that if m represents any whole Number, the Fluxion of x^m will be $mx^{m-1}\dot{x}$.

If the Exponent be negative, the Fluxion of x^{-m} or of $\frac{1}{x^m}$ shall be $\frac{-mx^{m-1}\dot{x}}{x^{2m}} = -mx^{-m-1}\dot{x}$.

2. When the Power is broken, that is, when the Exponent is a Fraction. It is requir'd to find the Fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ ($\frac{m}{n}$ being any Fraction.) Suppose $x^{\frac{m}{n}} = z$, and by raising both Sides of the Equation to the Power n, we shall have $x^m = z^n$; and finding the Fluxions of both Sides, (as in the first Case) then will $mx^{m-1}\dot{x} = nz^{n-1}\dot{z}$, and $\dot{z} = \frac{mx^{m-1}\dot{x}}{nz^{n-1}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$, or $\frac{m}{n}\dot{x}\sqrt[n]{x^{m-n}}$, by substituting $nx^{m-\frac{m}{n}}$ for its Value nz^{n-1}. If the Exponent be negative, the Fluxion of $x^{-\frac{m}{n}}$ or of $x^{\frac{1}{\frac{m}{n}}}$ shall be $\frac{-\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}}{x^{\frac{2m}{n}}} = -\frac{m}{n}x^{-\frac{m}{n}-1}\dot{x}$. From hence we have the following general

Rule IV.

For the Fluxions of all Kinds of Powers.

The Fluxion of any Power (whole or broken) of a variable Quantity, is equal to the Product

Product of the Exponent of that Power made by that same Quantity, raised to a Power lessen'd by 1, and multiply'd by its Fluxion.

As if m expresses any whole Number or Fraction positive or negative, and x any variable Quantity, the Fluxion of x^m will be always $mx^{m-1}\dot{x}$.

EXAMPLES.

The Fluxion of the Cube of $ay-xx$, that is, of $\overline{ay-xx}^3$, is $3\times\overline{ay-xx}^2\times a\dot{y}-2x\dot{x}$ $=3a^3yy\dot{y}-6aaxxyy\dot{y}+3ax^4\dot{y}-6aayyx\dot{x}$ $+12ayx^3\dot{x}-6x^5\dot{x}$.

The Fluxion of $\sqrt{xy+yy}$ or of $\overline{xy+yy}^{\frac{1}{2}}$, is $\frac{1}{2}\times\overline{xy+yy}^{-\frac{1}{2}}\times y\dot{x}+x\dot{y}+2y\dot{y}$, or $\dfrac{y\dot{x}+x\dot{y}+2y\dot{y}}{2\sqrt{xy+yy}}$.

The Fluxion of $\sqrt{a^4+axyy}$ or of $\overline{a^4+axyy}^{\frac{1}{2}}$, is $\frac{1}{2}\times\overline{a^4+axyy}^{-\frac{1}{2}}\times ayy\dot{x}+2axy\dot{y}$, or $\dfrac{ayy\dot{x}+2axy\dot{y}}{2\sqrt{a^4+axyy}}$.

The Fluxion of $\sqrt[3]{ax+xx}$ or of $\overline{ax+xx}^{\frac{1}{3}}$ is $\frac{1}{3}\times\overline{ax+xx}^{-\frac{2}{3}}\times a\dot{x}+2x\dot{x}$, or $\dfrac{a\dot{x}+2x\dot{x}}{3\sqrt[3]{\overline{ax+xx}^2}}$.

The Fluxion of $\sqrt{ax+xx+\sqrt{a^4+axyy}}$ or of $\overline{ax+xx+\sqrt{a^4+axyy}}^{\frac{1}{2}}$, is $\frac{1}{2}\times\overline{ax+xx+\sqrt{a^4+axyy}}^{-\frac{1}{2}}\times a\dot{x}+2x\dot{x}+\dfrac{ayy\dot{x}+2axy\dot{y}}{2\sqrt{a^4+axyy}}$, or

$$\frac{a\dot{x}+2x\dot{x}}{2\sqrt{ax+xx+\sqrt{a^4+axyy}}}+\frac{ayy\dot{x}+2axy\dot{y}}{2\sqrt{a^4+axyy}\times2\sqrt{ax+xx+\sqrt{a^4+axyy}}}.$$

Lastly,

Lastly, the Fluxion of $\frac{\sqrt[3]{ax+xx}}{\sqrt{xy+yy}}$ according to this Rule shall be*

$$\frac{\frac{a\dot{x}+2x\dot{x}}{3\sqrt[3]{ax+xx}}\times\sqrt{xy+yy}-\frac{y\dot{x}+x\dot{y}+2y\dot{y}}{2\sqrt{xy+yy}}\times\sqrt[3]{ax+xx}.}{xy+yy.}$$

* Art. 6, 7.

SCHOLIUM.

8. HERE we are to observe, that in finding the Fluxions of Quantities, we have hitherto supposed one of the variable Quantities x as increasing; while the others y, z, &c. do so likewise: That is, when the x's become $x+\dot{x}$, the y's and z's, &c. become $y+\dot{y}$, $z+\dot{z}$, &c. So that if it should happen that some of them do decrease, while others increase, then the Fluxions of the former must be look'd upon as negative Quantities, with respect to the Fluxions of the latter increasing Quantities; and consequently the Signs of the Terms affected with the Fluxions of these decreasing Quantities, must be changed.

To make this plain, let x increase while y and z decrease; that is, when x becomes $x+\dot{x}$, let y and z become $y-\dot{y}$ and $z-\dot{z}$: First, get the Fluxion $xy\dot{z}+xz\dot{y}+yz\dot{x}$ (by *Art.* 5.) and change the Signs of the Terms affected with $\dot{y}$ and $\dot{z}$. And this Expression thus alter'd, *viz.* $yz\dot{x}-xy\dot{z}-xz\dot{y}$, shall be the Fluxion sought.

SECT.

SECT. II.

Of the Use of Fluxions in drawing Tangents to all kinds of Curve Lines.

DEFINITION.

IF one of the small Sides Mm of a Polygon, whereof a Curve Line consists *, be continued out; the said small Side thus continued, is a Tangent to the Curve in the Point M or m. Fig. 2. * Art. 3.

PROP. I.

9. *It is required to draw a Tangent* MT *from a given Point* M, *in a Curve Line* AM, *whose Nature is expressed by any Equation, representing the relation of an Absciss* AP *to its Correspondent ordinate* PM. Fig. 3.

Draw the Ordinate MP, and conceive the right Line MT meeting the Diameter in the Point T, to be the Tangent sought; moreover, let mp be an Ordinate infinitely near MP, and draw the small right Line MR parallel to AP. Now call the given Lines AP, x; PM, y; and then will Pp or MR be $= \dot{x}$, and $Rm = \dot{y}$, and the similar Triangles mRM, MPT, will give this Proportion $mR\ (\dot{y}) : RM\ (\dot{x}) :: MP\ (y) : TP = \frac{y\dot{x}}{\dot{y}}$. Now by Means of the Fluxion of the

the given Equation, we can get a Value of $\dot{x}$, expressed in Terms that will be all affected with $\dot{y}$; which being multiplied by y, and divided by $\dot{y}$, will give us the Value of the Subtangent PT in known Terms, freed from Fluxions; by which the sought Tangent MT may be drawn.

SCHOLIUM.

FIG. 4. 10. WHEN the Point T falls on the contrary Side of the Point A, where the x's begin, with respect to P, then it is plain, that while x increases, y does decrease; and consequently, the Signs of all the Terms of the Fluxion of the given Equation affected with $\dot{y}$, must be * changed: Otherwise, the Value of $\dot{x}$ in $\dot{y}$ will be negative; and therefore that of PT $\left(\frac{y\dot{x}}{\dot{y}}\right)$. Notwithstanding, to avoid Trouble, it will be best to get the Fluxion of the given Equation, according to the Rules laid down in *Sect.* I. without any Alteration: For, if at the End of the Operation, it happens that the Value of PT be positive, then the Point T must be assumed on the same Side the Vertex A of the Diameter, as was supposed in the Operation; but if it be negative, it must be taken on the other Side the Point A. All this will be plain by the following Examples.

* *Art.* 8.

EXAMPLE I.

FIG. 3. II. 1. IF $ax = yy$ expresses the Relation of AP to PM, the Curve AM will be a Parabola, the given Quantity a, being the Parameter;

Parameter; and if you throw both Sides of the Equation into Fluxions, then will $a\dot{x}$ be $=2y\dot{y}$, and $\dot{x}=\frac{2y\dot{y}}{a}$, and $PT\left(\frac{y\dot{x}}{\dot{y}}\right)=\frac{2yy}{a}=2x$, by substituting ax for yy the Value thereof. Hence, if you make PT equal to the double of AP, and draw the right Line MT, this shall be a Tangent to the Curve in the Point M.

2. If $aa=xy$, be an Equation expressing the Nature of an Hyperbola between the Asymtotes. By throwing both Sides of the Equation into Fluxions, we shall have $x\dot{y}+y\dot{x}=0$; and therefore $PT\left(\frac{y\dot{x}}{\dot{y}}\right)=-x$. Whence, if you take $PT=PA$ on the other Side of the Point A, and draw the right Line MT, this will be a Tangent to the Hyperbola in the Point M. FIG. 4.

3. Let $y^m=x$ express the Nature of all Kinds of Parabolas, where the Exponent m, represents a positive whole Number or Fraction. As also of all Kinds of Hyperbola's, when that represents a negative Number or Fraction. Now the Equation being thrown into Fluxions, will be $my^{m-1}\dot{y}=\dot{x}$; and therefore $PT\left(\frac{y\dot{x}}{\dot{y}}\right)=my^m=mx$, by substituting x for y^m, which is equal to it. If m be $=\frac{3}{2}$, the Equation will be $y^3=ax$, expressing the Nature of one of the Cubick Parabola's, and the Sub-tangent $PT=\frac{3}{2}x$. If $m=-2$, the Equation will be $a^3=xyy$, expressing the Nature of a Cubick Hyperbola, and the Sub-tangent $PT=-2x$.

If

If a Tangent be to be drawn to the Point *A* the Vertex of the Diameter of a Parabola, you must find the Ratio of $\dot{x}$ to $\dot{y}$ in that Point. For it is plain, when that Ratio is known, that the Angle made by the Tangent and the Diameter, will also be determined. In this Example, $\dot{x} : \dot{y} :: m y^{m-1} : 1$. Whence it appears, that y being $= o$ in *A*, the Ratio of $\dot{y}$ to $\dot{x}$ will be then infinitely great, when m is greater than 1; and infinitely small when the same is less: That is, the Tangent to the Curve in *A*, must, in the former Case, be parallel to the Ordinates, and coincide with the Diameter in the latter Case.

Example II.

Fig. 5. 12. If AMB be a Curve Line of such a Nature, that $AP \times PB \left(x \times \overline{a-x}\right) : \overline{PM}^2 \left(yy\right) :: AB\ (a) : AD\ (b)$. Then $\frac{ayy}{b} = ax - xx$; and throwing both Sides of the Equation into Fluxions, we have $\frac{2ay\dot{y}}{b} = a\dot{x} - 2x\dot{x}$; whence $PT \left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2ayy}{ab-2bx} = \frac{2ax-2xx}{a-2x}$, by substituting $ax - xx$ for $\frac{ayy}{b}$; and $PT - AP$, or $AT = \frac{ax}{a-2x}$.

Now if $\overline{AP}^3 \times \overline{PB}^2 \left(x^3 \times \overline{a-x}^2\right) : \overline{PM}^5 \left(y^5\right) :: AB\ (a) : AD\ (b)$; then will $\frac{ay^5}{b}$ be $= x^3 \times \overline{a-x}^2$, and throwing both Sides of this Equation

Equation into Fluxions, we have $\frac{5ay^4\dot{y}}{b} = 3xx\dot{x}\times\overline{a-x}^2 - \overline{2a\dot{x}+2x\dot{x}}\times x^3$; and so $\frac{y\dot{x}}{\dot{y}} = \frac{5x^3\times\overline{a-x}^2}{3xx\times\overline{a-x}^2 - \overline{2a+2x}\times x^3} = \frac{5x\times\overline{a-x}}{3a-3x-2x}$, or $\frac{5ax-5xx}{3a-5x}$, and $AT = \frac{2ax}{3a-5x}$.

AND generally, if m be the Exponent of the Power of AP, and n that of the Power of PB, we shall have $\frac{ay^{m+n}}{b} = x^m\times\overline{a-x}^n$, which is a general Equation for all Kinds of Ellipses. And throwing both Sides of this Equation into Fluxions, we have $\frac{\overline{m+n}\,ay^{m+n-1}\dot{y}}{b} = mx^{m-1}\dot{x}\times\overline{a-x}^n - n\overline{a-x}^{n-1}\dot{x}\times x^m$, and so by substituting $x^m\times\overline{a-x}^n$ for $\frac{ay^{m+n}}{b}$, the Value thereof, there will come out $PT\ \left(\frac{y\dot{x}}{\dot{y}}\right)$

$$= \frac{\overline{m+n}\,x^m\times\overline{a-x}^n}{mx^{m-1}\times\overline{a-x}^n - n\overline{a-x}^{n-1}\times x^m} = \frac{\overline{m+n}\,x\times\overline{a-x}}{m\overline{a-x}-nx},$$ or $$PT = \frac{\overline{m+n}\times\overline{ax-xx}}{ma-\overline{m-n}\,x},$$

and $AT = \frac{nax}{ma-\overline{m-n}\,x}$.

EXAMPLE III.

13. THE same Things being supposed as in the foregoing Example, only here the Point B falls on the other Side A, with re- FIG. 6.

ſpect to the Point P, and we ſhall have this Equation $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}^n$, which expreſſes the Nature of all Hyperbolas with reſpect to their Diameters. Whence, as above, we get $PT = \frac{\overline{m+n} \times \overline{ax+xx}}{ma + \overline{m+n}x}$, and $AT = \frac{nax}{ma+\overline{m+n}x}$.

Now if AP be ſuppoſed to be infinitely great, the Tangent TM will meet the Curve at an infinite Diſtance; that is, it will become the Aſymtote CE; in which Caſe AT $\left(\frac{nax}{ma+\overline{m+n}x}\right)$ will be $= \frac{n}{m+n} a = AC$; becauſe a being infinitely leſs than x, the Term ma will be o in regard to $\overline{m+n}x$. By the ſame Reaſon, the Equation of the Curve here will be $ay^{m+n} = bx^{m+n}$. And making $m+n=p$, for Brevity ſake, and extracting the Root p of both Sides, then will $y\sqrt[p]{a} = x\sqrt[p]{b}$, the Fluxion of which will be $\dot{y}\sqrt[p]{a} = \sqrt[p]{}\dot{x}\,b$. So that if you draw AE parallel to the Ordinates, and conceive a ſmall Triangle to be at the Point wherein the Aſymtote CE meets the Curve, the following Proportion will ariſe. $\dot{x} : \dot{y} ::$ or $\sqrt[p]{a} : \sqrt[p]{b} :: AC$ $\left(\frac{n}{p}a\right) : AE = \frac{n}{p}\sqrt[p]{ba^{p-1}}$. Now the Value of CA and CE being thus determined, the Aſymtote CE may be drawn.

If $m=1$, and $n=1$, the Curve will be the common Hyperbola, and AC will be $=\frac{1}{2}a$; and $AE = \frac{1}{2}\sqrt{ab}$; that is, one half of the conjugate Diameter, which is a known Truth eſtabliſhed from other Principals.

EXAMPLE IV.

14. Let there be an Equation $y^3 - x^3 = axy$, (*AP* being $= x$, $PM = y$, and a is a given right Line) expressing the Nature of the Curve *AM*. The Fluxion of this shall be $3yy\dot{y} - 3xx\dot{x} = ax\dot{y} + ay\dot{x}$. Whence $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - axy}{3xx + ay}$, and *AT* $\left(\frac{y\dot{x}}{\dot{y}} - x\right) = \frac{3y^3 - 3x^3 - 2axy}{3xx + ay} = \frac{axy}{3xx + ay}$ by substituting $3axy$ for $3y^3 - 3x^3$. FIG. 6.

Now if *AP* and *PM* be supposed infinitely great, the Tangent *TM* will become the Asymptote *CE*, and the right Lines *AT*, *AS*, will become *AC*, *AE*, and these will determine the Position of the Asymptote. Now *AT*, which call t, is $= \frac{axy}{3xx + ay}$, from whence we get $y = \frac{3txx}{ax - at} = \frac{3tx}{a}$ when *AT* becomes *AC*, because then at is 0 with respect to ax. And putting down $\frac{3tx}{a}$ for y in $y^3 - x^3 = axy$, and there will come out $27t^3x^3 - a^3x^3 = 3a^2txx$, and striking out the Term $3a^2txx$, since x being infinite, it is 0 with regard to the two others $27t^3x^3$ and a^3x^3; then will *AC* (t) be $= \frac{1}{3}a$. In like manner *AS* $\left(y - \frac{x\dot{y}}{\dot{x}}\right)$ which call s, is $= \frac{axy}{3yy - ax}$; whence we get $x = \frac{3syy}{ay + as} = \frac{3sy}{a}$, since y being infinite with respect to s, the Term as will be 0 compar'd to ay; and putting that Value

Value in the Equation of the Curve, we shall get $AE\ (s) = \frac{1}{3}a$. Whence if you take $AC = AE = \frac{1}{3}a$, and draw the right Line CE, it will be the Asymptote to the Curve AM.

These two latter Examples will serve as Guides in finding the Asymptotes of other Curve Lines.

PROP. II.

FIG. 7. 15. *If in the Proposition aforegoing, the Absciſſes* AP *be conceived to be Parts of a Curve Line that we know how to draw the Tangents* (PT) *of. It is requir'd to draw the Tangent* MT *from the given Point* M *in the Curve* AM.

Draw the Ordinate MP, and the Tangent PT, and suppose the right Line MT, which cuts it in T, to be the Tangent sought: likewise suppose another Ordinate mp infinitely near to the former, and let the little right Line MR be parallel to PT: then call the given Quantities AP, x, and PM, y: and as before we shall have Pp or $MR = \dot{x}$, $Rm = \dot{y}$, and because of the similar Triangles mRM, MPT, we have this Proportion, *viz.* $mR\ (\dot{y}) : RM\ (\dot{x}) :: MP\ (y) : PT = \frac{y\dot{x}}{\dot{y}}$. Now proceed with the Equation expressing the Relation of the Absciſſes $AP\ (x)$ to the Ordinates $PM\ (y)$, as in the aforegoing Examples, and moreover the following ones.

EXAMPLE I.

16. Let $\frac{yy}{x}$ be $= \frac{x\sqrt{aa+yy}}{a}$: this thrown

into

into Fluxions will be $\frac{2xy\dot{y}-yy\dot{x}}{xx}=\frac{\dot{x}\sqrt{aa+yy}}{a}+\frac{xy\dot{y}}{a\sqrt{aa+yy}}$, and reducing the same into an Analogy $\dot{y}:\dot{x}$ $(MP:PT)::\frac{\sqrt{aa+yy}}{a}+\frac{yy}{xx}:\frac{2xy}{xx}-\frac{xy}{a\sqrt{aa+yy}}$. And therefore the Relation of the given Ordinate MP to the Subtangent PT sought, is expressed in known Terms freed from Fluxions. Which is what was proposed to be done.

EXAMPLE II.

17. LET x be $=\frac{ay}{b}$. This thrown into Fluxions, and we have $\dot{x}=\frac{a\dot{y}}{b}$; and PT $\left(\frac{y\dot{x}}{\dot{y}}\right)=\frac{ay}{b}=x$. If the Curve APB be a Semicircle, and the Ordinates MP, continued out to Q, be perpendicular to the Diameter AB; then the Curve AMC shall be a Semi-Cycloid. And when $b=a$, the Cycloid will be a common one: when b is greater than a, the Cycloid will be a *Prolate* one, and when it is less, a *Curtate* one.

COROL.

18. WHEN the generating Point of the Cycloid is in the Periphery of the Circle, if you draw the Chord AP. I say, it will be parallel to the Tangent MT. For the Triangle MPT being then an Isosceles one,

the external Angle TPQ shall be the Double of the internal opposite Angle TMQ. But the Angle APQ is equal to the Angle APT, because half of the Arch AP is the Measure of each of them; and therefore it is the one half of the Angle TPQ. Whence the Angles TMQ, APQ, shall be equal to each other; and consequently the Lines MT, AP shall be parallel.

Prop. III.

Fig. 7. 19. *LET* AP *be any Curve Line, and the right Line* KNAQ *a Diameter of it; and supposing the Method of drawing Tangents* (PK) *to it known; likewise let* AM *be another Curve such, that drawing any how the Ordinate* MQ *cutting the former Curve in the Point* P, *the relation of the Arch* AP *to the Ordinate* MQ *be expressed by any Equation. It is requir'd to draw the Tangent* MN *from a given Point* M.

Call the known Quantities PK, t; KQ, s; the Arch AP, x; MQ, y: then supposing another Ordinate mq infinitely near MQ, and drawing PO, MS parallel to AQ, we shall have $Pp = \dot{x}, mS = \dot{y}$, and since the Triangles KPQ and Ppo, mSM and MQN are similar; therefore $PK(t) : KQ(s) :: Pp(\dot{x}) : PO$ or $MS = \frac{s\dot{x}}{t}$. And $mS(\dot{y}) : SM\left(\frac{s\dot{x}}{t}\right) :: MQ (y) : QN = \frac{sy\dot{x}}{t\dot{y}}$. Now by means of the Fluxion of the given Equation find a Value of $\dot{x}$ in Terms that are all affected by $\dot{y}$, and if you substitute this Value in $\frac{sy\dot{x}}{t\dot{y}}$ for $\dot{x}$, the $\dot{y}$'s will

vanish,

vanish, and the Quantity of the Subtangent *QN* sought, will be expressed in known Terms. Which was to be found.

PROP. IV.

20. *LET there be two Curves* AQC, BCN, *the right Line* TEABF *being a Diameter; and suppose the Method of drawing the Tangents* QE, NF, *to be known; moreover, let there be another Curve Line* MC *such, that the Relation of the Ordinates* MP, QP, NP, *be expressed by any given Equation. It is required to draw the Tangent* MT *from a given Point* M *in this latter Curve.* FIG. 8.

IMAGINE the small Triangles *QOq*, *MRm*, *NSn*, at the Points Q, M, N, and call the known Quantities *PE*, s; *PF*, t; *PQ*, x; *PM*, y; *PN*, z; then will *Oq* be $=\dot{x}$, $Rm=\dot{y}$, $Sn=-\dot{z}$, because * when x and y increase, z decreases. And since the Triangles *QPE* and *qOQ*, *NPF* and *nSN*, *MPT* and *mRM* are similar; therefore *QP* $(x):PE(s)::qO(\dot{x}):OQ$ or *MP* or $SN=\frac{s\dot{x}}{x}$. And $NP(z):PF(t)::nS(-\dot{z}):SN=\frac{-t\dot{z}}{z}=\frac{s\dot{x}}{x}$. $\left(\text{Whence arises } \dot{z}=\frac{-sz\dot{x}}{tx}\right)$ And $mR(\dot{y}):RM\left(\frac{s\dot{x}}{x}\right)::MP(y):PT=\frac{sy\dot{x}}{x\dot{y}}$. Now if $-\frac{sz\dot{x}}{tx}$ be substituted for $\dot{z}$ in the Equation of the Curve thrown into Fluxions, we shall have a Value of $\dot{x}$ in $\dot{y}$; * Art. 8.

which being put in $\frac{sy\dot{x}}{x\dot{y}}$, and the $\dot{y}$'s will destroy one another; and so the Value of the Subtangent PT will be had in known Terms.

EXAMPLE.

21. Let $yy = xz$; this thrown into Fluxions, is $2y\dot{y} = z\dot{x} + x\dot{z} = \frac{tz\dot{x} - sz\dot{x}}{t}$ by putting $-\frac{sz\dot{x}}{tz}$ for $\dot{z}$; whence we get $\dot{x} = \frac{2ty\dot{y}}{tz - sz}$; and therefore $PT\left(\frac{sy\dot{x}}{x\dot{y}}\right) = \frac{2styy}{txz - sxz} = \frac{2st}{t-s}$, by substituting xz for yy.

Again, let there be given this general Equation, *viz.* $y^{m+n} = x^m z^n$; this thrown into Fluxions will be $\overline{m+n}\, y^{m+n-1}\dot{y} = mz^n x^{m-1}\dot{x} + nx^m z^{n-1}\dot{z} = \frac{mtz^n x^{m-1}\dot{x}}{t} - \frac{nsz^n x^{m-1}\dot{x}}{t}$, by putting $\frac{-sz\dot{x}}{tx}$ for $\dot{z}$; whence we get $PT \left(\frac{sy\dot{x}}{x\dot{y}}\right) = \frac{\overline{mst + nst}\, y^{m+n}}{mtz^n x^m - nszx^m} = \frac{mst + nst}{mt - ns}$, if $x^m z^n$ be put for y^{m+n}.

Here you may observe, that if the Curves *AQC*, *BCN*, become right Lines, the Curve *MC* will be one of the Conick Section kind, *viz.* an *Ellipsis*, when the Ordinate *CD*, drawn from the Point of Concurrence *C*, falls between the Extremities *A* and *B*; an *Hyperbola*, when it falls on either Side; and a *Parabola*, when one of the Extremities *A* or *B* is infinitely distant from the other, or when one of

2 the

the right Lines *CA* or *CB* is parallel to the Diameter *AB*.

PROP. V.

22. *LET* APB *be a Curve beginning at the Point* A, *and suppose the Method of drawing Tangents* (PH) *to it known; and let the Point* F *be assumed without this Curve, always having the same Situation, and if there be another Curve* CMD *such, that any right Line* FMP *being drawn, the Relation of the Part* FM *thereof, to the Part* AP *of the Curve, is expressed by any given Equation. It is required to draw the Tangent* MT *from the given Point* M. FIG. 9.

Draw *FH* perpendicular to *FP*, meeting the given Tangent *PH* in the Point *H*, and the sought Tangent *MT* in the Point *T*, suppose a right Line *FRmOp* making an Angle infinitely small with *FP*, and from the Centre *F* describe the small Arches *PO*, *MR*, of a Circle; the little Triangle *pOP* shall be similar to the right-angled Triangle *PFH*; for the Angles *HPF*, *HpF*, are * equal, because they differ only by the Angle *PFp*, which is supposed to be infinitely small; and moreover the Angle *pOp* is a right Angle, since the Tangent in *O* (which is the Continuation of the little Arch *PO* consider'd as a right Line) is perpendicular to the Radius *FO*. By the same Reason, the Triangles *mRM*, *MFT*, will be similar. Now it is evident, that the little Triangles or Sectors *FPO*, *FMR* are similar. And call the known Quantities *PH*, t; *HF*, s; *FM*, y; *FP*, z; and the Arch *AP*, x; then

* *Art.* 2.

then shall $PH(t):HF(s)::Pp(\dot{x}):PO=\frac{s\dot{x}}{t}$. And $FP(z):FM(y)::PO\left(\frac{s\dot{x}}{t}\right):MR=\frac{ys\dot{x}}{tz}$. And $mR(\dot{y}):RM\left(\frac{sy\dot{x}}{tz}\right)::FM(y):FT=\frac{syy\dot{x}}{tz\dot{y}}$. And by throwing the given Equation into Fluxions, what is still to be done may be effected.

EXAMPLE.

FIG. 10. 23. If the Curve APB be a Circle, the Point F being the Centre; it is plain that the Tangent PH does become parallel and equal to the Subtangent FH, because HP shall be also perpendicular to PF; and so, in this Case, $FT=\frac{yy\dot{x}}{z\dot{y}}=\frac{yy\dot{x}}{a\dot{y}}$, by calling $FP(z)$, a; since it is now a constant Quantity. This being supposed, if the whole Periphery, or any determinate Part thereof, be called b, and you make $b:x::a:y$, the Curve CMD, which in this Case is FMD, will be the Spiral of *Archimedes*, and $y=\frac{ax}{b}$, which thrown in Fluxions, will be $\dot{y}=\frac{a\dot{x}}{b}$, whence arises $y\dot{x}=\frac{by\dot{y}}{a}=x\dot{y}$, by putting $\frac{ax}{b}$ for y; and therefore $FT\left(\frac{yy\dot{x}}{a\dot{y}}\right)=\frac{xy}{a}$. And so we get the following Construction.

From

From the Centre F with the Radius FM, describe the circular Arch MQ, bounded in Q by the Radius FA joining the Points A, F; and take FQ = Arch MQ. I say, the right Line MT will be a Tangent in M. For because of the similar Sectors FPA, FMQ, the following Proportion will arise $FP\ (a) : FM\ (y) :: AP\ (x) : MQ = \frac{yx}{a} = FT$.

If you suppose generally, that $b : x :: a^m : y^m$ (the Exponent m expressing any whole Number or Fraction) the Curve FMD will be a Spiral of all Kinds *ad infinitum*. And then $y^m = \frac{a^m x}{b}$, and this thrown into Fluxions, and $my^{m-1}\dot{y} = \frac{a^m \dot{x}}{b}$; whence arises $y\dot{x} = \frac{mby^m\dot{y}}{a^m} = mx\dot{y}$, by putting $\frac{a^m x}{b}$ for y^m; and therefore $FT\left(\frac{yy\dot{x}}{a\dot{y}}\right) = \frac{mxy}{a} = m \times MQ$.

PROP. VI.

24. *LET there be a Curve Line* APB, *the Method of drawing Tangents* (PH) *to it being known, and let* F *be a given Point taken without the same, always keeping its Situation. Let there be likewise another Curve* CMD *such, that any how drawing the right Line* FPM, *the Relation of* FP *to* FM *be expressed by any given Equation. It is required to draw a Tangent* MT *from the given Point* M. FIG. 11.

Draw the right Line FHT perpendicular to FM, and suppose (as in the last Proposition)

tion) the little Triangles POp, MRm similar to the Triangles HFP, TFM; then calling the known Quantities FH, s; FP, x; FM, y; and we shall have $PF\ (x) : FH\ (s) :: pO\ (\dot{x}) : OP = \frac{s\dot{x}}{x}$. And $FP\ (x) : FM\ (y) :: OP\left(\frac{s\dot{x}}{x}\right) : RM = \frac{sy\dot{x}}{xx}$. And $mR\ (\dot{y}) : RM\left(\frac{sy\dot{x}}{xx}\right) :: FM\ (y) : FT = \frac{syy\dot{x}}{xx\dot{y}}$. What is farther to be done, may be effected by throwing the given Equation into Fluxions.

Example.

25. If instead of the Curve APB, you would have a straight Line PH, and the Equation expressing the Relation of FP to PM be $y - x = a$; that is, if PM be always equal to the same given right Line a; then will $\dot{y}$ be $= \dot{x}$; and therefore $FT\left(\frac{syy\dot{x}}{xx\dot{y}}\right) = \frac{syy}{xx}$. Whence we get the following Construction.

Draw ME parallel to PH, and MT parallel to PE: I say, the same will touch the Curve in M.

For $FP\ (x) : FH\ (s) :: FM\ (y) : FE = \frac{sy}{x}$.

And $FP\ (x) : FE\left(\frac{sy}{x}\right) :: FM\ (y) : FT = \frac{syy}{xx}$.

It is therefore manifest, that the Curve CMD is the Conchoid of *Nicomedes*, the right Line PH being the Asymptote, and the Point F the Pole.

PROP.

PROP. VII.

26. *LET* ARM *be a Curve, the Method of drawing Tangents* (MH) *to it being known, and let the right Line* EPAHT *be a Diameter thereof, without which let the Point* F *have a constant Position; and from the same let the indefinite right Line* FPSM *issue, cutting the Diameter in* P, *and the Curve in* M. *Now if you conceive the right Line* FPM *to revolve about the Point* F, *and at the same time to move the Plane* PAM *constantly parallel to itself along the indefinite immoveable right Line* ET, *so that the Distance* PA *be every where the same; it is evident that* (M) *the continual Intersection of the Lines* FM, AM, *by this Motion, will describe the Curve* CMD. *From the given Point* M *of which, it is required to draw* MT *to touch the Curve.* FIG. 12.

The Plane PAM being supposed to be come to a Situation pam infinitely near, and the right Line mRS drawn parallel to AP, it is evident (from the manner of Generation) that $Pp = Aa = Rm$; and therefore that $RS = Sm - Pp$. Now if you call the known Quantities FP or Fp, x; FM or Fm, y; PH, s; MH, t; and the Fluxion Pp, $\dot{z}$: Then from the similar Triangles FPp and FSm, MPH and MSR, MHT and MRm, we shall have $Fp\,(x) : Fm\,(y) :: Pp\,(\dot{z}) : Sm = \frac{y\dot{z}}{x}$ (whence $SR = \frac{y\dot{z} - x\dot{z}}{x}$). And $PH\,(s) : HM\,(t) :: SR \left(\frac{y\dot{z} - x\dot{z}}{x}\right) : RM = \frac{ty\dot{z} - tx\dot{z}}{sx}$. And MR

$\left(\frac{ty\dot{z}-tx\dot{z}}{sx}\right) : Rm\ (\dot{z}) :: MH\ (t) : HT = \frac{sx}{y-x}$.
Whence if *FE* be drawn parallel to *MH*, and you take *HT*=*PE*; the Line *MT* shall be the Tangent sought.

If *AM* were a right Line, the Curve *CMD* would be an Hyperbola, and the Line *ET* would be one of the Asymptotes. And if it were a Circle, the Point *P* would be the Centre, and the Curve *CMD* would be the Conchoid of *Nicomedes*, the Line *ET* being its Asymptote, and the Point *F* the Pole of it. But if it were a Parabola, the Curve *CMD* would be one of the parabolick Kind, mentioned by *Descartes* in *Lib.* 3. *Geom.* and at the same time would be described below the right Line *ET*, by the Intersection of *FM* with the other half of the Parabola.

Prop. VIII.

Fig. 13. 27. *L*ET AN *be a Curve, whose Diameter is* AP, *and let* F *be a Point without them, having a constant Situation: Moreover let* CMD *be another Curve such, that any how drawing the right Line* FMPN, *the Relation of the Parts* FN, FP, FM *of it, is expressed by any given Equation. It is required to draw a Tangent* MT *from any given Point* M *in it.*

Thro' the Point *F* draw the Line *HK* perpendicular to *FN*, meeting the Diameter *AP* in *K*, and the given Tangent *NH* in *H*. From the Centre *F*, with the Distances *FN*, *FP*, *FM*, describe the small Arches *NQ*, *PO*, *MR*, terminated by the right Line *Fn* supposed to make an infinitely small Angle with *FN*. This being supposed.

Call

Call the known Quantities FK, s; FH, t; FP, x; FM, y; FN, z; then because of the similar Triangles PFK and pOP, FMR and FPO, FPO and FNQ, HFN and NQn, mRM and MFT, we shall have the following Proportions: $PF(x):FK(s)::pO(\dot{x}):OP=\frac{s\dot{x}}{x}$. And $FP(x):FM(y)::PO\left(\frac{s\dot{x}}{x}\right):MR=\frac{sy\dot{x}}{xx}$. And $FP(x):FN(z)::PO\left(\frac{s\dot{x}}{x}\right):NQ=\frac{sz\dot{x}}{xx}$. And $HF(t):FN(z)::NQ\left(\frac{sz\dot{x}}{xx}\right):Qn(-\dot{z})=\frac{szz\dot{x}}{txx}$ And $mR(\dot{y}):RM\left(\frac{sy\dot{x}}{xx}\right)::FM(y):FT=\frac{syy\dot{x}}{xx\dot{y}}$. Now by throwing the given Equation into Fluxions, we shall find a Value of $\dot{y}$ in $\dot{x}$ and $\dot{z}$; in which substituting $\frac{-szz\dot{x}}{txx}$ for $\dot{z}$; because when x increases, z decreases; and then all the Terms will be affected with $\dot{x}$: So that at length this Value being put in $\frac{syy\dot{x}}{xx\dot{y}}$, the $\dot{x}$'s will go out; and therefore the Value of FT will be expressed in known Terms freed from Fluxions.

If the right Line AP were supposed a Curve, and the Tangent PK had been drawn, we should find that FT would always have the same Value, and the Reasoning would have been the same.

EXAMPLE

EXAMPLE.

FIG. 14. 28. LET the Curve AN be a Circle passing thro' the Point F (so situated with respect to the Diameter AP, that the Line FB perpendicular to the same Diameter, passes thro' G the Centre of the Circle) and let PM be always equal to PN; it is manifest that the Curve CMD, which in this Case becomes FMA, will be the Cissoid of *Diocles*, and the Equation thereof $z+y=2x$; which thrown into Fluxions, will be $\dot{y}=2\dot{x}-\dot{z}=\frac{2txx\dot{x}+szz\dot{x}}{txx}$, by substituting $-\frac{szz\dot{x}}{txx}$ found above (*Art.* 27.) for $\dot{z}$. And therefore FT $\left(\frac{syy\dot{x}}{xx\dot{y}}\right)=\frac{styy}{2txx+szz}$.

If the Point M coincides with the Point A, the Lines FM, FN, FP, will be each equal to FA; as also the right Lines FK, FH; and therefore we shall have in this Case $FT=\frac{x^4}{3x^3}=\frac{1}{3}x$; that is, take $FT=\frac{1}{3}AF$, and draw the right Line AT, the same will touch the Curve in the Point A.

Tangents may be drawn likewise to the Cissoid by means of the first Proposition, by drawing NE, ML, perpendicular to the Diameter FB, and seeking an Equation expressing the Relation between the Ordinate LM, and correspondent Abscis FL. And this may be thus done; first call the known Quantities FB, $2a$; FL or BE, x; LM, y; then from the Similarity of the Triangles FEN, FLM, and the Nature of the Circle, we have FL $(x):LM\,(y)::FE:FN::EN\,(\sqrt{2ax-xx}):EB\,(x)$.

EB (x). Whence we get $yy = \frac{x^3}{2a-x}$; which thrown into Fluxions, will be $2y\dot{y} = \frac{6axx\dot{x} - 2x^3\dot{x}}{\overline{2a-x}^2}$; and therefore *LO* * $\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{yy \times \overline{2a-x}}{3axx - x^3} = \frac{2ax - xx}{3a-x}$, by putting $\frac{x^3}{2a-x}$ for yy. * Art. 9.

PROP. IX.

29. *LET there be two Curves* ANB, CPD, *and a straight Line* FKT, *in which are taken the three Points* A, C, F; *and let there be some other Curve* EMG *such, that a right Line* FMN *being drawn from any Point* M *of it, and the right Line* MP *parallel to* FK *the Relation of the Arch* AN *to the Arch* CP, *is expressed by any given Equation. It is required to draw the Tangent* MT *from a given Point* M *in the Curve* EG. FIG. 15.

Thro' the Point *T* sought, draw the right Line *TH* parallel to *FM*; and thro' the given Point *M*, the right Lines *MRK*, *MOH*, parallel to the Tangents in *P* and *N*, and draw *FmOn* infinitely near to *FMN*, and *mRp* parallel to *MP*.

Now call the known Quantities *FM*, s; *FN*, t; *MK*, u; *CP*, x; *AN*, y; (then will *Pp* or $MR = \dot{x}$, $Nn = \dot{y}$). And because of the similar Triangles *FNn*, and *FMO*; *MOm*, and *MHT*; *MRm*, and *MKT*; therefore

$$FN(t) : FM(s) :: Nn(\dot{y}) : MO = \frac{s\dot{y}}{t}.$$

And $MR(\dot{x}) : MO\left(\frac{s\dot{y}}{t}\right) :: MK(u) : MH = \frac{su\dot{y}}{t\dot{x}}$.

Now by help of the Fluxion of the given Equation, we shall have a Value of $\dot{y}$ in Terms, every of which will be affected with $\dot{x}$; which being substituted in $\frac{su\dot{y}}{t\dot{x}}$, and the $\dot{x}$"s do destroy one another. And therefore the Quantity of *MH* will be expressed in known Terms. From whence we have the following Construction.

Draw *MH* parallel to the Tangent in *N*, and equal to the Expression before found; draw *HT* parallel to *FM*, meeting the right Line *FK* in *T*; through which, and the given Point *M*, you must draw the Tangent *MT*; which will be that sought.

Example.

Fig. 16. 30. If the Curve *ANB* be a Quadrant of a Circle, the Point *F* being the Centre; and the Curve *CPD* becomes the Radius *APF*, perpendicular to the right Line *FKGQTB*, and the Arch *AN* (y), be always to the right Line *AP* (x), as the Quadrant *ANB* (b), to the Radius *AF* (a); then the Curve *EMG*, shall become *AMG* the *Quadratrix* of *Dinostratus*; and $MH\left(\frac{su\dot{y}}{t\dot{x}}\right)$ will be $=\frac{as\dot{y}-sx\dot{y}}{a\dot{x}}$, since *FP* or *MK* ($u$) $=a-x$, and *FN* (t) $=a$. But from the supposed Analogy $ay=bx$, and $a\dot{y}=b\dot{x}$. Now putting $\frac{ay}{b}$ and $\frac{b\dot{x}}{a}$ for their Equals x and $\dot{y}$, in the Value of *MH*, and there arises $MH=\frac{bs-ys}{a}$. From whence we get the following Construction.

Draw

Draw MH perpendicular to FM, and equal to the Arch MQ described from the Centre F, and draw HT parallel to FM. I say, the Line MT will touch the Curve in M: For because of the similar Sectors FNB, FMQ, we have this Proportion, *viz.* $FN\,(a) : FM\,(s) :: NB\,(b-y) : MQ = \frac{bs-sy}{a}$.

COROLLARY.

31. If you desire the Determination of the Point G, where the *Quadratrix* AMG meets the Radius FB, you must conceive another Radius Fgb infinitely near FGB; and then drawing gf parallel to FB, from the Nature of the *Quadratrix*, and the similar Triangles FBb, gfF, right-angled at B and f, we shall have this Proportion $AB : AF :: Bb : Ff :: FB$ or $AF : gf$ or FG. FIG. 17.

Whence if you take a third Proportional to the Quadrant AB, and the Radius AF, it shall be equal to FG; that is, $FG = \frac{aa}{b}$. By which Means the Construction of Tangents may be shortened thus:

Draw TE parallel to MH; then from the similar Triangles FMK, FTE, we have $MK\,(a-x) : MF\,(s) :: ET$ or $MH\left(\frac{bs-sy}{a}\right) : FT = \frac{bss-yss}{aa-ax} = \frac{bss}{aa}$, by putting $\frac{ay}{b}$ for its Equal x, and afterwards dividing the whole by $b-y$. From whence it is manifest, that the Line FT is a third Proportional to FG and FM. FIG. 16.

PROP. X.

FIG. 18. 32. *LET* AMB *be a Curve such, that the Relation of the right Lines* MF, MG, MH, *&c. drawn from any Point* M *taken in it, to the Foci* F, G, H, &c. *is expressed by any given Equation. It is required to draw the Perpendicular* MP *from the given Point* M, *to the Tangent in that Point.*

Take the infinitely small Arch *Mm* in the Curve *AB*, and draw the right Lines *FRm*, *GmS*, *HmO*; and from the Centres *F*, *G*, *H*, describe the small Arches *MR*, *MS*, *MO*; likewise from the Centre *M*, with any Distance, describe the Circle *CDE* cutting the Lines *MF*, *MG*, *MH*, in the Points *C*, *D*, *E*; from which let fall the Perpendiculars *CL*, *DK*, *EI*, to *MP*. This being done, you may observe,

1°, That the right-angled Triangles *MRm*, *MLC*, are similar; for if from the right Angles *LMm*, *RMC*, the common Angle *LMR* be taken away, the Angles remaining *RMm*, *LMC* are equal; and they are right-angled at *R* and *L*. After the same way we prove that the right-angled Triangles *MSm* and *MKD*, *MOm* and *MIE*, are similar. Therefore since *Mm* is a common Hypothenuse to the little Triangles *MRm*, *MSm*, *MOm*, and the Hypothenuses *MC*, *MD*, *ME*, of the Triangles *MLC*, *MKD*, *MIE*, are equal; it is evident that the Perpendiculars *CL*, *DK*, *EI*, have the same Relation to each other, as the Fluxions *Rm*, *Sm*, *Om*.

2°, That

2°, That the Lines issuing from the Foci situate on the same Side the Perpendicular *MP*, do increase at the same time the others decrease, or contrariwise. As (*in Fig.* 18.) *FM* increases by its Fluxion *Rm*, while *GM*, *HM*, decrease by theirs, *viz.* *Sm*, *Om*.

Now if the Equation $ax+xy-zz=0$, be supposed to express, for Examples sake, the Relation of the right Lines *FM* (x), *GM* (y), *HM* (z). This Equation thrown into Fluxions, will be $a\dot{x}+y\dot{x}+x\dot{y}-2z\dot{z}=0$. And then it will follow that the Tangent in *M* (which indeed is only the Continuation of the little Side *Mm* of the Polygon, that the Curve *AMB* is conceiv'd * to be made of) must be so situate, that if the Parallels *mR*, *mS*, *mO*, to the right Lines *FM*, *GM*, *HM*, be drawn from any Point *m* in it, terminated in *R*, *S*, and *O*, by Perpendiculars *MR*, *MS*, *MO*, to the said right Lines, we shall always have this Equation $\overline{a+y}\times Rm+x\times Sm-2z\times Om=0$; or (which comes to the same, in putting *CL*, *DK*, *EI*, for their Proportionals *Rm*, *Sm*, *Om*) the Perpendicular (*MP*) to the Curve, must be situate so, that $a+y\times CL+x\times DK-2z\times EI=0$. From whence we have the following Construction.

* *Art.* 3.

If the Point *C* be conceived to be laden with a Weight $a+y$, which multiples the Fluxion $\dot{x}$ of the right Line *FM* on which it is situate, and the Point *D* with the Weight x, and the Point *E*, taken on the contrary Side *M* with regard to the Focus *H* (since the Term $-2z\dot{z}$ is negative) with the Weight $2z$. I say, the right Line *MP* passing thro' the common Centre of Gravity of the Weights supposed to

FIG. 18, 19.

be in the Points C, D, E, will be the Perpendicular required.

For it is plain (by the Principles of Mechanicks) that every right Line paſſing thro' the Centre of Gravity of ſeveral Weights, ſo ſeparates them, that the Weights on one Side, each multiplied by its Diſtance from that Line, are equal to the Weights on the other Side, each multiplied alſo by its Diſtance from the ſaid Line. Whence, ſuppoſing x, y, z to increaſe together; that is, conceiving the Foci
Fig. 19. F, G, H, to be all on one ſide MP, as we have always done, in throwing the given Equation into Fluxions, according to the Rules before delivered: Therefore the Line MP will leave on one Side the Weights in C and D, and on the other the Weight in E, and ſo we ſhall have $\overline{a+y}\times CL+x\times DK-2z\times EI=0$, which is the Equation to be conſtructed.

Now ſince the Conſtruction is juſt in this Caſe, I ſay it is likewiſe ſo in any other Caſe. For Example, if the Situation of the Point M in the Curve be ſo alter'd, that x increaſes while y and z decreaſe; that is, that the Foci
Fig. 18. G and H fall on the other Side MP. Then it
* Art. 8. follows, 1°.* That the Signs of the Terms of the given Equation affected with $\dot{y}$ and $\dot{z}$, or their Proportionals DK, EI, muſt be changed. Whence the Equation to be conſtructed in this latter Caſe, will be $\overline{a+y}\times CL-x\times DK+2z\times EI=0$. 2°. That the Weights in D and E will change their Situation with regard to MP; and ſo from the Nature of the Centre of Gravity, we have $\overline{a+y}\times CL-x\times DK+2z\times EI=0$, which is the Equation to be con-

constructed. And since this is so in all the Cases possible, therefore, *&c.*

Hence it appears, that the Reasoning is the same, let what will be the Number of the Foci, and the given Equation, so that we thus denounce the general Construction.

Find the Fluxion of one Side of the given Equation (the other Side being *o*) and from the Centre *M*, with any convenient Distance, describe the Circle *C D E* intersecting the right Lines *M F*, *MG*, *MH*, in the Points *C*, *D*, *E*, in which are conceived the Weights that have the same Relation as the Quantities, multiplying the Fluxions of the Lines on which they are situate. I say, the Line *MP* passing through their common Centre of Gravity, shall be the Perpendicular required. But here we must observe, that if one of the Weights be negative in the Fluxion of the given Equation, the same must be supposed to be on the contrary Side of the Point *M* with regard to the Focus.

If instead of the Foci *F*, *G*, *H*, you suppose right Lines or Curves, to which the right Lines *MF*, *MG*, *MH*, fall at right Angles, the same Construction still takes place. FIG. 20.

For drawing from the Point *m* taken infinitely near *M* the Perpendiculars *mf*, *mg*, *mh*, to the Foci, *viz.* the Lines in the Foci; and from the Point *M*, the little Perpendiculars *MR*, *MS*, *MO*, to those Lines. It is evident that *Rm* will be the Fluxion of *MF*, because the right Lines *MF*, *Rf*, being Perpendiculars between the Parallels *Ff*, *MR*, are equal to one another; and in like manner *Sm* is the Fluxion of *MG*, and *Om* that of *MH*; and what remains may be proved as above.

FIG. 21. Instead of all or some of the Foci F, G, H, we may suppose Curves beginning in F, G, H. And the Curve AMB such, that the Tangents MV, MX, and the right Line MG being drawn from any Point M in it; the Relation of the mixed Lines FVM, HXM, and the right Line GM be expressed by a given Equation. For by drawing the Tangent mu from the Point m taken infinitely near M, it is manifest that the same will meet the other Tangent in the Point V, (it being only the Continuation of the little Arch Vu, consider'd as a little right Line.) And therefore if from the Centre V be described the small Circular Arch MR; Rm shall be the Fluxion of the mixed Line FVM, which becomes $FVuRm$. And all the rest may be demonstrated as aforegoing.

This Problem was first started by Mr. Tschirnhausen, *in his Book* de la Medecine de l'esprit; *and Mr.* Fatio *in the* Journal des Sçavans, *has given a very ingenious Solution thereof. But their Solutions are only particular Cases of the general Construction here laid down.*

EXAMPLE I.

FIG. 22. 33. LET $axx + byy + czz - f = 0$, (the right Lines a, b, c, f being given) the Fluxion of which is $ax\dot{x} + by\dot{y} + cz\dot{z} = 0$. Now supposing the Weight ax in C, the Weight by in D, and the Weight cz in E, *viz.* Weights that are to one another, as those Rectangles. The Line MP going through the common Centre of Gravity of them, shall be perpendicular to the Curve in the Point M.

But

But if FO be drawn parallel to CL, and the Radius MC be $=1$, then (becauſe of the ſimilar Triangles, MCL, MFO) FO will be $=x \times CL$; and in like manner drawing GR parallel to DK, and HS to EI, we ſhall have $GR = y \times DK$, and $HS = z \times EI$: ſo that by imagining the Weights a, b, c in the Foci F, G, H, the Line MP paſſing through the Centre of Gravity of the Weights ax, by, cz conceiv'd in C, D, E, ſhall alſo paſs through the Centre of Gravity of theſe latter Weights. Now this Centre is an invariable Point as to Situation, becauſe the Weights in F, G, H, viz. a, b, c are conſtant ſtraight Lines, being the ſame in all Poſitions of the Point M. Therefore the Curve AMB muſt be ſuch, that all the Perpendiculars to it do cut each other in one Point; that is, it is a Circle having that Point for its Centre. From whence we have the following remarkable Property of a Circle.

If there be ever ſo many Weights a, b, c, on the ſame Plane ſituate in F, G, H, and a Circle AMB be deſcribed about their common Centre of Gravity. And if the right Lines MF, MG, MH, &c. be drawn from any Point M in it: I ſay, the Sum of their Squares, each multiplied by the correſpondent Weight, will always be equal to the ſame Quantity.

EXAMPLE II.

34. LET the Curve AMB be ſuch, that the right Line MF being drawn from any Point M in it to the Focus F of a ſtanding Poſition, and the Perpendicular MG to the Focus FIG. 23.

Focus G, taken as a straight Line; the Relation of *MF* to *MG* will be always the same, as of the given Quantity *a* to the given one *b*.

Call *FM*, x; and *GM*, y; then will $x : y :: a : b$. And therefore $ay = bx$. And this thrown into Fluxions will be $a\dot{y} - b\dot{x} = 0$. Now conceiving the Weight *b* in *C* taken on the other Side *M* with respect to *F*, and the Weight *a* in *D* (at a like Distance from *M*) and drawing the Line *MP* through the common Centre of Gravity of those Weights; and it will be the Perpendicular requir'd.

For from the Nature of the Balance it is plain, if the Cord *CD* be so divided in *P* that $CP : DP :: a : b$, and the Point will be the common Centre of Gravity of the Weights in *C* and *D*.

The Curve *AMB* is a Conick Section; *viz.* a Parabola, when $a = b$; an Hyperbola, when *a* exceeds *b*; and an Ellipsis, when the same is less.

EXAMPLE III.

FIG. 24. 35. If you fasten the Ends of a Thread *FZVMGMXYH* in the Points *F* and *H*, fixing a little Peg in the Point *G*, and then extend the Thread by means of the Pin *M* so, that the Parts *FZV*, *HVX* wrap about the Curves beginning in *F* and *H*; and if the Part *MG* be doubled or folded together, the Motion of the Pin *M* will describe a Curve *AMB*. Now it is required to draw a Perpendicular *MP* to this Curve from a given Point *M* in it. The Position of the Thread aforesaid being given in that Point.

Here

Here I observe, that the straight Parts *MV*, *MX* of the Thread are always Tangents in *V* and *X*, and if the mixed Lines *FZVM*, and *HYXM* be called x and z, the right Line *MG*, y; and a right Line equal to the Length of the Thread, a. Then we shall always have $x+2y+z=a$: whence we know that the Curve *AMB* is contained under the general Construction. Therefore finding the Fluxion of the Equation, *viz.* $\dot{x}+2\dot{y}+\dot{z}=0$, and conceiving the Weights 1 in *C*, 2 in *D*, and 1 in *E*. I say, the Line *MP* passing through the common Centre of Gravity of those Weights, will be the Perpendicular requir'd.

PROP. XI.

36. *LET* APB, EQF *be any two Lines, the Method of drawing Tangents* PG, QH, *to the same being known; and let* PQ *be a right Line, having a Point* M *given in it. Now if the Ends* P *and* Q *of this right Line move along the Lines* AB, EF; *it is plain that the Point* M *by that Motion will describe a Curve* CD. *It is requir'd to draw a Tangent* MT *to the same from any given Point* M. FIG. 25.

Imagine the moveable right Line *PMQ* to come to a Situation infinitely near *pmq*, and draw the small right Lines *PO*, *MR*, *QS*, perpendicular to *PQ*, and we shall have the small right Angles *pOP*, *mRM*, *qSQ*; likewise take *PK*=*MQ*, and draw the right Line *HKG* perpendicular to *PQ*, and continue out *OP* to *T*, whereat it is supposed to intersect the Tangent sought *MT*. This being done, it is evident that the little right Lines

Lines Op, Rm, Sq, will be equal to each other, because by Construction PM and MQ are every where the same.

Call the known Quantities PM or KQ, a; MQ or PK, b; KG, f; KH, g; and the little right Line Op, Rm, or Sq, $\dot{y}$. Then the similar Triangles PKG and pOP, QKH and qSQ will give $PK(b):KG(f)::pO(\dot{y}):OP = \frac{f\dot{y}}{b}$. And $QK(a):KH(g)::qS(\dot{y}):SQ = \frac{g\dot{y}}{a}$. Now from common Geometry we know that $MR = \frac{OP \times MQ = QS \times PM}{PQ} = \frac{f\dot{y}+g\dot{y}}{a+b}$. So the similar Triangles mRM, MPT will give $mR(\dot{y}):RM\left(\frac{f\dot{y}+g\dot{y}}{a+b}\right)::MP(a):PT = \frac{af+ag}{a+b}$. Which was to be found.

Prop. XII.

Fig. 26. 37. *Let* BN, FQ *be any two Lines, the right Lines* BC, ED, *cutting each other at right Angles in the Point* A, *being the Axes of them; and let* LM *be a Curve such, that the right Lines* MGQ, MPN *drawn from any Point* M *taken in it, parallel to* AB, AE, *the Relation of the Spaces* EGQF, (*the Point* E *being a Stable, one in the right Line* AE, *and the Line* EF *parallel to* AC) APND, *and the right Lines* AP, PM, PN, GQ, *be expressed by a given Equation. It is required to draw the Tangent* MT *to the given Point* M *in the Curve* LM.

Call

Call AP or GM, x; PM or AG, y; PN, u; GQ, z; the Space $EGQF$, s; the Space $APND$, t; and the given Subtangents PH, a; GK, b. Then will Pp or NS or $MR = \dot{x}$, Gg or Rm or $OQ = -\dot{y}$; $Sn = -\dot{u} = \frac{u\dot{x}}{a}$ because of the similar Triangles HPN, NSn; $OQ = \dot{z} = -\frac{z\dot{y}}{b}$, $NPpn = \dot{t} = u\dot{x}$, and $QGgq = \dot{s} = -z\dot{y}$; where we are to take notice that the Values of Rm and Sn are negative, because when $AP(x)$ increases, $PM(y)$ and $PN(u)$ do decrease. This being supposed, throw the given Equation into Fluxions, wherein substitute $u\dot{x}$, $-z\dot{y}$, $-\frac{u\dot{x}}{a}$, $\frac{z\dot{y}}{b}$ for their Equals $\dot{t}$, $\dot{s}$, $\dot{u}$ and $\dot{z}$, and we shall have a new Equation expressing the Relation (sought) of $\dot{y}$ to $\dot{x}$, or MP to PT.

EXAMPLE I.

38. LET $s + zz$ be $= t + ux$; this thrown into Fluxions is $\dot{s} + 2z\dot{z} = \dot{t} + u\dot{x} + x\dot{u}$, and putting for $\dot{s}$, $\dot{t}$, $\dot{z}$, $\dot{u}$ their Equals, we shall find $-z\dot{y} - \frac{2zz\dot{y}}{b} = 2u\dot{x} - \frac{ux\dot{x}}{a}$; from whence we get $PT\left(\frac{y\dot{x}}{\dot{y}}\right) = \frac{2ayzz + aybz}{bax - 2abu}$.

EXAMPLE II.

39. LET s be $= t$: then $\dot{s} = \dot{t}$; that is, $-z\dot{y} = u\dot{x}$; and therefore $PT\left(\frac{y\dot{x}}{\dot{y}}\right) = -\frac{yz}{u}$. Now since this Quantity is negative,

there-

* *Art.* 10. therefore the Point *T* must * be taken on the other Side the Point *A*, where the *x*'s begin. If we suppose the Line *FQ* to be an Hyperbola, whose Asymptotes are *AC*, *AE*, so that *GQ* $(z) = \frac{cc}{y}$, and the Line *BND* a straight Line parallel to *AB*; whence *PN* (u) will be always equal to the given right Line *c*. Then it is plain that the right Line *AB* will be an Asymptote to the Curve *LM*, and the Subtangent *PT* $\left(-\frac{yz}{u}\right) = -c$; that is, equal to a standing Quantity.

And so in this Case *LM* will be the *Logarithmetick* Curve.

PROP. XIII.

FIG. 27. 40. *LET* B N, F Q *be any two Lines, the right Line* B A *being their Axis, in which take the two Points* A *and* E; *and let* L M *be a Curve such, that drawing a right Line* A N *from any Point* M *taken in it, describing the circular Arch* M G *from the Centre* A, *and drawing the Line* G (*parallel to* E F) *perpendicular to* A B, *the Relation of the Spaces* E G Q F, s; A N B, t; *and the right Lines* A M *or* A G, y; A N, z; G Q, u; *is expressed by any given Equation. It is required to draw the Tangent* M T *from a given Point* M *in the Curve* L M.

Draw the right Line *ATH* perpendicular to *AMN*, and conceive another right Line *Amn* infinitely near *AMN*, another Arch *mg*, another Perpendicular *gg*, and the little Arch *NS* described from the Centre *A*. Now

 call

call the given Subtangents AH, a; and GK, b; then will Rm or $Gg = \dot{y}$, $Sn = \dot{z}$: And because of the similar Triangles HAN, NSn; KGQ and QOq; therefore $SN = \frac{a\dot{z}}{z}$, $Oq = -\dot{u} = \frac{u\dot{y}}{b}$, $GQqg = -\dot{s} = u\dot{y}$, ANn or $AN \times \frac{1}{2} NS = -\dot{t} = \frac{1}{2} a\dot{z}$. All these Values must be put in the Fluxion of the given Equation, with which a new one will be formed; from whence we get a Value of $\dot{z}$ in $\dot{y}$. Now because of the Sectors and similar Triangles ANS and AMR, mRM and MAT, $AN (z) : AM (y) :: NS \left(\frac{a\dot{z}}{z}\right) : MR = \frac{ay\dot{z}}{zz}$. And $mR (\dot{y}) : RM \left(\frac{ay\dot{z}}{zz}\right) :: AM (y) : AT = \frac{ayy\dot{z}}{zz\dot{y}}$. Now if instead of $\dot{z}$ you substitute its Equal in $\dot{y}$, the Fluxions will vanish, and the Quantity of the Subtangent AT sought, will be expressed in known Terms. Which was to be found.

EXAMPLE I.

41. LET $uy - s$ be $= zz - t$; this thrown into Fluxions, is $u\dot{y} + y\dot{u} - \dot{s} = 2z\dot{z} - \dot{t}$; and substituting as necessary we have $\dot{z} = \frac{4bu\dot{y} - 2uy\dot{y}}{4bz + ab}$. And lastly, putting this Value in $\frac{ayy\dot{z}}{zz\dot{y}}$, there arises $AT = \frac{4abuyy - 2auy^3}{4bz^3 + abzz}$.

EXAMPLE II.

42. Let $s=2t$; then $\dot{s}=2\dot{t}$, *viz.* $-u\dot{y}=-a\dot{z}$, or $\dot{z}=\frac{u\dot{y}}{a}$; therefore $AT\left(\frac{ayy\dot{z}}{zz\dot{y}}\right)=\frac{uyy}{zz}$.

If the Line BN be a Circle, the Point A being the Centre, and the Radius be $AB=AN=c$; and if FQ be an Hyperbola such, that $GQ(u)=\frac{ff}{y}$; then it is manifest, that the Curve LM makes an infinite Number of Revolutions about the Centre A before it comes to the same; because the Space $FEGQ$ becomes infinite when the Point G falls in A, and AT is $=\frac{ffy}{cc}$. Whence we may observe, that the Ratio of AM to AT is constant; and therefore the Angle AMT is a standing Quantity E; and so the Curve LM is the *Logarithmical Spiral.*

PROP. XIV.

FIG. 28. 43. Let AMD, BMC *be any two Curves touching one another in the Point* M, *and let* L *be a given stable Point in the Plane of the Curve* BMC. *Now if you conceive the Curve* BMC *to revolve on (or roll along) the Curve* AMD *so, that the revolving Parts* AM, BM, *be always equal to each other. Now it is manifest that the Point* L, *moving along in the Plane* BMC, *will describe a kind of Cycloid* ILK. *This being premised, I say, if the right Line* LM *be drawn from the describing Point* L,

L, to the Point of Contact M in every different Position of the Curve BMC, the same will be perpendicular to the Curve ILK.

For the infinitely small equal Parts *Mm*, *Mm* of the Curves *AMD*, *BMC*, may be taken * for two little right Lines making an infinitely small Angle at the Point *M*. Now that the little Side *Mm* of the Curve or Polygon *BMC*, may fall upon, or coincide with the little Side *Mm* of the Curve or Polygon *AMD*; it is necessary for the Point *L* to describe a little Arch *Ll*, about the Point of Contact *M*, as a Centre; and consequently the said small Arch will be a part of the Curve *ILK*; and the right Line *ML*, which is perpendicular to it, will also be perpendicular to the Curve (*ILK*) in the Point *L*. Which was to be demonstrated.

* *Art.* 3.

PROP. XV.

44. LET MLN *be any right-lined Angle, whose Sides* LM, LN, *touch any two Curves* AM, BN. *If these Sides move along the Curves, so as to touch them continually, it is evident, that* L *the Vertex of the Angle will describe the Curve* ILK. *Now it is requir'd to draw* LC *perpendicular to this Curve, the Position of the Angle* MLN *being given.* FIG. 29.

Describe a Circle thro' the Vertex *L*, and the Points of Contact *M*, *N*, and draw the right Line *CL* thro' the Centre of the same; I say, it will be perpendicular to the Curve *ILK*.

For conceiving the Curves *AM*, *BN*, as Polygons of infinite Numbers of Sides, of which *Mm* and *Nn* are each one; it is plain that if the Sides *LM*, *LN*, of the right-lined Angle *LMN*, of a given Quantity, move about the stable Points *M*, *N*, (the Tangents *LM*, *LN*, being supposed the Continuations of the little Sides *Mf*, *Ng*) until *LM*, the Side of the Angle, falls on the little Side *Mm* of the Polygon *AM*, and the other Side *LN* upon the little Side *Nn* of the Polygon *BN*; the Vextex *L* will describe a little Part *Ll* of the circular Arch *MLN*. Therefore the said small Part *Ll* will be common to the Curve *ILK*; and consequently the right Line *CL*, which is perpendicular to it, will be perpendicular to the Curve in the Point *L*. Which was to be demonstrated.

PROP. XVI.

FIG. 30. 45. LET ABCD *be a flexible Cord having different Weights* A, B, C, *&c. hung to it at any given Distances* AB, BC, *&c. Now if this Cord be drawn along an Horizontal Plane by one End* D *thereof, which End moves along a given Curve* DP *in the said Horizontal Plane, it is manifest, that the said Weights will stretch the Cord while it is drawing along, and will describe the Curves* AM, BN, CO, *&c. It is required to draw Tangents to them, the Magnitude of the Weights, and the Position of the Cord* ABCD *being given.*

In the first Particle of Time, the End *D* moves forwards towards *P*, the Weights *A*, *B*, *C*, *B*, *C*, will describe, or endeavour to describe the little Sides *Aa*, *Bb*, *Cc*, of Polygons, which the Curves *AM*, *BN*, *CO*, are suppos'd to be; and therefore to draw the Tangents *AB*, *BG*, *CK*, to them, is only the Determination of the Weights *A*, *B*, *C*, in the said first Instant or Particle of Time, or the Position of the right Lines they endeavour to describe: To find which, we must observe,

1°, That the Weight *A*, in the first Particle of Time, is drawn in the Direction *AB*; and since there is nothing to divert it from that Direction, because it draws no other Weight itself, it must keep to that Direction; and so the right Line *AB* will touch the Curve *AM* in *A*.

2°, That the Weight *B* is drawn according to the Direction *BC*; but because it draws the Weight *A* after it, which does not move in that Direction, and so must induce some Alteration to that Direction; the Direction of *B* will not be in the Line *BC*, but in another Line *BG*, whose Position may be thus found.

Describe a Rectangle *EF*, with *BC* for the Diagonal, and having one Side *BF* in the Continuation of *AB*; then if the Force wherewith the Weight *B* is drawn according to the Direction *BC*, be expressed by *BC*; it follows by the Laws of Mechanicks, that the Force *BC* may be divided into two others *BE* and *BF*, *viz.* when the Weight *B* is drawn by the Force *BC* according to the Direction *BC*, it is the same, as if it was drawn at the same time by the Force *BE* in the Direction *BE*, and by the Force *BF* in the Direction *BF*. Now the Weight *A* gives no Disturbance to the Direction *BE*, it being drawn perpendicular to it; and conſequently the Force *B E* in that Direction will receive no Alteration; but it oppoſes the Force *B F* in the Direction *BF* by the whole Weight thereof. Therefore in order for the Weight *B* with the Force *B F* to overcome the Reſiſtance of the Weight *A*, the Force *B F* muſt be divided into two Parts, having the ſame Proportion to each other, as the Weights *A* and *B*. Whence divide *E C* in the Point *G* ſo, that *C G* be to *G E*, as the Weight *A* to the Weight *B*; then it is plain that *E G* will expreſs the remaining Force wherewith the Weight *B* endeavours to move in the Direction *B F*, when it has overcome the Reſiſtance of the Weight *A*. Therefore the Weight *B* is drawn in the ſame Time by the Force *B E* in the Direction *B E*; and by the Force *E G* in the Direction *B F* or *E C*; and ſo it will endeavour to move along *B G* with the Force *B G*: That is, *B G* will be the Direction, and conſequently will touch the Curve *B N* in *B*.

3°. To find the Tangent *C K*. With *C D* as a Diagonal, make the Rectangle *H I*, the Side *C I* being in *B C* continued. Now the Weight *B* does not at all diſturb the Force *C H*, wherewith the Weight *C* is drawn in the Direction *C H*; but the Force *C I* in the Direction *C I*, is diſturbed the greateſt poſſible by the Weight *B*, and in ſome meaſure by the Weight *A* alſo. To find out the Quantity of theſe, draw *A L* perpendicular to *B C* continued out. (Here we may obſerve, that if *A B* expreſſes the Force wherewith the Weight *A* is drawn according to the Direction *A B*; *B L* will expreſs the Force wherewith the ſaid Weight *A* is drawn in the Direction *B C*.) So

that

that the Weight C, together with the Force CI, must overcome the whole Weight B, together with a Part of the Weight A, which is to the Weight A as BL to BA, or BF to BC. Therefore if $B + \frac{A \times BF}{BC} : C :: DK : KH$, it is plain that CK will be the Direction of the Weight C, and consequently will be a Tangent to the third Curve CO in the Point C.

If there were a greater Number of Curves, the Tangents to the fourth, fifth, *&c.* might have been found after the same way; and the Tangents of the Curves described by the intermediate Points between the Weights, may be found by *Art.* 36.

SECT. III.

Of the Use of Fluxions in finding the greatest and least Ordinates in a Curve, to which the Solution of Problems de MAXIMIS & MINIMIS *may be reduced.*

DEFINITION I.

FIG. 31, 32, 33, 34. LET *MDM* be a Curve, whose Ordinates *PM*, *ED*, *PM* are parallel to each other; and let this Curve be such, that while the Abscifs *AP* continually increafes, the Ordinate *PM* increafes likewife, until it comes to a certain Point *E*, and afterwards decreafes; or, on the contrary, if the fame decreafes until it comes to a certain Point *E*, and afterwards increafes.

Then the Line *ED* is called the *greateft* or *leaft* Ordinate, or a *maximum* or *minimum*.

DEFIN. II.

IF a Quantity, as *PM*, be propofed, confifting of one or more indeterminate Quantities, as *AP*; and while *AP* continually increafes, the faid Quantity *PM* increafes likewife until it comes to a certain Point *E*, after which it conftantly decreafes, or contrariwife; and

and if it be required to find such a Value or Expression of *AP*, that the Quantity *ED*, of which it consists, may be greater or less than any other Quantity *PM* formed in like manner from *AP*. This is called a Problem *de maximis* and *minimis*.

GENERAL PROPOSITION.

46. THE *Nature of the Curve* MDM *being given: to find* AE *such a Value of* AP, *that the Ordinate* ED *be greater or lesser than any other* PM *of the same nature.*

When *AP*, *PM* increase together, it is evident * that *Rm* the Fluxion of *Pm* will be positive with regard to the Fluxion of *AP*. And on the contrary, if *M* decreases while the Ordinate *AP* increases, the Fluxion of *PM* will be negative. Now every Quantity that continually increases or decreases cannot from being positive become negative, without first passing thro' Infinity or nothing; *viz.* thro' o when the Quantity in the Beginning constantly goes on decreasing, and thro' Infinity when it continually increases in the Beginning. Therefore the Fluxion of a Quantity expressing a *maximum* or *minimum*, must be equal to o, or Infinity. Now because the Nature of the Curve *MDM* is given, we can find (by *Sect.* 1 or 2.) a Value of *Rm*, which being first made equal to o, and afterwards to Infinity; from thence in both the Suppositions the requir'd Value of *AE* will be had.

* *Art.* 8, 10.

SHOLIUM.

FIG. 31, 32. 47. THE Tangent in D is parallel to the Axis AB, when the Fluxion $R\dot{m}$ becomes o in that Point: But when it becomes FIG. 33, 34. infinite, the Tangent coincides with the Ordinate ED. Whence we may observe, that the Ratio of mR to RM, *viz.* that of the Ordinate to the Subtangent in the Point D, is o or Infinite.

It easily appears that a Quantity continually decreasing, from positive cannot become negative without first passing thro' o; but that a Quantity continually increasing must pass thro' Infinity to become negative, does not so easily appear. And therefore to assist the Imagination, let Tangents be conceived to issue FIG. 31, 32. from the Points M, D, M; now in Curves, where the Tangent in D is parallel to the Axis AB, it is manifest that the Subtangent PT increases so much the more, as the Points M and P accede to D, E; and so when M coincides with D, the same becomes infinite; and when at length AB is greater than AE, the Subtangent PT from positive becomes * negative, or contrariwise.

* *Art.* 10.

EXAMPLE I.

FIG. 35. 48. LET $x^3+y^3=axy$ (AP being $=x$, $PM=y$, $AB=a$) express the Nature of the Curve MDM. The Fluxion of the same will be $3xx\dot{x}+3yy\dot{y}=ax\dot{y}+ay\dot{x}$, and $\dot{y}=\frac{ay\dot{x}-3xx\dot{x}}{3yy-ax}=o$, when the Point P coincides with the Point E sought. Whence we

we get $y = \frac{2xx}{a}$. And putting this latter Part of the Equation for y in the Equation of the Curve $x^3 + y^3 = axy$, and there will arise $AE\,(x) = \frac{1}{3}a\sqrt[3]{2}$. Being such that the Ordinate ED will be a *maximum*, or the greatest of any other Ordinate PM to the Diameter AB.

EXAMPLE II.

49. LET $y - a = a^{\frac{1}{3}} \times \overline{a - x}^{\frac{2}{3}}$, express the Nature of the Curve MDM. This thrown into Fluxions will be $\dot{y} = \frac{2\dot{x}\sqrt[3]{a}}{3\sqrt[3]{a-x}}$, which I first make equal to 0; but because this Supposition gives us $-2\dot{x}\sqrt[3]{a} = 0$, from which the Value of AE cannot be known, I afterwards make $\frac{-2\dot{x}\sqrt[3]{a}}{3\sqrt[3]{a-x}}$ infinite; and so $3\sqrt[3]{a-x} = 0$. Consequently $AE\ (x) = a$. FIG. 33.

EXAMPLE III.

50. LET AMF be a Semicycloid, whose Base BF is less than half the Circumference ANB of the generating Circle, and Centre is C. It is required to find the Point E in the Diameter AB being such, that the Ordinate ED shall be a *maximum*, or the greatest possible. FIG. 36.

Draw the Ordinate PM at pleasure, intersecting the Semicircle in N, and at the Points M, N of the Ordinate conceive the little Triangles MRm, NSn; and calling the indeterminate Quantities AP, x; PN, z; the Arch AN, u;

AN, u; and the given Quantities ANB, a; BF, b; CA or CN, c. Then from the Nature of the Cycloid $ANB\,(a) : BF\,(b) :: AN\,(u) : NM = \frac{bu}{a}$. Whence $PM = z + \frac{bu}{a}$, and the Fluxion thereof $Rm = \frac{a\dot{z} + b\dot{u}}{a} = 0$, when the Point P coincides with E the Point sought. Now the right-angled Triangles NSn, NPC are similar. For if the common Angle CNS be taken away from the right Angles CNn, PNS the remaining Angles SNn, PNC shall be equal. And therefore $CN\,(c) : GP\,(c - x) :: Nn\,(\dot{u}) : Sn\,(\dot{z}) = \frac{c\dot{u} - x\dot{u}}{c}$. Whence putting this Value for $\dot{z}$ in $a\dot{z} + b\dot{u} = 0$, and there will arise $\frac{ac\dot{u} - ax\dot{u} + bc\dot{u}}{c} = 0$; and so we get x (which in this Case is AE) $= c + \frac{bc}{a}$.

Therefore assume CE towards B, a fourth Proportional to the Semi-circumference ANm, the Base BF, and the Radius CB, and the Point E will be that sought.

EXAMPLE IV.

FIG. 35. 51. To cut or divide the given right Line AB in the Point E so, that the Product of the Square of AE, one of the Parts into the other Part, be the greatest of any Product of the like nature.

Call the unknown Quantity AE, x; and the given Quantity AB, a; then must $\overline{AE}^2 \times EB = axx - x^3$ be a *maximum*. Now a Curve

MDM

MDM muſt be ſuppoſed ſuch, that the Relation of the Ordinate *MP* (y) to the correſpondent Abſciſs *AP* (x) is expreſſed by $y = \frac{axx - x^3}{aa}$, and the Point *E* muſt be found ſuch, that the Ordinate *ED* be a *maximum*; and ſo $\dot{y} = \frac{2ax\dot{x} - 3xx\dot{x}}{aa} = 0$, from whence we get *AE* $(x) = \frac{2}{3}a$.

And generally if you would have $x^m \times \overline{a-x}^n$ (where m and n expreſs any Numbers at pleaſure) to be a *maximum*, the Fluxion of it muſt be made equal to 0, or Infinity. Whence $mx^{m-1}\dot{x} \times \overline{a-x}^n - n\overline{a-x}^{n-1}\dot{x} \times x^m = 0$; which divided by $x^{m-1} \times \overline{a-x}^{n-1}\dot{x}$, and there comes out $am - mx - nx = 0$, and *AE* $(x) = \frac{m}{m+n}a$.

If $m = 2$, and $n = -1$; then will $AE = 2a$, and the Problem may be thus laid down.

Continue out the given Line *AB* (towards *B*) to the Point *E*, in ſuch manner that the Quantity $\frac{\overline{AE}^2}{BE}$ be a *minimum*, and not a *maximum*; for the Equation of the Curve *MDM* will be $\frac{xx}{x-a} = y$, wherein if we ſuppoſe $x = a$, the Ordinate *PM*, which becomes *BC*, will be $\frac{aa}{0}$, that is, infinite; and conceiving x infinite one ſhall have $y = x$, viz. the Ordinate will be alſo infinite. FIG. 37.

If $m = 1$, and $n = -2$; then will $AE = -a$: whence the Problem may be after this manner ſtated.

Con-

FIG. 38. Continue out the given right Line AB (towards A) to the Point E so, that the Quantity $\frac{AE \times \overline{AB}^2}{\overline{BE}^2}$ be the greatest of any other the like Quantity $\frac{AP \times \overline{AB}^2}{\overline{BP}^2}$.

EXAMPLE V.

FIG. 39. 52. THE right Line AB being divided into three Parts AC, CF, FB. To divide or cut the middle Part CF in the Point E being such, that the Ratio of the Rectangle $AE \times EB$ to the Rectangle $CE \times EF$, be less than any other Ratio formed in like manner.

Call the given Quantities AC, a; CF, b; CB, c; and the unknown Quantity CE, x: then will $AE = a + x$, $EB = c - x$, $EF = b - x$, and therefore the Ratio of $AE \times EB$, to $CE \times EF$ will be $\frac{ac + cx - ax - xx}{bx - xx}$, which must be a *minimum*. Whence if a Curve MDM be supposed such, that the Relation of the Ordinate $PM (y)$ to the Absciss $CP (x)$ be expressed by $y = \frac{aac + acx - aax - axx}{bx - xx}$, the Problem will be brought to this, *viz.* to find such a Value CE for x, that the Ordinate ED be less than any other PM of like sort. Therefore throwing the said Equation into Fluxions, and afterwards dividing by $a\dot{x}$, there will come out $cxx - axx - bxx + 2acx - abc = 0$; and one Root of this Equation will solve the Problem.

If $c = a + b$; then will $x = \frac{1}{2}b$.

EXAM-

EXAMPLE VI.

53. Of all the Cones that can be inscribed within a given Sphere, to find that whose convex Surface is the greatest. FIG. 40.

This Problem, in other Words, is to find the Point E in AB the Diameter of the Simicircle AFB such, that drawing the Perpendicular EF, and joining AF, the Rectangle $AF \times FE$ be greater than any other Rectangle ($AN \times NP$) like it. For if the Semicircle AFB be conceived to make an entire Revolution about the Diameter AB, it is evident, that it shall describe a Sphere, and the right-angled Triangles AEF, APN, will generate Cones inscribed in the Sphere; the convex Surfaces of which described by the Chords AE, AN, will be to one another as the Rectangles $AF \times FE$, $AN \times NP$.

Now let the unknown Quantity $AE = x$, and the given one $AB = a$. Then from the Nature of the Circle $AF = \sqrt{ax}$, $EF = \sqrt{ax - xx}$; and therefore $AF \times FE = \sqrt{aaxx - ax^3}$ which must be a *Maximum*. Whence we must conceive a Curve MDM to be such, that the Relation of the Ordinate PM (y) to the correspondent Abscifs AP (x) is expressed by $\frac{\sqrt{aaxx - ax^3}}{a} = y$; and find the Point E so, that the Ordinate ED be greater than any other (PM) of like sort. And making the Fluxion of the Equation $\frac{2az\dot{x} - 3xx\dot{x}}{2\sqrt{aaxx - ax^3}} = 0$, we get AE (x) $= \frac{2}{3}a$.

EXAMPLE

EXAMPLE VII.

54. AMONG all the Parallelepipedons equal to a given Cube a^3, and having the given right Line b for one of the Sides, to find that which has the least Superficies.

Call one of the two Sides sought x, and the other will be $\frac{a^3}{bx}$; then assuming the alternate Planes b, x, $\frac{a^3}{bx}$ of the Parallelepipedon, and their Sum, *viz.* $b+x+\frac{a^3}{x}+\frac{a^3}{b}$ will be half of the Superficies, which must be a *Minimum*. And so (as all along) conceiving a Curve expressed by $\frac{bx}{a}+\frac{aa}{x}+\frac{aa}{b}=y$, the Fluxion of this Equation must be equal to o; that is $\frac{b\dot{x}}{a}-\frac{aa\dot{x}}{xx}=o$, from whence there comes out $xx=\frac{a^3}{b}$, and $x=\sqrt{\frac{a^3}{b}}$: Consequently the three Sides of the Parallelepipedon required, will be b, $\sqrt{\frac{a^3}{b}}$, and $\sqrt{\frac{a^3}{b}}$. Whence you may observe, that two Sides are equal.

EXAMPLE VIII.

FIG. 41. 55. AMONG all the Parallelepipedons that are equal to a given Cube a^3, to find that which has the least Superficies.

Call one of the unknown Sides x; then by the last Example it is plain, that the two other

ther Sides will be each equal to $\frac{\sqrt{a^3}}{b}$; and therefore the Sum of the alternate Planes, which is the half of the Superficies, will be $\frac{a^3}{x} + 2\sqrt{a^3 x}$, which must be a *Minimum*. Whence the Fluxion of this must be equal to o, *viz.* $\frac{a^3\dot{x}}{xx} + \frac{a^3\dot{x}}{\sqrt{a^3 x}} = o$; and so $x = a$; and consequently the two Sides shall also be equal to a; so that the Cube itself solves the Problem.

EXAMPLE IX.

56. THE Line AEB being given in Position on a Plane, together with two stable Points C, F; and if two right Lines CP (u), PF (z), be drawn from any Point P in it; and if a Quantity be made up of these indeterminate ones u and z, and other given right Lines a, b, &c. at pleasure. It is required to find such a Position of the right Lines CE, EF, that the Quantity given, which is made up of them, be greater or lesser than that same Quantity when it is made up of the right Lines CP, PE. FIG. 41.

Let us suppose the Lines CE, EF, to have the requisite Situation: Join CF, and conceive the Curve DM to be such, that drawing PQM at pleasure perpendicular to CF, the Ordinate QM may express the Quantity given. Now it is manifest, that when the Point P falls in the Point E, the Ordinate QM, which then becomes QD, must be the greatest or least of all others of like sort. Therefore the Fluxion of it must be equal to o, or infi-

2 nite:

nite: Whence if the given Quantity, for Example, is $auxzz$; then will $a\dot{u}xzz\dot{z}=o$, and consequently $\dot{u}:-\dot{z}::zz:a$. Wherefore we may already perceive that $\dot{z}$ must be negative with respect to $\dot{u}$; that is, the right Lines *CE*, *EF*, must have such a Position that *z* decreases at the same time as *u* increases.

Now if *EG* be drawn perpendicular to the Line *AEB*, and from any Point *G* therein, the Lines *GL*, *GI*, be likewise drawn perpendicular to *CE*, *EF*, and the right Lines *CKe*, *FeH*, be drawn from the Point *e* infinitely near *E*, and from the Centres *C*, *F*, be described the small Arches *EK*, *EH*; the right-angled Triangles *ELG* and *EKe*, *EIG* and *EHF*, will be similar. For if the Angle *LEe* be taken from the right Angles *GEe*, *LEK*, the remaining Angles *LEG*, *KBe*, shall be equal. Whence $GL:GI::KB\ (\dot{u}):He\ (-\dot{z})::zz:a$. Therefore the Position of the right Lines *CE*, *EF*, must be such, that when *EG* is drawn perpendicular to the Line *AEB*, the Sine (*GL*) of the Angle *GEC*, is to the Sine (*GI*) of the Angle *GEF*, as the Quantities drawn into $\dot{z}$, to the Quantities drawn into $\dot{u}$. Which was to be found.

Corol.

57. If the right Line *CE* be given both in Position and Magnitude, and *EF* only in Magnitude, and the Position thereof be required, it is evident that the Angle *GEC* being given, its Sine (*GL*) will be given likewise, and consequently the Sine (*GI*) of the Angle (*GEF*) sought. Therefore if a Circle

be described with EG as a Diameter, and the Value of GI be laid off in the Circumference from G to I; the right Line EF passing thro' the Point I, will have the requisite Position.

Let $au+bz$ be the given Quantity; then will GI be $=\frac{a\times GL}{b}$: and so it appears, that let EC and EF have what Length soever, the Position of this latter shall be always the same, since they do not come into the Value GI, which consequently does not vary. If $a=b$, it is plain that the Position of EF must be in CE continued out from E; because $GL=GI$, when the Points C and F fall on each Side the Line AB: But when they fall on the same Side, the Angle FEG must be assumed equal to the Angle CEG. FIG. 42.

EXAMPLE X.

58. THE Circle AEB being given in Position, as also the Points C and F without the same: to find the Point E in the Periphery being such, that the Sum of the right Lines CE, EF may be a *minimum*. FIG. 42.

Suppose the Point E to be that sought, and draw the Line OEG from the Centre O; which will be perpendicular to the Circumference AEB; and so * the Angles FEG, CEG will be equal. Therefore if EH be drawn so, that the Angle EHO be equal to the Angle CEO, and likewise EK so, that the Angle EKO be equal to the Angle FEO, and the Parallels ED, EL to OF, OC; and there will be formed the similar Triangles OCE and OEH, OFE and OEK, HDE and KLE; and calling the known Quantities

* Art. 57.

OE, OA or OB, a; OC, b; OF, c; and the unknown ones OD or LE, x; DE or OL, y. Then will $OH=\frac{aa}{b}$, $OK=\frac{aa}{c}$, and HD $\left(x-\frac{aa}{b}\right):DE\,(y)::KL\left(y-\frac{aa}{c}\right):LE\,(x)$.

Whence $xx-\frac{aax}{b}=yy-\frac{aay}{c}$; and this is an Equation appertaining to an Hyperbola, which may be easily constructed, and will cut the Circle in the Point E sought.

EXAMPLE XI.

FIG. 43. 59. A TRAVELLER setting out from a Place C to go to a Place F, must cross two Countries divided from each other by the right Line AEB. Now suppose him to go the Length a in the time c in the Country adjoining to C, and the Length b in the same time c in the other Country adjoining to F: it is required to find the Point E in the right Line AEB thro' which he is to pass in the shortest time possible from C to F.

Make $a:CE\,(u)::c:\frac{cu}{a}$. And $b:EF\,(z)::c:\frac{cz}{b}$. Then it is plain, that $\frac{cu}{a}$ expresses the Time of the Travellers going the Length CE, and $\frac{cz}{b}$ the Time of his going the Length EF; so that $\frac{cu}{a}+\frac{cz}{c}$ must be a *minimum*. Whence * drawing EG perpendicular to AB, the Sine of the Angle GEC must be to the Sine of the Angle GEF, as a to b.

* Art. 56.

This

This being premised, if the Circle CGH be described with the Radius EC from the Point sought E as a Centre, and the Perpendiculars CA, HD, FB be drawn to the right Line AEB, and the Perpendiculars GL, GI to the right Lines CE, EF, we shall have $a:b::GL:GI$. But $GL=AE$, and $GI=ED$, because the right-angled Triangles GEL and ECA, GEI and EHD are equal and similar, as may be easily proved. Therefore if the unknown Line AE be called x, we shall have $ED=\frac{bx}{a}$: and calling the known Lines AB, f; AC, g; BF, b: from the Similarity of the Triangles EBF, EDH, $EB\ (f-x)$: $BF\ (b)::ED\left(\frac{bx}{a}\right):DH=\frac{bbx}{af-ax}$. But because of the right-angled Triangles EDH, EAC, of equal Hypothenuses EH, EC, $\overline{ED}^2+\overline{DH}^2$ will be $=\overline{EA}^2+\overline{AC}^2$, that is, $\frac{bbxx}{aa}+\frac{bbbbxx}{aaff-2aafx+aaxx}=xx+gg$: and freeing the Equation from Fractions, and afterwards duly ordering the same, there will arise

$$\begin{array}{llllll} aax^4 & -2aafx^3 & +aaffxx & -2aafggx & +aaffgg & =0. \\ -bb & +2bbf & +aagg & & & \\ & & -bbff & & & \\ & & -bbbb & & & \end{array}$$

This Equation may be gotten after the following manner, without having recourse to the 9th Example.

Having named the known Lines as before, *viz.* AB, f; AC, g; BF, b; and the unknown one AE, x. Make $a:CE\ (\sqrt{gg+xx})::c:\frac{c\sqrt{gg+xx}}{a}=$ to the Time the Traveller is going the Length CE. And in like manner $b : EF\ (\sqrt{ff - 2fx + xx + bb}) :: c : \frac{c\sqrt{ff - 2fx + xx + bb}}{b}$ = Time of his going the Length EF: so that $\frac{c\sqrt{gg + xx}}{a} + \frac{c\sqrt{ff - 2fx + xx + bb}}{b}$ = a *minimum*: and therefore $\frac{cx\dot{x}}{a\sqrt{gg + xx}} + \frac{cx\dot{x} - cf\dot{x}}{b\sqrt{ff - 2fx + xx + bb}} = 0$. Whence dividing by $c\dot{x}$, and freeing the Equation from Surds, we shall have the same Equation as before; one Root of which will express AE the *minimum* sought.

EXAMPLE XII.

FIG. 44. 60. LET F be a Pulley hanging freely from the End of a Cord CF fasten'd in C, and let D be a Weight suspended by the Cord DFB put over the Pulley F, which Cord is fasten'd in B; so that the Points C and B lie in the same horizontal Line CB. Now if the Pulley and Cords be supposed to have no Weight, it is required to find in what Place the Weight D, or Pulley F, will settle or come to rest.

By the Principles of Mechanicks, it is plain that the Weight D will descend as far as possible below the horizontal Line CB: therefore the Plumb Line DFE will be a *maximum*. And therefore calling the given Quantities CF, a; DFB, b; CB, c; and the unknown Quantity CE, x; there will arise $EF = \sqrt{aa - xx}$, $FB = \sqrt{aa + cc - 2cx}$, and $DFE = b - \sqrt{aa + cc - 2cx} + \sqrt{aa - xx}$, which must be a *maximum*. And so the Fluxion

xion of it will be $\frac{c\dot{x}}{\sqrt{aa+cc-2cx}} - \frac{x\dot{x}}{\sqrt{aa-xx}}$ $=0$. Whence there arises $2cx^3 - 2ccxx - aaxx + aacc = 0$, and dividing by $x-c$, there comes out $2cxx - aax - aac = 0$, one Root of which will express CE such, that the Perpendicular ED passes by the Pulley F and the Centre D of the Weight, when the same is at rest.

Here follows another Solution of this Problem.

Call EF, y; BF, z; then will $b-z+y=$ *maximum*, and so $\dot{y}=\dot{z}$. Now it is evident, that the Pulley F does describe a Circle CFA about the Point C as a Centre: and if fR be drawn from the Point f, infinitely near to F, parallel to CB, and fS perpendicular to BF; therefore will $FR=\dot{y}$, and $FS=\dot{z}$. Which are consequently equal to each other: and so the little right-angled Triangles FRf, FSf, having the common Hypothenuse Ff, are equal and similar: whence the Angle RFf is equal to the Angle SFf, that is, the Point F must be so situate in the Periphery FA, that the Angles made by the right Lines EF, FB, with the Tangents in F, be equal to each other: or else (which comes to the same) the Angles BFC, DFC equal.

This being premised, if you draw FH so, that the Angle FHC be equal to the Angle CFB or CFD; the Triangles CBF, CFH will be similar; as also the right-angled Triangles ECF, EFH, since the Angle CFE is equal to the Angle FHE, each of them being the Complement of the equal Angles FHC, CFD to two right Angles; and consequently

$CH = \frac{aa}{c}$, and $HE\left(x - \frac{aa}{c}\right) : EF(y) :: EF(y) : EC(x)$. Whence $xx - \frac{aax}{c} = yy = aa - xx$ from the Nature of the Circle: from whence arises the same Equation as at first.

EXAMPLE XIII.

FIG. 45. 61. THE Elevation of the Pole being given: To find the Day of the Year wherein the Twilight will be the shortest possible.

Let *C* be the Centre of the Sphere; *APTOBHQ* the Meridian; *HDdO*, the Horizon; *QEeT* the Crepuscular Circle parallel to the Horizon; *AMNB* the Equator; *FEDG* that Part of the Parallel to the Equator (the Sun describes the Day wherein the Twilight is shortest) contained between the Planes of the Horizon and the Crepuscular Circle; *P* the South Pole; *PEM*, *PDM* Quadrants of Circles of Declination. Now the Arch *HQ* or *OT* of the Meridian comprehended between the Horizon and the Crepuscular Circle, and the Arch *OP* of the Elevation of the Pole are given: and consequently their Sines *CI* or *FL* or *QX*, and *OV*. Now we must find *CK* the Sine of the Arch *EM* or *DN* of the Sun's Declination, when he describes the Parallel *ED*.

If you suppose another Part *fedg* of a Parallel to the Equator infinitely near to *FEDG*, and draw the Quadrants *Pem*, *Pdn*: it is manifest, that the Time the Sun takes up in describing the Arch *ED* must be a *minimum*, and the Fluxion of the Arch *MN* being the Measure of it, and becoming *mn* when *ED* becomes

comes ed, must be equal to nothing; whence the small Arches Mm, Nn, and consequently the little Arches Re, Sd, will be equal to each other. Now the Arches RE, SD being contained between the same Parallels ED, ed, are equal likewise, and the Angles at S and R are right ones. Therefore the little right-angled Triangles ERe, DSd (consider'd as right * lin'd Triangles, on account of their Sides being infinitely small) will be equal and similar: and consequently the Hypothenuses Ee, Dd shall be equal likewise. * *Art.* 3.

This being premised, the right Lines DG, EF, dg, ef, the common Sections of the Planes $FEDG$, $fedg$, parallel to the Equator, and the Horizon and Crepuscular Circle, will be perpendicular to the Diameters HO, QT, because the Planes of all these Circles are perpendicular to the Plane of the Meridian; and the little right Lines Gg, Ff, will be equal to each other, since the right Lines FG, fg are parallel. Therefore $\sqrt{\overline{Dd}^2 - \overline{Gg}^2}$, or $DG - dg$ $= \sqrt{\overline{Ee}^2 - \overline{Ff}^2}$, or $fe - FE$. Now it is plain (from *Art.* 50.) if two Ordinates in a Semicircle be drawn infinitely near, the small Arch intercepted between them will be to their Fluxion, as the Radius is to the Part of the Diameter intercepted by the Centre and those Ordinates. Whence (because of the Circles HDO, QET) $CO : CG :: Dd$ or $Ee : DG - dg$ or $fe - FE :: IQ : IF :: CO + IQ$ or OX : $CG + IF$ or GL. But because of the right-angled similar Triangles CVO, CKG, FLG; therefore $CO : CG :: OV : GK$. And GK : $GL :: CK : FL$ or QX. Whence $OV : CK$ $:: OX : XQ :: XQ : XH$ by the Nature of the

 Circle:

Circle: that is, if QX be taken for the Radius in the right-angled Triangle QXH, and the Angle HQX be 9 Degrees, (because the Arch HQ by Astronomers is supposed to be 18 Degrees) and you make as the Radius is to the Tangent of an Angle of 9 Degrees, so is the Sine of the Elevation of the Pole to a fourth Sine; this will be the Sine of the Sun's *Southern* Declination that Day of the Year the Twilight is the shortest possible. So that if you take 0.8002875 from the Logarithm of the Sine of the Elevation of the Pole, the Remainder will be the Logarithm of the Sine sought.

SECT. IV.

Of the Use of Fluxions in finding the Points of Inflexion and Retrogression of Curves.

BECAUSE second, third, &c. Fluxions are used hereafter; before we go further, we think it necessary to define them.

DEFINITION I.

THE infinitely small Part generated by the continual increasing or decreasing of the Fluxion of a variable Quantity, is called the * *Fluxion* of the *Fluxion* of that Quantity, or *second Fluxion*. So if a third Ordinate *nq* be suppos'd infinitely near the second *mp*, and *mS* be drawn parallel to *AB*, and *mH* to *RS*. *Hn* is the *Fluxion of the Fluxion Rm*, or *the second Fluxion* of *PM*. FIG. 46.

In like manner, if a fourth Ordinate *of* be suppos'd infinitely near the third *nq*, and *nT* be drawn parallel to *AB*, and *nL* to *ST*, the Difference of the small right Lines *Hn*, *Lo*, is the *Fluxion of the second Fluxion*, or the *third Fluxion* of *PM*. Understand the same of others.

* *See the Translator's* Preface.

OBSER-

Observation.

Fluxions of Fluxions are denoted with Dots over the Letters, being so many in Number as the Order of the Fluxion is. For Example, $\ddot{x}$ *is the second Fluxion of* x; $\dddot{x}$ *the third Fluxion of* x; $\ddddot{x}$ *the fourth Fluxion of* x, *&c. Understand the same of* y, *or any other variable Quantity. And so if* MP=y, *then will* $\ddot{y}$ *express* Hn; $\dddot{y}$, Lo—Hn *or* Hn—Lo, *&c.*

Corollary I.

62. If each of the Abscisses *AP*, *Ap*, *Aq*, *Af*, be called *x*, and each of the correspondent Ordinates *PM*, *pm*, *qn*, *fo*, *y*; and every of the Parts of the Curve *AM*, *Am*, *An*, *Ao*, *u*; then it is plain that $\dot{x}$ will be the Fluxions *Pp*, *pq*, *qf*, of the Abscisses; $\dot{y}$ the Fluxions *Rm*, *Sn*, *To*, of the Ordinates; and $\dot{u}$ the Fluxions *Mm*, *mn*, *no*, of the Parts of the Curve *AMD*. Now in order (for Example) to get the second Fluxion *Hn* of the variable Quantity *PM*, we must suppose two little Parts *Pp*, *pq*, in the Axis, and two others *Mm*, *mn*, in the Curve, to get the two Fluxions *Rm*, *Sn*; and therefore if the two small Parts *Pp*, *pq*, be conceived as equal to one another, it is evident that $\dot{x}$ will be a standing Quantity with respect to $\dot{y}$ and $\dot{u}$, because *Pp* which becomes *pq*, will not vary while *Rm*, which becomes *Sn*, and *Mm* which becomes *mn*, does. If the little Parts *Mm*, *mn*, of the Curve, be supposed to be equal to each other, then will $\dot{u}$ be a standing Quantity with respect to $\dot{x}$ and $\dot{y}$ And lastly, if *Rm* and *Sn* be sup-

supposed equal, then will $\dot{y}$ be a standing Quantity with regard to $\dot{x}$ and $\dot{u}$, and the Fluxion *Hn* ($\ddot{y}$) of it will be equal to nothing.

In like manner, to get the third Fluxion of *PM*, or the Fluxion of the second Fluxion *Hn*, we must imagine three little Parts *Pp*, *pq*, *qf*, in the Axis; three others *Mm*, *mn*, *no*, in the Curve, and three others *Rm*, *Sn*, *To*, in the Ordinates: Then will $\dot{x}$, $\dot{u}$, or $\dot{y}$, be a constant or standing Quantity, according as the small Parts *Pp*, *pq*, *qf*, or *Mm*, *mn*, *no*, or *Rm*, *Sn*, *To*, be supposed equal. The same must be conceived of the fourth, fifth, *&c.* Fluxions.

All this is to be understood also in Curves, as *AMD*, whose Ordinates *BM*, *Bm*, *Bn*, all issue from a stable Point *B*. As, for Example, to get the second Fluxion of *BM*, we must conceive two other Ordinates *Bm*, *Bn*, making infinitely small Angles *MBm*, *mBn*; then if the small Arches of Circles *MR*, *mS*, be described from the Centre *B*, the Difference of the small right Lines *Rm*, *Sn*, shall be the second Fluxion of *BM*. And the small Arches *MR*, *mS*, or the small Parts *Mm*, *mn*, of the Curve, or, finally, the little right Lines *Rm*, *Sn*, may be taken as standing Quantities. Understand the same of third, fourth, *&c.* Fluxions of the Ordinate *BM*. FIG. 47.

SCHOLIUM.

63. HERE we are to observe, 1°, That if there be several Orders of infinitely small Quantities: For Example, *RM* will be infinitely small with respect to *PM*, and infinitely great with respect to *Nn*. In like manner, FIG. 46.

ner, the Space $MPpm$ is infinitely small with regard to the Space APM, and infinitely great with regard to the Triangle MRm.

2°, That the whole Fluxion Pf is likewise infinitely small with regard to AP; because every Quantity which is the Sum of a finite Number of infinitely small Quantities, as Pp, pq, qf, with respect to another AP, will be infinitely small with regard to the said Quantity: And that in order for it to become of the same Order, it is necessary for the Number of Quantities of the Order next below it, and of which it consists, to be infinite.

Corol. II.

Fig. 48, 49. 64. The second Fluxions, in all the Cases possible, may be represented thus:

1°, In Curves whose Ordinates mR, nS, are parallel, continue out the small Side Mm to intersect the Ordinate Sn in H, and describe the Arch nk from the Centre m with the Distance mn, and draw the little right Lines nl, li, kcg, parallel to mS and Sn. This being done, if $\dot{x}$ be supposed a standing Quantity, *viz.* $MR = mS$, it is plain that the Triangle mSH is similar and equal to the Triangle MRm; and so Hn is $= \ddot{y}$; that is, the Difference of Rm and Sn, and $Hk = \ddot{u}$. But if $\dot{u}$ be conceived to be constant, *viz.* $Mm = mn$ or mk, then it is plain that the Triangle mgk is similar and equal to the Triangle MRm; and so $kc = \ddot{y}$, and Sg or $cn = \ddot{x}$. Lastly, if $\dot{y}$ be supposed invariable, *viz.* $mR = nS$, then will the Triangle mil be equal and similar to the Triangle MRm; and so iS or $nl = \ddot{x}$, and $lk = \ddot{u}$.

2°, In

2°, In Curves whose Ordinates *BM*, *Bm*, *Bn*, issue from a Point *B*. Describe the Arches *MR*, *mS*, from the Centre *B*; these may be looked * upon as small right Lines perpendicular to *Bm*, *Bn*. Continue out *Mm* to *E*, and describe the small Arch *nkE* from the Centre *m*, with the Distance *mn*; make the Angle $EmH = mBn$, and draw the little right Lines *nl*, *li*, *kcg* parallel to *mS* and *Sn*. This being done, because the Triangle *BSm* is right-angled at *S*, the Angle $BmS + mBn$, or $+ EmH =$ a right Angle; and therefore the Angle $BmE =$ a right Angle $+ SmH$: It is also equal to the right Angle $MRm + RMm$, since it is external to the Triangle *RmM*: Therefore the Angle $SmH = RMm$. FIG. 50, 51. * Art. 3.

1. Hence, if you suppose $\dot{x}$ invariable, that is, the little Arches *MR*, *mS*, equal, the Triangle *SmH* will be similar and equal to the Triangle *RMm*, and so $Hn = \dot{y}$, and $HK = \ddot{u}$. 2. If $\dot{u}$ be supposed invariable, the Triangle *gmk* will be equal and similar to the Triangle *RMm*; and so *kc* will express $\ddot{y}$ and *Sg* or *cn*, $\ddot{x}$. Lastly, if $\dot{y}$ be supposed invariable, the Triangles *iml*, *RMm*, will be equal and similar; and so *iS* or $ln = \ddot{x}$, and $lk = \ddot{u}$.

PROP. I.

65. *TO find the Fluxion of a Quantity consisting of Fluxions.*

Consider any one of the Fluxions of the given Expression as invariable, and proceed with the others as variable Quantities, according to the Rules laid down in Section the first.

For

For Example, if $\dot{x}$ be ſuppoſed invariable, the Fluxion of $\frac{y\dot{y}}{\dot{x}}$ will be $\frac{\dot{y}\dot{y}+y\ddot{y}}{\dot{x}}$, and taking $\dot{y}$ as invariable, it will be $\frac{\dot{x}\dot{y}\dot{y}-y\dot{y}\ddot{x}}{\dot{x}\dot{x}}$. In like manner, taking $\dot{x}$ as invariable, the Fluxion of $\frac{z\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}$, will be $\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}+\frac{z\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$, and dividing it by $\dot{x}$, there will ariſe $\frac{\dot{z}\dot{x}\dot{x}+\dot{z}\dot{y}\dot{y}+z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$; and ſuppoſing $\dot{y}$ invariable, it will be $\dot{z}\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}+\frac{z\dot{x}\ddot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}-z\ddot{x}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$ the whole divided by $\dot{x}\dot{x}$, and this will be $\frac{\dot{z}\dot{x}\dot{x}\dot{x}+\dot{z}\dot{x}\dot{y}\dot{y}-z\dot{y}\dot{y}\ddot{x}}{\dot{x}\dot{x}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$.

Taking $\dot{x}$ as conſtant, the Fluxion of $\frac{y\dot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$ will be $\overline{\dot{y}\dot{y}+y\ddot{y}}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}-\frac{y\dot{y}\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$, which divided by $\dot{x}\dot{x}+\dot{y}\dot{y}$, and then it will be $\frac{\dot{x}\dot{x}\dot{y}\dot{y}+\dot{y}\dot{y}\dot{y}\dot{y}+y\dot{x}\dot{x}\ddot{y}}{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$; and ſuppoſing $\dot{y}$ to be invariable, it will be $\frac{\dot{x}\dot{x}\dot{y}\dot{y}+\dot{y}\dot{y}\dot{y}\dot{y}-y\dot{y}\dot{x}\ddot{x}}{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$.

In like manner, the Fluxion of $\frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{-\dot{x}\ddot{y}}$, or $\frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, ſuppoſing $\dot{x}$ invariable, will be $-\frac{3\dot{x}\dot{y}\ddot{y}\ddot{y}\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{\frac{1}{2}}+\dot{x}\dddot{y}\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{\frac{3}{2}}}{\dot{x}\dot{x}\ddot{y}\ddot{y}}$.

Here we muſt obſerve, that in the laſt Example, $\dot{y}$ cannot be ſuppoſed invariable; for if it be, its Fluxion $\ddot{y}$ will be nothing; and conſequently muſt not come into the Quantity propoſed.

DEFIN-

DEFINITION II.

WHEN a Curve AFK is partly convex, and partly concave, to a right Line AB or Point B; the Point F dividing the convex from the concave Part, or the End of the one, and the Beginning of the other, is called the Point of *Inflexion*, when the Curve being come to F, continues its Course towards the same Parts; and the Point of *Retrogression*, when the Curve returns back again towards the Place of its beginning. FIG. 52, 53, 54, 55.

PROP. II.

A General PROBLEM.

66. *THE Nature of the Curve* AFK *being given, to determine the Point of Inflexion or Retrogression* F.

First, Let us suppose the Curve AFK to have the right Line AB as a Diameter, and the Ordinates PM, EF, &c. parallel to each other. Now if you draw the Ordinate FE from the Point F, and the Tangent FL; and another Ordinate MP from the Point M in the concave Part AF of the Curve, as likewise a Tangent MT. Then it is evident, FIG. 52, 53.

1°, In Curves that have a Point of Inflexion, that while the Absciss AP constantly increases, the Part AT of the Diameter, intercepted between the Vertex of the Diameter A and the Point T, where the Tangent meets the Diameter, does likewise increase until the Point P falls in E, after which it continually decreases:

ses: therefore AT must become a *maximum* AL, when the Point P falls in E the Point sought.

2°, That in those Curves that have a Point of Retrogression, the Part AT continually increases, and the Abscis AP, till the Point T falls or coincides with L; and afterwards it constantly decreases; whence AE must be a *maximum*, when the Point T coincides with L.

Now call AE, x; EF, y; then will $AL = \frac{y\dot{x}}{\dot{y}} - x$, and the Fluxion of this will be $\frac{\dot{y}\dot{y}\dot{x} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}} - \dot{x}$, supposing $\dot{x}$ invariable: which (being divided by $\dot{x}$ the Fluxion of AE) must be $= 0$*, or infinite: whence $-\frac{y\ddot{y}}{\dot{y}\dot{y}}$ is $= 0$ or Infinity: and so multiplying by $\dot{y}\dot{y}$, and dividing by $-y$, there comes out $\ddot{y} = 0$, or infinite. Now with this last Expression, and the general Equation of a Curve, the Point of Inflexion or Retrogression F may be found. For the Nature of the Curve AFK being given, we shall have a Value of y in x, and throwing that Value into Fluxions, supposing $\dot{x}$ invariable, we shall get a Value of $\ddot{y}$ in $\dot{x}\dot{x}$, which being made equal to 0, and afterwards to Infinity: by means thereof, in either of these Suppositions, we may find AE so expressed, that its correspondent Ordinate EF shall intersect the Curve in the Point of Inflexion or Retrogression F.

* *Art.* 47.

The Point A whereat the x^s begin, may be so situate that $AL = x - \frac{y\dot{x}}{\dot{y}}$, instead of $\frac{y\dot{x}}{\dot{y}} - x$, and AL or AE a *minimum* instead of a *maximum*. But as the Conse-

Consequence is always the same, and there is no Difficulty arising from it, I shall say no more thereof.

Here we must observe that AL can never be $= x + \frac{y\dot{x}}{\dot{y}}$, for when the Point T falls on the other Side the Point P, with respect to A, the Value of $\frac{y\dot{x}}{\dot{y}}$ will be negative, (by *Art.* 10.) and consequently the Value of $-\frac{y\dot{x}}{\dot{y}}$ will be positive; so that in this Case we have still $AE + EL$ or $AL = x - \frac{y\dot{x}}{\dot{y}}$.

The same things may be found after a different way. It is plain, taking x as invariable, and supposing the Ordinate y to increase, that Sn is less than SH or Rm in the concave Part of the Curve, and greater in the convex Part: therefore Hn ($\ddot{y}$) which is positive, must become negative in the Point of Inflexion or Retrogression F; and consequently * in that Point it must be nothing or infinite. FIG. 48, 49.

* *Art.* 47.

Secondly, Let the Ordinates BM, BF, BM of the Curve AFK all meet in the same Point B. If any Ordinate BM be drawn, and the Tangent MT meeting BT perpendicular to BM in the Point T; and if m be taken infinitely near M, and the Ordinate Bm, Tangent mt, the Perpendicular Bt to Bm meeting MT in O, be all drawn: then supposing the Ordinate BM, becoming Bm to increase, it is plain, in the concave Part of the Curve, that Bt is greater than BO, and in the convex Part the same is less. So that under the Point of Inflexion or Retrogression F, Ot must change from negative to positive. FIG. 54, 55. FIG. 56, 57.

G This

FIG. 56. This being premised, about the Centre B describe the small Arches MR, TH, and there will be form'd the similar Triangles mRM, MBT, THO, and the little similar Sectors BMR, BTH. Now if you make $BM = y$; $MR = \dot{x}$, then will $mR\ (\dot{y}) : RM\ (\dot{x}) :: BM\ (y) : BT = \frac{y\dot{x}}{\dot{y}} :: MR\ (\dot{x}) : TH = \frac{\dot{x}\dot{x}}{\dot{y}} :: TH\left(\frac{\dot{x}\dot{x}}{\dot{y}}\right)$ $: HO = \frac{\dot{x}\dot{x}\dot{x}}{\dot{y}\dot{y}}$. Now if you throw $BT\left(\frac{y\dot{x}}{\dot{y}}\right)$ into Fluxions, supposing $\dot{x}$ invariable, there comes out $Bt - BT$ or $Ht = \frac{\dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$; and therefore $OH + Ht$ or $Ot = \frac{\dot{x}^3 + \dot{x}\dot{y}^2 - y\dot{x}\ddot{y}}{\dot{y}^2}$. Whence multiplying by $\dot{y}^2$, and dividing by $\dot{x}$, the Expression $\dot{x}^2 + \dot{y}^2 - y\ddot{y}$ must be nothing or infinite in the Point of Inflexion or Retrogression F. Now when the Nature of the Curve (*Fig.* 54, 55.) AFK is given, we shall have Values of $\dot{y}$ in $\dot{x}$, and of $\ddot{y}$ in $\dot{x}^2$, which being substituted in $\dot{x} + \dot{y}^2 - y\ddot{y}$, there will arise a Quantity, which being made equal first to 0, and afterwards to Infinity; by means thereof we may get such a Value of BF, that if a Circle be described from B with the same as a Radius, it shall cut the Curve AFK in the Point of Inflexion or Retrogression F.

FIG. 50, 51. *Lastly*, To find the same things moreover another way, you must consider that the Angle BmE is greater than the Angle Bmn in the concave Part of the Curve; and the same is less in the convex Part: and therefore the FIG. 50. Angle $BmE - Bmn$ or Emn, that is, the Arch En, being the Measure of it, changes from

from positive to negative in the sought Point F. Now taking $\dot{x}$ as invariable, because of the right-angled similar Triangles HmS, Hnk, we have $Hm\ (\dot{u}) : mS\ (\dot{x}) :: Hn\ (-\ddot{y}) : nk =$ $-\frac{\dot{x}\ddot{y}}{\dot{u}}$. Where you must observe, that Hn is negative, because as $Bm\ (y)$ increases, $Rm\ (\dot{y})$ decreases. But because of the similar Sectors BmS, mEk, therefore will $Bm\ (y) : mS\ (\dot{x}) ::$ $mE\ (\dot{u}) : Ek = \frac{\dot{x}\dot{u}}{y}$; and so $Ek + kn$ or $En =$ $\frac{\dot{x}\dot{u}^2 - y\dot{x}\ddot{y}}{y\dot{u}}$. Whence multiplying by $y\dot{u}$, and dividing by $\dot{x}$, it follows that $\dot{u}^2 - y\ddot{y}$ or $\dot{x}^2 + \dot{y}^2 - y\ddot{y}$ must change from positive to negative under the sought Point F. FIG. 54, 55.

If y be supposed to become infinite, the Terms $\dot{x}^2$ and $\dot{y}^2$ will be nothing with respect to $y\ddot{y}$; and consequently the Form $\dot{x}^2 + \dot{y}^2 - y\ddot{y} = 0$, or infinite, will be changed into this $-y\ddot{y} = 0$, or infinite; that is, dividing by $-y$, $\ddot{y}$ will be $= 0$, or infinite, which is the Form of the first Case; and must be so likewise, because the Ordinates BM, BF, EM do then become parallel.

COROL.

67. WHEN $\ddot{y} = 0$, it is plain that the Fluxion of AL must be nothing with regard to the Fluxion of AE; and therefore the two infinitely near Tangents FL, fL, will coincide and make but one right Line fFL. FIG. 52. But when $\ddot{y} =$ Infinity, the Fluxion of AL must be infinitely great in respect to that of AE, or (which is the same thing) the Fluxion of AE is infinitely little with respect to the FIG. 53.

 the

the Fluxion of AL; and consequently two Tangents FL, Fl may be drawn from the Point F, making an infinitely small Angle LFL.

FIG. 56, 57. In like manner, when $\dot{x}^2+\dot{y}^2-y\ddot{y}=0$, it is plain that Ot must become nothing in regard to MR; and so the two infinitely near Tangents MT, mt, must coincide, when the Point M becomes a Point of contrary Flexion or Retrogression; but on the contrary, when $\dot{x}^2+\dot{y}^2-y\ddot{y}=$ Infinity, Ot must be infinite with respect to MR, or (which is the same thing) MR infinitely small with regard to Ot: and consequently the Point m must coincide with M, that is, one may draw two Tangents thro' the Point M, making an infinitely small Angle with each other, when that Point becomes a Point of Flexion, and contrary Retrogression.

Hence it follows, that the Tangent to the Point of Inflexion or Retrogression F, continued out, does both touch and cut the Curve AFK in that Point.

EXAMPLE I.

FIG. 58. 68. Let AFK be a Curve, the right Line AB being a Diameter of it, and let the Relation of any Abscifs AE (x) to its correspondent Ordinate EF (y) be expressed by this Equation $axx=xxy+aay$. It is required to find such an Expression for AE, that the Ordinate EF shall intersect the Curve AFK in the Point of Inflexion F.

The Equation of the Curve is $y=\frac{axx}{xx+aa}$; and

and therefore $\dot{y} = \frac{2a^3x\dot{x}}{\overline{xx+aa}^2}$, and finding the Fluxion of this Quantity, supposing $\dot{x}$, to be invariable, and making it equal to nothing, there will arise $\frac{2a^3\dot{x}^2 \times \overline{xx+aa}^2 - 8a^3xx\dot{x}^2 \times \overline{xx+aa}}{\overline{xx+aa}^4} = 0$.

This multiply'd by $\overline{xx+aa}^4$, and afterwards divided by $2a^3\dot{x}^2 \times \overline{xx+aa}$, will be $xx+aa-4xx = 0$. Whence AE $(x) = \frac{a}{\sqrt{3}}$.

If $\frac{1}{3}aa$ be put for its Equal xx in the Equation of the Curve $y = \frac{axx}{xx+aa}$, we shall have EF $(y) = \frac{1}{4}a$; so that the Point of Inflexion F may be determined without supposing the Curve AFK to be described.

If AC be drawn parallel to the Ordinates EF, and made equal to the given right Line a, and CG be drawn parallel to AB, the same will be an Asymptote to the Curve AFK. For if x be supposed infinite, we may take xx for $xx+aa$; and therefore the Equation of the Curve $y = \frac{axx}{xx+aa}$ will be changed into this $y = a$.

EXAMPLE II.

69. Let $y - a = \overline{x-a}^{\frac{2}{3}}$. Then $\dot{y} = \frac{2}{3}\overline{x-a}^{-\frac{1}{3}}\dot{x}$, and $\ddot{y} = -\frac{6}{27}\overline{x-a}^{-\frac{4}{3}}\dot{x}^2 = \frac{-6\dot{x}^2}{27\sqrt[3]{\overline{x-a}^4}}$, supposing $\dot{x}$ invariable. Now if the latter Expression be made equal to 0, there will come out $-6\dot{x}^2 = 0$. Which determines nothing; therefore this last Expression must be supposed infinitely great; and consequently the Deno-

minator $2s\sqrt[3]{x-a}$ thereof infinitely small, or $=0$. Whence the unknown Quantity *AE* $(x)=a$.

EXAMPLE III.

FIG. 59. 70. LET *AFK* be a Semi-cycloid, the Base *BK* of which let be greater than the Semi-circumference *ADB* of the generating Circle, whose Centre is *C*. It is required to find the Point *E* in the Diameter *AB*, from which the Ordinate *EF* being drawn, will cut the Cycloid in the Point of Inflexion *F*.

Call the known Quantities *ADB*, a; *BK*, b; *AB*, $2c$; and the unknown ones *AE*, x; *ED*, z; the Arch *AD*, u; *EF*, y. Then from the Nature of the Cycloid $y=z+\frac{bu}{a}$; and therefore $\dot{y}=\dot{z}+\frac{b\dot{u}}{a}$. Now by the Nature of the Circle we have $z=\sqrt{2cx-xx}$, $\dot{z}=\frac{c\dot{x}-x\dot{x}}{\sqrt{2cx-xx}}$, and $\dot{u}\,(\sqrt{\dot{x}^2+\dot{z}^2})=\frac{c\dot{x}}{\sqrt{2cx-xx}}$. Whence substituting for $\dot{z}$ and $\dot{u}$ their Equals, we have $\dot{y}=\frac{ac\dot{x}-ax\dot{x}+bc\dot{x}}{a\sqrt{2cx-xx}}$. The Fluxion of which (supposing $\dot{x}$ invariable) will be $\frac{bcx-acc-bcc\times\dot{x}^2}{2cx-xx\times\sqrt{2cx-xx}}=0$. From whence arises *AE* $(x)=c+\frac{ac}{b}$, and $CE=\frac{ac}{b}$.

It is manifest, if there be a Point of Inflexion *F*, that *C* must be greater than *a*; for if it be less, *CE* will be greater than *CB*.

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EXAM-

EXAMPLE IV.

71. LET *AFK* be the Conchoid of *Nicomedes*, and the Point *P* the Pole, and the right Line *BC* the Asymptote. It is required to find the Point of Inflection *F*. Now the Nature of this Curve is such, that a right Line *PF* drawn from any Point *F* in it to the Pole *F* meeting the Asymptote *BC* in *D*; the Part *DF* thereof is always equal to a standing Line *a*. FIG. 60.

Draw *PA* perpendicular, and *EF* parallel to *BC*, and call the known Quantities *AB* or *FD* a; *BP*, b; and the unknown ones *BE*, x; *EF*, y; and draw *DL* parallel to *BA*. Then because of the similar Triangles *DLF*, *PEF*, $DL\ (x) : LF\ (\sqrt{aa-xx}) :: PE\ (b+x) : EF\ (y) = \frac{\overline{b+x}\sqrt{aa-xx}}{x}$. The Fluxion of this will be $\dot{y} = \frac{x^3\dot{x}+aab\dot{x}}{xx\sqrt{aa-xx}}$. Now if you take the Fluxion of it again, and make the same equal to nothing, there will arise $\frac{\overline{2a^4b-aax^3-3aabxx}\times\dot{x}^2}{\overline{aax^3-x^5}\times\sqrt{aa-xx}}=0$, which may be brought down to $x^3+3bxx-2aab=0$; and one of the Roots thereof will be the Quantity of *BE* sought.

If $a=b$, the Equation aforegoing will become $x^3+3axx-2a^3=0$; which divided by $x+a$, and there comes out $xx+2ax-2aa=0$; and so $BE\ (x) = -a+\sqrt{3aa}$.

Otherwise.

Conceive the Lines *PF* issuing from the Pole *F* as Ordinates, and use the Form *$y\ddot{y}=\dot{x}^2+\dot{y}^2$, where $\dot{x}$ is supposed invariable. Now if you * *Art.* 66.

 you

you conceive another Ordinate Pf making an infinitely small Angle FPf with PF, and describe the small Arches FG, DH, from the Centre P. Call the known Quantities AB, a; BP, b; and the unknown ones PF, y; PD, z. Then from the Nature of the Conchoid $y=z+a$, and so $\dot{y}=\dot{z}$. Now from the right-angled Triangle DBP, $DB=\sqrt{zz-bb}$, and because of the similar Triangles DBP, dHD; PDH, PFG, $DB(\sqrt{zz-bb}):BP\ (b)::dH(\dot{z}):HD$ $=\frac{b\dot{z}}{\sqrt{zz-bb}}$; and $PD\ (z):PF\ (z+a)::HD$ $\frac{b\dot{z}}{\sqrt{zz-bb}}:FG\ (\dot{x})=\frac{bz\dot{z}+ab\dot{z}}{z\sqrt{zz-bb}}$. Whence we get $\dot{z}$ or $\dot{y}=\frac{z\dot{x}\sqrt{zz-bb}}{bz+ab}$, the Fluxion of which, supposing $\dot{x}$ invariable, will be $\ddot{y}=$ $\frac{bz^2+2abzz-ab^3\times\dot{z}\dot{x}}{\overline{bz+ab}^2\times\sqrt{zz-bb}}=\frac{bz^4+2abz^3-ab^3z\times\dot{x}^2}{\overline{bz+ab}^3}$, substituting for $\dot{z}$ what is equal to it. Now

* *Art.* 66. if in the general Form * $y\ddot{y}=\dot{x}^2+\dot{y}^2$, you put $z+a$ for y, and for $\dot{y}$ and $\ddot{y}$, the Values already found in $\dot{x}$ and $\dot{x}^2$, we shall have $\frac{z^4+2az^3-abbz\times\dot{x}^2}{\overline{bz+ab}^2}=\frac{z^4+2abbz+aabb\times\dot{x}^2}{\overline{bz+ab}^2}$, which may be brought down to $2z^3-3bbz-abb=0$; and one of the Roots of this *Plus* a, will express PF sought.

If $a=b$, then will $2z^3-3aaz-a^3=0$, which divided by $z+a$, will be $zz-az-\frac{aa}{2}=0$; whence $PF\ (z+a)=\frac{3}{2}a+\frac{1}{2}a\sqrt{3}=\frac{3a+a\sqrt{3}}{2}$.

EXAMPLE

EXAMPLE V.

72. LET AFK be a Conchoid of another Kind, the Nature whereof is such, that if a right Line PF be drawn from any Point F taken in it to the Pole P, intersecting the Asymptote BC in D, the Rectangle $PD \times DF$, will be always = to a constant Rectangle $PB \times BA$. It is required to find the Point of Inflexion F. FIG. 60.

Call the unknown Quantities BE, x; EF, y; and the known ones AB, a; BP, b; then will $PD \times DF = ab$: And because of the Parallels BD, EF, $PD \times DF\,(ab) : PB \times BE\,(bx) :: \overline{PF}^2\,(bb+2bx+xx+yy) : \overline{PE}^2\,(bb+2bx+xx)$. Whence $bbx+2bxx+x^3+yyx=abb+2abx+axx$, or $yy=\frac{abb+2abx+axx-bbx-2bxx-x^3}{x}$, and $y=\overline{b+x}\sqrt{\frac{a-x}{x}}=\sqrt{ax-xx}+b\sqrt{\frac{a-x}{x}}$, the Fluxion of which will be $\dot{y}=\frac{ax\dot{x}+2xx\dot{x}+ab\dot{x}}{2x\sqrt{ax-xx}}$. And again, throwing this into Fluxions, there will arise $\frac{3aab-aax-4abx \times \dot{x}^2}{4axx-4x^3 \times \sqrt{ax-xx}}=0$, which may be brought down to $x=\frac{3ab}{a+4b}=BE$.

If you make $\frac{-ax\dot{x}+2xx\dot{x}+ab\dot{x}}{2x\sqrt{ax-xx}}$ the Value of $\dot{y}=0$; then will $xx-\frac{1}{2}ax+\frac{1}{2}ab=0$, the two Roots $\frac{a+\sqrt{aa-8ab}}{4}$ and $\frac{a-\sqrt{aa-8ab}}{4}$ whereof, will be such Expressions of BH and BL, when a exceeds $8b$, that the Ordinate FIG. 61.

HM ſhall be leſs than thoſe near it, and the Ordinate *LN* greater, *viz.* the Tangents to the Curve in *M* and *N*, will be parallel to the Axis *AB*; and then the Point *E* will fall between *H* and *L*.

FIG. 62. But when $a=8b$, the Lines *BH*, *BE*, *BL*, will be equal each to $\frac{1}{4}a$. And then the Tangent in the Point of Inflexion *F*, will be parallel to the Axis *AB*. Laſtly, when *a* is leſs than $8b$, the two Roots are imaginary, and ſo no Tangent can be parallel to the Axis.

FIG. 60. This Problem may be ſolved likewiſe in taking the Lines *PF*, *Pf*, iſſuing from the Pole *P* as Ordinates, and uſing the Form $y\ddot{y} = \dot{x}^2+\dot{y}^2$, as in the laſt Example.

EXAMPLE VI.

FIG. 63. 73. LET *AED* be a Circle, whoſe Centre is the Point *B*; and let *AFK* be a Curve of ſuch a Nature, that any Radius *BFE* being drawn, the Square of *FE* be equal to the Rectangle under the Arch *AE*, and a given right Line *b*. It is required to find the Point of Inflexion *F* of that Curve.

Make the Arch $AE=z$, the Radius *BA* or $BE=a$; and the Ordinate $BF=y$; then will $bz=aa-2ay+yy$, and (throwing it into Fluxions) $\frac{2y\dot{y}-2a\dot{y}}{b}=\dot{z}=Ee$. Now becauſe of the ſimilar Sectors *BEe*, *BFG*, $BE\,(a) : BF\,(y) :: Ee\left(\frac{2y\dot{y}-2a\dot{y}}{b}\right) : FG\,(\dot{x})=\frac{2yy\dot{y}-2ay\dot{y}}{ab}$. The Fluxion of this, ſuppoſing $\dot{x}$ invariable, will be $4y\dot{y}^2-2a\dot{y}^2+2yy\ddot{y}-2ay\ddot{y}=0$; and therefore $y\ddot{y}=\frac{a\dot{y}^2-2y\dot{y}^2}{y-a}$. If now for $\dot{x}^2$ and $y\ddot{y}$, you

you put their Equals in $\dot{y}^2$, in the general Form * $y\ddot{y}=\dot{x}^2+\dot{y}^2$, there will arise $\frac{a\dot{y}^2-2y\dot{y}^2}{y-a}$ * Art. 66.

$=\frac{4y^4\dot{y}^2-8ay^3\dot{y}^2+4aayy\dot{y}^2+aabb\dot{y}^2}{aabb}$; which may be brought to $4y^5-12ay^4+12aay^3-4a^3yy+3aabby-2a^3bb=0$; and so one of the Roots of this will express BF sought.

It is manifest, that the Curve AFK, which may be called a *Parabolical Spiral*, must have a Point of Inflexion F: For because the Circumference AED does not at first sensibly differ from the Tangent in A, it follows, from the Nature of the Parabola, that it must at first be concave towards that Tangent; and afterwards, when the Curvature of the Circumference becomes sensible about that Centre, it must become concave towards the said Centre.

EXAMPLE VII.

74. Let AFK be a Curve, whose Axis is the right Line AB; and let the Nature of it be such, that any Tangent FB being drawn meeting AB in the Point B, the intercepted Part AB will be always to the Tangent BF, in the given Ratio of m to n. It is required to determine the Point of Retrogression F. Fig. 64.

Call the variable Quantities AE, x; EF, y; then $EB=-\frac{y\dot{x}}{\dot{y}}$ (because when x increases, y decreases) and $FB=\frac{y\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{y}}$. Now from the Nature of the Curve, $AE+EB$ or AB $\left(\frac{x\dot{y}-y\dot{x}}{\dot{y}}\right):BF\left(\frac{y\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{y}}\right)::m:n$. Therefore

fore $m\sqrt{\dot{x}^2+\dot{y}^2}=\frac{nx\dot{y}}{y}-n\dot{x}$; and the Fluxion will be $\frac{m\dot{y}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}=\frac{-ny\dot{x}\dot{y}+nxy\ddot{y}-nx\dot{y}^2}{yy}$, supposing $\dot{x}$ invariable and negative; whence we get $\ddot{y}=\frac{ny\dot{x}\dot{y}-nx\dot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}}{myy\dot{y}-nxy\sqrt{\dot{x}^2+\dot{y}^2}}$. Now making this last fractional Expression equal to nothing, and there will be had $-y\dot{x}-x\dot{y}=0$, from which nothing can be determined. But if the said fractional Expression be made infinite, *viz.* the Denominator equal to o, then we shall have $\sqrt{\dot{x}^2+\dot{y}^2}=\frac{my\dot{y}}{nx}=\frac{nx\dot{y}-ny\dot{x}}{my}$ because of the Equation of the Curve; and consequently $\dot{x}=nnxx\dot{y}-mmyy\dot{y}$. Now squaring both Sides of the Equation $my\dot{y}=nx\sqrt{\dot{x}^2+\dot{y}^2}$, and we shall have likewise $\dot{x}=\frac{\dot{y}\sqrt{mmyy-nnxx}}{nx}=\frac{nnxx\dot{y}-mmyy\dot{y}}{nnxy}$. From whence, at length, there comes out $y\sqrt{mm-nn}=nx$; and so we get the following Construction.

On the Diameter $AD=m$, describe a Semicircle AID, and take the Cord $DI=n$, and draw the Line AI. This will intersect the Curve AFK in the Point of Retrogression F.

For if IH be drawn perpendicular to AB, because of the similar right-angled Triangles DIA, IHA, FEA, $DI\,(n):IA\,(\sqrt{mm-nn})::IH:HA::FE\,(y):EA\,(x)$. And therefore $y\sqrt{mm-nn}=nx$.

It is manifest that BF is parallel to DI; because $AB:BF::AD\,(m):DI\,(n)$. Whence the

the Angle AFB is a right Angle, and so the Lines AB, BF, BE, are continued Proportionals.

This Property will appear from other Principals thus. Conceive * FB, Fb, two Tangents infinitely near each other in the Point F of Retrogression, making the infinitely small Angle BFb. Then if the small Arch BL be described from the Centre F, there will be had $m : n :: Ab : bF :: AB : BF :: Ab - AB$ or $Bb : bF - BF$ or $bL :: BF : BE$. Because of the right-angled similar Triangles BbL, FBE; whence, &c.

* *Art.* 67.

If $m = n$, it is plain that the right Line AF will become perpendicular to the Axis AB; and so the Tangent FB will be parallel to the same; which that it is so, is otherwise evident, because in this Case the Curve AF must be a Semicircle, a Diameter whereof is perpendicular to the Axis AB. But if m be less than n, the Curve cannot have a Point of Retrogression, because then $y\sqrt{mm - nn} = nx$ is an impossible Equation.

SECT.

SECT. V.

Use of Fluxions in the Doctrine of Evolute and Involute Curves.

DEFINITION.

FIG. 65. If one End F of the Thread $ABDF$ be first fixed to the Point F in a Curve BDF concave the same way; and afterwards the Thread be put about the said Curve, so as to touch it in every Part. Then if the other End A of the Thread be tightly moved in the same Plane with the Curve, so as to continually disengage itself from the Curve BDF, the said End A of the Thread, will describe a Curve AHK, which is called an *Involute* Curve or *Figure*.

And the Curve BDF is called the *Evolute* of the Curve AHK, or the *Evolute Curve*.

And the straight Parts AB, HD, KF, of the Thread, are each called the *Radius* of *Evolution*, or of the *Evolute*.

COROL. I.

75. Because the Length of the Thread $ABDF$ does not vary, it is plain that the Length of the Part BD of the Curve, is equal to the Difference between the Radii HD and AB issuing from the Ends thereof. In like

like manner, the Length of the Part *DF*, will be equal to the Difference between the Radii *FK*, *DH*; and the Length of the whole Curve *BDF*, to the Difference of the Radii *FK*, *BA*. Whence if the Radius *BA* of the Curve be *o*, or the End *A* of the Thread falls in *B*, the Origin of the Curve *BDF*; then will the Radii of the Evolute *DH*, *FK*, be equal to the Lengths of the Parts *BD*, *BDF*, of the Curve *BDF*.

COROL. II.

76. If the Curve *BDF* be conceived as a Polygon *BCDEF* of an infinite Number of Sides, it is evident, that *A* the End of the Thread *ABCDEF* will describe the small Arc *AG*, whose Centre is the Point *C*, until the said Radius *CG* coincides with the Continuation of the little Side *CD* adjoining to *CB*; and then it will describe the small Arch *GH*, having the Point *D* as a Centre, until *DH* coincides with the Continuation of the little Side *DE*; and so on, until the Thread be quite disengaged from the Curve *BCDEF*. Therefore the Curve *AHK* must be consider'd as the Assemblage of an infinite Number of small circular Arches *AG*, *GH*, *HI*, *IK*, &c. having the Points *C*, *D*, *E*, *F*, &c. as Centres. Whence, FIG. 66.

1°, The Radii of Evolution touch the Curve continually, *viz.* *DH* in *D*, *KF* in *K*, &c. For *DH*, for Example, is perpendicular to the small Arches *GH* and *HI*, because it passes thro' their Centres *D*, *E*. Whence we may observe, 1°, That the evolute Curve *BDF*, terminates the Space wherein all the Perpendiculars FIG. 65.

diculars to the Curve AHK fall. 2°, That if any Radius HD be continued out, intersecting the Radius AB in R, until it meets any other Radius KF in S. We can draw always from all the Points in the Part RS, two Perpendiculars to the Curve AHK, except from the Point of Contact D; from whence but one Tangent DH can be drawn. For it is evident that R, the Intersection of the Radii AB, DH, runs thro' all the Points in the Part RS, while the End A, of the Radius AB, describes the Line AHK, to which it is continually perpendicular: And that the Radii AB, HD, do not coincide, but when the Intersection R falls in the Point of Contact D.

FIG. 66. 2°, If the little Arches be continued out, *viz.* HG to l, IH to m, KI to n, &c. towards A. Every little Arch, as IH, will outwardly touch HG the little Arch next to it, because the Radii CA, DG, EH, FI, increase so much the more as the small Arches, the Curve AHK consists of, are farther from the Point A. In like manner, if you continue out the little Arches AG to o, GH to p, HI to q, towards contrary Parts from the Point A; every little Arch, as HI, will touch inwardly the small Arch IK next to it. Now because the Points H and I, D and E, on account of the infinitely Smallness of the Arch HI and the Side DE, may be consider'd as coinciding: Therefore, if from any intermediate Point D of the evolute BDF, as a Centre, with the Radius DH, you describe a Circle mHp, it will outwardly touch the Part HA, which will entirely fall within that Circle, and inwardly the other Part HK, which will fall quite without the said Circle: That

is,

is, it will both touch and cut the Curve *AHK* in the Point *H*; just as the Tangent in the Point of Inflection does cut the Curve in that Point.

3°, Because the Radius *HD* of the little Arch *HG*, differs from the Radii *CG*, *EH*, of the Arches *GA*, *HI*, next to it, only by the infinitely small Quantity *CD* or *DE*; therefore if the Radius *HD* be lessen'd never so little, it shall be less than *CG*; and so its Circle will inwardly touch the Part *HA*; and, contrariwise, if it be never so little increased, it will be greater than *HE*, and so the Circle thereof will touch the Part *HK* outwardly. Whence the Circle *mHp* is the least of all those that touch the Part *HA* outwardly, and the greatest of all those that touch the Part *HK* inwardly; that is, no Circle can be drawn between this and the Curve.

4°, Because the Curvature of Circles increases in the same Proportion as their Radii decrease; therefore the Curvature of the small Arch *HI*, will be to the Curvature of the small Arch *AG* reciprocally as the Radius *BA* or *CA* of this latter, to its Radius *DH* or *EH*: That is, the Curvature in *H* of the Curve *AHK*, will be to the Curvature in *A*, as the Radius *BA* to the Radius *DH*; and in like manner, the Curvature in *K* is to the Curvature in *H*, as the Radius *DH* is to the Radius *FK*. Whence it is manifest, that the Curvature of the Line *AHK*, decreases so much the more as the Radius of the evolute Curve *BDF* is greater; so that in the Point *A*, the beginning of the Curve, it will be a *Maximum*, and at the Point *K*, where it is supposed to end, a *Minimum*.

5°, That the Points of the evolute Curve, are only the Intersections of the Perpendiculars drawn from the Ends of little Arches, whereof the Curve *AHK* is supposed to consist. For Example, the Point *D* or *E* is the Intersection of the * Perpendiculars *HD*, *IE*, to the small Arch *HI*; so that if the Curve *AHK* be given together with the Position of one of the Perpendiculars *HD* to it, and it be required to find the Point *D* or *E*, wherein the Perpendicular touches the evolute Curve; the Business is only to find the Point wherein the Perpendiculars *HD*, *IE*, infinitely near each other do meet: And this is the Work of the following general Problem.

PROP. I.

FIG. 67. 77. *THE Nature of the Curve* AMD *being given, together with the Position of a Perpendicular* MC *to it; to find the Length of the Radius* MC *of the evolute Curve; or, which is the same thing, to find the Point wherein the Perpendiculars* MC, mC, *infinitely near each other, meet.*

First, Let the Ordinates *PM* of the Curve *AMD*, be perpendicular to the Axis *AB*; and let the Ordinate *mp* be infinitely near *MP*, because the Point *m* is supposed infinitely near *M*. From the Point of Intersection *C*, draw *CE* parallel to the Axis *AB*, meeting the Ordinates *MP*, *mp*, in the Points *E*, *e*. Lastly, draw *MR* parallel to *AB*, and then the right-angled Triangles *MRm*, *MEC*, will be similar: Since the Angles *EMR*, *CMm*, being right Angles, and the Angle *CMR* being common

* *These Perpendiculars are called the* Radii *of the* Curvature.

mon to them, the Angle EMC shall be equal to the Angle RMm.

Now call the given Quantities AP, x; PM, y; and the unknown one ME, z. Then will Ee, or Pp, or $MR = \dot{x}$, $Rm = \dot{y} = \dot{z}$, $Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$; and $MR\ (\dot{x}) : Mm\ (\sqrt{\dot{x}^2 + \dot{y}^2})$ $:: ME\ (z) : MC = \frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$. Now since the Point C is the Centre of the small Arch Mm, the Radius CM thereof, which becomes Cm while EM increased by its Fluxion Rm continues invariable; therefore the Fluxion of it will then be nothing; and so making $\dot{x}$ invariable, $\frac{\dot{z}\dot{x}\dot{x} + \dot{z}\dot{y}\dot{y} + z\dot{y}\ddot{y}}{\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = 0$. Whence we get $ME\ (z) = \frac{\dot{z}\dot{x}\dot{x} + \dot{z}\dot{y}\dot{y}}{-\dot{y}\ddot{y}} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$ by substituting $\dot{y}$ for $\dot{z}$.

2°, Let the Ordinates BM, Bm, all issue from the same Point B. Draw the Perpendiculars GE, Ce, from the sought Point C to the Ordinates, which suppose to be infinitely near each other, and describe the small Arch MR from the Centre B; then the right-angled Triangles RMm and EMC, BMR, BEG and GeG will be similar. And making $BM = y$, $ME = z$, $MR = \dot{x}$, and we shall have $Rm = \dot{y}$, $Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$, CE or $Ce = \frac{z\dot{y}}{\dot{x}}$, and $MC = \frac{z\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}$. FIG. 68.

And, as before, we shall get $z = \frac{\dot{z}\dot{x}^2 + \dot{z}\dot{y}^2}{-\dot{y}\ddot{y}}$.

Now $BM\ (y) : Ce\ \left(\frac{z\dot{y}}{\dot{x}}\right) :: MR\ (\dot{x}) : Ge = \frac{z\dot{y}}{y}$, and $me - ME$ or $Rm - Ge = \dot{z} = \frac{y\dot{y} - z\dot{y}}{y}$. Whence substituting this for $\dot{z}$,

 and

and ME (z) will be $=\frac{y\dot{x}+y\dot{y}^2}{\dot{x}^2+\dot{y}^2-y\ddot{y}}$.

If y be supposed infinite, the Terms $\dot{x}^2$ and $\dot{y}^2$, will be nothing in respect of $y\ddot{y}$; and consequently this last Form will fall into that of the Case aforegoing. This must be so, because the Ordinates then do become parallel, and the Arch MR a right Line perpendicular to the Ordinates.

Now if the Nature of the Curve AMD be given, we may get the Values of $\dot{y}^2$ and $\ddot{y}$ in $\dot{x}^2$, or of $\dot{x}^2$ and $\ddot{y}$ in $\dot{y}^2$; which being substituted in the precedent Forms, and there will arise an Expression for ME freed from Fluxions. And then drawing EC perpendicular to ME, it shall cut MC perpendicular to the Curve, in the sought Point C.

Corol. I.

Fig. 67, 68. 78. Because of the right-angled similar Triangles MRm and MEC, in the former Case $MC=\frac{\overline{\dot{x}^2+\dot{y}^2}\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}$, and in the latter $MC=\frac{\overline{y\dot{x}^2+y\dot{y}^2}\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}$.

Scholium.

79. There are other ways of finding the Radii of Evolution; some of which I shall mention for the sake of those who have not been acquainted with Investigations of this kind.

Case I. *In Curves whose Ordinates are perpendicular to the Axis.*

1st Way.

1st Way. Continue out MR to G, wherein it intersects the Perpendicular mC. Because the Angles MRm, MmG are right ones, RG will be $=\frac{\dot{y}^2}{\dot{x}}$; and so $MG=\frac{\dot{x}^2+\dot{y}^2}{\dot{x}}$. Now since the Triangles MRm, MPQ (the Points Q, q being the Intersections of the infinitely near Perpendiculars, with the Axis AB) are similar, there arises $MQ=\frac{y\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}$, $PQ=\frac{y\dot{y}}{\dot{x}}$; and therefore $AQ=x+\frac{y\dot{y}}{\dot{x}}$, the Fluxion whereof (supposing $\dot{x}$ invariable) brings out $Qq=\dot{x}+\frac{\dot{y}^2+y\ddot{y}}{\dot{x}}$; and because of the similar Triangles CMG, CQq, $MG-Qq$ $\left(\frac{-y\ddot{y}}{\dot{x}}\right):MG\left(\frac{\dot{x}^2+\dot{y}^2}{\dot{x}}\right)::MQ\left(\frac{y\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}\right)$ $:MC=\frac{\dot{x}^2+\dot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}$. FIG. 67.

2d Way. From the Centre C, describe the small Arch QO; then the little right-angled Triangles QOq, MRm, will be similar, because Mm, QO and MR, Qq, are parallel. Therefore Mm $(\sqrt{\dot{x}^2+\dot{y}^2}):MR$ $(\dot{x})::Qq$ $\left(\frac{\dot{x}^2+\dot{y}^2+y\ddot{y}}{\dot{x}}\right):QO=\frac{\dot{x}^2+\dot{y}^2+y\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$. Now because of the similar Sectors CMm, CQO, $Mm-QO$ $\left(\frac{-y\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right):Mm(\sqrt{\dot{x}^2+\dot{y}^2})::$ $MQ\left(\frac{y\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}}\right):MC=\frac{\dot{x}^2+\dot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}$.

3d Way. If the Tangents MT, mt, be drawn infinitely near each other, then will

$PT - AP$ or $AT = \frac{y\dot{x}}{\dot{y}} - x$, the Fluxion whereof will be $Tt = -\frac{y\dot{x}\ddot{y}}{\dot{y}^2}$; and if the small Arch TH be described from the Centre m, the right-angled Triangle HTt will be similar to the right-angled Triangle RmM; for the Angles HTt, RMm or PTM are equal, since their Difference is the Angle Tmt, which is infinitely small. Whence $Mm\ (\sqrt{\dot{x}^2 + \dot{y}^2}) : mR\ (\dot{y}) :: Tt \left(\frac{-y\dot{x}\ddot{y}}{\dot{y}^2}\right) : TH = \frac{-y\dot{x}\ddot{y}}{\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}}$. Now the Sectors TmH, MCm, are similar; for the Angles $TMt + MmC =$ one right Angle, and the Angles $MmC + MCm$ are likewise equal to one right Angle, because the Triangle CMm is consider'd as right-angled at M. Therefore $TH \left(\frac{-y\dot{x}\ddot{y}}{\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2}}\right) : Mm\ (\sqrt{\dot{x}^2 + \dot{y}^2}) :: Tm$ or $TM \left(\frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{y}}\right) : MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$.

FIG. 69. *4th Way*. Taking $\dot{x}$ as invariable, the second Fluxions must be denoted*; and because of the right-angled similar Triangles HmS, Hnk, Hm or $Mm\ (\sqrt{\dot{x}^2 + \dot{y}^2}) : mS$ or $MR\ (\dot{x}) :: Hn\ (-\ddot{y}) : nk = -\frac{\dot{x}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. Now the Angle kmn is equal to that made by the Tangent at the Points M, m, and therefore equal to the Angle MCm; whence the Sectors nmk, MCm, are similar; and so $nk \left(\frac{-\dot{x}\ddot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) : mk$ or *$Mm\ (\sqrt{\dot{x}^2 + \dot{y}^2}) : MC = \frac{\dot{x}^2 + \dot{y}^2 \sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$. Note,

* Art. 64.

* Art. 2.

mH or Mm is taken for mk, because their Difference is only the short Line Hk infinitely less than either of them. In like manner, as Hn is infinitely less than Rm or Sn.

Case II. *In Curves whose Ordinates issue all from a given Point.*

1st Way. Draw the Perpendiculars BF, Bf, from the Point B to the infinitely-near Radii CM, Cm; then because the right-angled Triangles mMR, BMF, are similar (since if the same Angle FMR be added to the Angles mMR, BMF, each of the Sums will be equal to a right Angle) MF or $MH = \frac{y\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$, and $BF = \frac{y\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$, the Fluxion whereof is $Bf - BF$, or $Hf = \frac{\dot{x}^2\dot{y}^2+\dot{y}^4+y\dot{x}^2\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}\times\sqrt{\dot{x}^2+\dot{y}^2}}$. Supposing $\dot{x}$ invariable. Now since the Sectors CMm, CHf, are similar, therefore $Mm - Hf : Mm :: MH : MC$; and so $MC = \frac{y\dot{x}^2+y\dot{y}^2\sqrt{\dot{x}^2+\dot{y}^2}}{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}$. Fig. 68.

2d Way. Denote * the second Fluxions in making $\dot{x}$ invariable; then because of the similar Sectors BmS, mEk, $BM\ (y) : mS\ (\dot{x}) :: mE\ (\sqrt{\dot{x}^2+\dot{y}^2}) : Ek = \frac{\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}}{y}$. Now because of the right-angled similar Triangles HmS, Hnk, Hm or $Mm\ (\sqrt{\dot{x}^2+\dot{y}^2}) : mS$ or $MR\ (\dot{x}) :: Hn\ (-\ddot{y}) : nk = -\frac{\dot{x}\ddot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$. And therefore $En = \frac{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}{y\sqrt{\dot{x}^2+\dot{y}^2}}$; and finding a third Proportional to En, Em or Mm, by means of the * Art. 64. Fig. 70.

the ſimilar Sectors *Emn*, *MCm*, the ſame Expreſſion for *MC*, as before, will be had.

If you make $Mm\ (\sqrt{\dot{x}^2+\dot{y}^2})=\dot{u}$; and $\dot{y}$ be taken inſtead of $\dot{x}$, as invariable. In the former Caſe, $MC=\frac{\dot{u}^3}{\dot{y}\ddot{x}}$, and in the other $MC=\frac{y\dot{u}^3}{\dot{x}\dot{u}^3+yy\dot{y}\ddot{x}}$. And laſtly, if $\dot{u}$ be ſuppoſed invariable, there comes out in the former Caſe $MC=\frac{\dot{x}\dot{u}}{-\ddot{y}}$ or $\frac{\dot{y}\dot{u}}{\ddot{x}}$ (becauſe the Fluxion of $\dot{x}^2+\dot{y}^2=\dot{u}^2$ is $\dot{x}\ddot{x}+\dot{y}\ddot{y}=0$; and ſo $\frac{\dot{x}}{-\ddot{y}}=\frac{\dot{y}}{\ddot{x}}$) and in the latter, $MC=\frac{y\dot{x}\dot{u}}{\dot{x}^2-y\ddot{y}}$ or $\frac{y\dot{y}\dot{u}}{\dot{x}\dot{y}+y\ddot{x}}$.

Corol. II.

Fig. 72. 80. Since *ME* or *MC* has been found to have but one Value, therefore the involute Curve *AMD* has but one evolute *BCG*.

Corol. III.

Fig. 67, 68. 81. If *ME* $\left(\frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}\right)$ or $\left(\frac{y\dot{x}^2+y\dot{y}^2}{\dot{x}^2+\dot{y}^2-y\ddot{y}}\right)$ be poſitive, the Point *E* muſt be taken on the ſame Side the Axis *AB* or Point *B*, as it was ſuppoſed in the Operation aforegoing. And ſo the Curve in that Caſe will be concave towards the Axis, or that Point. But if *ME* be negative, the Point *E* muſt be taken on the contrary Side, and ſo the Curve will be then convex towards the Axis, or that Point. Therefore it is plain, that in the Point of Inflexion

or

or Retrogreſſion, which ſeparates the concave from the convex Part, *ME* from being poſitive will become negative. Whence the contiguous or infinitely-near Perpendiculars from converging become diverging. Which can happen but two ways only: for as they go on increaſing ſtill the more, as they accede to the Point of Inflexion or Retrogreſſion, they muſt become at laſt parallel, that is, the Radius of Evolution will be infinite: and where they conſtantly continue decreaſing, they muſt at laſt coincide, that is, the Radius of Evolution will be nothing. All this is correſpondent to what has been demonſtrated in the foregoing Section.

SCHOLIUM.

82. BECAUSE hitherto the Radius of Evolution has been conſider'd as infinitely great in the Point of Inflexion; I ſhall here ſhew, that in numberleſs Species of Curves the Radius of Evolution in the Point of Inflexion is equal to nothing: and that there is but one Species where the ſaid Radius is infinite.

Let *BAC* be a Curve, whoſe Radius of Evolution in the Point of Inflexion *A* is infinite. FIG. 71.

Now if the Parts *BA*, *AC* be conſider'd as evolute Curves, the Point *A* being ſuppoſed their Beginning, and the Curve *DAE* as an involute Curve formed from them: it is plain that this latter Curve will have a Point of Inflexion in *A*, but the Radius of Evolution in that Point will be equal to nothing. And if a third involute Curve be formed from the ſecond *DAE* as an Evolute, and a fourth Involute

lute

lute from the third as an Evolute, and so on; it is manifest that the Radius of Evolution in the Point of Inflexion *A* in every of those Curves, will always be equal to nothing. Whence, &c.

PROP. II.

FIG. 72. 83. LET AMD *be an involute Curve, whose Axis* AB *is at right Angles to the Tangent in* A: *to find the Point* B *wherein the said Axis touches the Evolute* BCG.

If the Point *M* be supposed to become infinitely near the Vertex *A*, it is plain that the Perpendicular *MQ* will intersect the Axis in the Point *B* sought. Whence if you seek the Value in general of *PQ* $\left(\frac{y\dot{y}}{\dot{x}}\right)$ in *x* or *y*, and afterwards you make *x* or $y = 0$, we may determine the Point *P*, which is to coincide with *A*, and the Point *Q* which coincides with the Point *B* sought; that is, *PQ* will then become equal to *AB* sought. This will be more plain by the following Examples.

EXAMPLE I.

FIG. 72. 84. LET the Involute *AMD* be a Parabola, whose Parameter is a right Line, suppose *a*. The Equation of the Curve is $ax = yy$, the Fluxion whereof is $\dot{y} = \frac{a\dot{x}}{2y} = \frac{a\dot{x}}{2\sqrt{ax}}$; and throwing this last Equation again into Fluxions, by making $\dot{x}$ invariable, there arises $\ddot{y} = \frac{-a\dot{x}^2}{4x\sqrt{ax}}$. Now substituting these Values for $\dot{y}$ and

$\dot{y}$ and $\ddot{y}$ in the general Form $\frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}$, and then will * $ME = \frac{a+4x\sqrt{ax}}{a} = \sqrt{ax} + \frac{4x\sqrt{ax}}{a}$. * *Art.* 77.

From whence arises the following Construction.

From the Point T, wherein the Tangent TM intersects the Axis, draw TE parallel to MC; I say, this shall meet MP continued out in the Point E sought. For because of the right Angles MPT, MTE, MP $(\sqrt{ax})$: PT $(2x)$: : PT $(2x)$: $PE = \frac{4xx}{\sqrt{ax}} = \frac{4x\sqrt{ax}}{a}$; and consequently $MP + PE = \sqrt{ax} + \frac{4x\sqrt{ax}}{a}$. Again, because of the right-angled Triangles MPQ, MEC, therefore PM $(\sqrt{ax})$: PQ $(\frac{1}{2}a)$: : $ME \left(\sqrt{ax} + \frac{4x\sqrt{ax}}{a}\right)$: EC or $PK = \frac{1}{2}a + 2x$. And therefore $QK = 2x$. From whence we get this new Construction:

Take QK the Double of AP, or (which is the same) take $PK = TQ$, and draw KC parallel to PM. This will meet the Perpendicular MC in the Point C, which will be in the Evolute BCG.

Otherwise. $yy = ax$, and $2y\dot{y} = a\dot{x}$ (and taking $\dot{x}$ as invariable) $2\dot{y}^2 + 2y\ddot{y} = 0$; whence there arises $-\ddot{y} = \frac{\dot{y}^2}{y}$. And putting this Value in the Form $\frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}$, there will arise * $ME = \frac{y\dot{y}^2 + y\dot{x}^2}{\dot{y}^2}$; * *Art.* 77.

and therefore EC or $PK = \frac{y\dot{y}^2 + y\dot{x}^2}{\dot{y}\dot{x}} = \frac{y\dot{y}}{\dot{x}} + \frac{y\dot{x}}{\dot{y}}$ $= PQ + PT$ or TQ. And so we get the same

same Constructions as before. For $MP : PT :: \dot{y} : \dot{x} :: PT\left(\frac{y\dot{x}}{\dot{y}}\right) :: PE = \frac{y\dot{x}^2}{\dot{y}^2} = \frac{4x\sqrt{ax}}{a}$.

Now to find the Point B, wherein the Axis AB touches the Evolute BCG: we have $PQ \left(\frac{y\dot{y}}{\dot{x}}\right) = \frac{1}{2}a$. And since this is invariable, where-ever the Point M is assumed; therefore when M coincides with the Vertex A, we still shall have PQ, which in this Case becomes $AB = \frac{1}{2}a$.

To find the Nature of the Involute BCG according to *Descartes*'s way. Call the Absciss BK, u; and the Ordinate KC or PE, t; then will $CK\,(t) = \frac{4x\sqrt{ax}}{a}$, and $AP + PK - AB\,(u) = 3x$. Now putting $\frac{1}{3}u$ for its Equal x in the Equation $t = \frac{4x\sqrt{ax}}{a}$, and there will come out $27att = 16u^3$ expressing the Relation of BK to KC. Whence if the Involute be a common Parabola, the Evolute BCG is a second cubical Parabola, whose Parameter is equal to $\frac{27}{16}$ of the Parameter of the Involute.

FIG. 73. It is evident, that if the Involute be the whole Parabola MAM, the Evolute CBC will consist of two Parts CB, BC, having contrary Concavities; so that B will be a Point of Retrogression.

DEFINITION.

FIG. 72. *BY Geometrical Curves, as* AMD, BCG, *I understand those, whereof the Relation of the Abscisses* AP, BK *to the correspondent Ordinates*

nates PM, KC, *being all right Lines, can be expressed by a finite Equation free from Fluxions. And whatever is effected by means of those Lines, is said to be Geometrical.*

COROLL.

85. WHEN the given Involute Curve *AMD* is a Geometrical one, it is plain that we can (as above) always find an Equation expressing the Nature of the Evolute *BCG*; and so the Evolute will be likewise a Geometrical Curve. I say moreover that it is to be rectified, *viz.* we can find geometrically straight Lines equal to the Length of any Part *BC* of it. For * by means of the Involute *AMD*, which is a Geometrical Curve, we can determine the Point *M* in the Tangent *CM* to the Part *BC*, such that the straight Line *CM* differs from the Part *BC* only by a given right Line *AB*. * *Art.* 75.

EXAMPLE II.

86. LET the Involute *MDM* be an Hyperbola within its Asymptotes. The Nature of this is $aa = xy$. FIG. 74.

Now $\frac{aa}{y} = x$, $\frac{aa\dot{y}}{yy} = \dot{x}$, and supposing $\dot{x}$ invariable * $\frac{-aayy\ddot{y} + 2aay\dot{y}^2}{y^4} = 0$; whence we get $\ddot{y} = \frac{2\dot{y}^2}{y}$, and putting this Value in $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$, there comes out * $ME = \frac{y\dot{x}^2 + y\dot{y}^2}{-2\dot{y}^2}$: so that EC or $PK = -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2\dot{y}}$. Hence the following Constructions are derived. * *Art.* 77. * *Art.* 77.

Thro'

Thro' the Point *T* wherein the Tangen *MT* intersects the Asymptote *AB*, draw *T* parallel to *MC* meeting *MP* continued out i *S*; and assume *ME* equal to $\frac{1}{2}$ *MS* on th other side the Asymptote (which is here take as the Axis) because the Value of it is nega tive; or else assume *PK* equal to $\frac{1}{2}$ *TQ*, o the same side as the Point *T* is: then, I say, i *EC* be drawn parallel, or *KC* perpendicula to the Axis, they will intersect *MC* in th Point *C* sought: for it is plain that $MS = \frac{y\dot{x}^2+\dot{y}y^2}{\dot{y}^2}$, and $TQ = \frac{y\dot{y}}{\dot{x}} + \frac{y\dot{x}}{\dot{y}}$.

From an Inspection of the Figure of th Hyperbola *MDM*, it will not be difficult to perceive that the Evolute *CLC* must have a Point *L* of Retrogression, as the Evolute in the last Example has. To determine which, we must observe that the Radius of Evolution *DL* is less than any other Radius *MC*; so that the Fluxion of the Expression thereof,

* *Art.* 78. *viz.* $\frac{\overline{\dot{x}^2+\dot{y}^2}\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}$ or $\overline{\dot{x}^2+\dot{y}^2}^{\frac{3}{2}}$ will be * nothing or infinite. And so taking $\dot{x}$ as invariable, the second Fluxion will be

$$\frac{-3\dot{x}\dot{y}\ddot{y}^2\,\overline{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}+\dot{x}\dddot{y}\,\overline{\dot{x}^2+\dot{y}^2}^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}=0,$$

or infinite; whence dividing by $\overline{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}$, and afterwards multiplying by $\dot{x}\ddot{y}^2$, there arises $\dot{x}^2\dddot{y}+\dot{y}^2\dddot{y}-3\dot{y}\ddot{y}^2=0$, or infinite; by which we may find such an Expression as *AH* for *x*, that by drawing the Ordinate *HD* and the Radius of Evolution *DL*, the Point *L* will be the Point of Retrogression sought.

In

In this Example $y = \frac{aa}{x}$, $\dot{y} = \frac{-aa\dot{x}}{xx}$, $\ddot{y} = \frac{2aa\dot{x}^2}{x^3}$, $\dddot{y} = \frac{-6aa\dot{x}^3}{x^4}$. Whence putting the latter Members of every of these Equations for the former ones in the Equation aforegoing, and there comes out AH $(x) = a$. Therefore the Point D is the Vertex of the Hyperbola, and the Lines AD, DL coincide with AL: Which is the Axis of the Curve.

EXAMPLE III.

87. LET $y^m = x$ generally express the Nature of all Parabola's, the Exponent m representing a positive whole Number or Fraction, and all Hyperbola's, when the same Exponent is a negative whole Number or Fraction. FIG. 72, 74.

Now $my^{m-1}\dot{y} = \dot{x}$, and the Fluxion of this again, taking $\dot{x}$ as invariable, will be $\overline{mm - m}\,y^{m-2}\dot{y}^2 + my^{m-1}\ddot{y} = 0$; and dividing by my^{m-1}, there comes out $-\ddot{y} = \frac{\overline{m-1}\,\dot{y}^2}{y}$; whence substituting this latter Expression in $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$, we * get $ME = \frac{y\dot{x}^2 + \dot{y}y^2}{\overline{m-1}\,\dot{y}^2}$; and therefore EC or $PK = \frac{y\dot{y}}{\overline{m-1}\,\dot{x}} + \frac{y\dot{x}}{\overline{m-1}\,\dot{y}}$. From whence the following Constructions are gained.

* Art. 77.

From T the Intersection of the Axis AP, with the Tangent MT draw the Line TS parallel to MC meeting MP continued out in the Point S: Assume $ME = \frac{1}{\overline{m-1}}\,MS$, or else

else take $PK = \frac{1}{m-1} TQ$; then if a parallel to the Axis be drawn thro' E, or a Perpendicular thro' K; these shall intersect MC in the Point C sought.

FIG. 74. If m be negative, as in the Hyperbola's, the Value of ME will be negative; and so the Curves will be convex towards their Axis, FIG. 75. which will be an Asymptote then. But in the Parabola's where m is positive, there may happen two Cases: for when m is less than 1, they will be convex next to the Axis, which will be a Tangent to the Vertex; and when FIG. 72. m exceeds 1, then they will be concave next to the Axis, which will be perpendicular in the Vertex.

Now in this latter Case, to find the Point B where the Axis AB touches the Evolute. $PQ\left(\frac{y\dot{y}}{\dot{x}}\right)$ is $=\frac{y^{2-m}}{m}$; from whence arises three Cases. For when $m=2$, as in the common Parabola, then the Exponent of y being nothing, that unknown Quantity vanishes; and consequently $AB=\frac{1}{2}$, *viz.* to one half the Parameter. When m is less than 2, then the Exponent of y being positive, it shall be found in the Numerator, and the Fraction will be- * *Art.* 83. come nothing by making * it equal to 0; that is, the Point B in this Case will coincide with A, as in the second Cubick Parabola $axx=y^3$. FIG. 76. Or lastly, when m is greater than 2, and then the Exponent of y being negative, it will be in the Denominator; so that when it becomes nothing, the Fraction will be infinite: *viz.* the Point B is infinitely distant from A, or (which is the same) the Axis AB is an Asymptote of the Evolute, as in the first Cubick Para-

Parabola $aax = y^3$. In this latter Case, we may observe that the Involute CLO, when the Evolute ADM is a Semi-parabola, will have a Point of Retrogression; so that tho' the Part LO of the Evolute be infinite, yet the Part DA of the Involute formed by it will be determinate or finite; and the other Part LC of the Evolute being infinite, will notwithstanding describe the infinite Part DM of the Involute. FIG. 77.

Now the Point L may be determined after the same way as in the Hyperbola. For Example: Let $aax = y^3$, or $y = x^{\frac{1}{3}}$; then will $\dot{y} = \frac{1}{3} x^{-\frac{2}{3}} \dot{x}$, $\ddot{y} = -\frac{2}{9} x^{-\frac{5}{3}} \dot{x}^2$, $\dddot{y} = \frac{10}{27} x^{-\frac{8}{3}} \dot{x}^3$, which Values being substituted in the Equation $\dot{x}^2 \dddot{y} + \dot{y}^2 \dddot{y} - 3\dot{y}\ddot{y}^2 = 0$, and there comes * out $AH\ (x) = \sqrt[4]{\frac{1}{91125}}$.

* Art. 86.

SCHOLIUM.

88. WHEN m is greater than 1; and so the Parabola's concave next to the Axis, there are several Cases. For if the Numerator of the Fraction denoted by m be even, and the Denominator odd, all the Parabola's will fall on each Side of their Axis just as the common Parabola does. But if the Numerator and Denominator be each odd, they have an inverted Position on each Side their Axis, so that the Vertex A is a Point of Inflexion; as the first Cubical Parabola $x = y^{\frac{1}{3}}$, or $aax = y^3$. Lastly, if the Numerator be odd, and the Denominator even, these Curves have an inverted Position on the same Side their Axis; so FIG. 78. FIG. 77. FIG. 76.

so that the Vertex A is a Point of Retrogression; as the second Cubical Parabola $x = y^{\frac{3}{2}}$ or $axx = y^3$. The Reason of all this is, because an even Power cannot have a negative Value. Whence it evidently follows,

FIG. 77. 1°, That in the Point of Inflexion A, the Radius of Evolution may be infinitely great, as in $aax = y^3$; or infinitely small, as in $aax^3 = y^5$.

FIG. 76. 2°, That in the Point of Retrogression A, the Radius of Evolution may be either infinite, as in $a^3xx = y^5$; or nothing, as in $axx = y^3$.

FIG. 73. 3°, That because the Radius of Evolution is infinite or 0, it does not from hence follow that the Curves have then a Point of Inflexion or Retrogression: for in $a^3x = y^4$ it is infinite, and in $ax^3 = y^4$ it is 0; and yet these Parabola's have the same Position with regard to their Axis, as the common Parabola.

EXAMPLE IV.

FIG. 78, 79. 89. Let the Involute AMD be an Hyperbola or Ellipsis, whose Axis is $AH (a)$, and Parameter $AF (b)$.

Then from the Nature of these Curves $y = \sqrt{\frac{abx \pm bxx}{\sqrt{a}}}$, $\dot{y} = \frac{ab\dot{x} \pm 2bx\dot{x}}{2\sqrt{aabx \pm abxx}}$, and $\ddot{y} = \frac{-a^3bb\dot{x}^2}{4aabx \pm 4abxx\sqrt{aabx \pm abxx}}$. Now if these Values be introduced into $\frac{\dot{x}^2 + \dot{y}^2\sqrt{\dot{x}^2 + \dot{y}^2}}{-\dot{x}\ddot{y}}$, the ge-

* Art. 78. neral Expression for * MC; then will $MC = \frac{aabb + 4abbx + 4bbxx \pm 4aabx \mp 4abxx\sqrt{aabb + 4abbx + 4bbxx \pm 4aabx \mp 4abxx}}{2a^3bb}$

$= \frac{4MQ^3}{bb}$, because on each Side $MQ \left(\frac{y\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}}\right) =$

$\sqrt{a}$

$$\frac{\sqrt{aabb + 4abbx + 4bbxx + 4aabx + 4abxx}}{2a}.$$

Hence we get the following Construction, which will serve for the Parabola also.

Assume *MC* equal to four times a fourth continual Proportional between the Parameter *AF*, and the Perpendicular *MQ* bounded by the Axis: then will the Point *C* fall in the Evolute.

If you make $x = 0$, then will * $AB = \frac{1}{2}b$. * *Art.* 83.
And if in the Ellipsis you make $x = \frac{1}{2}a$; then FIG. 79.
will *DG* be $= \frac{a\sqrt{ab}}{2b}$, *viz.* equal to half the Parameter of the conjugate Axis. And so it appears that the Evolute *BCG*, when the Involute is an Ellipsis, terminates in the Point *G* of the conjugate Axis *DO*; wherein is formed a Point of Retrogression; but in the Parabola and Hyperbola it runs out *ad infinitum*.

If $a = b$ in the Ellipsis, there comes out $MC = \frac{1}{2}a$; from whence it follows, that the Radii of Evolution are all equal to one another; and so the Evolute will become here a Point; that is, the Involute will be a Circle, and the Evolute the Centre of it. Which is an establish'd Truth from other Principles.

EXAMPLE V.

90. LET *AMD* be a Logarithmick Curve FIG. 80.
of such a Nature, that if from any Point *M* in the same be drawn the Perpendicular *MP* to the Asymptote *KP* and the Tangent *MT*; the Subtangent be $PT = a$ a given right Line.

Now $PT\left(\frac{y\dot{x}}{\dot{y}}\right)=a$; whence we get $\dot{y}=\frac{y\dot{x}}{a}$, and the Fluxion of this again, supposing $\dot{x}$ invariable, will be $\ddot{y}=\frac{\dot{y}\dot{x}}{a}=\frac{y\dot{x}^2}{aa}$; and putting these Values in $\frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}$; and there will arise * $ME=\frac{-aa-yy}{y}$; and therefore EC or $PK=\frac{-aa-yy}{y}$. From whence arises this Construction.

* Art. 77.

Assume $PK=TQ$ on the same Side T, because the Value of it is negative; and draw KC parallel to PM: I say the same will intersect the Perpendicular MC in the Point C sought. For $TQ=\frac{aa+yy}{a}$.

If you have a mind that the Point M be in the greatest Part possible of the Curvature, the general Expression $\dot{x}^2\dddot{y}+\dot{y}^2\dddot{y}-3\dot{y}\ddot{y}\ddot{y}=0$, (of *Art.* 86.) must be used: then if $\frac{y\dot{x}}{a}, \frac{y\dot{x}^2}{aa}, \frac{y\dot{x}^3}{a^3}$ be put for $\dot{y}$, $\ddot{y}$, $\dddot{y}$, PM (y) will be $=a\sqrt{\frac{1}{2}}$.

Now when $\dot{x}$ is invariable, it is plain that the Ordinates y are to each other as their Fluxions $\dot{y}$ or $\frac{y\dot{x}}{a}$; from whence it follows, that they will be in a geometrical Progression also. For if the Asymptote, or Axis PK, be supposed to be divided into an infinite Number of small equal Parts Pp or MR, pf or mS, fg or nH, &c. intercepted by the Ordinates PM, pm, fn, go, &c. then will $PM:pm::Rm:Sn::Rm::PM+Rm$, or $pm:pm+Sn$ or Fn.

In like manner we prove that $pm:fn::fn:go$, and so on. Therefore the Ordinates PM, pm, fn, go, &c. are in a geometrical Progression.

EXAMPLE VI.

91. LET AMD be a Logarithmetical Spiral, of such a Nature, that if you draw a right Line MA from any Point M in it, to the Point A being the Centre, and the Tangent MT; the Angle AMT is of a given Quantity, *viz.* always the same. FIG. 81.

Because the Angle AMT or AmM is invariable, the Ratio of $mR(\dot{y})$ to $RM(\dot{x})$ will be also invariable; therefore the Fluxion of $\frac{\dot{y}}{\dot{x}}$ must be nothing; so that (supposing $\dot{x}$ invariable) $\ddot{y}=0$. Therefore if the Term $y\ddot{y}$ be struck out of $\frac{y\dot{x}^2+y\dot{y}^2}{\dot{x}^2+\dot{y}^2-y\ddot{y}}$, the general Expression * for ME, when the Ordinates issue from the same Point, ME will be $=y$, viz. $ME=AM$. From whence comes the following Construction. * *Art.* 77.

Draw AC perpendicular to AM, meeting MC the Perpendicular to the Curve in C, which will be a Point in the Evolute ACB.

The Angles AMT, ACM are equal, because by adding AMC to each of them, the wholes will be right Angles. Therefore the Evolute ACG will be a Logarithmick Spiral differing only from the Involute AMD in Position.

If the Point C in the Evolute ACG be given, and it be requir'd to find the Length of CM the Radius of the Evolute in that Point, which is * equal to the Part AC of the Spiral * *Art.* 75.

 making

making an infinite Number of Revolutions before its Accession to *A*; it is manifest, that *AM* need only be drawn perpendicular to *AC*. So that if *AT* be drawn perpendicular to *AM*, the Tangent *MT* will be equal likewise to the Part *AM* of the Logarithmick Spiral *AMD* given.

If you imagine an infinite Number of Ordinates *AM*, *Am*, *An*, *Ao*, &c. making infinitely small equal Angles; it is manifest, that the Triangles *MAm*, *mAn*, *nAo*, &c. will be similar, because the Angles at *A* are equal, as likewise are the Angles at *m*, *n*, *o*, &c. from the Nature of the Spiral. Therefore *AM*: *Am*::*Am*:*An*; and *Am*:*An*::*An*:*Ao*, &c. Whence the Ordinates *AM*, *Am*, *An*, *Ao*, &c. are in a geometrical Progression, if they make equal Angles with each other.

EXAMPLE VII.

FIG. 82. 92. LET *AMD* be one Spiral (of an infinite Number) formed in the Sector *BAD*, of such a Nature, that any Radius *AMP* being drawn, and calling the whole Arch *BPD*, b; the Part *BP*, z; the Radius *AB* or *AP*, a; and the Part *AM*, y; there is always the following Proportion, $b:z::a^m:y^m$.

The Equation of the Spiral *AMD* is $y^m = \frac{a^m z}{b}$, which thrown into Fluxions is $my^{m-1}\dot{y} = \frac{a^m \dot{z}}{b}$. Now because of the similar Sectors *AMR*, *APp*, *AM* (y) : *AP* (a) :: *MR* ($\dot{x}$): *Pp* ($\dot{z}$) $= \frac{a\dot{x}}{y}$. Now putting this Value for $\dot{z}$ in the Equation afore found, and $my^m\dot{y} = a^{m-}$

$\frac{a^{m-1}\dot{x}}{b}$; which again being thrown into Fluxions making $\dot{x}$ invariable, and $mmy^{m-1}\dot{y}^2 + my^m\ddot{y} = 0$; whence dividing by my^{m-1}, we get $-y\ddot{y} = m\dot{y}^2$; and therefore * ME * *Art.* 77.
$\left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \dot{y}^2 - y\ddot{y}}\right) = \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \overline{m+1}\,\dot{y}^2}$. So that the following Construction comes out thus.

Thro' A draw TAQ perpendicular to AM meeting the Tangent MT in T, and the Perpendicular MQ in Q; make $TA + \overline{m+1}\,AQ : TQ :: MA : ME$. Then if EC be drawn parallel to TQ, it will meet MQ in a Point, as C, which will be in the Evolute.

For because MRG, TAQ are parallel, $MR\ (\dot{x}) + \overline{m+1}\ RG\ \left(\frac{\dot{y}^2}{\dot{x}}\right) : MG\ \left(\dot{x} + \frac{\dot{y}^2}{\dot{x}}\right) :: TA + \overline{m+1}\ AQ : TQ :: AM\ (y) : ME = \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 + \overline{m+1}\,\dot{y}^2}$.

EXAMPLE VIII.

93. LET AMD be Semi-cycloid, whose Base BD is equal to ½ the Circumference BEA of the generating Circle. FIG. 83.

Call AP, x; PM, y; the Arch AE, u; and the Diameter AB, $2a$. Then $PE = \sqrt{2ax - xx}$, from the nature of the Circle, and $y = u + \sqrt{2ax - xx}$, from the nature of the Cycloid; which last Equation thrown into Fluxions will be $\dot{y} = \dot{u} + \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}} = \frac{2a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$, or $\frac{\dot{x}\sqrt{2a - x}}{x}$, putting $\frac{a\dot{x}}{\sqrt{2ax - xx}}$ for $\dot{u}$; and supposing

$\dot{x}$ invariable, $\ddot{y} = \frac{-a\dot{x}^2}{x\sqrt{2ax-xx}}$; and substituting these Expressions in $\frac{\overline{\dot{x}^2+\dot{y}^2}\sqrt{\dot{x}^2+\dot{y}^2}}{-\dot{x}\ddot{y}}$, and there comes * out $MC = 2\sqrt{4aa-2ax}$, that is, $2BE$ or $2MG$.

* Art. 78.

If you make $x=0$; then will $AN=4a$ be the Radius of Evolution in the Vertex A. But if $x=2a$, the Radius of Evolution in D will become nothing. From whence we are assured, that D is the Beginning of the Evolute, and the Point N the End, so that $BN = BA$.

Now to find the Nature of this Involute, you need only compleat the Rectangle BS, describe the Semicircle DIS on DS, as a Diameter, and draw DI parallel to MC or BE. This being done, it is plain that the Angle BDI is equal to the Angle EBD; and consequently the Arches DI, BE are equal to one another; whence the Chords DI, BE or GC are likewise equal. And therefore if IC be drawn, it will be equal and parallel to DG, which by the Generation of the Cycloid is equal to the Arch BE or DI; and consequently the Evolute DCN is a Semi-cycloid, having the right Line NS for its Base, being equal to ½ the Circumference DIS of the generating Circle: that is, it will be the same Semi-cycloid $AMDB$ having an inverted Situation.

Corol.

COROL.

94. It is * evident that the Part *DC* of the Cycloid is the Double of its Tangent *CG*, or correspondent Chord *DI*; and the Semi-cycloid *DCN*, is the Double of the Diameter *BN* or *DS* of the generating Circle. * *Art.* 75.

ANOTHER SOLUTION.

95. The Length of the Radius *MC* may be found without any Calculus, thus: Conceive the Perpendicular *mC* to be infinitely near the former one, another Parallel *me*, another Chord *Be*, and from the Centres *C*, *B*, describe the small Arches *GH*, *EF*; then will the right-angled Triangles *GHg*, *EFe*, be equal and similar; for $Gg = Ee$, since *BG* or *ME* is equal to the Arch *AE*, and in like manner, *Bg* or *me* is equal to the Arch *Ae*; moreover Hg or $mg - MG = Fe$ or $Be - BE$; whence *GH* will be equal to *EF*. Now since the Perpendiculars *MC*, *mC*, are parallel to the Chords *EB*, *eB*, the Angle *MCm* shall be equal to the Angle *EBe*. Therefore since the Arches *GH*, *EF*, that are the Measures of those Angles, are equal, it follows that the Radii *CG*, *BE* shall be equal likewise; and so *MC* must be assumed the Double of *MG* or *BE*.

LEMMA.

96. *If there be any Number of Quantities* a, b, c, d, e, &c. *either finite or infinite, the Sum of their Differences,* viz. a—b+b—c+c—d+d—e, *&c. is equal to the greatest Quantity*

tity a Minus *the least* e, *or else equal to the greatest, when the least is* o.

Corol. I.

97. Because the Sectors CMm, CGH are similar, Mm is $= 2GH = 2EF$; and since this is always so, be the Point M where it will; therefore the Sum of all the little Arches Mm, that is, the Part Am of the Semi-cycloid AMD, is the Double of the Sum of all the little Arches EF. But the little Arch EF, being a Part of the Chord AE perpendicular to BE, is the Difference of the Chords AE, Ae, since the small right Line eF perpendicular to Ae, may be consider'd as a small Arch described about the Centre A. And therefore the Sum of all the little Arches EF in the Arch AZE, will be the Sum of the Differences of all the Chords AE, Ae, &c. in that Arch; that is (by the Lemma above) equal to the Chord AE. Therefore it is evident, that the Part AM of the Semi-Cycloid AMD, is the Double of the correspondent Chord AE.

Corol. II.

98. The Space $MGgm$*, or the Trapezium $MGHm = \frac{1}{2}\overline{Mm + \frac{1}{2}GH} \times MG = \frac{3}{2}EF \times BE$; that is, it is the Triple of the Triangle EBF or EBe; whence the Space $MGBA$, or Sum of all the said Trapezia, is the Triple of the circular Space $BEZA$, being the Sum of all the said Triangles.

* Art. 2.

Exam-

EXAMPLE III.

99. CALL BP, z; the Arch AZE or EM or BG, u; and the Radius KA, a; then will the Parallelogram $MGBE$ be $=uz$. Now the cycloidal Space $MGBA = 3BEZA = 3EKB + \frac{1}{2}au$; and therefore the Space $AMEB$ contained under the Part AM of the Cycloid, the Parallel ME, the Chord BE, and the Diameter AB, is $= 3EKB + \frac{1}{2}au - uz$. Whence if you assume BP $(z) = \frac{1}{2}a$, the Space $AMEB$ will be the Triple of the correspondent Triangle EKB; and so we have the Quadrature thereof independent of the Quadrature of the Circle; (which Mr. *Hugens* first observed) as likewise the Quadrature of the following Space.

If the Segment $BEZA$ be taken away from the Space $AMEB$, there will remain the Space $AZEM = 2EKB + au - uz$; therefore when the Point P coincides with K, the Space $AZEM$ will be then equal to the Square of the Radius.

It is plain, that among all the Spaces $AMEB$, $AZEM$, there are but these two only that can be squared independent of the Quadrature of the Circle.

EXAMPLE IX.

100. LET AMD be a Semi-cycloid described by the Rotation of the Semicircle AEB about or along another Circle BGD at rest. It is requir'd to find that Point in the Perpendicular MG of a given Position that touches the Evolute. FIG. 84.

Because

Because in the Use of general Expressions or Formules, we must first take right Lines perpendicular to the Axis *AO*, as Ordinates to the Curve *AMD*, and then find an Equation expressing the Relation of the Ordinates to the Abscisses, or of their Fluxions, which often is a very operose Performance; therefore in Occurrences of this Kind, it is much better to endeavour at the Solution from the Generation itself.

When the Semi-circle *AEB* is come to the Situation *MGB*, wherein it touches the Base *DB* in *G*, and the describent Point *A*, falls on the Point *M* of the Semi-cycloid *AMD*, it is plain,

1°, That the Arch *GM* is equal to the Arch *GD*; as also the Arch *GB* of the moveable Circle, to the Arch *GB* of the immoveable one.

* Art. 43. 2°, That *MG* is * perpendicular to the Curve: For if you consider the Semi-circumference *MGB* or *AEB*, and the Base *BGD*, as the Assemblage of an infinite Number of little equal right Lines, each equal to its Correspondent, it is manifest, that the Semi-cycloid *AMD* will be the Assemblage of an infinite Number of little Arches, whose Centres are successively the Contact Points *G*, each being described thro' the same Point *M* or *A*.

3°, That if from *O* the Centre of the immoveable Circle, the Arch *ME* be described, then the Arches *MG*, *EB*, of the moveable Circle, will be equal to one another, as well as their Chords *MG*, *EB*, and the Angles *OGM*, *OBE*. For the Lines *OK*, *OK*, joining the Centres of the Circles are equal, because they pass thro' the Points of Contact *B*, *G*. Therefore drawing the Radii

dii OM, OE, and KE, the Triangles OKM, OKE, will be equal and ſimilar. Whence becauſe the Angle OKM is equal to the Angle OKE, the Arches MG, BE, of the equal Semi-circles MGB, BEA, being the Meaſures of theſe Angles, ſhall be equal; as likewiſe their Chords MG, EB; and ſo the Angles OGM, OBE, are equal likewiſe.

This being laid down, let mC be another Perpendicular infinitely near the firſt, me another concentrick Arch, and Be another Chord; and from the Centres C and B, deſcribe the ſmall Arches GH, EF. Now the right-angled Triangles GHg, EFe, are equal and ſimilar; for Gg or $Dg - DG = Ee$, or the Arch $Be -$ the Arch BE: Moreover, Hg or $mg - MG = Fe$ or $= Be - BE$. Whence the little Arch GH will be equal to the little Arch EF; and ſo the Angle GCH, is to the Angle EBF, as BE to CG. And therefore the whole Difficulty is brought to this, *viz.* to find the Relation between thoſe Angles; which may be effected thus: FIG. 85.

Draw the Radii OG, Og, KE, Ke, and call OG or OB, b; KE or KB, or KA, a; then it is manifeſt, that the Angle $EBe = OBe - OBE = Ogm - OGM =$ (drawing GL, GV, parallel to Cm, Og) $LGM - OGV = GCH - GOg$. Therefore the Angle GCH ſhall be $= GOg + EBF$. But becauſe the Arches Gg, Ee, are equal, $GOg : EKe$ or $2EBF :: KE (a) : OG (b)$; and conſequently the Angle $GOg = \frac{2a}{b} EBF$, and $GCH = \frac{2a+b}{b} EBF$. Whence $GCH : EBF$ or BE

$:: CG$

$:: CG : \frac{2a+b}{b} : 1$. And so the unknown Quantity $CG = \frac{b}{2a+b} BE$ or MG.

FIG. 86. Whence if you make $OA (2a+b) : OB (b) :: MG : GC$, the Point C will be in the Evolute.

It is evident, 1°, That the Evolute begins at the Point D, in which it touches the Base BGD; because the Arch GM becomes infinitely small in that Point. 2°, And that it ends in the Point N; so that $OA : OB :: AB : BN :: OA - AB$ or $OB : OB - BN$ or ON; that is, OA, OB, ON, are continual Proportionals. 3°, If the Circle NSQ be described from the Centre O, then, I say, the Evolute DCN is generated by the Rotation of the moveable Circle GCS, having GS or BN, as a Diameter, along the immoveable one NSQ: That is, the said Evolute is a Semi-cycloid, (because the Diameters AB, BN, of the moveable Circles, are to one another, as the Radii OB, ON, of the immoveable one) having an inverted Situation with respect to that of the other, the Vertex being in D.

To prove this, let us suppose the Diameters of the moveable Circles to fall in the right Line OT drawn at pleasure from the Centre O; this shall pass thro' the Points of Contact S, G; and making AB or $TG : BN$ or $GS :: MG : GC$, the Point C will be in the Evolute, as likewise in the Circumference of the Circle GCS; for the Angle GMT being a right Angle, the Angle GCS shall be so likewise. But because of the equal Angles MGT, CGS, the Arch TM or GB is to the Arch CS, as the Diameter GT to the Diameter $GS :: OG : OS$

$:: GB$

:: $GB:NS$; and therefore the Arches CS, SN, are equal. Whence, *&c.*

COROL. I.

101. It is * plain that the Part DC of the Cycloid is equal to the right Line CM, and therefore DC is to the Tangent $CG : AB + BN : BN :: OB + ON : ON$; that is, as the Sum of the Diameters of the generating Circles, or of the moveable and immoveable ones, is to the Radius of the immoveable Circle. The Truth of this may be shewn otherwise, thus: * *Art.* 75.

Because of the Similarity of the Triangles CMm, CGH, $Mm : GH$ or $EF :: MC : GC :: OA + OB$, $(2a+2b) : OB (b)$. Whence (as in *Art.* 97.) the Part AM of the Cycloid is to the correspondent Chord AE, as the Sum of the Diameters of the generating Circle, and the Base is to the Semi-diameter of the Base. FIG. 85.

COROL. II.

102. THE Trapezium $MGHm = \frac{1}{2}GH + \frac{1}{2}Mm \times MG$. Now $CG\left(\frac{b}{2a+b}MG\right) : CM\left(\frac{2a+2b}{2a+b}MG\right) :: GH : Mm = \frac{2a+2b}{b}GH$. Therefore because $GH = EF$, and $MG = EB$, $MGHm$ shall be $= \frac{2a+3b}{2b} EF \times EB$. That is, the Trapezium $MGHm$ will be always to the correspondent Triangle $EBF :: 2a+3b : b$. FIG. 85.

Whence

Whence the Space $MGBA$ contained under MG, AB, Perpendiculars to the Cycloid, the Arch BG, and the Part MA of the Curve, is to the correspondent Segment $BEZA$ of the Circle, as $2a+3b$ is to b.

Corol. III.

Fig. 87. 103. It is evident that the Quadrature of any Part of the Cycloid depends on the Quadrature of the Circle; but if OQ be taken as a mean Proportional between OK, OA, and the Arch QEM being described with that as a Semi-diameter. I say, the Space $ABEM$, contained under the Diameter AB, the Chord BE, the Arch EM, and the Part AM of the Cycloid, is to the Triangle $EKB :: 2a+3b : b$. For call the Arch AE or GB, u; and the Radius OQ, z; then will $OB\ (b) : OQ\ (z) :: GB\ (u) : RQ$ or $ME = \frac{uz}{b}$. And therefore the Space $RGBQ$ or $MBGE$, that is, $\overline{\tfrac{1}{2}GB + \tfrac{1}{2}RQ} \times BQ = \frac{zzu - bbu}{2b}$. Now the Space

* Art. 102. * $MGBA = \frac{2a+3b}{b} \times BEZA = \frac{2a+3b}{b} \times EKB + \frac{2a+3b}{b} \times KEZA \left(\frac{au}{2}\right)$. Now if the Space aforesaid be taken from this, there shall remain $ABEM = \frac{2aau+3abu+bbu-zzu}{2b} + \frac{2a+3b}{b} \times EKB = \frac{2a+3b}{b} EKB$; because by Construction $zz = 2aa+3ab+bb$. Whence the said Space is the only one, among others like it, whose Quadrature is independent of that of the Circle.

Moreover,

Moreover, we may have another mixted-lined Space, whose Quadrature is independent of that of the Circle. For if from the Space $ABEM$, the Segment $BEZA$ ($\frac{1}{2}au+EKB$) be taken away, there will remain $AZEM=$

$$\frac{2aau+2abu+bbu-zzu}{2b}+\frac{2a+2b}{b}\times EKB=$$

$\frac{2a+2b}{b}EKB$, by making $zz=2aa+2ab+bb$: That is, if the Semi-circumference be bisected in the Point E, the Space $AZEM$ shall be to the Double of the Triangle EKB, viz. to the Square of the Semidiameter : : OK $(a+b)$: OB (b).

COROL. IV.

104. WHEN the moveable Circle AEB revolves within the immoveable one BGD, the Diameter AB thereof, which before was positive, is here negative; and therefore the Signs of the Terms affected with it when its Dimensions are odd, must be changed. Whence, FIG. 88.

1°, If you draw MG at pleasure perpendicular to the Cycloid, and make OA $(b-2a)$: OB (b) : : $MG:GC$, the Point C will be * in the Evolute DCN described by the Rotation of the Circle (having BN as a Diameter) within the Circumference NS concentrick to BD. *Art. 100.

2°, If the Arch ME be described from the Centre O, the Part AM of the Cycloid shall be * to the Chord AE: : $2b-2a:b$. *Art. 101.

3°, The Space $MGBA$ is * to the Segment $BEZA$: : $3b-2a:b$. *Art. 102.

4°, If we assume $OQ=\sqrt{2aa-3ab+bb}$, viz. a mean Proportional between OK, OA;

then the Space $ABEM$ comprehended under the Part AM of the Cycloid, the Arch ME, the Chord EB, and the Diameter AB, shall be * to the Triangle $EKB :: 3b-2a : b$. But if we make OQ or $OE = \sqrt{2aa-2ab+bb}$; that is, the Arch AE equal to a Quadrant; the Space $AZBM$ comprehended under the Part AM of the Cycloid, and the Arches ME, AE, shall be * to the Triangle EKB, (which, in this Case, is the half of the Square of the Radius) $:: 2b-2a : b$.

* Art. 103.

* Ibid.

Corol. V.

* Fig. 86, 88.

105. If the Radius OB of the immoveable Circle be conceived to become infinite, then will the Arch BGD become a straight Line, and the Curve AMD will be the common Cycloid. Now since, in this Case, AB the Diameter of the moveable Circle, is nothing with respect to that of the immoveable one. Therefore,

1°, $MG : GC :: b : b$, because $b \pm 2a = b$, that is, $MG = GC$; and consequently, if you assume $BN = AB$, and draw the right Line NS parallel to BD, the Evolute DCN shall be generated by the Rotation of the Circle, having BN as a Diameter, on or along the Base NS.

Fig. 85, 88.

2°, The Part AM of the Cycloid is to the correspondent Chord $AE :: 2b : b$. And the Space $MGBA$ is to the Segment $BEZA :: 3b : b$.

Fig. 87, 88.

3°, Because BQ or $\pm OQ \mp OB$, which I call x, is $= \mp b \pm \sqrt{2aa \mp 3ab + bb}$, there comes out $xx \mp 2bx = 2aa \pm 3ab$, and $x = \frac{1}{2}a$, since all the Terms affected with b vanish, they being

ing infinitely less than the other Terms. Consequently if in the common Cycloid you assume $BP = \frac{3}{4} AB$, and draw *PEM* parallel to the Base *mD*; the Space *AMEB* shall be the Triple of the Triangle *EKB*. In like manner, you will find that when *P* coincides with the Centre *K*, the Space *AZEM* contained under the Part *AM* of the Cycloid, the right Line *ME*, and the Arch *AE*, shall be equal to the Square of the Radius. Which has been demonstrated before (*Art.* 99.) FIG. 83.

SCHOLIUM.

106. BECAUSE the Arches *DG*, *GM*, are always equal to each other; the Angle *DOG* shall likewise be always to the Angle *GKM* :: *GK* : *OG*. Therefore when we have given *D* the beginning of the Cycloid *DMA*, the Radii *OG*, *GK*, of the generating Circles, and the Point of Contact *G*, if you have a mind to determine the Point *M* in this Position which describes the Cycloid, it is only drawing the Radius *KM* so, that the Angle *GKM* be to the given Angle *DOG* :: *OG* : *GK*. Now this may be done geometrically always, when the Relation of the said Radii can be expressed in Numbers; and so the Cycloid *DMA* will then be a geometrical Curve. FIG. 84.

For let, for Example, *OG* : *GK* :: 13 : 5. It is manifest that the Angle *MKG* must be $2\frac{3}{5}$ *DOG* the given Angle. And so the whole Matter consists in dividing the Angle *DOG* into five equal Parts. But every Geometrician knows, that a given Angle or Arch may be divided into any Number of equal Parts geometrically; because there always comes out an Equation containing ſtraight Lines only. Whence, &c.

I ſay, moreover, that the Cycloid *DAM* is a mechanical Curve; or, which is the ſame thing, that the Points *M* in it cannot be geometrically determined, when the Ratio of *OG* to *KG* can be expreſſed only by Surds.

FIG. 89. For every Line, whether mechanical or geometrical, returns into itſelf, or elſe goes on (or is extended) *ad infinitum*; ſince the Generation thereof may be continued at pleaſure. Now when the Point *A*, in the moveable Circle *ABC*, in one Revolution, has deſcribed the Cycloid *ADE*, this will be but the firſt Part of the Curve; ſince as the Circle rolls on, there will be deſcribed a ſecond Part *EFG*, and a third *GHI*, &c. until the deſcribent Point *A*, after ſeveral Revolutions, returns again to the ſame Point in the immoveable Circle from whence it went. So that if you again revolve the Circle *ABC* as before, the ſame Curve Line *ADEFGHI* will again be deſcribed by the Point *A*. Now when the Radii of the generating Circles are incommenſurable, their Circumferences are ſo likewiſe; and conſequently the deſcribing Point *A* in the moveable Circle *ABC*, can never return again to the Point *A* in the moveable Circle from whence it went, be the Number of Revolutions never ſo many; therefore there may be an infinite Number of Cycloids that all together make up but one Curve *ADEFGHI*, &c. Now if an indefinite right Line be drawn thro' the immoveable Circle, it is plain that it ſhall cut the Curve continued *ad infinitum* in an infinite Number of Points. But becauſe the Equation expreſſing the Nature of a geo-

metrical

metrical Line, must at least be of as many Dimensions as is the Number of Points that a straight Line can cut the Curve in. Therefore the Equation that expresses the Nature of the Curve *ADEFGHI*, &c. must be of an infinite Number of Dimensions: Which being impossible, it is plain that the said Curve is a mechanical or transcendent one.

PROP. III.

107. TO *find any Number of Involutes* AM, BN, EFO, *to the given Evolute* BFC. FIG. 90.

It is manifest that the Points *A*, *B*, *F*, of the Thread *ABFC* of the Evolute *BFC*, will describe the Curves *AM*, *BN*, *FO*, to which the given Curve *BFC* is the common Evolute. But because the Curve *FO* is described by the Evolution of the Part *FC*, it does not begin in *F*. And in order to find where it does begin, the Part *BF* remaining, must be taken as the Evolute, the Point *B* being that to which the Thread is fixed; and beginning at *F*, the Part *EF* of the Involute *EFO* must be described, which begins in *E*, and is the Involute to the whole Curve *BFC*.

If you have a mind to find the Points *M*, *N*, O, without the Thread *ABFC*, you need only assume the Parts *CM*, *CN*, *CO*, equal to *ABFC*, *BFC*, *FC*, in any Tangent *CM* excepting *BA*.

COROL.

108. HENCE it follows, 1°, That the involute Curves *AM*, *BN*, *EFO*, differ very much from each other. I mean as to their Na-

ture; since in the Vertex *A* of the Curve *AM*, the Radius of Evolution is equal to *AB*, whereas that of the Curve *BN* is nothing. It is likewise evident from the very Figure of the Curve *EFO*, that it differs very much from the Curves *AM*, *BN*.

2°, That the Curves *AM*, *BN*, *EFO*, are geometrical ones only, when the given Evolute *BFC*, is a geometrical Curve, and rectifiable: For if the same be not a geometrical Curve, when *BK* is assumed for an Abscifs, the Ordinate *KC* cannot be determined geometrically; And if it be not rectifiable, when the Tangent *CM* is drawn, the Points *M*, *N*, *O*, cannot be determined in the Curves *AM*, *BN*, *EFO*, because straight Lines cannot be found equal to the Curve *BFC*, and the Parts *BF*, *FC* thereof.

SCHOLIUM.

FIG. 91. 109. If the Evolute *BAC* has a Point of Inflexion in *A*; then from the Evolution of the Part *BAD* thereof, beginning at the Point *D* (not the Point of Inflexion) there will be formed the Part *DEF* of the Involute, and from the Evolution of the Part *DC*, will be generated the remaining Part *DG* of the Involute: So that the whole involute Curve formed from the Evolution of the Curve *BAC* will be *FEDG*. Now it is evident that this Curve has two Points *D* and *E* of Retrogression, with this Difference, that at *D* the Parts *DE*, *DG*, have opposite Convexities; and at *E*, the Concavities of *DE*, *EF*, lie the same way. Now in the Section aforegoing, the Determination of Points of Retrogression

of

of the ſame Nature with *D* was handled, therefore I ſhall here ſhew the Manner of finding the Points *E*, which may be called Points of Retrogreſſion of the ſecond Kind; and this is what no body, that I know of, has hitherto conſider'd.

In order to which, draw at pleaſure two Perpendiculars *MN*, *mn*, to the Part *DE* terminating in the Points *N*, *n*, of the Evolute; from which draw two other Perpendiculars *NH*, *nH* to *NM*, *nm*; then the ſmall Sectors *MNm*, *NHn*, will be ſimilar, becauſe the Angles *MNn*, *NHn*, are equal. Therefore $Nn : Mm :: NH : NM$. Now in the Point *A* of Inflexion, the Radius *NH* becomes * infinite or nothing; and the Radius *MN*, which becomes *AE*, continues finite. Therefore in the Point *E* of Retrogreſſion of the ſecond Kind, the Ratio of *Nn*, the Fluxion of the Radius of Evolution *MN*, to *Mm* the Fluxion of the Curve, muſt become infinitely great or or infinitely ſmall. And becauſe * $Nn =$ * Art. 81. * Art. 86.

$$\frac{-3\dot{x}\dot{y}\ddot{y}^2\overline{\dot{x}^2+\dot{y}^2}^{\frac{1}{2}}+\dot{x}\dddot{y}\,\overline{\dot{x}^2+\dot{y}^2}^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2},$$ and $Mm =$

$\sqrt{\dot{x}^2+\dot{y}^2}$, therefore $\frac{\dot{x}^2\dddot{y}+\dot{y}^2\dddot{y}-3\dot{y}\ddot{y}^2}{\dot{x}\ddot{y}^2}=0$, or Infinity; and multiplying by $\dot{x}\ddot{y}^2$, we have the following general Expreſſion $\dot{x}^2\dddot{y}+\dot{y}^2\dddot{y}-3\dot{y}\ddot{y}^2=0$, or Infinity, for determining the Points of Retrogreſſion of the ſecond Kind.

If an Involute *DEF*, or *HDEFG*, has a Point of Retrogreſſion of the ſecond Kind, the Evolute *BAC* may have a Point of Retrogreſſion of the ſecond Kind alſo; ſo that the ſecond Point *A* of Retrogreſſion anſwers to the ſecond Point *E*, *viz.* both lie in the Radius of FIG. 92, 93.

Evolution issuing from the Point *E*. Now in this Supposition it is evident, that the Radius of Evolution *EA*, will be always a *Minimum* or *Maximum*. And therefore the Fluxion of $\frac{\overline{\dot{x}^2 + \dot{y}^2}^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ the general Expression * for the Radii of Evolution, must be 0, or infinite in the Point *E* sought; from whence we get the same general Expression as before; which must be used to investigate the Points of Retrogression of the second Kind.

* Art. 78.

SECT.

SECT. VI.

The Use of Fluxions in finding of Caussticks by Reflection.

DEFINITION.

If an infinite Number of Rays *BA*, *BM*, *BD*, issuing from a luminous Point *B*, be reflected by the Curve *AMD* in such manner, that the Angles of Reflexion be equal to the Angles of Incidence; the Line *HFN* that touches the reflected Rays, or their Continuations *AH*, *MF*, *DN*, is called a *Caustick by Reflexion*. FIG. 94, 95.

COROL. I.

110. If *HA* be continued out to *I*, so that *AI*=*AB*, and the Caustick *HFN* be taken as an Evolute, and *IA* as the first Radius of Evolution, then will the Involute *ILK* to the same, be of such a Nature, that the Tangent *FL* shall be* constantly equal to the Part *FH* of the Caustick *Plus* the right Line *HI*. And if *Bm*, *mF*, be supposed two reflected Rays infinitely near *BM*, *MF*; and if *Fm* be continued out to *l*, and the little Arches *MO*, *MR*, be described from the Centres *F*, *B*; the small right-angled Triangles *MOm*, *MRm*, will be equal and similar: FIG. 94.

* *Art.* 75.

Since

Since the Angle $OmM = FmD = RmM$, and the Hypothenuse Mm is common; so that $Om = Rm$. Now because Om is the Fluxion of LM, and Rm the Fluxion of BM, and this is always so, wherever the Point M be taken: Therefore $ML - IA$, or $AH + HF - MF$,

* Art. 96. the Sum of all * the Fluxions Om in the Part AM of the Curve, is $= BM - BA$, the Sum

* Art. 96. * of all the Fluxions Rm in the same Part AM; and consequently the Part HF of the Caustick HFN, will be equal to $BM - BA + MF - AH$.

There may happen several Cases, according as the incident Ray BA is greater or less than BM, and the reflected Ray HA, as a Radius of Evolution disengages itself, from the Part HF of the Curve, to become MF: But we can always prove, as we have already, that the Difference of the Radii of Incidence, is equal to the Difference of the reflected Rays *Plus* the Part of the Caustick taken as an Evolute that one of the Rays disengages itself from, before it coincides with the other. For Example,

Fig. 95. $BM - BA = MF + FH - AH$; and consequently $FH = BM - BA + AH - MF$.

Fig. 94, 95. If the Arch AP be described from the Centre B, it is plain that PM shall be the Difference of the incident Rays BM, BA. And if the luminous Point B becomes infinitely di-

Fig. 96. stant from the Curve AMD, the incident Rays BA, BM, will become parallel, and the Arch AP a straight Line perpendicular to those Rays.

Corol. II.

Fig. 94. 111. If the Figure $BAMD$ be supposed to be inverted on the same Plane, so that the Point

Point B coincides with I, and the Tangent to the Curve AMD in its first Situation, still touches it in this latter one; and if the Curve aMd revolves on AMD, viz. on itself, so that the Parts aM, AM, be always equal. I say, by this Motion the Point B will describe a kind of Cycloid ILK, which will be an Involute to the Caustick HFN, taken as an Evolute.

For from the Generation it follows, 1°, That the Line LM drawn from the describent Point L, to the Point of Contact M, will be * perpendicular to the Curve ILK. 2°, That La or $IA = BA$, and $LM = BM$. 3°, That the Angles made by the right Lines ML, BM, with the common Tangent in M are equal. And therefore if LM be continued out to F, MF shall be the Incident Ray BM reflected. Whence the Perpendicular LF touches the Caustick HFN; and since this is so always, let the Point L be taken where it will, it is plain that ILK is the Involute to the Caustick HFN, the right Line HI being the Radius of Evolution.

* Art. 43.

Hence it follows, that the Part FH or $FL - HI = BM + MF - BA - AH$. Which is what has been otherwise demonstrated in the Corollary aforegoing.

COROL. III.

112. WHEN the Tangent DN is infinitely near the Tangent FM, it is manifest that the Point of Contact N, and the Point V of Intersection, will both coincide with F the other Point of Contact: So that the Point F, wherein the reflected Ray MF touches

touches the Caustick *HFN* is determin'd, with only seeking the Concurrence of the infinitely near reflected Rays *MF*, *mF*. Consequently if we suppose an infinite Number of infinitely near incident Rays, the Intersections of these Rays reflected, will form a Polygon of an infinite Number of Sides, *viz.* the Caustick *HFN*.

PROP. I.

FIG. 97. 113. *THE Nature of the Curve* AMD, *the luminous Point* B, *and the incident Ray* BM *being given; To find the Point* F *in the reflected Ray* MF *given in Position, wherein it touches the Caustick.*

FIND (by Section aforegoing) *MC* the Length of the Radius of Evolution in the Point *M*, assume the infinitely small Arch *Mm*, draw the right Lines *Bm*, *Cm*, *Fm*, from the Centres *B*, *F*, describe the small Arches *MR*, *MO*, draw *CE*, *Ce*, *CG*, *Cg*, perpendicular to the incident and reflected Rays; and lastly, call the given Quantities *BM*, y; *ME* or *MG*, a.

*Art. 110. Now we prove (as in * *Coroll.* 1.) that the Triangles *MRm*, *MOm*, are equal and similar; and so $MR=MO$. But because of the Equality of the Angles of Incidence and Reflexion, we have also $CE=CG$, $Ce=Cg$; and therefore $CE-Ce$ or $EQ=CG-Cg$ or SG. Whence because of the similar Triangles *BMR*, *BEQ*; *FMO*, *FGS*, $BM+BE\ (2y-a) : BM\ (y) :: MR+EQ$ or $MO+GS : MR$ or $MO :: MG\ (a) : MF=\frac{ay}{2a-y}$.

If the luminous Point *B* falls on the other Side the Point *E*, with respect to the Point *M*,

or

or (which is the same thing) if the Curve *AMD* be convex next to the luminous Point *B*; then will *y* be negative, and consequently $MF = \frac{-ay}{-2y-a}$ or $\frac{ay}{2y+a}$.

If *y* be infinite, that is, if the Point *B* be at an infinite Distance from the Curve *AMD*, then the incident Rays will be parallel, and *MF* will be $=\frac{1}{2}a$, because *a* is nothing with respect to $2y$. FIG. 96.

COROL. I.

114. BECAUSE *MF* has but one Value, wherein is the Radius of Evolution; therefore the Curve *AMD* can have only one Caustick by Reflexion *HFN*, since there is * but one Evolute to it. FIG. 94, 95.

* *Art.* 80.

COROL. II.

115. WHEN the Involute *AMD* is a geometrical Curve, it is manifest * the Evolute is one likewise; that is, any Point *C* in it may be determined geometrically; and consequently any Point *F* in the Caustick thereof, may be determined geometrically, or (which is the same) the Caustick *HFN* will be a geometrical Curve. I say, moreover, that the said Caustick is always rectifiable; because by means of the Curve *AMD*, which is supposed to be a geometrical one, it is * manifest that straight Lines may be found equal to any Part thereof. FIG. 97. FIG. 94, 95.

* *Art.* 85.

* *Art.* 110.

COROL.

Corol. III.

Fig. 97. 116. If the Curve AMD be convex next to the luminous Point B; the Value of $ME\left(\frac{ay}{2y+a}\right)$ will always be positive; and the Point F must of consequence be assumed on the same Side the Point M as the Point C, as we have supposed in the Investigation aforegoing. Therefore the infinitely-near reflected Rays do diverge.

But if the Curve AMD be concave next to the luminous Point B; $MF\left(\frac{ay}{2y-a}\right)$ will be positive, when y is greater than $\frac{1}{2}a$, negative when the same is less, and infinite when it is $=\frac{1}{2}a$. Whence if a Circle be described with a Diameter equal to $\frac{1}{2}MC$ the Radius of Evolution, the infinitely-near reflected Rays will converge, when the luminous Point B falls without the Circumference thereof, and diverge when the same falls within; and finally will be parallel when it falls in the Circumference.

Corol. IV.

117. If the incident Ray BM touches the Curve AMD in the Point M, then will $ME(a)=0$; and therefore $MF=0$. But because the reflected Ray is then in the Direction of the incident Ray, and it is the Nature of the Caustick to touch all the reflected Rays; therefore it shall touch likewise the incident Ray BM in the Point M; that is,

is, the Caustick and the given Curve will have the same Tangent in the common Point M.

If the Radius CM of Evolution be nothing, then still will ME $(a) = 0$; and consequently $MF = 0$. Therefore the given Curve, and the Caustick, make an Angle with each other in the common Point M, equal to the Angle of Incidence.

If the Radius of Evolution CM be infinite, the small Arch Mm will become a straight Line, and $MF = \mp y$; because ME (a) being infinite, y will be nothing with regard to a. Now since this Expression or Value is negative when the Points B, C, are both on the same Side the Line AMD, and positive when one is on one side, and the other on the other. Therefore the infinitely near reflected Rays will always diverge when AMD is a right Line.

COROL. V.

118. It is evident, that when any two of the Points B, C, F, are given, the third is easily found.

1°, Let the involute Curve AMD be a Parabola, and the luminous Point B the Focus. Then (from the Conick Elements) it is manifest, that all the reflected Rays will be parallel to the Axis; and so in every Position of the Point M, MF will be infinite; and consequently $a = 2y$. Whence if you assume $ME = 2MB$, and draw the Perpendicular EC, this will intersect the Perpendicular MC to the Curve AMD, in the Point C, which will be in the Evolute. FIG. 98.

2°, Let

FIG. 99. 2°, Let *AMD* be an Ellipsis, and the luminous Point *B* one of the Foci. Then it is evident, that all the reflected Rays *MF* will coincide in the other Focus *F*. And if you call

* Art. 113. *MF*, z; then will * $z = \frac{ay}{2y-a}$; from whence comes out $ME\ (a) = \frac{2yz}{y+z}$. Which was sought.

FIG. 110. But if the Curve *AMD* be an Hyperbola, the Focus *F* will fall the contrary way, or without the Curve; and therefore $MF(z)$ will become negative; and consequently $ME\ (a) = \frac{-2yz}{y-z}$ or $\frac{2yz}{z-y}$. Whence arises the following Construction, which will serve for the Ellipsis also.

FIG. 99, 100. Assume *ME* a fourth Proportional to half the transverse Axis, and the incident and reflected Rays, and draw the Perpendicular *EC*: This shall intersect the Line *MC* perpendicular to the Curve in the Point *C*, which shall be in the Evolute.

EXAMPLE I.

FIG. 101. 119. Let *AMD* be a Parabola, and let the incident Rays *PM* fall perpendicular to the Axis. It is required to find the Points *F* in the reflected Rays *MF*, wherein they touch the Caustick *AFK*.

If we draw *MC* the Radius of Evolution, and the Line *CG* perpendicular to the reflected Ray *MF*, it is manifest * that we must

* Art. 113. assume $MF = \frac{1}{2}MG$. But this Construction may be shorten'd, if you consider that when *MN* is drawn parallel to the Axis, *AP*, and the

the right Line ML to the Focus L, the Angles LMP, FMN shall be equal, since from the Nature of the Parabola $LMQ = QMN$, and by the Supposition $PMQ = QMF$. If then the common Angle PMF be added to both, the Angle LMF shall be equal to the Angle PMN, that is, a right Angle. Now it has been demonstrated (*Art.* 118. *Num.* 1.) that LH perpendicular to ML shall intersect the Radius of Evolution MC in H the middle thereof. Therefore if MF be drawn equal and parallel to LH, it will be one of the reflected Rays, and will touch the Caustick AFK in the Point F. Which was to be found.

If the reflected Ray MF be supposed parallel to the Axis AP, it is manifest that the Point F of the Caustick will be at the greatest Distance possible from AP, because the Tangent to that Point will be parallel to the Axis. And consequently in order to determine that Point in all the Causticks, as AFK, formed by incident Rays parallel to the Axis of the given Curve, we need only consider that then MQ must be equal to PQ. And therefore $\dot{y} = \dot{x}$.

Now let $ax = yy$. Then $\dot{y} = \frac{a\dot{x}}{2\sqrt{ax}} = \dot{x}$, and so $AP\,(x) = \frac{1}{4}a$: that is, when the Point P falls in the Focus L, the reflected Ray MF will be parallel to the Axis; which is a known Truth otherwise establish'd; since in this Case MP coinciding with LM, MF must also coincide with MN, and LH with LQ. Whence MF will then be equal to ML; and therefore if FR be drawn perpendicular to the Axis, AR or $AL + ME$ will be $= \frac{3}{4}a$. We may ob-

ſerve, that in this Caſe the Part AF of the Cauſtick is equal to the Parameter, ſince it is
*Art. 110. * always equal to $PM+MF$.

To determine the Point K wherein the Cauſtick AFK interſects the Axis AP, the Value of MO muſt be found, and made equal to that of MF; for it is plain, when the Point F falls in K, the Lines MF, MO become equal to each other. For when the Point F coincides with K, it is manifeſt that the Lines MF, MO will become equal to one another. Therefore if the unknown Quantity MO be called t; from the Biſection of the Angle PMO by MQ perpendicular to the Curve, we ſhall have $MP\ (y)$: $MO\ (t) :: PQ \left(\frac{y\dot{y}}{\dot{x}}\right) : OQ = \frac{t\dot{y}}{\dot{x}}$. And therefore $OP = \frac{t\dot{y}+y\dot{y}}{\dot{x}} = \sqrt{tt-yy}$, becauſe of the right-angled Triangle MPO; and dividing both Sides by $t+y$, there comes out $\frac{\dot{y}}{\dot{x}} = \sqrt{\frac{t-y}{t+y}}$, from whence we get $MO\ (t) =$
* Art. 77. $\frac{y\dot{x}^2+y\dot{y}^2}{\dot{x}^2-\dot{y}^2} = MF\ (\frac{1}{2}a) = \frac{\dot{x}^2+\dot{y}^2}{-2\ddot{y}}$, ſince * ME $(a) = \frac{\dot{x}^2+\dot{y}^2}{-\ddot{y}}$. And ſo by means of $\dot{y}^2 - 2y\ddot{y} + \dot{x}^2$, the Point P may be determined ſo, that drawing the incident Ray PM, and the reflected Ray MF, this latter will touch the Cauſtick AFK in the Point K, wherein it interſects the Axis AP.

Now in the Parabola $y = x^{\frac{1}{2}}$, $\dot{y} = \frac{1}{2}x^{-\frac{1}{2}}\dot{x}$, $\ddot{y} = -\frac{1}{4}x^{-\frac{3}{2}}\dot{x}^2$, and ſubſtituting theſe Expreſſions in the foregoing Equation, there will come

come out $\frac{1}{4}x^{-1}\dot{x}^2 + \frac{1}{2}x^{-1}\dot{x}^2 = \dot{x}^2$; and ſo we ſhall get $AP\,(x) = \frac{3}{4}$ of the Parameter.

Now to find the Nature of the Cauſtick AFK after the manner of *Deſcartes*, we muſt get an Equation expreſſing the Relation of the Abſciſs $AR\ (u)$ to the Ordinate $RF\ (z)$; which may be done thus. Becauſe $MO\,(t) = \frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \dot{y}^2}$, therefore $PO\left(\frac{t\dot{y} + y\dot{y}}{\dot{x}}\right) = \frac{2y\dot{x}\dot{y}}{\dot{x}^2 - \dot{y}^2}$; and becauſe of the ſimilar Triangles MPO, MSF, therefore $MO\ \left(\frac{y\dot{x}^2 + y\dot{y}^2}{\dot{x}^2 - \ddot{y}}\right) : MF\ \left(\frac{\dot{x}^2 + \dot{y}^2}{-2\ddot{y}}\right)$ or $-2y\ddot{y} : \dot{x}^2 - \dot{y}^2 :: MP\ (y) : MS\ (y - z) = \frac{\dot{x}^2 - \dot{y}^2}{-2\ddot{y}} :: PO\left(\frac{2y\dot{x}\dot{y}}{\dot{x}^2 - \dot{y}^2}\right) : SF$ or $PR\ (u - x) = \frac{\dot{x}\dot{y}}{-\ddot{y}}$. Therefore with $z = y + \frac{\dot{y}^2 - \dot{x}^2}{-2\ddot{y}}$, and $u = x + \frac{\dot{x}\dot{y}}{-\ddot{y}}$ together with the Equation of the given Curve, we get a new Equation freed from x and y, which expreſſes the Relation of $AR\,(u)$ to $FR\,(z)$.

When the Curve AMD is a Parabola, as in the Example, we ſhall get $z = \frac{3}{2}x^{\frac{1}{2}} - 2x^{\frac{3}{2}}$, or (ſquaring each Side) $\frac{9}{4}x - 6xx + 4x^3 = zz$, and $u = 3x$. Whence we get the Equation ſought $azz = \frac{4}{27}u^3 - \frac{2}{3}auu + \frac{3}{4}aau$ expreſſing the Nature of the Cauſtick AFK. Here we may obſerve, that PR is always the Double of AP, becauſe $AR\,(u) = 3x$. From whence we get moreover another way of determining the ſought Point F in the reflected Ray MF.

Example II.

Fig. 104. 120. Let *AMD* be a Semicircle, the Line *AD* a Diameter, and *C* the Centre, and let the incident Rays *PM* be perpendicular to *AD*.

Because the Evolute of the Circle is a Point, *viz.* the Centre of it, therefore if *CM* be bisected in *H*, and *HF* be drawn perpendicular to the reflected Ray *MF*, it shall * cut the said Ray in the Point *F* wherein it touches the Caustick *AFK*. Now because the reflected Ray *MF* is equal to $\frac{1}{2}$ of the incident Ray *PM*; therefore when the Point *P* coincides with *C*, it is plain the Point *F* will coincide with *K* the middle of *CB*. And the Part *AF* is the triple of *MF*, and the Caustick *AFK* the triple of *BK*. We may likewise observe, that when the Angle *ACM* is made one half a right Angle, the reflected Ray *MF* shall be parallel to *AC*; and therefore the Point *F* will be the highest Point of the Caustick above the Diameter *AD*.

* Art. 113.

The Circle whose Diameter is the Line *MH* passes thro' the Point *F*; because the Angle *HFM* is a right Angle. And if from the Centre *C*, with the Radius *CK* or *CH* the half of *CM*, the Circle *KHG* be described; the Arch *HF* shall be equal to the Arch *HK*: for since the Angle *CMF* is equal to *CMP* or *HCK*, the Arches $\frac{1}{2}$ *HF* and *HK* that measure the said Angles in the Circles *MFH*, *KHG* shall be to each other as the Radii $\frac{1}{2}$ *MH*, *HC* of these Circles. Whence it appears, that the Caustick *AFK* is a Cycloid generated by the Rotation of the Circle MFH along the immoveable Circle KHG, the Beginning thereof being K, and the Vertex A.

EXAMPLE III.

121. LET AMD be a Circle, the Line AD the Diameter, and the Point C the Centre; and let A, one End of that Diameter, be the luminous Point from whence all the Rays of Incidence AM issue. FIG. 103.

From the Centre C draw CE perpendicular to the incident Ray AM; then from the Nature of the Circle, it is manifest that the Point E bisects the Chord AM; and so ME $(a) = \frac{1}{2}y$. Whence MF $\left(\frac{ay}{2y-a}\right) = \frac{1}{3}y$: that is, we must take the reflected Ray $MF = \frac{1}{3}AM$ the incident Ray. Consequently $DK = \frac{1}{3}AD$, $CK = \frac{1}{3}CD$, and the * Caustick $AFK = \frac{2}{3}AD$, like as its Part $AF = \frac{2}{3}AM$. If you assume $AM = AC$, the reflected Ray MF will be parallel to the Diameter AD; and consequently the Point F will be the highest Point possible above the said Diameter. *Art. 110.

If you take $CH = \frac{1}{3}CM$, and draw HF perpendicular to MF; the Point F shall be in the Caustick: for drawing HL perpendicular to AM, it is manifest that $ML = \frac{2}{3}ME = \frac{1}{3}AM$, because $MH = \frac{2}{3}CM$. Therefore the Circle having MH as a Diameter, shall pass thro' the Point F of the Caustick; and if from the Centre C with the Radius CK or CH another Circle KHG be described, it shall be equal to the former one, and the Arch HK shall be equal to the Arch HF: For in the Isosceles Triangle CMA the external Angle

$KCH = 2CMA = AMF$; and therefore the Arches *HK*, *HF* being the Measures of those Angles in the equal Circles, shall likewise be equal. Whence it follows, that the Caustick *AFK* is likewise here a Cycloid generated by the Rotation of the Circle *MFH* along the immoveable one *KHG*, the Beginning being *K*, and the Vertex *A*.

This may be demonstrated otherwise thus. If a Cycloid be described by the Revolution of a Circle equal to the Circle *AMD* about this same, beginning at *A*, we have demonstrated (*Art.* 111.) that the Caustick *AFK* will be the Evolute to the said Cycloid taken
*Art. 110. as an Involute. But * the said Evolute is a Cycloid of the same Kind, *viz.* the Diameters of the generating Circles of them shall be equal; and the Point *K* will be determined by taking *CK* a third Proportional to $CD+DA$ and *CD*, *viz.* equal to $\frac{1}{3}$ *CD*. Whence, &c.

Example IV.

Fig. 104. 122. Let the Curve *AMD* be one half of a common Cycloid described by the Rotation of the Semicircle *NCM* along the right Line *BD*, whose Vertex is *A*, and the Beginning thereof *D*; and let the incident Rays *KM* be parallel to the Axis *AB*.

Fig. 104. Because * *MG* is equal to the half of the
* Art. 95. Radius of Evolution; therefore * if *GF* be
* Art. 113. drawn perpendicular to the reflected Ray *MF*, the Point *F* shall be in the Caustick *DFB*. And so *MF* must be assumed equal to *KM*.

If

If from H the Centre of the generating Circle MGN to the Point of Contact G, and to the describent Point M be drawn the Radii HG, HM; it is evident that HG will be perpendicular to BD, and the Angle $GMH = MGH = GMK$: whence the reflected Ray MF passes thro' the Centre H. Now the Circle having GH as a Diameter, passes thro' the Point F likewise: because the Angle GFH is a right Angle. Therefore the Arches GN, $\frac{1}{2}GF$ the Measures of the same Angle GHN shall be to one another, as the Diameters MN, GH of their Circles. And consequently the Arch $GF = GN = GB$. Whence it is evident, that the Caustick DFB is a Cycloid described by the entire Rotation of the Circle GFH along the right Line BD.

EXAMPLE V.

123. LET the Curve AMD be still the half of a common Cycloid, the Base BD whereof is equal to one half the Circumference ANB of the generating Circle; and let the incident Rays PM be parallel to the Base BD. FIG. 105.

If GQ be drawn perpendicular to PM, the right-angled Triangles GQM, BPN will be equal and similar; and therefore $MQ = PN$. Whence it follows, * that MF must be assumed equal to the correspondent Ordinate PN in the generating Circle ANB.

* Art. 95, 113.

When the Point F is at the greatest Distance possible from the Axis AB, the Tangent MF in that Point must be parallel to the Axis. And consequently the Angle PMF will be a right Angle, and PMG or PNB half a right

Angle. Whence the Point P falls in the Centre of the Circle AND.

Here it is worth observing, that as the Point P afterwards accedes towards the Extremity B of the Diameter, so does the Point F likewise accede towards the Axis AB, until it comes to a certain Point K; after which it recedes therefrom till it comes to D. So that the Caustick $AFKFD$ has a Point of Retrogression in K.

* Art. 110, 111. To determine which, we must observe* that the Part $AF = PM + MF$, the Part $AFK = HL + LK$, and the Part KF of KFD is $= HL + LK - PM - MF$: whence $HL + LK$ must be a *maximum*. And if you call AH, x; HI, y; the Arch AI, u; then will $HL + LK = u + 2y$, the Fluxion whereof is $\dot{u} + 2\dot{y} = 0$, and $\frac{a\dot{x}}{y} + 2\dot{y} = 0$, by substituting $\frac{a\dot{x}}{y}$ for $\dot{u}$: from whence we get $a\dot{x} = -2y\dot{y} = 2x\dot{x} - 2a\dot{x}$ because of the Circle: and therefore $AH(x) = \frac{3}{2}a$.

Corol.

124. The Space AFM or $AFKFM$ contained under the Parts of the Curves AF or $AFKE$, AM, and the reflected Ray MF is equal to the half of the circular Space APN. For the Fluxion thereof, *viz.* the Sector FMO is equal to the half of the Rectangle $PpSN$, the Fluxion of the Space APN; since the right-angled Triangles MOm, MRm being similar and equal, MO shall be equal to MR or NS or Pp, and moreover $MF = PN$.

Exam-

EXAMPLE VI.

125. LET the Curve AMD be the half of a Cycloid generated by the Rotation of the Circle MGN about AGK equal to it, and let A be the beginning thereof, and D the Vertex. And let the incident Rays AM all issue from the Point A. The Line BH joining the Centres of the generating Circles constantly passes thro' the Point of Contact G, and the Arches GM, GA, as also their Chords, are always equal; also the Angle HGM $=BGA$, and the Angle $GMA=GAM$. Now the Angle $HGM+BGA=GMA+GAM$; because if the Angle AGM be added to each Side, there will be two right Angles made. Whence the Angle HGM shall be always equal to the Angle GMA; and so likewise to the Angle of Reflection GMF. Whence it follows, that MF always passes thro' H the Centre of the moveable Circle. FIG. 106.

Now if CE, GO, be drawn perpendicular to the incident Ray AM, it is manifest that $MO=OA$, and $OE=\frac{1}{3}OM$; since the Point C being * in the Evolute, $GC=\frac{1}{3}GM$; therefore $ME=\frac{2}{3}AM$; that is, $a=\frac{2}{3}y$; and consequently $MF\left(\frac{ay}{2y-a}\right)=\frac{1}{2}y$. Whence if you draw GF perpendicular to MF, the Point F will be in the Caustick AFK. * Art. 100.

The Circle whereof GH is a Diameter, does pass thro' the Point F; and since the Arches GM, $\frac{1}{2}GF$, the Measures of the Angle GHM are to one another, as the Diameters MN, GH, of their Circles; the Arch GF shall be equal to GM, and consequently to the Arch GA.

From

From whence it is evident, that the Caustick *AFK* is a Cycloid generated by the Rotation of the moveable Circle *HFG* about or along the immoveable one *AGK*.

Corollary.

126. If a Circle be described about the Centre *B*, with a Radius equal to *BH* or *AK*; and an infinite Number of right Lines parallel to *BD* falls on the Circumference of
* *Art.* 120. it: it is then manifest * that by their Reflection they will form the same Caustick *AFK*.

Example VII.

Fig. 107. 127. Let *AMD* be a Logarithmick Spiral, and let the incident Rays (*AM*) all issue from the Centre *A*.

If the right Line *CA* be drawn from *C*, the End of the Radius of Evolution, perpendicular to the incident Ray *AM*, it shall meet
* *Art.* 91. * the same in the Centre *A*. Therefore *AM* $(y) = a$; and consequently $MF\left(\frac{ay}{2y-a}\right) = y$. Whence *AMF* shall be an Isosceles Triangle; and since the Angles of Incidence and Reflection *AMT*, *FMS*, are equal to one another, the Angle *AFM* is equal to the Angle *AMT*. Whence it is manifest that the Caustick *AFK* will be a Logarithmick Spiral, differing only from the proposed one *AMD* in Position.

PROP. II.

128. *THE Caustick* HF *by Reflection, and the luminous Point* B *being given, to find an infinite Number of Curves, as* AM, *whereof the same is the Caustick by Reflection.* FIG. 108.

Take the Point A at pleasure in any Tangent HA, for one of the Points of the Curve AM sought; and from the Centre B, with the Distance BA, describe the circular Arch AP; and with any other Distance BM, another circular Arch. Then assume $AH+HE=BM-BA$ or PM; and by the Evolution of the Caustick HF, beginning at E, describe the involute Curve, cutting the circular Arch described with the Radius BM in the Point M, which will be * in the Curve AM. For by Construction $PM+MF=AH+HF$. *Art. 110.

Or else take a Thread BMF, and having fixed one End in B, and the other in F, stretch the Thread by Means of a Pin at M, which so move along, that the Part MF of the Thread wraps about the Caustick HF: Then it is evident that the Pin M in thus moving, describes the Curve MA sought.

Otherwise thus:

129. DRAW any Tangent (FM) excepting HA, at pleasure, and in the same find the Point M such, that $BM+MF=BA+AH+HF$; which may be done thus:

Assume $FK=BA+AH+HF$, and bisecting BK in G, draw the Perpendicular GM. This

This shall intersect the Tangent FM in the sought Point M: for $BM = MK$.

Fig. 109. If the Point B be supposed at an infinite Distance from the Curve AM; that is, if the incident Rays BA, BM, be parallel to a right Line given in Position; the former Construction will serve likewise in this Case, in conceiving the Arches described from the Centre B, to become right Lines perpendicular to the incident Ray. But the latter Construction will not do here; for which we shall lay down the following one.

Assume $FK = AH + HF$. And find the Point M such, that MP (a Parallel to AB) perpendicular to AP, be equal to MK; then
* Art. 110. it is plain * that M will be the Point sought in the Curve AM, because $PM + MF = AH + HF$. Now this is done thus.

Draw KG perpendicular to AP; and assuming $KO = KG$, draw KP parallel to OG, and PN parallel to GK: I say the Point M will be that required. For because of the similar Triangles GKO, PMK, PM will be $= MK$; since $GK = KO$. If the Caustick HF degenerates into a Point, the Curve AM will become a Conick Section.

Corol. I.

130. Hence it is manifest, that the Curve, which passes thro' all the Points K, is generated by the Evolution of the Curve HF, beginning at A, and that its Nature varies according as the Situation of the Point A in the Tangent AH varies. Wherefore because the Curves (AM) are all generated by the same geometrical Construction from the said

ſaid Curves: It is manifeſt * that the Nature *Art.* 108.
of them differ, and they are only geometrical when the Cauſtick HF is a geometrical Curve and rectifiable.

COROL. II.

131. A CURVE DN, together with the luminous Point C being given, to find any Number of Lines (as AM) being ſuch, that they may cauſe all the Rays DA, NM, reflected from them, to unite in a given Point B. FIG. 110.

If the Curve HF be ſuppoſed to be the Cauſtick of the given Curve DN, formed by the luminous Point C; it is manifeſt that the ſaid Line HF muſt likewiſe be the Cauſtick of the Curve AM, having the given Point B as a luminous Point: So that $FK=BA+AH+HF$, and $NK=BA+AH+HF+FN=BA+AD+DC-CN$, ſince * $HD+DC=HF+FN+NC$. Which gives this Conſtruction. *Art.* 110.

In any reflected Ray aſſume the Point A at pleaſure, for one Point in the ſought Curve AM; and in any other reflected Ray MN, aſſume the Part $NK=BA+AD+DC-CN$, and the Point M will be found as above (*Art.* 129.)

SECT.

SECT. VII.

The Use of Fluxions in finding of Causticks by Refraction.

DEFINITION.

FIG. 111. If the Directions of an infinite Number of Rays BA, BM, BD, all issuing from the same luminous Point B, be alter'd after their Concurrence with a Curve AMD, either nearer or farther from the Perpendiculars MC to it: And if the Law of Alteration be constantly such, that (CE) the Sine of the Angles of Incidence (CME) be to (CG) the Sines of the Angles (CMG) of Refraction, in the given
FIG. 112. Ratio of m to n; the Curve HFN that touches all the broken or refracted Rays, or the Continuations of them, AH, MF, DN, is called the *Caustick by Refraction.*

COROLLARY.

132. If the Caustick HFN be taken as an Evolute, and the Involute ALK be described from it beginning at A; the Line LF *Plus* FH, a Part of the Caustick, will be always equal to AH. And if you conceive another Tangent Fml infinitely near to FML, and another Ray of Incidence Bm; and from the Centres F, B, be described the small Arches MO,

MO, MR; then the little right-angled Triangles MRm, MOm, will be similar, the former to MEC, and the latter to MGC; because if the Angle EMm be taken from the right Angles RME, CMm, the Angles RMm, EMC, remaining, shall be equal: and, in like manner, if the Angle GMm be taken away from the right Angles GMO, CMm, the remaining AMm, GMC, will be equal. Therefore $Rm : Om :: CE : CG :: m : n$. Now since Rm is the Fluxion of BM, and Om the Fluxion of LM; $BM-BA$* the Sum of all the Fluxions (Rm) in the Part of the Curve AM, will be* to ML or $AH-MF-FH$, the Sum of all the Fluxions (Om) in the same Part, as m to n. And therefore the Part $FH=AH-MF+\frac{n}{m}BA-\frac{n}{m}BM$. * Art. 96.

There are several Cases, according as the incident Ray BA is greater or less than BM, and the refracted Ray AH wraps about, or unwraps (itself from) the Part HF: But we prove, as already, that the Difference between the incident Rays, is always to the Difference of the refracted Rays, (*Plus* the Part of the Caustick that one of these Rays disengages itself from before it falls on the other) as m to n. For Example, $BA-BM : AH-MF-FH :: m : n$, From whence we get $FH=AH-MF+\frac{n}{m}BM-\frac{n}{m}BA$. FIG. 112.

If the Arch AP be described from the Centre B, it is manifest that PM will be the Difference of the incident Rays BM, BA. And if the luminous Point B becomes infinitely distant from the Curve AMD, the incident Rays BA, BM, will be parallel, and the Arch FIG. 111.

AP will become a right Line perpendicular to the said Rays.

Prop. I.

FIG. III. 133. THE *Nature of the Curve* AMD, *the luminous Point* B, *and the Ray of Incidence* BM, *being given, to find the Point* F *in the refracted Ray* MF *given in Position, wherein the same touches the Caustick by Refraction.*

*Sect. V. FIRST, find * the Length *MC* of the Radius of Evolution at the given Point *M*, assume the infinitely small Arch *Mm*, draw the right Lines *Bm*, *Cm*, *Fm*, from the Centres *B*, *F*, describe the little Arches *MR*, *MO*, draw *CE*, *Ce*, *CG*, *Cg*, perpendicular to the incident and refracted Rays; and call the given Quantities *BM*, y; *ME*, a; *MG*, b; and the little Arch *MR*, $\dot{x}$. This done,

Because of the right-angled similar Triangles *MEC* and *MRm*, *MGC* and *MOm*, *BMR* and *BQe*, $ME\ (a) : MG\ (b) :: MR\ (\dot{x}) : MO = \frac{b\dot{x}}{a}$. And $BM\ (y) : BQ$ or $BE\ (y+a) :: MR\ (\dot{x}) : Qe = \frac{a\dot{x}+y\dot{x}}{y}$. Now from the Nature of Refraction $Ce : Cg :: CE : CG :: m : n$. And therefore $m : n :: Ce - CE$ or $Qe \left(\frac{a\dot{x}+y\dot{x}}{y}\right) : Cg - GC$ or $Sg = \frac{an\dot{x}+ny\dot{x}}{my}$. Wherefore because of the right-angled similar Triangles *FMO* and *FSg*, $MO - Sg \left(\frac{bmy\dot{x} - any\dot{x} - aan\dot{x}}{amy}\right)$:

$MO\left(\frac{bx}{a}\right) : MS$ or $MG\ (b) : MF = \frac{bbmy}{bmy - any - aan}$. From whence comes out the following Construction.

Towards CM make the Angle $ECH = GCM$, and towards B take $MK = \frac{aa}{y}$. Then if you make $HK : HE :: MG : MF$. I say the Point F will be in the Caustick by Refraction. Fig. 113.

For because of the Similarity of the Triangles CGM, CEH, $CG : CE :: n : m :: MG\ (b) : EH = \frac{bm}{n}$. Whence $HE - ME$ or $HM = \frac{bm - an}{n}$, $HM - MK$ or $HK = \frac{bmy - any - aan}{ny}$. And therefore $HK\left(\frac{bmy - any - aan}{ny}\right) : HE\left(\frac{bm}{n}\right) :: MG\ (b) : MF = \frac{bbmy}{bmy - any - aan}$.

It is manifest, that when the Value of HK is negative, that of MF shall be so likewise: Whence it follows, that when the Point M falls between the Points G, F, the Point H falls between the Points K, E.

If the luminous Point B be next to E, or (which is the same thing) if the Curve AMD be concave next to B, then will y be changed from positive to negative; and consequently, $MF = \frac{-bbmy}{-bmy + any - aan}$ or $\frac{bbmy}{bmy - any + aan}$. And the Construction will be the same.

If y be supposed infinite; that is, when the luminous Point B is at an infinite Distance from the Curve AMD, the incident Rays will be then parallel, and $MF = \frac{bbm}{bm-an}$, since the Term aan is o with respect to bmy, any; and because $MK\left(\frac{aa}{y}\right)$ then vanishes, we need only make $HM : HE :: MG : MF$.

Corol. I.

134. It may be demonstrated after the same * manner, as in the Causticks by Reflection, that the Curve AMD has but one Caustick by Refraction, the Ratio of m to n being given; which Caustick is always a geometrical Curve and rectifiable, when the proposed Curve AMD is a geometrical Curve.

* Art. 114, 115.

Corol. II.

135. If the Point E falls on the other Side the Perpendicular MC with regard to the Point G, and CE be $= CG$; it is evident that the Caustick by Refraction will become a Caustick by Reflection. For MF $\left(\frac{bbmy}{bmy-any+aan}\right) = \frac{ay}{2y+a}$; because $m = n$, and a is here negative and equal to b. And this agrees with what has been before demonstrated in the Section aforegoing.

If m be infinite with respect to n, it is evident that the refracted or broken Ray MF will coincide with the Perpendicular CM: So that the Caustick by Refraction will be the Evolute. For MF will be $= b$, which in this Case becomes MC: That is, the Point F will coincide with the Point C, which is in the Evolute.

Corol.

COROL. III.

136. If the convex Side of the Curve AMD be next to the luminous Point B, and the Value of $MF\left(\frac{bbmy}{bmy-any-aan}\right)$ be positive; it is evident that the Point F must be assumed on the same Side of M as G is, which was supposed in the Operation of the Problem; and on the contrary, if it be negative, the same must be assumed on the contrary Side. The same is to be understood when the concave Part of the Curve AMD is towards the Point B. But we must observe that then $MF = \frac{bbmy}{bmy-any-ann}$. From whence it follows, that the infinitely near refracted or broken Rays do converge when the Value of MF is positive in the first Case, and negative in the second; and on the contrary, they diverge when it is negative in the first Case, and positive in the second. This being premised, it is evident,

1°, That if the convex Side of the Curve AMD be next to the luminous Point B, and m less than n; or if the concave Part thereof be next to the said luminous Point, and m greater than n; then the infinitely near refracted or broken Rays will always diverge.

2°, If the convex Side of AMD be turned towards the luminous Point B, and m exceeds n, or the concave Side, and m be less than n; the infinitely near refracted Rays do converge when $MK\left(\frac{aa}{y}\right)$ is less than MH

$\left(\frac{bm}{n} - a \text{ or } a - \frac{bm}{n}\right)$; diverge, when it is greater; and are parallel, when it is equal. Now since $MK = 0$, when the Rays of Incidence are parallel, it follows that in this Case the infinitely near refracted Rays do always converge.

Corol. IV.

137. If the Ray of Incidence *BM* touches the Curve *AMD* in the Point *M*, then will *ME* $(a) = 0$; and therefore $MF = b$. And so the Point *F* will coincide with the Point *G*.

If the Ray of Incidence *BM* be perpendicular to the Curve *AMD*, the right Lines *ME* (a) and *MG* (b) will be each equal to the Radius *CM* of Evolution; because they coincide with it. Therefore $MF \frac{bmy}{my - ny + bn}$, which becomes $\frac{bm}{m-n}$ when the Rays of Incidence are parallel to one another.

If the refracted Ray *MF* touches the Curve *AMD* in the Point *M*; then will *MG* $(b) = 0$. And consequently the Caustick does then touch the given Curve in the Point *M*.

If *CM* the Radius of Evolution be nothing, the straight Lines *ME* (a), *MG* (b) shall likewise be equal to nothing; and so the Terms aan, $bbmy$ are nothing with respect to the Terms bmy, any. Whence it follows that $MF = 0$; and so the Point *M* is both in the Caustick and given Curve.

If *CM* the Radius of Evolution be infinite, the right Lines *ME* (a), *MG* (b) shall be infinite

finite also; and consequently the Terms bmy, any shall be nothing with regard to aan, $bbmy$: So that MF will be $= \frac{bbmy}{+aan}$. Now since this Quantity is * negative when the Point F falls not on the same side the Line AMD as B; and on the contrary is positive, when F and B are both on the same side AMD; therefore the Point F must be * assumed next to the Point B, that is, the infinitely refracted Rays are diverging. It is evident that the small Arch Mm then becomes a right Line, and the Construction above will not do here. And so we shall lay down the following one, which determines Points in Caustick by Refraction when AMD is a right Line.

*Art. 133.

*Art. 136.

Draw BO perpendicular to the Ray of Incidence BM, intersecting the right Line MC perpendicular to AD in the Point O; likewise draw OL perpendicular to the refracted Ray MG, make the Angle BOH equal to the Angle LOM, and also $BM : BH :: ML : MF$. I say, the Point F will be in the Caustick by Refraction. FIG. 114.

For let MC be of what Magnitude you please, the right-angled Triangles MEC and MBO, MGC and MLO will always be similar: and therefore when it becomes infinite, we still have $ME (a) : MG (b) :: BM (y) : ML = \frac{by}{a}$. And because of the similar Triangles OLM, OBH we have likewise $OL : OB (n : m) :: ML \left(\frac{by}{a}\right) : BH = \frac{bmy}{an}$. Whence $BM (y) : BH \left(\frac{bmy}{an}\right) :: ML \left(\frac{by}{a}\right) : MF \left(\frac{bbmy}{aan}\right)$.

 COROL.

Corol. V.

138. When any two of the three Points B, C, F are given, it is manifest that the third may be easily found.

Example I.

Fig. 115. 139. Let AMD be a Quadrant of a Circle, whereof the Point C is the Centre; and let the Rays of Incidence BA, BM, BD be parallel to each other, and perpendicular to CD; and finally let the Ratio of m to n be as 3 to 2, which is the Ratio between the Sine of the Angle of Incidence and Refraction in the Passage of the Rays of Light from Air into Glass. Now because the Evolute of the Circle AMD is the Centre C thereof; if a Semicircle MEC be described, with the Radius CM, as a Diameter, and you assume the Chord $CG = \frac{2}{3}CE$, it follows that the Line MG will be the refracted Ray, in which the Point F may be determined as above (Art. 133.)

To find the Point H, wherein the Ray of Incidence BA perpendicular to AMD touches

*Art. 137. the Caustick by Refraction, we have * AH $\left(\frac{bm}{m-n}\right) = 3b = 3CA$. And if a Semicircle CND be described with CD as a Radius, and

*Art. 137. you take the Chord $CN = \frac{2}{3}CD$; it is * manifest that the Point N will be in the Caustick by Refraction, because the Radius of Incidence BD touches the Circle AMD in the Point D,

*Art. 132. If AP be drawn parallel to CD, it is * manifest that the Part $FH = AH - MF - \frac{2}{3}PM$: so so that the whole Caustick $HFN = 3\,CA - DN = \frac{7 - \sqrt{5}}{3}\,CA$.

If the concave Part of the Circle AMD be turned next to the incident Rays BM, and the Ratio of m to n be as 2 to 3; then on the Semi-circumference CEM having CM as a Diameter, assume the Chord $CG = \frac{2}{3}CE$; and draw the refracted Ray MG, in which the Point M may be determined according to the general Construction (*Art.* 133.) Fig. 116.

We shall have * $AH \left(\frac{bm}{m-n}\right) = -2b$, that is, AH will be * next to the convex Part of the Quadrant AMD, and the Double of the Radius AC. And if we suppose CG or $\frac{2}{3}CE = CM$, it is manifest that the refracted Ray MF will touch the Circle AMD in M, because then the Point G will coincide with the Point M. Whence if you take $CE = \frac{2}{3}CD$, the Point M will coincide with the Point N, wherein the Caustick HFN touches * the Circle AMD: but when CE is greater than $\frac{2}{3}CD$, the Rays of Incidence (BM) can be no more refracted, that is, pass out of Glass into Air; because it is impossible for CG, perpendicular to the refracted Ray MG, to be greater than CM: so that all the Rays that fall on the Part DN will be reflected.

* *Art.* 137.

* *Art.* 136.

* *Art.* 137.

If AP be drawn parallel to CD, it is manifest * that the Part $FH = AH - MF + \frac{2}{3}PM$: so that drawing NK parallel to CD, the whole Caustick $HFN = 2\,CA + \frac{2}{3}AK = \frac{7 - \sqrt{5}}{2}\,CA$.

* *Art.* 137.

EXAMPLE II.

FIG. 117. 140. LET the Curve *AMD* be a *Logarithmical Spiral*, the Centre thereof being the Point *A*, from which let all the incident Rays (*AM*) issue.

*Art. 91. It is manifest * that the Point *E* coincides with *A*, *viz.* $a=y$. Therefore if you substitute y for a in $\frac{bbmy}{bmy-any+aan}$ the Value * of *MF* when the Concavity of the Curve is next the luminous Point; we shall have $MF=b$: Whence the Point *F* coincides with *G*.

*Art. 113.

If you draw the right Line *AG*, and the Tangent *MT*, the Angle *AGO*, the Complement of the Angle *AGM* to two right Angles, will be equal to the Angle *AMT*. For since the Circle, whereof *CM* is the Diameter, passes thro' the Points *A* and *G*, the half of the Arch *AM* is the Measure of each of the Angles *AGO*, *AMT*. Therefore it is evident that the Caustick *AGN* is a Logarithmical Spiral, the same as the given one, differing from it only in Position.

PROP. II.

FIG. 118. 141. *THE Caustick* HF, *the luminous Point* B, *and the Ratio of* m *to* n, *being given: To find any Number of Curves (as* AM) *whereof it may be the Caustick by Refraction.*

Take the Point *A* at pleasure in any Tangent for one Point in the Curve *AM*, and from the Centre *B*, with the Interval *BA*, describe the Arch *AP*, and another Arch with any other Interval.

terval. Then take $AE = \frac{n}{m} PM$, and with the Caustick HF as an Evolute, describe the Involute EM, which will intersect the Arch described with the Distance BM in the Point M; and this will be in the Curve sought: For * $PM : AE$ or $ML :: m : n$. *Art. 132.

Otherwise.

142. IN any Tangent FM, excepting HA, find the Point M with this Condition, that $HF + FM + \frac{n}{m} BM = HA + \frac{n}{m} BA$. And then if you take $FK = \frac{n}{m} BA + AH - FH$, and find the Point M such, in the Line FK, that $MK = \frac{n}{m} BM$; the said Point will be * that sought. Now this may be effected *Art. 132. in describing a Curve GM of such a Nature, that the right Lines MB, MK, drawn from FIG. 119. any Point in it to the given Points B and K, may be to each other always in the constant Ratio of m to n. Now this Curve may be thus found.

Draw MR perpendicular to BK, and call the given Quantity BK, a; and the indeterminate Quantities BR, x; RM, y. Then because of the right-angled Triangles BRM, KRM, we shall have $BM = \sqrt{xx+yy}$, and $KM = \sqrt{aa - 2ax + xx + yy}$. So that to fulfil the Conditions of the Problem $\sqrt{xx+yy} : \sqrt{aa - 2ax + xx + yy} :: m : n$. Whence $yy = \frac{2ammx - aamm}{mm - nn} - xx$, which is a *Locus ad Circulum*;

culum; and may be thus described.

Assume $BG = \frac{am}{m+n}$, and $BQ = \frac{am}{m-n}$, and with GQ as a Diameter, describe the Semicircle GMQ. I say, this will be the *Locus* sought. For since QR or $BQ - BR = \frac{am}{m-n} - x$, and RG or $BR - BG = x - \frac{am}{m+n}$; from the Nature of the Circle, where $QR \times RG = \overline{RM}^2$, we have $yy = \frac{2ammx - aamm}{mm - nn} - xx$.

FIG. 120. If the Rays of Incidence BA, BM, be parallel to a right Line given in Position, the former Solution will still take place: But the latter one is of no Effect, and instead of it we may use the following one.

Assume $FL = AH - HF$, draw LG parallel to AB, and perpendicular to AP; assume $LO = \frac{n}{m} LG$, and draw LP parallel to GO, and PM parallel to GL. Then it is manifest

*Art. 132. * that the Point M will be that sought. For since $LO = \frac{n}{m} LG$, $ML = \frac{n}{m} PM$.

If the Caustick by Refraction FH becomes a Point, the Curves (AM) will then be the famous OVALES of *Descartes*.

COROL. I.

143. AFTER the same manner as in Causticks by Reflexion, we demonstrate * that the Curves AM are of a Nature very different from one another; and that they are not geometrick Curves but when the Caustick by Re-

*Art. 130.

fraction

fraction HF is a geometrical Curve, and rectifiable also.

COROL. II.

144. A CURVE AM, the luminous Point B, and the Ratio of m to n being given: To find any Number of Lines DN of such a Nature, that they may again refract the refracted Rays MN, so as to unite them at length in a given Point C. FIG. 121.

If we imagine the Curve HF to be the Caustick by Refraction of the given Curve AM, formed by means of the luminous Point B; it is manifest that the said Line HF must likewise be the Caustick by Refraction of the Curve DN sought, having the given luminous Point C. Therefore* $\frac{n}{m}BA + AH = \frac{n}{m}BM + MF + FH$, and $NF + FH - \frac{n}{m}NC = HD - \frac{n}{m}DC$; and therefore $\frac{n}{m}BA + AH = \frac{n}{m}BM + MN + HD - \frac{n}{m}DC + \frac{n}{m}NC$. And by the usual Transposition $\frac{n}{m}BA - \frac{n}{m}BM + \frac{n}{m}DC + AD = MN + \frac{n}{m}NC$. Which gives the following Construction. *Art. 132.

First assume the Point D in any refracted Ray AH, as one Point in the sought Curve DN; then on any other refracted Ray MF, take the Part $MK = \frac{n}{m}BA - \frac{n}{m}BM + \frac{n}{m}$

DC

DC+*AD*; and having found the Point *N*
Art. 142. (as above) of ſuch Condition, that $NK = \frac{n}{m} NC$, it is evident * that the ſame ſhall be
*Art. 132. in the Curve *DN*.

A General Corollary

For the three laſt Sections.

*Art. 80, 85, 107, 108, 114, 115, 128, 129, 134, 143.

145. It is manifeſt * that a Curve can have but one Evolute, one Cauſtick by Reflection, and one Cauſtick by Refraction, when the luminous Point, the Ratio of the Sines of the Angles of Incidence and Refraction is given; which Lines are always geometrical and rectifiable, when that Curve is geometrical. Whereas the ſame Curve may be an Evolute or a Cauſtick by Reflection or Refraction (in a given Ratio of the Sines, and Poſition of the luminous Point) to an infinite Number of Lines of different Natures, which will not be geometrical, but when that Curve is geometrical, and withal rectifiable.

SECT.

SECT. VIII.

The Use of Fluxions in finding the Points of Curves touching an infinite Number of Curves, or Right Lines given in Position.

PROP. I.

146. *LET* AMB *be any given Curve, whereof the right Line* AP *is the Axis; and let us conceive an infinite Number of Parabola's* AMC, AmC, *all to pass thro' the Point* A, *and to have the Ordinates* PM, pm, *as Axes. It is required to find the Curve touching all the said Parabola's.* Fig. 122.

It is manifest that the Point of Contact of every Parabola *AMC*, is *C* the Point wherein the Parabola *AmC*, which is infinitely near it, intersects the same. This being supposed, draw *CK* parallel to *MP*; call the given Quantities *AP*, x; *PM*, y; and the unknown Quantities *AK*, u; *KC*, z. Then from the Nature of the Parabola $\overline{AP}^2$ (xx) : $\overline{PK}^2$ $(uu - 2ux + xx)$ :: MP (y) : $MP - CK$ $(y - z)$. Whence $zxx = 2uxy - uuy$; which is an Equation common to all the Parabola's as *AMC*. Here I observe that the unknown Quantities do not vary, while the given ones *AP* (x), *PM* (y) do, *viz.* become *Ap* and *pm*; and *KC* (z) is never

ver invariable, but when the Point C is the Intersection aforesaid: For it is manifest that every where else the right Line KC intersects the Parabola's AMC, AmC, in two different Points, and consequently it will have two different Values correspondent to the same Value of AK. Therefore if the Equation above found be thrown into Fluxions, with u and z, taken as constant Quantities, the Point C will be determined to be that of the Intersection aforesaid. Whence $2zx\dot{x} = 2ux\dot{y} + 2uy\dot{x} - uu\dot{y}$: And so $AK\,(u) = \frac{2xx\dot{y} - 2yx\dot{x}}{x\dot{y} - 2y\dot{x}}$, by substituting $\frac{2ux\dot{y} - uu\dot{y}}{xx}$ for z; and the Nature of the Curve AMB being given, we may find a Value of $\dot{y}$ in $\dot{x}$; which being put in the Value of AK, the said unknown Quantity will at length be expressed in known Terms freed from Fluxions. Which was proposed to be done.

If other Curves or right Lines of a determinate Position, be proposed instead of the Parabola's (AMC), the Solution of the Problem will be much the same, as will appear in the following Example.

EXAMPLE.

147. LET $xx = 4ay - 4yy$ express the Nature of the Curve AMB: This will be half an Ellipsis, whose conjugate Axis is AB equal to a, and perpendicular to AP, and transverse Axis the Double of the Conjugate.

Now $x\dot{x} = 2a\dot{y} - 4y\dot{y}$; and therefore AK $\left(\frac{2xx\dot{y} - 2xy\dot{x}}{x\dot{y} - 2y\dot{x}}\right) = \frac{ax}{y} = u$. Whence if AK

be

be assumed a fourth Proportional to *MP*, *PA*, *AB*, and *KC* be drawn perpendicular to *AK*; it shall intersect the Parabola *AMC* in the Point *C* sought. Again; to find the Nature of the Curve touching all the Parabola's, or which passes thro' all the Points *C* thus found, we must find an Equation expressing the Relation of *AK* (u) to *KC* (z) after this manner. Substitute $\frac{ax}{y}$ for u its Equal in $zxx = 2uxy - auy$, and we get $y = \frac{aa}{2a-z}$; and therefore x or $\frac{uy}{a}$ $= \frac{au}{2a-z}$. Now if these Values be put for x and y in $xx = 4ay - 4yy$, there will arise $uu = 4aa - 4az$, wherein x and y are got out, and which expresses the Relation of *AK* to *KC*. Therefore it is manifest, that the Curve sought is a Parabola, whereof the Line *BA* is the Axis, the Point *B* the Vertex, the Point *A* the Focus, and consequently the Parameter is four times *AB*.

We have found $y = \frac{aa}{2a-z}$, and so we get *KC* $(z) = \frac{2ay-aa}{y}$. But since this Expression is positive when $2y$ is greater than a, negative when it is less, and nothing when it is equal; therefore in the first Case the Point of Contact *C* falls above *AP*, as it was supposed to do in the Investigation; in the second, it falls below; and in the last Case it falls in *AP*.

If the right Line *AC* be drawn to intersect *MP* in *G*; I say *MG* = *BQ*, and the Point *G* is the Focus of the Parabola *AMG*. For, 1°, *AK* $\left(\frac{ax}{y}\right)$: *KC* $\left(\frac{2ay-aa}{y}\right)$ :: *AP* (x) :

PG

$PG = 2y - a$. And therefore $MG = a - y =$ BQ. 2°, The Parameter of the Parabola AMC, is $4a - 4y$, putting $4ay - 4yy$ for xx; and therefore $MG = a - y$ is the fourth Part of the Parameter. Whence it is evident, that the Point G is the Focus of the Parabola; and consequently the Angle BAC must be bisected by a Tangent to the Curve in A.

Hence the Parameter of the Parabola AMC is four times BQ; and when the Vertex M falls in A, the Parameter will be four times AB; consequeutly the Parabola, whereof the Point A is the Vertex, is Asymtotick to that passing thro' all the Points C.

Because the Parabola BC touches all the Parabola's (as AMC), it is manifest that all the said Parabola's will cut the determinate Line AC in Points, which will all be nearer to A than the Point C. Now it is shewn in the Doctrine of Projecties, that (AK being a Horizontal Line) all the Parabola's, as AMC, will be the Paths of Bombs thrown out of a Mortar with a given Force placed in A, having all possible Elevations. And consequently if a right Line be drawn bisecting the Angle BAC, the same will be the Position of the Axis of the Mortar; so that a Bomb thrown out of it, will fall on the Plane AC given in Position to a Point C, which will be farther from the Mortar than when it has any other Elevation.

PROP. II.

FIG. 123. 148. *LET any Curve* AM *be given, whereof* AP *is its Axis: To find another Curve* BC *of such a Nature, that if any Ordinate* PM *be*

be drawn, and the Perpendicular P C *to the sought Curve; these Lines* P M, P C, *may be always equal to one another.*

If an infinite Number of circular Arches be supposed to be described from the Centres P, p, with the Radii PC, pC, equal to PM, pm. It is evident that the Curve BC sought, must touch all the said Circles; and that C, the Point of Contact of every Circle, is the Point wherein the Circle infinitely near it cuts it. This being premised, draw CK perpendicular to AP, and call the given and variable Quantities AP, x; PM or PC, y; and the unknown and constant Quantities AK, u; KC, z. Then from the Nature of the Circle $\overline{PC}^2 = \overline{PK}^2 + \overline{KC}^2$ viz. $yy = xx - 2ux + uu + zz$, which is the Equation common to all the said Circles. This thrown into Fluxions, will be $2y\dot{y} = 2x\dot{x} - 2u\dot{x}$; and so we get $PK\ (x-u) = \frac{y\dot{y}}{\dot{x}}$. From whence comes the following general Construction.

Draw MQ perpendicular to the Curve AM, take $PK = PQ$, and draw KC parallel to PM. I say, this will meet the Circle described from the Centre P, with a Radius $PC = PM$ in the Point C, wherein it touches the Curve CB sought. This is evident, because $PQ = \frac{y\dot{y}}{\dot{x}}$.

The Value of PK may be found otherwise thus:

Draw PO perpendicular to Cp; then the right-angled Triangles pOP, PKC, will be

 similar.

similar. Therefore $Pp(\dot{x}) : Op(\dot{y}) :: PC(y) : PK = \frac{y\dot{y}}{\dot{x}}$.

When $PQ = PM$, it is manifest that the Circle described with the Radius PC, will touch KC in the Point K; so that the Point C will then coincide with K, and consequently will fall in the Axis.

But when PQ is greater than PM, the Circle described with the Radius PC, cannot touch the Curve BC; because it cannot at all intersect the right Line KC.

Example.

Fig. 123. 149. Let the given Curve AM be a Parabola, the Equation whereof is $ax = yy$. Now PQ or PK $(x-u) = \frac{1}{2}a$; and consequently $x = \frac{1}{2}a + u$, and $yy = \frac{1}{4}aa + zz$, because of the right-angled Triangle PKC. If these Values be put in $ax = yy$, we shall have $\frac{1}{2}aa + au = \frac{1}{4}aa + zz$, or $\frac{1}{4}aa + au = zz$, expressing the Nature of the Curve BC. Whence we may perceive that the said Curve is also a common Parabola as well as AM, because they have both the same Parameter a, and the Vertex B is distant from the Vertex A, by the Distance $BA = \frac{1}{4}a$.

Prop. III.

Fig. 124. 150. *Let* AM *be any given Curve, the right Line* AP *being a Diameter, and the Ordinates* PM, pm, *parallel to a right Line* AQ *given in Position; and having drawn* MQ, mq, *parallel to* AP, *then draw the right Lines* PQC, pqC. *It is required to find the Curve* AC

AC of such a Nature, that all these last mentioned Lines may be Tangents too; or, which is the same thing, to find the Point of Contact C *in every right Line* PQC.

Conceive another Tangent pqC to be infinitely near PQC, and draw CK parallel to AQ; then call the given and variable Quantities AP, x; PM or AQ, y; the unknown and variable Quantities AK, u; KC, z. Now because of the similar Triangles PAQ, PKC, $AP\ (x) : AQ\ (y) :: PK\ (x+u) : KC\ (z) = y + \frac{uy}{x}$. Which is an Equation common to all the right Lines as KC. The Fluxion thereof is $\dot{y} + \frac{ux\dot{y} - uy\dot{x}}{xx} = 0$. From whence arises $AK\ (u) = \frac{xx\dot{y}}{y\dot{x} - x\dot{y}}$; and so the following general Construction.

Draw the Tangent MT, in which assume AK a third Proportional to AT, AP: Then if KC be drawn parallel to AQ, it shall cut the right Line PQC in the Point C sought.

For $AT\left(\frac{y\dot{x} - x\dot{y}}{\dot{y}}\right) : AP\ (x) :: AP\ (x) : AK = \frac{xx\dot{y}}{y\dot{x} - x\dot{y}}$.

EXAMPLE I.

151. LET the given Curve AM be a Parabola; so that the Equation thereof will be $ax = yy$. Then $AT = AP$; whence $AK\ (u) = x$; that is, the Point K falls in T. Now to have an Equation expressing the Relation of $AK\ (u)$ to $KC\ (z)$; we shall have $KC\ (z) = 2y$, because PK is the Double of AP. Fig. 124.

Now putting u and $\frac{1}{2}z$ for their Equals x and y in $ax = yy$, and then will $4au = zz$: Whence we may perceive that the Curve AC is a Parabola, whose Vertex is A, and Parameter a Line equal to four times the Parameter of the Parabola AM.

Example II.

Fig. 125. 152. Let the given Curve AM be a Quadrant BMD, whose Centre is the Point A, and Semidiameter the Line AB or AD, which I call a. It is manifest that PQ is always equal to AM or AB, *viz.* is an invariable Quantity. So that the Ends P, Q may be supposed to slide or move along the Sides BA, AD of the right Angle BAD. Now AK (u) will be $= \frac{x^3}{aa}$, because $AT = \frac{aa}{x}$; and on account of the Parallels KC, AQ; $AP\,(x) : PQ\,(a) :: AK\left(\frac{x^3}{aa}\right) : QC = \frac{xx}{a}$. Whence we may perceive that to determine the Point of Contact C; we need only assume QC a third Proportional to PQ and AP. If you seek the Equation of the Curve BCD, the same will be found this here,

$$\begin{aligned} u^6 - 3aau^4 &+ 3a^4uu - a^6 = 0. \\ +3zz &+ 21aazz + 3a^4zz \\ &+ \quad 3z^4 - 3aaz^4 \\ &\qquad + \quad z^6 \end{aligned}$$

Corol. I.

153. If the Relation of DC the Part of the Curve BCD to its Tangent CP be required, you must first conceive another Tangent cp infinitely near CP; then describing the

the little Arch PO from the Centre C, and $cp - CP$ or $Op - Cc$ will be $= \frac{2x\dot{x}}{a}$, which is the Fluxion of $CP = \frac{aa - xx}{a}$: whence there comes out $Cc = Op + \frac{2x\dot{x}}{a}$. Now because of the right-angled similar Triangles QPA, PpO, $PQ(a) : AP(x) :: Pp(\dot{x}) : Op = \frac{x\dot{x}}{a}$. And therefore $Cc = \frac{3x\dot{x}}{a} = DC - Dc$. Whence it is evident, that where-ever the Point C be assumed, this Proportion will always be had, *viz.* $DC - Dc \left(\frac{3x\dot{x}}{a}\right) : CP - cp \left(\frac{2x\dot{x}}{a}\right) :: 3 : 2$. And consequently the Sum of all the Fluxions $DC - Dc$ corresponding to the same right Line PD, that is, * the Part DC of the Curve (BCD) is to the Sum of all the Fluxions $CP - cp$ answerable to the same right Line PD, *viz.* * to the Tangent $CP :: 3 : 2$. And also the whole Curve BCD is to its Tangent $BA :: 3 : 2$.

* *Art.* 96.

* *Art.* 96.

COROL. II.

154. IF the Curve BCD be taken as an Evolute, the Involute DNF formed from it beginning at D, will be of such a Nature, that $CN : CP :: 3 : 2$. Because CN is always equal to DC the Part of the Curve BCD. Whence it follows, that the similar Sectors CNn, CPO, are to one another $:: 9 : 4$. And therefore the Space DCN comprehended under the Curves DC, DN, and the right Line CN, which is

a Tangent in C, and perpendicular at N, is to the Space DCP, contained under the Curve DC and the Tangents DP, CP, as 9 to 4.

Corol. III.

155. The Centre of Gravity of the Sector CNn must be in the Arch PO; because $CP = \frac{2}{3} CN$. And since the said Arch is infinitely small, it follows that the Centre of Gravity must be in the right Line AD; and therefore the Centre of Gravity of the Spaces DCN, BDF, made up of all those Sectors, must be in the right Line AD. Consequently if a Figure be described on the other side of BDF equal and similar to BDF, the Centre of Gravity of the entire Figure shall be in the Point A.

Corol. IV.

156. Because of the right-angled similar Triangles PQA, pPO, PQ (a) : AQ or PM $(\sqrt{aa-xx})$: : Pp $(\dot{x})$: $PO = \frac{\dot{x}\sqrt{aa-xx}}{a}$. And because of the similar Sectors CPO, CNn, $CP : CN$ or $2 : 3$: : PO $\left(\frac{\dot{x}\sqrt{aa-xx}}{a}\right)$: $Nn = \frac{3\dot{x}\sqrt{aa-xx}}{2a}$. Now the Rectangle $MP \times Pp$, *viz.* * the little circular Space $MPpm = \dot{x}\sqrt{aa-xx}$. Whence $AB \times Nn = \frac{3}{2} MPpm$: and consequently the Part ND of the Curve DNF drawn into the Radius AB, is the Sesquialter of the circular Segment DMP, and the whole Curve DNF is equal to $\frac{3}{2}$ of the circular Quadrant BMD.

* Art. 2.

Prop.

PROP. IV.

157. *LET* AM *be any given Curve, whereof the right Line* AP *is the Axis; and let there be an infinite Number of Perpendiculars* MC m C *drawn to the same. It is required to find that Curve which all these Perpendiculars are Tangents to; or, which is the same thing, to find the Point of Contact* C *in every Perpendicular* MC. FIG. 126.

First imagine another Perpendicular mC infinitely near to MC; let MP be an Ordinate, and through the Point of Interfection C draw the right Lines CK perpendicular, and CE parallel to the Axis: then call the given and variable Quantities AP, x; PM, y; and the unknown and invariable ones AK, u; KC, z. This done, PQ will be $= \frac{y\dot{y}}{\dot{x}}$, PK or $CE = u - x$, $ME = y + z$; and because of the right-angled similar Triangles MPQ, MEC, MP $(y) : PQ \left(\frac{y\dot{y}}{\dot{x}}\right) :: ME\ (y+z) : EC\ (u-x)$ $= \frac{y\dot{y} + z\dot{y}}{\dot{x}}$. Which is an Equation common to all the Perpendiculars as MC, and the Fluxion thereof (supposing $\dot{x}$ invariable) will be $-\dot{x} = \frac{y\ddot{y} + \dot{y}^2 + z\ddot{y}}{\dot{x}}$: from whence comes out $ME\ (z+y) = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$. Now the Nature of the Curve AM being given, we shall have Values of $\dot{y}^2$ and $\ddot{y}$ in $\dot{x}^2$, which being put in $\frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}}$, will give a known Value of ME

 freed

freed from Fluxions; which is what was proposed.

It is manifest that the Curve passing thro' all the Points C, is the Evolute of the Curve AM: and because in the fifth Section these Curves are fully handled, it is needless to give again new Examples of them here.

Prop. V.

Fig. 127. 158. *ANY two Lines* AM, BN *being given, together with a right Line* MN *continuing always of the same Length. Now if the Ends* M, N *of this Line continually move along the two former Lines, it is required to find the Curve that it always touches.*

First draw the Tangents MT, NT, and conceive another right Line mn infinitely near MN, and which by consequence cuts it in the Point C, wherein the same touches the Curve in which we are now determining Points. Now it is plain, that while the right Line MN is moving to the Situation mn, the Ends thereof will describe, or continually be in the small Parts Mm, Nn of AM, BN, which on account of their being infinitely small, are common to the Tangents TM, TN: So that while the right Line MN is moving to the infinitely-near Situation mn, the Ends thereof may be concived as moving along the right Lines TM, TN given in Position.

This being well understood, draw MP, CK perpendicular to NT, and call the given and variable Quantities TP, x; PM, y; the unknown and constant Quantities TK, u; KC, z; and the stable Line MN, a. Now because of of the right-angled Triangle MPN, $PN = \sqrt{aa-yy}$; and becauſe of the ſimilar Triangles NPM, NKC, NP $(\sqrt{aa-yy}) : PM (y) :: NK$ $(u - x - \sqrt{aa-yy}) : KC$ $(z) = \frac{uy - xy}{\sqrt{aa-yy}} - y$. And this thrown into Fluxions will be $aau\dot{y} - aax\dot{y} - aay\dot{x} + y^3\dot{x} = aa\dot{y} - yyy\dot{y}\sqrt{aa-xx}$: and making $\sqrt{aa-yy} = m$ for brevity's ſake, there comes out PK $(u-x) = \frac{m^3\dot{y} + mmy\dot{x}}{aa\dot{y}} = \frac{m^3 + mmx}{aa}$, by ſubſtituting $x\dot{y}$ for its Equal $y\dot{x}$, becauſe of the ſimilar Triangles mRM, MPT; and therefore $MC = \frac{mm+mx}{a}$: from whence comes the following Conſtruction.

Draw TE perpendicular to MN, and aſſume $MC = NE$: I ſay, the Point C will be that ſought. For ſince the right-angled Triangles MNP, TNE are ſimilar, MN $(a) : NP$ $(m) :: NT$ $(m+x) : NE$ or $MC = \frac{mm+mx}{a}$.

Otherwiſe. Draw TE perpendicular to MN, and deſcribe the ſmall Arches MS, NO, from the Centre C, call the given Quantities NE, r; ET, s; MN, a; and the unknown Quantity CM, t. Then will Sm or $On = \dot{t}$; and becauſe of the right-angled ſimilar Triangles MET and mSM, NET and nON, CMS and CNO, $ME (r-a) : ET (s) :: mS (\dot{t}) : SM = \frac{s\dot{t}}{r-a}$. And NE $(r) : ET$ $(s) :: nO$ $(\dot{t}) : ON = \frac{s\dot{t}}{r}$. And $MS - NO$ $\left(\frac{as\dot{t}}{rr-ra}\right) : MS$ $\left(\frac{s\dot{t}}{r-a}\right) :: MN (a) : MC (t) = r$. From whence ariſes the ſame Conſtruction as above. If

If AM, BN be supposed right Lines at right Angles to each other; it is manifest that the Curve sought is the same as that of *Article* 152.

Prop. VI.

FIG. 128. 159. *LET* L, M, N *be any three given Lines, and from every Point* L, l *in the Curve* L, *let two Tangents* LM *and* LN, lm *and* ln *be conceived to issue, to the Curves* M *and* N, *one to each. It is required to find a fourth Curve* C, *to which all the right Lines* MN, mn *joining the Points of Contact of the Curves* M, N *may be Tangents.*

Draw the Tangent LE, and thro' any Point E therein draw EF, EG perpendicular to the two other Tangents ML, NL, let the Point l be supposed infinitely near L, draw the little right Lines LH, LK perpendicular to ml, nl; as likewise MP, mP, NQ, nQ perpendicular to the Tangents ML, ml, NL, nl, which will intersect each other in the Points P and Q. Then the right-angled Triangles EFL and LHl, EGL and LKl will be similar; as likewise the Triangles LMH and MPm, LnK and NQn right-angled at H and m, K and N; because each of the Angles LMH, MPm added to the Angle PMm, makes a right Angle. And in like manner we prove that the Angles LnK, NQn are equal to one another.

This being premised, call the little side Mm of the Polygon, the Curve M is conceived to be, u; and the given Quantities EF, m; EG, n; MN or mn, a; ML or ml, b;

NL

NL or nl, c; MP or mP, f; NQ or nQ, g; (I here take the right Lines MP, NQ for given ones, because the Nature of the Curves M, N being given by Supposition, they may always * be found); then we shall have, 1°, MP (f) : ML (b) :: Mm ($\dot{u}$) : $LH = \frac{b\dot{u}}{f}$. 2°, EF (m) : EG (n) :: $LH \left(\frac{b\dot{u}}{f}\right) : LK = \frac{bn\dot{u}}{mf}$. 3°, LN or Ln (c) : nQ (g) :: $LK \left(\frac{bn\dot{u}}{mf}\right) : nN = \frac{bgn\dot{u}}{cfm}$. 4°, (Drawing MR parallel to NL or nl) ml (b) : ln (c) :: mM ($\dot{u}$) : $MR = \frac{c\dot{u}}{b}$. 5°, $MR + Nn \left(\frac{c\dot{u}}{b} + \frac{bgn\dot{u}}{cfm}\right) : MR \left(\frac{c\dot{u}}{b}\right) :: MN\ (a) : MC = \frac{accfm}{ccfm + bbgn}$, which was to be found.

* Art. 78.

When the Tangent EL coincides with the Tangent ML, it is manifest that EF (m) will become nothing; and therefore the Point sought C will coincide with M. Likewise when the Tangent EL coincides with the Tangent LN; then EG (n) will become nothing; and consequently $MC = a$. Whence it is evident, that the Point C sought will coincide also with N. Lastly, if the Tangent EL falls in the Angle GLI; in this Case EG (n) will be negative: so that then $MC = \frac{accfm}{ccfm - bbgn}$; and the Point sought C will fall without the Points M and N.

EXAM-

EXAMPLE I.

FIG. 129. 160. SUPPOSE the Curves M, N, to be Parts of the same Circle; then it is plain, that in this Case $b=c$, and $f=g$: So that $MC = \frac{am}{m+n}$; whence it follows, that the Point C sought, is determined by dividing the right Line MN into two Parts, being to one another in the given Ratio of m to n, viz. so that $MC : NC :: m : n$:

EXAMPLE II.

161. SUPPOSE the Curves M and N be any Conick Section. Here the general Construction will be changed into a far more simple one, from the Consideration of a Property of the Conick Sections demonstrated in Treatises of those Curves, *viz.* That *if from every Point* L, l, *of a right Line* EL, *be drawn two Tangents* LM *and* LN, lm, ln, *to any Conick Section, all the Right Lines* MN, mn, *joining the Points of Contact, will intersect one another in one and the same Point* C, *thro' which the Diameter* AC, *to Ordinates that are parallel to the Right Line* EL *passes.* For it follows from thence, that the Point C is determined by only drawing a Diameter whose Ordinates are parallel to the Tangent EL.

In the Circle it is manifest that the Diameter must be perpendicular to the Tangent EL; that is, a Perpendicular AB drawn from the Centre A to the Tangent, will intersect the right Line MN in the Point C sought.

SCHO-

SCHOLIUM.

162. THE Solution of the following Problem depending on the Method of Tangents, will be had by means of the aforesaid Problem. FIG. 128.

The three Curves *C*, *M*, *N*, being given, and a right Line *MN* being continually moved about the Curve C, so as to always touch it: And if from the Points *M*, *N*, wherein it cuts the Curves *M* and *N*, be drawn the Tangents *ML*, *NL*, intersecting each other in the Point *L*, which by the Motion describes a fourth Curve *Ll*. It is required to draw *LE* the Tangent to this Curve, the right Lines *MN*, *ML*, together with the Point of Contact *C* being given.

For it is manifest that this Problem is but the Inverse of the aforegoing one, and here *MC* is given: So that we are to find the Ratio of *EF* to *EG*, which determines the Position of the Tangent *EL*. Therefore if you call the given Quantity *MC*, *b*, we shall have $\frac{accfm}{ccfm+bbgn}=b$. Whence comes out $m=\frac{bbgbn}{accf-ccfb}$; and consequently the Tangent *LE* must be so situate in the given Angle *MLG*, that if from any Point *E* in it, be drawn *EF*, *EG*, perpendicular to the Sides of that Angle, they may be always to each other in a given Ratio, *viz.* of *bbgb* to *accf*—*ccfb*. Now this is done by drawing *MD* parallel to *NL* and $=\left(\frac{bbgb}{accf-ccfb}\right)$.

When

FIG. 129. When the Curves *M* and *N*, are Parts of the
*Art. 161. same Conick Section, it is * manifest that then you need but draw the Tangent *LE* parallel to the Ordinates to the Diameter passing thro' the Point C.

SECT.

SECT. IX.

The Solution of some Problems depending on the Methods aforegoing.

PROP. I.

163. *LET* AMD *be a Curve* (AP$=x$, PM $=y$, AB$=a$) *of such a Nature, that the Value of the Ordinate* y *is expressed by a Fraction, the Numerator and Denominator of which, do each of them become* o *when* $x=a$, viz. *when the Point* P *coincides with the given Point* B. *It is required to find what will then be the Value of the Ordinate* BD. FIG. 130.

Let *ANB*, *COB*, be two Curves (having the Line *AB* as a common Axis) of such a Nature, that the Ordinate *PN* expresses the Numerator, and the Ordinate *PO* the Denominator of the general Fraction representing any Ordinate *PM*: So that $PM=\frac{AB\times PN}{PO}$. Then it is manifest, that these two Curves will meet one another in the Point *B*; since by the Supposition *PN*, *PO*, do each become o when the Point *P* falls in *B*. This being supposed, if an Ordinate *bd* be imagined infinitely near to *BD*, cutting the Curves *ANB*, *COB*,

COB, in the Points f, g; then will $bd = \frac{AB \times bf}{bg}$, which will be* equal to BD. Now
* Art. 2.
our Business is only to find the Relation of bg to bf. In order thereto it is manifest, when the Abscis AP becomes AB, the Ordinates PN, PO, will be o, and when AP becomes Ab, they do become bf, bg. Whence it follows, that the said Ordinates bf, bg, themselves, are the Fluxions of the Ordinates in B and b, with regard to the Curves ANB, COB; and consequently, if the Fluxion of the Numerator be found, and that be divided by the Fluxion of the Denominator, after having made $x = a = Ab$ or AB, we shall have the Value of the Ordinates bd or BD sought. Which was to be found.

EXAMPLE I.

164. LET $y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$. Now it is manifest when $x = a$, that the Numerator and Denominator of the Fraction will each be equal to o. Therefore we must assume the Fluxion of the Numerator $\frac{a^3\dot{x} - 2x^3\dot{x}}{\sqrt{2a^3x - x^4}} - \frac{aa\dot{x}}{3\sqrt[3]{aax}}$, and divide it by the Fluxion of the Denominator $-\frac{3a\dot{x}}{4\sqrt[4]{a^3x}}$, after having made $x = a$, *viz.* divide $-\frac{4}{3}a\dot{x}$ by $-\frac{3}{4}\dot{x}$; and there comes out $\frac{16}{9}a = BD$.

EXAMPLE II.

165. LET $y=\frac{aa-ax}{a-\sqrt{ax}}$. Then when $x=a$, y will be $=2a$.

This Example may be solved without Fluxions thus.

Taking away the Surds, and then will $aaxx+2aaxy-axyy-2a^3x+a^4+aayy-2a^3y=0$, which being divided by $x-a$, will be brought down to $aax-a^3+2aay-ayy=0$; and substituting a for x, we have as before $y=2a$.

LEMMA.

166. *LET* BCG *be any Curve, which the right Line* AE *touches in the Point* B, *in which take two Points (having an invariable Position)* A, E. *Now if this right Line moves about the Curve so as to touch it continually, it is evident that the stable Points* A, E, *by the said Motion will describe two Curves* AMD, ENH. *Then if* DL *be drawn parallel to* AB, *and which consequently makes with* DK *(in which the right Line* AE *is supposed to be when it touches the Curve* BCG *in* G) *the Angle* KDL *equal to the Angle* AOD *formed by the Tangents in* B *and* D; *and from the Centre* D *be described any Arch* KFL. FIG. 131.

I say DK : KFL :: AE : AMD $\pm$ ENH, *viz.* + *when the Point of Contact falls between the describent Points, and* — *when it does not.*

For suppose the right Line AE in its Motion about the Curve BCG to be come to the Situations or Positions MCN, mCn infinitely

 near

near each other, and draw the Radii *DF*, *Df*, parallel to *GM*, *Cm*. Then it is manifest that the Sectors *DFf*, *CMm*, *CNn*, are similar; and so $DF : Ff :: CM : Mm :: CN : Nn :: CM + CN$ or $AE : Mm + Nn$. Now since this is always so, let the Point of Contact *C* be any where taken, it follows that the Radius *DK* is to the Arch *KFL* the Sum of all the small Arches $Ff : AE :: AMD + ENH$ the Sum of all the little Arches $Mm + Nn$. Which was to be demonstrated.

Corol. I.

167. It is plain that the Curves *AMD*, *ENH*, are formed by the Evolution of the Curve *BCG*; and so the right Line *AE* is always perpendicular to those two Curves in all the various Positions of it; so that their Distance is every where the same; which is the Nature of parallel Lines. Whence it appears that a Curve Line *AMD* being given, we can find an infinite Number of Points in the Curve *ENH* without the use of the Evolute *BCG*, by drawing an infinite Number of Perpendiculars to that Curve, and taking them all equal to the right Line *AE*.

Corol. II.

168. If *BC*, *CG*, the Halves of the Curve *BCG*, be similar and equal, it is manifest that the Curves *AMD*, *ENH*, shall be similar and equal; so that they only differ in Position. Whence it follows, that the Curve *AMD* will be to the circular Arch $KFL :: \frac{1}{2} AE : DK$: That is, in a given Ratio.

Prop.

PROP. II.

169. LET *there be any two Curves* AEV, BCG, *together with a third Curve* AMD *of such a Nature, that a Part of a Curve* EM *being described from the Evolution of the Curve* BCG, *the Relation of the Portions or Parts* AE, EM, *to the Radii of Evolution* EC, MG, *be expressed by any given Equation. It is required to draw the Tangent* MT *from the given Point* M *in the Curve* AMD. FIG. 132.

Conceive another Part or Portion *em* of the Curve infinitely near to *EM*, and the Radii of Evolution *CeF*, *GmR*, to be drawn. Then, 1°, Let *CH* be perpendicular to *CE*, meeting *EH* the Tangent to the Curve *AEV* in *H*. 2°, *ML* parallel to *CE*, meeting the Arch *GL* described from the Centre *M* with the Radius *MG* in the Point *L*. 3°, *GC* be perpendicular to *MG* meeting the sought Tangent *MT* in *T*.

This being done, make $AE=x$, $EM=y$, $CE=u$, $GM=z$, $CH=s$, $EH=t$, the Arch $GL=r$. Then will $Ee=\dot{x}$, Fe or $Rm=\dot{u}=\dot{z}$; and because of the right-angled similar Triangles eFE and ECH, $CE\,(u):CH(s)::Fe\,(\dot{z}):FE=\frac{s\dot{z}}{u}$. And $CE\,(u):EH\,(t)::Fe\,(\dot{z}):Ee\,(\dot{x})=\frac{t\dot{z}}{u}$. Now by the *Lemma** $RF-me=\frac{r\dot{z}}{z}$; and therefore $RM\,(\overline{RF-me}+\overline{me-ME}+\overline{ME-MF})=\frac{r\dot{z}}{z}+\dot{y}+\frac{s\dot{z}}{u}$. Whence because of the similar Triangles *mRM*,

 MGT,

MGT, $mR\,(z) : RM\left(\frac{r\dot{z}}{z}+\frac{s\dot{z}}{u}+\dot{y}\right) ::$ $MG\,(z) : GT = r+\frac{sz}{u}+\frac{z\dot{y}}{\dot{z}}$; but if $\dot{z}$ and $\frac{t\dot{z}}{u}$ be put for their Equals $\dot{u}$ and $\dot{x}$ in the Fluxion of the given Equation, we shall get a Value of $\dot{y}$ in $\dot{z}$; which being substituted in $\frac{z\dot{y}}{\dot{z}}$, there will come out a known Value (freed from Fluxions) of the Subtangent GT sought. Which was to be done.

FIG. 133. If the Curve BCG degenerates into a Point O; it is manifest that the Portion of the Curve $ME\,(y)$ will then be an Arch of a Circle equal to the Arch $GL\,(r)$, and the Radii $CE\,(u)$, $GM\,(z)$ of Evolution will be equal to each other: So that GT, which in this Case becomes OT, will be $= y+s+\frac{z\dot{y}}{\dot{z}}$.

EXAMPLE.

FIG. 133 170. LET $y=\frac{xz}{a}$; this thrown into Fluxions will be $\dot{y}=\frac{z\dot{x}-x\dot{z}}{a}$ ($-x\dot{z}$ being taken negative *, because while x and y increase, z decreases) $=\frac{t\dot{z}-x\dot{z}}{a}$, by substituting $\frac{t\dot{z}}{z}$ for its Equal $\dot{x}$; and therefore $OT\left(y+s+\frac{z\dot{y}}{\dot{z}}\right)=y+s+\frac{tz-xz}{a}=\frac{as+tz}{a}$, putting y for its Equal $\frac{xz}{a}$.

* Art. 8.

SCHO-

SCHOLIUM.

171. If the Point O falls in the Axis AB, and the Curve AEV be a Semicircle, the Curve AMD will be half a Cycloid generated by the Rotation of a Semicircle BSN along an equal Arch BGN of a Circle described from the Centre O, the generating Point A falling without, within, or upon the Circumference of the moveable Circle BSN, according as the given Quantity a is greater, equal to, or less than OV. To prove this, and withal determine the Point B, FIG. 134.

Let us suppose the thing to be so, *viz.* that the Curve AMD is a Semi-cycloid, generated by the Rotation of the Semicircle BSN, (whose Centre is K the Centre of the Semicircle AEV) along the Arch BGN described from the Centre O; and conceiving the said Semicircle BSN to remain in such a Situation BGN, that the describing Point A falls in the Point M, draw the right Line OK thro' the Centres of the generating Circles; which, by consequence, will pass thro' the Point of Contact G; then drawing KSE, we may observe that the Triangles OKE, OKM, are equal and similar, because the three Sides of the one are each equal to the three Sides of the other. Whence it follows, 1°, That the extreme Angles MOK, EOK, are equal; and so likewise the Angles MOE, GOB. Whence $GB : ME :: OB : OE$. 2°, That the Angles MKO, EKO, are moreover equal: And consequently the Arches GN, BS, measuring them, are equal also. The same may be said of their Complements GB, SN, to two right Angles, because they appertain to equal Circles. Now by the Generation of the Cycloid, the Arch GB of the moveable Circle is equal to the Arch GB of the immoveable one. Wherefore $SN:ME::OB:OE$. This being premised,

Call the known Lines OV, b; KV or KA, c; and the unknown one KB, u. Then will $OB = b + c - u$; and because of the similar Sectors KEA, KSN, $KE\ (c) : KS\ (u) :: AE\ (x) : SN = \frac{ux}{c}$. And therefore $OB\ (b+c-u) : OE\ (z) :: SN\left(\frac{ux}{c}\right) : EM\ (y) = \frac{uxz}{bc+cc-cu} = \frac{xz}{a}$. Whence we get $KB\ (u) = \frac{bc+cc}{a+c}$. Wherefore if you assume $KB = \frac{bc+cc}{a+c}$, and from the Centres K and O do describe the Semicircle BSN and Arch BGN; it is evident that the Curve AMD will be half a Cycloid, described by the Rotation of the Semicircle BSN along the Arch BGN, the describing Point A falling without, within, or on the Circumference of the said Circle, according as $KV\ (c)$ is greater, less than, or equal to $KB \left(\frac{bc+cc}{a+c}\right)$, that is, according as a is greater, less than, or equal to $OV\ (b.)$

Corol. I.

172. It is manifest that $EM\ (y) : AE\ (x) :: KB \times OE\ (uz) : OB \times KV\ (bc+cc-uc)$. Now if OB be supposed to become infinite, the right Line OE will be so also, and parallel to OB, because it will never meet the same; the

the Concentrick Arches *BGN*, *EM* will become parallel right Lines, perpendicular to *OB*, *OE*; and the right Line *EM* will be to the Arch *AE*::*KB*:*KV*. Because the infinite right Lines *OE*, *OB*, which differ from each other by a finite Magnitude only, may be looked upon as equal.

COROL. II.

173. BECAUSE the Angles *MKO*, *EKO* are equal, the Triangles *MKG*, *EKB* shall be equal and similar; and so the right Lines *MG*, *EB* are equal to each other. Whence * if it be required to draw *MG* from a given Point *M* in the Curve of the Cycloid perpendicular to the same, you need only describe the Arch *ME* from the Centre *O*, and from the Centre *M* with the Distance *EB*, an Arch which will cut the Base *BGN* in the Point *G*, thro' which and the given Point *M* you may draw the Perpendicular requir'd.

* *Art.* 43.

COROL. III.

174. THE Point *G* being given in the Circumference of the moveable Semicircle *BGN*; and it be requir'd to find the Point *M* in the Cycloid wherein the describing Point *A* falls, when the given Point *G* touches the Base; you must assume the Arch *SN*=*BG*, and drawing the Radius *KS* meeting the Circumference *AEV* in *E*, describe the Arch *EM* from the Centre *O*. Then it is manifest that the said Arch shall cut the Cycloid in the Point *M* sought.

PROP. III.

FIG. 135, 136. 175. *LET* AMD *be a Semi-cycloid described by the Rotation of the Semi-circle* BGN *along an Arch* BGN *equal to it; so that the contiguous Parts* BG, BG, *as they still increase, are constantly equal to one another: And let the describing Point* M *be assumed in the Diameter* BN, *either without, within, or on the Circumference of the moveable Circle* BGN. *It is requir'd to find the Point* M *in the Semi-cycloid, whose Distance from the Axis* OA *thereof shall be a* maximum.

If the Point *M* be supposed to be that sought, it is * manifest that the Tangent in *M* will be parallel to the Axis *OA*; and therefore the Perpendicular *MG* to the Cycloid, must likewise be perpendicular to the Axis which it meets in the Point *P*. This being supposed, if *OK* be drawn thro' the Centres of the generating Circles, it will pass thro' the Point of Contact *G*; and if *KL* be drawn perpendicular to *MG*, the equal Angles *GKL*, *GOB* will be formed; and therefore the Arch *IG*, which is the Double of the Measure of the Angle *GKL*, will be to the Arch *GB*, the Measure of the Angle *GOB*, as the Diameter *BN* is to the Radius *OB*. Whence the Determination of the Point *G*, in the Arch of the Semicircle *BGN*, wherein it touches the Arch serving as a Base to it, when the Distance of the describing Point *M* from *AO* is a *maximum*, will be had by so dividing the Semicircle *BGN* in *G*, that drawing the Chord *IG* thro' the given Point *M*, the Arch *IG* is

* Art. 47.

to

to the Arch *BG*, in the given Ratio of *BN* to *OB*. Therefore the Problem is brought to a common geometrical one, and may be always ſolved geometrically, when the given Ratio can be expreſſed in whole Numbers; but by means of Lines repreſented by Equations of ſo many Dimenſions the more, as the Ratio is more compounded.

If the Radius *OB* be ſuppoſed to become infinite, as it will be when the Baſe *BGN* is a right Line; then the Arch *IG* ſhall be infinitely ſmall with reſpect to the Arch *GB*. Therefore the Secant *MIG* then will become the Tangent *MT*, when the deſcribing Point *M* falls without the moveable Circle; and it is manifeſt that when the deſcribing Point *M* falls within the Circle, there will be no *maximum* in the Caſe aforeſaid.

When the Point *M* falls in *N* on the Circumference, you need only divide the Semicircumference *BGN* in the Point *G* in the given Ratio of *BN* to *OB*. For the Point G ſo found will be that wherein the moveable Circle *BGN* touches the Baſe, when the deſcribing Point falls in the Point ſought.

LEMMA.

176. *IN every Triangle* BAC, *the Angles thereof* ABC, ACB, *and* CAD *the Complement of the obtuſe Angle* BAC *to two right Angles being infinitely ſmall, are in the ſame Proportion as the oppoſite Sides* AC, AB, BC. FIG. 137.

For if a Circle be deſcribed about the Triangle *BAC*, the Arches *AC*, *AB*, *BAC* being the Meaſures of the Doubles of thoſe Angles,

gles, will be infinitely small; and so will not
* Art. 3. * differ from their Chords or Subtenses.

If the Sides *AC*, *AB*, *BC* of the Triangle *BAC* be not infinitely little, *viz.* are of a finite Magnitude; it follows, that the circumscribing Circle must be infinitely great; because the Arches *AC*, *AB*, *BAC* having a finite Magnitude, and being the Measures of the infinitely small Angles, must be infinitely small with respect to that Circle.

Prop. IV.

Fig. 135, 136. 177. *THE same Things being supposed; it is required to determine the Point* C *in every Perpendicular* MG, *wherein it touches the Evolute of the Cycloid.*

First conceive another Perpendicular *mg* infinitely near *MG*, and which by consequence cuts the same in the sought Point C, draw the right Line *Gm*, assume the small Arch *Gg* in the Circumference of the moveable Circle, equal to the Arch *Gg* in the moveable Circle, and draw the right Lines *Mg*, *Ig*, *Kg*, *Og*. This being done, if the small Arches *Gg*, *Gg* be conceived as straight Lines perpendicular to the Radii *Kg*, *Og*, it is evident that when the little Arch *Gg* in the moveable Circle coincides with the Arch *Gg* of the immoveable one, the describing Point *M* will coincide with *m*; so that the Triangle *GMg* will coincide with the Triangle *Gmg*. Whence it appears that the Angle *MGm* is equal to the Angle $gGg = GKg + GOg$; because the same Angles *KGg*, *OGg* being added to both Sides, will make up two right Angles.

Now

Now calling the given Lines OG, b; KG, a; GM or Gm, m; GI or Ig, n; and we shall have, * 1°, $OG : KG :: GKg : GOg$. And *Art. 176. $OG (b) : OG + GK$ or $OK (b+a) :: GKg : GKg + GOg$ or $MGm = \frac{a+b}{b} \times GKg$.

2°, * $Ig : MI :: GMg : MgI$. And $Ig \pm MI$ * Ibid. or $MG (m) : Ig (n) :: GMg \pm MgI$ or GIg, or $\frac{1}{2} GKg : GMg$ or $Gmg = \frac{n}{2m} GKg$. 3°, The Angle * MCm or $MGm - Gmg \left(\frac{a+b}{b} - \frac{n}{2m}\right.$ * Ibid. $\left. GKg\right) : Gmg \left(\frac{n}{2m} GKg\right) :: Gm (m) : GC = \frac{bmn}{2am + 2bm - bn}$. And consequently the Radius MC of Evolution sought will be $= \frac{2amm + 2bmm}{2am + 2bm - bn}$.

If the Radius OG (b) of the immoveable Circle be supposed to become infinite, the Circumference thereof will be a right Line; and striking out the Terms $2amm$, $2am$, as being infinitely little with respect to the others $2bmm$, $2bm - bn$, we shall have $MC = \frac{2mm}{2m - n}$.

COROL. I.

178. BECAUSE the Angle $MGm = \frac{a+b}{b} GKg$, and Arches of different Circles are to one another in a Ratio compounded of the Radii and Angles that they measure: therefore $Gg : Mm :: KG \times GKg : MG \times \frac{a+b}{b} GKg$.

And consequently also $KG \times Mm = \frac{a+b}{b} MG \times Gg$; or (which is the same thing) $KG \times Mm : MG \times Gg :: OK (a+b) : OG (b)$. Which is a constant or standing Ratio. Hence it appears, that the Dimension of the Part or Portion of the Cycloid *AMD*, depends on the Sum of all the Rectangles $MG \times Gg$ in the Arch *GB*; which is what Mr. *Pascal* has demonstrated in common Cycloids.

Mr. *Varignon* found out this Property after a manner very different from this here.

Corol. II.

Fig. 155. 179. When the describing Point *M* falls without the Circumference of the moveable Circle, there will of necessity happen one of the three following Cases. For drawing the Tangent *MT*, the Point of Contact *G* will fall (1°.) in the Arch *TB*, as it is supposed to do in the Figure for the Investigation; and then $MC \left(\frac{2amm+2bmm}{2am+2bm-bn}\right)$ will be always greater than $MG (m)$. 2°, In the Point of Contact *T*; and then $MC \left(\frac{2am+2bmm}{2amm+2bm-bn}\right) = m$, because $IG (n)$ vanishes. 3°, In the Arch *TN*; and then the Value of $GI (n)$ being now negative, we shall have $MC = \frac{2amm+2bmm}{2am+2bm+bn}$: So that *MC* will be less than $MG (m)$, and always positive. Therefore in all those Cases it is evident, that the Value of (MC) the Radius of Evolution is always positive.

Corol.

COROLL. III.

180. **W**HEN the describing Point M falls within the Circumference of the moveable Circle, we have always $MC = \frac{2amm + 2bmm}{2am + 2bm - bn}$; and it may happen that bn is greater than $2am + 2bm$, and so the Value of (MC) the Radius of Evolution negative: Whence it appears, that when it ceases to be positive to become negative, as it happens when * the Point M becomes a Point of Inflexion, then of necessity we must have $bn = 2am + 2bm$; and therefore $MI \times MG$ $(mn - mm)$ $= \frac{2amm + bmm}{b}$. Now if the given Line KM be called c; from the Nature of the Circle we shall have $MI \times MG \left(\frac{2amm + bmm}{b}\right) = BM \times MN$ $(aa - cc)$; and so the unknown Quantity MG $(m) = \frac{\sqrt{aab + bcc}}{2a + b}$. Whence if from the given Point M as a Centre, with the Distance $MG = \frac{\sqrt{aab - bcc}}{2a + b}$ you describe a Circle; this will cut the moveable Circle in the Point G, wherein it touches the immoveable Circle serving as its Base, when the describing Point M falls in the Point of Inflexion F. FIG. 136.

* Art. 81.

If MR be drawn perpendicular to BN, it is evident that the said $MG \left(\frac{\sqrt{aab - bcc}}{2a + b}\right)$ will be less than MR $(\sqrt{aa - cc})$, and that it must be equal to the same when b becomes infinite, *viz.* when the Base of the Cycloid becomes a right Line.

Note,

Note, That in order for the Circle described with the Radius MG to intersect the moveable Circle, it is necessary for MG to exceed MN, that is, for $\frac{\sqrt{aab-bcc}}{2a+b}$ to exceed $a-c$; and so KM (c) exceeds $\frac{aa}{a+b}$. Whence it is manifest, that in order to have a Point of Inflexion in the Cycloid AMD, KM must be less than KN, and greater than $\frac{aa}{a+b}$.

Lemma III.

Fig. 138. 181. *Let two Triangles* ABb, CDd *each have one Side* (Bd, *and* Dd) *infinitely small with respect to the others: I say, the Triangle* ABb *is to the Triangle* CDd *in a Ratio compounded of the Angle* BAb *to the Angle* DCd, *and of the Square of the Side* AB *or* Ab *to the Square of the Side* CD *or* Cd.

For if from the Centres A, C, and with the Distances AB, CD the Arches BE, DF be described; it is manifest, * that the Triangles ABb, CDd do not at all differ from the Sectors of the Circles ABE, CDF. Whence, &c.

* Art. 2.

If the Sides AB, CD are equal; the Triangles ABb, CDd shall be to each other as their Angles BAb, DCd.

Prop. V.

Fig. 135. 182. *The same things being supposed; it is required to square the Space* MGBA *comprehended under the Perpendiculars* MG, BA *to*

to the Cycloid, the Arch GB, *and the Portion* AM *of the Semi-cycloid* AMD *granting the Quadrature of the Circle.*

The Angle $GMg\left(\frac{n}{2m}GKg\right)$ is to the Angle MGm $\left(\frac{a+b}{b}GKg\right)$, as the * little Triangle MGg, whose Base is Gg the Arch of the moveable Circle, is to the little Triangle or Sector GMm; and therefore the Sector $GMm = \frac{2m}{n}MGg \times \frac{a+b}{b} = \frac{2a+2b}{b}$ $MGg + \frac{2ap+2bp}{bn}MGg$ by calling MI, p; and putting $p+n$ for m. Now * the little Triangle or Sector KGg is to the little Triangle MGg in a Ratio compounded of the Square of KG to the Square of MG, and of the Angle GKg to the Angle GMg; that is, $::aa \times GKg : mm \times \frac{n}{2m}GKg$. And therefore the little Triangle $MGg = \frac{mn}{2aa}GKg$. Now putting this Value in $\frac{2ap+2bp}{bn}MGg$ for the Triangle MGg, and there will come out the Sector $GMm = \frac{2a+2b}{b}MGg + \frac{\overline{a+b} \times pm}{aab}$ KGg. But because of the Circle $GM \times MI$ $(pm) = BM \times MN$ $(cc-aa)$, which is an invariable Quantity, being so in all Situations of the describing Point M; and consequently $GMm + MGg$ or mGg, that is, the small cycloidal Space $GMmg = \frac{2a+2b}{b}MGg +$

*Art. 181.

*Art. 187.

$a+b$

$\frac{\overline{a+b}\times\overline{cc-aa}}{aab}KGg$. Therefore because $GMmg$ is the Fluxion of the cycloidal Space $MGBA$, and MGg the Fluxion of the circular Space MGB contained under the right Lines MG, MB, and the Arch GB, and likewise since the little Sector KGg is the Fluxion of the Sector

* *Art.* 96. KGB; it follows * that the cycloidal Space $MGBA = \frac{2a+3b}{b}\times MGB + \frac{\overline{a+b}\times\overline{cc-aa}}{aab}$ KGB. Which was to be found.

Fig. 139. When the describing Point M falls without the Circumference BGN of the moveable Circle, and the Point of Contact G in the

* *Art.* 180. Arch NT; it is manifest * that the Perpendiculars MG, mg intersect each other in the Point C, and then will $m = p - n$. Wherefore the little Sector $GMm = -\frac{2a-2b}{b}$ $MGg + \frac{2ap+2bp}{bn}MGg = -\frac{2a-2b}{b}\times$ $MGg + \frac{amp+bmp}{aab}KGg$, by putting (as before) $\frac{mn}{2aa}KGg$ for its Equal the little Triangle MGg; and therefore $GMm - MGg$ or mGg, that is, $MCm - GCg = -\frac{2a-3b}{b}\times MGg +$ $\frac{\overline{a+b}\times\overline{cc-aa}}{aab}KGg$, by substituting $cc-aa$ for pm its Equal. Now if TH be supposed to be the Position of (TM) the Tangent to the moveable Circle, when the Point T thereof touches the Base in the Point T; then it is evident that $MCm - GCg = MGTH - mgTH$, viz.

viz. the Fluxion of the Space *MGTH*, and that *MGg* is the Fluxion of *MGT*, and likewise *KGg* the Fluxion of *KGT*. Therefore* the Space $MGTH = -\frac{2a-3b}{b} MGT + \frac{\overline{a+b} \times \overline{cc-aa}}{aab} \times KGT$. But, as we have already demonstrated, the Space $HTBA = \frac{2a+3b}{b} MTB + \frac{\overline{a+b} \times \overline{cc-aa}}{aab} KTB$. Whence in all Cases we have always the Space *MGBA* $(MGTH + HTBA) = \frac{2a+3b}{b} \overline{MTB - MGT}$ or $MGB + \frac{\overline{a+b} \times \overline{cc-aa}}{aab} \overline{KGT + KTB}$ or *KGB*.

* Art. 96.

Wherefore the whole Space *DNBA* contained under (*DN*, *BA*) two Perpendiculars to the Cycloid, the Arch *BGN*, and the Secycloid *AMD*, is $= \overline{\frac{2a+3b}{b} + \frac{\overline{a+b} \times \overline{cc-aa}}{aab}} \times KNGB$; because the Sector *KGB*, and the circular Space *MGB*, do each become the Semicircle *KNGB*, when the Point of Contact *G* falls in the Point *N*. FIG. 135.

When the describing Point *M* falls within the moveable Circle, we must put $aa-cc$, instead of $cc-aa$ in the foregoing Expressions, because then $BM \times MN = aa-cc$. FIG. 136.

If you make $c=a$, we shall have the Quadrature of Cycloids, whose generating Points are in the Circumference of the moveable Circle; and if b be supposed infinite, we shall have the Quadrature of Cycloids whose Bases are right Lines.

Another Solution.

Fig. 140. 183. With the Radius OD describe the Arch DV, and with the Diameters AV, BN, the Semicircles AEV, BSN; and draw at pleasure from the Centre O the Arch EM between the Semicircle AEV, and the Semi-cycloid AMD, as likewise the Ordinate EP. Now it is required to find the Quadrature of the Space AEM comprehended under the Arches AE, EM, and the Portion AM of the Semi-cycloid AMD.

To do this, let there be another Arch em concentrick and infinitely near to EM, another Ordinate ep, as likewise another Oe intersecting the Arch ME continued out (if necessary) in the Point F. Now call what is variable, *viz.* Oe, z; VP, u; the Arch AE, x; and (as before) the invariable Lines OB, b; KB or KN, a; KV or KA, c. Then will $Fe = \dot{z}$, $Pp = \dot{u}$, $OP = a + b - c + u$, $\overline{PE}^2 = 2cu - uu$, the Arch EM* $= \frac{axz}{bc}$; and therefore the Rectangle under the Arch EM, and small right Line Fe, *viz.* * the little Space $EMme = \frac{axz\dot{z}}{bc}$. Now because of the right-angled Triangle OPE; $zz = aa + 2ab + bb - 2ac - 2bc + cc + 2au + 2bu$; which thrown into Fluxions, and $z\dot{z} = a\dot{u} + b\dot{u}$. Now putting this Value for $z\dot{z}$ in $\frac{axz\dot{z}}{bc}$, and the little Space $EMme$ will be $= \frac{aax\dot{u} + abx\dot{u}}{bc}$.

*Art. 172.

* Art. 2.

Now

Now if the Semi-cycloid *AHT* be described by the Rotation of the Semi-circle *AEV* along the right Line *VT* perpendicular to *VA*, and the Ordinates *PE*, *pe*, be continued out meeting the same in the Points *H*, *h*: It is manifest * that *EH* × *Pp*; that is, the little Space *Art. 172

EHhe is $= x\dot{u}$; and so $EMme \left(\frac{aax\dot{u} + abx\dot{u}}{bc}\right)$

$: EHhe\ (x\dot{u}) :: aa + ab : bc$. Which is a standing Ratio. But because this is always so, let the Arch *EM* be where it will. Therefore the Sum of all the little Spaces *EMme*; that is, the Space *AEM* is to the Sum of all the little Spaces *EHhe*, that is, the Space *AEH* $:: aa + ab : bc$. But we have * the Quadrature of the Space *AEH* by means of the Quadrature of the Circle; and therefore also the Quadrature of the Space *AEM* sought. *Art. 99

This may be demonstrated without any analytical Investigation, as I have shewn in the *Acta Eruditorum* for *August*, in the Year 1695.

The Quadrature of the Space *AEH*, may be had otherwise than from *Art.* 99. For compleating the Rectangles *PQ*, *pq*, we shall have *Qq* or *HR* : *Pp* or *Rh* :: *EP* : *PA* or *HQ*. Because * the Tangent in *H* is parallel to the Chord *AE*; and therefore *HQ* × *Qq* = *EP* × *Pp*; that is, the small Spaces *HQqh*, *EPpe*, are always equal to each other. Whence it follows, that the Space *AHQ* contained under the Perpendiculars *AQ*, *QH*, and the Portion *AH* of the Semi-cycloid *AHT*, is equal to the Space *APE* contained under the Perpendiculars *AP*, *PE*, and the Arch *AE*. Therefore the Space *AEH* will be equal to the Rectangle *PQ* *minus* twice the circular Space *APE*; that is, to the Rectangle under *PE* and *KA* *plus* or *minus* the Rectangle under *Art. 18

der KP and the Arch AE, according as the Point P falls below or above the Centre. And consequently the sought Space $AEM = \frac{aa+ab}{bc} \times \overline{PE \times KA \pm KP \times AE}$.

COROL. I.

184. When the Point P falls in K, the Rectangle $KP \times AE$ vanishes, and the Rectangle $PE \times KA$ becomes equal to the Square of KA. Whence it appears that the Space AEM is then $= \frac{aac+abc}{b}$; and consequently it may be squared without the Quadrature of the Circle.

COROL. II.

185. If the Sector AKE be added to the Space AEM, the Space $AKEM$ contained under the Radii AK, KE, the Arch EM, and the Portion AM of the Semi-cycloid AMD will be (when the Point P falls above the Centre) $= \frac{bcc+2aac+2abc-2aau-2abu}{2bc} AE + \frac{aa+ab}{bc} PE \times KA$; and therefore if you assume VP $(u) \frac{2aac+2abc+bcc}{2aa+2ab}$ (which makes the Value of $\frac{bcc+2aac+2abc-2aau-2abu}{2bc} AE$ nothing,) we shall have the Space $AKEM = \frac{aa+ab}{bc} PE \times KA$. Therefore it appears still that the Quadrature thereof is had independent on that of the Circle.

Hence

Hence it is plain, that among all the Spaces *AEM* and *AKEM*, there are only those two above-mentioned that can be squared.

Note, *What has been demonstrated of exterior Cycloids, extends likewise to interior Cycloids, viz. those that are generated by the Rotation of the moveable Circle along the concave Part of the immoveable one; but then the Radii* KB (a), KV (c) *will be negative; and so the Signs of the Terms in the foregoing Forms wherein* a *or* c *are found of odd Dimensions, must be changed.*

SCHOLIUM.

186. THERE are some Curves which seem to have a Point of Inflexion, and yet have not; which I think proper to explain by an Example, because some Difficulty may arise about this Matter.

Let *NDN* be a geometrical Curve, the Nature whereof is expressed by $z = \frac{xx - aa}{\sqrt{2xx - aa}}$ ($AP = x$, $PN = z$), wherein it is evident, 1°, That x being $= a$, PN (z) vanishes. 2°, That when x exceeds a, the Value of z is positive; and when the same is less, the said Value is negative. 3°, That when $x = \sqrt{\frac{1}{2}aa}$, the Value of PN is infinite. Whence it appears, that the Curve *NDN* extends itself on each Side the Axis, cutting the same in the Point *D* such, that $AD = a$; and that the Asymptote thereof is the Perpendicular *BG* drawn thro' the Point *B* so, that $AB = \sqrt{\frac{1}{2}aa}$. Fig. 141.

Now if another Curve *EDF* be described of such a Nature, that drawing the Perpendicular *MPN* at pleasure, the Rectangle under the Ordinate PM, and the standing Quantity AD, be equal to the correspondent Space DPN; then if $PM = y$, it is manifest that $AD \times Rm$ $(a\dot{y}) = NPpn$ or $NP \times Pp$ $\left(\frac{xx\dot{x} - aa\dot{x}}{\sqrt{2xx - aa}}\right)$; and therefore $Rm\,(\dot{y}) : Pp$ or $RM\,(\dot{x}) :: PN : AD$. Whence it follows, that the Curve EDF touches the Asymptote BG continued out in the Point E, and the Axis AP in the Point D; and so it ought to have a Point of Inflexion in D. Yet the Value of the Radius of the Evolute of it will be

* Art. 78. found* $-\frac{x^3}{2aa}$, being always negative, and becomes equal to $-\frac{1}{2}a$ when the Point M falls

* Art. 81. in D. Whence we infer*, that the whole Curve passing thro' all the Points M, is convex next to the Axis AP; and so has not a Point of Inflexion in D. Now to unravel this.

If you assume PM on the same Side as PN, there will be formed another Curve GDH, which will be in all respects similar to EDF, and must be a Part thereof, since its Generation is the same. This being so, we must conceive the Parts of which the whole Curve consist, not to be EDF, GDH, but EDH, GDF, which touch in the Point D; for by this the above mentioned Difficulty is solved. For Example.

FIG. 142. Let DMG be a Curve, whose Nature is expressed by $y^4 = x^4 + aaxx - b^4$ ($AP = x$, $PM = y$). From this Equation it is manifest, that the whole Curve has two Parts EDH, GDF, opposite to each other, as the common Hyperbola; so that the Distance DD or $2AD$ $= \sqrt{-2aa + 2\sqrt{a^4 + 4b^4}}$.

If

If b be supposed to vanish, the Distance DD will vanish likewise; and therefore the two Parts EDH, GDF, will touch one another in the Point D: So that one would think the said Curve had a Point of Inflexion or Retrogression in D, according as the Parts thereof were supposed to be EDF, GDH or EDG, HDF. But this Deception will easily appear, by finding the Radius of Evolution; which will be positive always, and equal to $\frac{1}{2}$ a in the Point D, as aforesaid. FIG. 143.

By the way, we may observe that the Quadrature of the Space DPN is dependent on that of the Hyperbola; or (which comes to the same thing) the Rectification of the Curve of the Parabola; and the Part of the Curve DMF, solves the Problem proposed by Mr. BERNOULLI, in *Tom.* 2. of the Supplements to the *Acta Eruditorum*, page 291. FIG. 141.

SECT. X.

The Use of Fluxions in Geometrical Curves after a new Manner; from whence is deduced the Method of DESCARTES *and* HUDDE.

DEFIN. I.

Fig. 144, 145, 146. LET *ADB* be a Curve such, that the Parallels *KMN* to the Diameter *AB* thereof, meet it in two Points *M*, *N*; and conceive the intercepted Part *MN* or *PQ* to become infinitely small; then is that called the *Fluxion* of the Abscis *AP* or *KM*.

COROL. I.

187. WHEN the Part *MN* or *PQ* becomes infinitely small; it is evident that the Abscisses *AP*, *AQ*, will each become equal to *AE*, and the Points *M*, *N*, will coincide in *D*; so that the Ordinate *ED* is the greatest or least of all the Ordinates *PM*, *NQ* like it.

COROL. II.

188. AMONG all the Abscisses *AP*, it is evident that *AE* only has a Fluxion, because there can be none, but in that Case where *PQ* becomes infinitely small.

COROL.

COROL. III.

189. IF the indeterminate Lines AP or KM be called x; and PM or AK, (y) be invariable, it is evident that x will have two different Values, *viz.* KM, KN or AP, AQ. Therefore the Equation expressing the Nature of the Curve ADB must be clear'd of Surds, that so the same unknown Quantity x expressing the Roots thereof (for y is looked upon as known) may have different Values. Which must be observed hereafter.

PROP. I.

190. *THE Nature of a Geometrick Curve* ADB *being given: To determine the greatest or least Ordinate thereof.*

If the Equation expressing the Nature of the Curve be thrown into Fluxions, with y as a standing Quantity, and x as a variable one; it is plain * that a new Equation will be had, one of whose Roots x, shall express such a Value AE, that the Ordinate ED will be a *Maximum* or *Minimum*. *Art. 188.

For Example, let $x^3+y^3=axy$. This thrown into Fluxions with x as variable, and y as a standing Quantity, and $3xx\dot{x}=ay\dot{x}$; and therefore $y=\frac{3xx}{a}$. Now if this Value of y be put for it in the Equation of the Curve $x^3+y^3=axy$; then will AE (x) $=\frac{1}{3}a\sqrt[3]{2}$ be such, that ED is a *Maximum*, as has been already shewn in *Art.* 48.

It

It is manifest that by this way we not only can determine the Points D, when the Ordinates ED are Perpendiculars or Tangents to the Curve ADB; but likewise when they are oblique to the Curve, *viz.* when the Points D are those of Retrogression of the first or second Kind. So that this new Manner of considering Fluxions in geometrical Curves, is more simple and less intricate in some Cases than the first *.

* Sect. 3.

SCHOLIUM.

Fig. 146. 191. In Curves that have Points of Retrogression, we may observe, that the PM^s parallel to AK, meet them in two Points M, O, just as the KM^s parallel to AP do in the Points M, N: So that AP (x) continuing the same, y has two Values PM, PO. Therefore in finding the Fluxion of the Equation of that Curve, x may be consider'd as invariable, and y as variable. Consequently if x and y be taken as variable Quantities, in throwing the aforesaid Equation into Fluxions, all the Terms affected with $\dot{x}$ on one Side, and all those with $\dot{y}$ on the other, must be equal to 0. But we must observe that $\dot{x}$ and $\dot{y}$ do here denote the Fluxions of two Ordinates issuing from the same Point, and not (as before, *Sect.* 3.) the Fluxions of two Ordinates infinitely near.

COROL.

192. If after having order'd the Equation expressing the Nature of the Curve wherein there is only the unknown Quantity x variable, the same be thrown into Fluxions; it is evident,

evident, 1°, That in doing this, we only multiply every Term by the Exponent of the Power of x, and by the Fluxion of $\dot{x}$, and afterwards divide it by x. 2°, That the Division by x, as well as the Multiplication by $\dot{x}$ may be omitted, since in every Term it is the same. 3°, That the Exponents of the Powers of x are in an arithmetical Progression, the first Term thereof being the Exponent of the greatest Power, and the last o; for the Terms that may be wanting in an Equation, we have represented by a Star.

For Example, let $x^3 * - ayx + y^3 = 0$. If every Term be multiplied by the Terms of the arithmetical Progression 3, 2, 1, 0, there will arise a new Equation $3x^3 - ayx = 0$.

$$\begin{array}{l} x^3 \quad * - ayx + y^3 = 0. \\ 3, \quad 2, \quad 1, \quad 0. \\ \hline 3x^3 * - ayx \quad * = 0. \end{array}$$

Whence there comes out $y = \frac{3xx}{a}$, the same as would be found by taking the Fluxion the common way.

This being premised, instead of the arithmetical Progression 3, 2, 1, 0, you may take any other at pleasure, as $m+3$, $m+2$, $m+1$, $m+0$, or m, (m being any positive or negative whole Number or Fraction). For multiplying $x^3 * - ayx + y^3 = 0$ by x^m, and we shall have $x^{m+3} *$, &c. $= 0$; every of the Terms of which must be multiplied by those of the Progression $m+3$, $m+2$, $m+1$, m; each by that answering thereto, to get the Fluxion thus.

x^m

$$x^{m+3} \quad * \quad -ayx^{m+1} \quad +y^3x^m = 0.$$
$$m+3, \; m+2, \; m+1, \; m.$$

$$\overline{m+3}x^{m+3} \quad * \quad -\overline{m+1}ayx^{m+1} + my^3x^m = 0.$$

Whence there arises $\overline{m+3}x^{m+3} - \overline{m+1}ayx^{m+1} + my^3x^m = 0$; and dividing by x^m, there comes out $\overline{m+3}x^3 - \overline{m+1}ayx + my^3 = 0$, as was at first found by only multiplying the proposed Equation only by the Progression $m+3, m+2, m+1, m$.

If $m = -3$, the Progression will be $0, -1, -2, -3$; and the Equation shall be $2ayx - 3y^3 = 0$. If $m = -1$, the Progression will be $2, 1, 0, -1$, and the Equation $2x^3 - y^3 = 0$.

The Signs of all the Terms of the Progression may be changed, *viz.* instead of $0, -1, -2, -3$, and $2, 1, 0, -1$, we may take $0, 1, 2, 3$, and $-2, -1, 0, 1$; for doing this only alters the Signs of the Terms of the new Equation, which is to be made equal to 0. That is, instead of $2ayx - 3y^3 = 0$, $2x^3 - y^3 = 0$, we shall have $-2ayx + 3y^3$, $-2x^3 + y^3 = 0$.

Now it is manifest, that what we have demonstrated in the aforesaid Example, may be apply'd after the same Manner to any other. Therefore if after an Equation having two equal Roots, be duly order'd, the Terms thereof be multiplied by the Terms of any arithmetical Progression taken at pleasure, it is evident that a new Equation will be formed, one of whose Roots shall be equal to one of the Roots of the first Equation. By the same Reason, if this new Equation has two equal Roots likewise, and it be multiplied by an arithmetical Progression, we shall form a third Equation,

Equation, having one Root thereof equal to one of the equal Roots of the second Equation, and so on. So that if an Equation of three equal Roots be multiplied by the Product of two arithmetical Progressions; by that means a new Equation will be formed, having one Root equal to one of the three equal Roots of the first Equation: And, in like manner, if the Equation has four equal Roots, it should have been multiplied by the Product of three arithmetical Progressions; if five, by the Product of four, *&c.*

This is what the Method of Mr. HUDDE precisely consists in.

PROP. II.

193. *TO draw a Tangent* THM *from a given Point* T, *in the Diameter* AB; *or a given Point* H *in* AH, *parallel to the Ordinates.* FIG. 147.

From the Point of Contact *M*, draw the Ordinate *MP*, and call *AT*, s; *AH*, t; (one or the other of which is given) and the unknown Lines *AP*, x; *PM*, y; then because of the Similiarity of the Triangles *TAH*, *TPM*, $y=\frac{st+tx}{s}$, $x=\frac{sy-st}{t}$; and putting these Values for y or x in the given Equation expressing the Nature of the Curve *AMD*, we shall have a new Equation freed from x or y.

Now if a right Line *TD* be drawn cutting the right Line *AH* in *G*, and the Curve *AMD* in two Points *N*, *D*, from which are let fall the Ordinates *NQ*, *DB*; it is manifest that when t expresses *AG* in the foregoing Equation,

tion, x or y will have two Values *AQ*, *A*[illegible] or *NQ*, *DB*, which become equal to o[illegible] another, *viz.* to *AP* or *PM* sought when [illegible] expresses *AH*; that is, when the Secant *TD*[illegible] becomes the Tangent *TM*. Whence it follows, that that Equation must have two equal Roots. And so we will multiply it by any arithmetical Progression at pleasure; which must be repeated, if necessary, by multiplying *de Nouveau*, the same Equation by any other arithmetical Progression; that so comparing the Equations arising therefrom, we may find one of them affected with either of the unknow Quantities x or y, and having one of the given Quantities s or t in it. The following Example will be sufficient for explaining this.

EXAMPLE.

194. LET $ax = yy$ express the Nature of the Curve *AMD*. If instead of x we put $\frac{sy - st}{t}$; then will tyy, &c. Which must have two equal Roots.

$$\begin{array}{ccc} tyy & -asy & +ast = 0. \\ 1, & 0, & -1. \\ \hline tyy & * & -ast = 0. \end{array}$$

Therefore multiplying orderly (as you see here) these Terms by those of the arithmetical Progression 1, 0, −1, we shall have $as = yy = ax$; and consequently *AP* (x) $= s$. Whence taking *AP* = *AT*, and drawing the Ordinate *PM*, the Line *TM* will touch the Curve in *M*. But if *AH* (t) be given instead of *AT* (s), we must multiply the same Equation tyy, &c. by by this other Progression 0, 1, 2, and the sought Tangent $PM(y)$ will be $=2t$.

The same Construction may be found by putting $\frac{st+tx}{s}$ for y, in $ax=yy$. For there arises $ttxx$, &c. the Terms whereof multiplied by 1, 0, —1, produce $xx=ss$, and consequently $AP(x)=s$.

COROL.

195. NOW if you suppose the Point of Contact M to be given, and the Point T or H, wherein the Tangent MT intersects the Diameter AB or the Parallel AH to the Ordinates, be sought; you need only, (in the latter Equation expressing the unknown Quantity x or y with respect to the given one s or t,) look upon this last as the unknown Quantity, and x or y as known.

PROP. III.

196. *THE Nature of the Geometrick Curve* AFD *being given: To determine its Point of Inflexion* F. FIG. 148.

Draw the Ordinate FE from the Point F sought, as likewise the Tangent FL, and thro' the Point A (the Origin of the x's) the Line AK parallel to the Ordinates. Likewise call what are unknown, *viz.* LA, s; AK, t; AE, x; EF, y. Then because of the similar Triangles LAK, LEF, $y=\frac{st+tx}{s}$, and $x=\frac{sy-st}{t}$; so that if these Values be put in the Equation of the Curve for y or x, we shall get a new Equation

Equation freed from x or y, as in the last Proposition.

Now if a right Line *TD* be drawn intersecting the right Line *AK* in *H*, which touches the Curve *AFD* in *M*, and the Abscifs in *D*, from which are let fall the Ordinates *MP*, *DB*; it is evident, 1°, That when s expresses *AT*; and t, *AH*; the Equation found as aforesaid, must have two equal Roots, *viz.* * each equal to *AP* or *PM*, according as y or x be made to vanish, and another Root *AB*, or *BD*. 2°, That when s expresses *AL*; and t, *AK*; the Point of Contact *M* coincides with the Point of Intersection *D* in the Point *F* sought: Because * the Tangent *LF* must both touch and cut the Curve in the Point of Inflexion *F*; and so *AP*, *AB*, the Values of x, or *PM*, *PD*, the Values of y become equal to one another, *viz.* equal to *AE* or *EF* sought. Whence the said Equation must have three equal Roots. Consequently it must be multiplied by the Product of two arithmetical Progressions at pleasure; which must be again repeated, if necessary, by multiplying it in like manner with another Product of any two arithmetical Progressions; that so by comparing those Equations resulting therefrom, the unknown Quantities s and t may vanish.

* Art. 193.

* Art. 67.

Example.

197. Let $ayy = xyy + aax$ express the Nature of the Curve *AFD*. If $\frac{sy - st}{t}$ be put for x, there will arise $sy^3 - styy - atyy$, &c.

$sy^3 -$

$$\begin{array}{llll} sy^3 - styy + aasy - aast = 0. \\ \quad - at \\ 1, & 0, & -1, & -2. \\ 3, & 2, & 1, & 0. \\ \hline 3sy^3 & * & -aasy & * = 0. \end{array}$$

Which being multiplied by 3, 0, −1, 0, the Product of two arithmetical Progressions 1, 0, −1, −2, and 3, 2, 1, 0, gives us $yy = \frac{1}{3}aa$; and putting this Value in the Equation of the Curve, the unknown Quantity *AE* (x) will be $= \frac{1}{4}a$.

Another Solution.

198. The aforesaid Problem may be solved likewise, in considering that but only one Tangent *LF* or *KF* can be drawn from the same Point *L* or *K*, since it outwardly touches the concave Part *AF*, and inwardly the convex Part *FD*; whereas from any other Point *T* or *H*, taken in *AL* or *AK* between *A* and *L*, or *A* and *K*, we can draw two Tangents *TM*, *TD* or *HM*, *HD*, the one to the concave, and the other to the convex Part; So that the Point of Inflexion (*F*) may be conceived as the Point of Coincision of the two Points of Contact *M* and *D*. If then *AT* (s) or *AH* (t) be supposed to be given, and you seek * the Value of x or y with respect to s or t; we shall get an Equation having two Roots *AP*, *AB*, or *PM*, *BD*, which will be each equal to the sought Quantity *AE* or *EF*, when s expresses *AL*, and t, *AK*. Therefore that Equation must be multiplied by any arithmetical Progression, &c. Fig. 149, 150.

* Art. 194.

EXAMPLE.

199. LET as before, $ayy = xyy + aax$; then again will $sy^3 - styy - atyy + aasy - aast = 0$, which being multiplied by the arithmetical Progression 1, 0, —1, —2, and there arises $y^3 * - aay - 2aat = 0$, freed from s, having two unequal Roots, *viz.* PM, BD, when t expresses AH, and two equal ones each to EF sought, when t expresses AK. Therefore multiplying *de nouveau* this latter Equation by the arithmetical Progression 3, 2, 1, 0, and there will arise $3yy - aa = 0$; and therefore EF $(y) = \sqrt{\frac{1}{3}aa}$. Which was to be found.

PROP. IV.

FIG. 151. 200. *FROM a given Point* C *without a Curve Line* AMD, *to draw* CM *perpendicular to that Curve.*

Draw MP, CK, perpendicular to the Diameter AB, and with the Distance CM describe a Circle from the Centre C; then it is manifest that it shall touch the Curve AMD in the Point M. Now calling the known Lines AP, x; PM, y; CM, r; and the unknown ones AK, s; KC, t; and we shall have PK or $CE = s - x$, $ME = y + t$; and because of the right-angled Triangle MEC, $y = -t + \sqrt{rr - ss + 2sz - xx}$, $x = s - \sqrt{rr - tt - 2ty - yy}$: So that putting these Values for y or x in the Equation of the Curve, we shall get a new Equation freed from y or x.

Now if another Circle be described from the Centre C, cutting the Curve in two Points N, D,

N, D, from which the Perpendiculars NQ, DB, are let fall: It is manifest that when r expresses the Radius CN, or CD, in the Equation aforegoing, x or y will have two Values AQ, AB or NQ, DB, which become equal to one another (*viz.*) to the Quantity AP or PM sought, when r expresses the Radius CM. Whence that Equation must have two equal Roots; therefore it must be multiplied, *&c.*

EXAMPLE.

101. LET $ax = yy$ express the Nature of the Curve AMD, in which putting $s - \sqrt{rr - tt - 2ty - yy}$ for its Equal x, and then $as - yy = a\sqrt{rr - tt - 2ty - yy}$: So that squaring each Side, and afterwards duly ordering the Equation, we shall get y^4, *&c.* which must have two equal Roots when y expresses PM sought.

$$\begin{array}{llll} y^4 & * \; -2asyy & +2aaty & +aass = 0. \\ & \;\; +aa & & -aarr \\ & & & -aatt \\ \hline 4, & 3, \;\; 2, & 1, & 0. \\ \hline 4y^4 & * \; -4asyy & +2aaty & * \;\; = 0. \\ & \;\; +2aa & & \end{array}$$

Therefore multiplying it by the arithmetical Progression 4, 3, 2, 1, 0, (as here you see it done) and there comes out $4y^3 - 4asy + 2aay + 2aat = 0$; and the Value of y will be $= PM$ sought.

If the given Point C falls in the Diameter AB, then will $t = 0$, and consequently all the Terms affected with it will go out. Whence $4as - 2aa = 4yy = 4ax$, by putting yy instead FIG. 152.

 of

of ax, which is equal to it. Therefore we get $x = s - \frac{1}{2}a$; that is, if CP be assumed equal to $\frac{1}{2}$ the Parameter, and the Ordinate PM be drawn perpendicular to AB, and the right Line CM be drawn, it will be perpendicular to the Curve AMD.

COROL.

FIG. 152. 202. Now if the Point M be given, and the Point C be supposed to be that sought; then in the last Equation expressing the Value of AC (s) with respect to AP (x) or PM (y), we must esteem these latter as known, and the other as unknown.

PROP. V.

FIG. 153. 203. *ANY Point* M *being given in a Curve* AMD, *whose Nature is also given: To find* MC *the Radius of the Curvature in that Point: Or to find the Point* C, *being the Centre of a Circle of equal Curvature with the Curve in the Point* M.

Draw MP, CK, perpendicular to the Axis; denote the Lines by the same Letters as in the Problem aforegoing. Then we shall get the same Equation as there, wherein we must observe that the Letter x or y does here denote a given Magnitude, tho' it did there an unknown one; and on the contrary s, t, esteemed as known ones there, are in Reality here unknown as well as r.

Now it is manifest, 1°, That C the Centre of the Circle of equal Curvature with the Curve in the Point M, must be in MG perpendicular

pendicular to the Curve. 2°, That a Circle can always be described which will touch the Curve in M, and at least cut it in two Points (the nearest of which is D, from which the Perpendicular DB is let fall); because we can always find a Circle that will intersect any Curve Line, except a Circle, in four Points at least, and the Point of Contact M is equivalent but to two Intersections. 3°, That the nearer the Centre G thereof is to C the extreme of the Radius sought, the nearer will the Point of Intersection D be to the Point of Contact M. So that when the Point G falls in C, the Point D falls in M; because * the Circle described with the Radius CM sought, must both touch and cut the Curve in the same Point M. Whence it is manifest, that s expressing AF, and t, FG, the Equation must have two equal Roots, (*viz.* * each equal to AP or PM according as y or x are made to vanish) and another AB or BD, which becomes likewise equal to AP or PM, when s and t express AK, KC, sought; and so the said Equation must have three equal Roots. * *Art.* 76. * *Art.* 200.

EXAMPLE.

204. LET $ax = yy$ express the Nature of the Curve AMD. We shall get * y^4, &c. which being multiplied by 8, 3, 0, —1, 0, the Product of two arithmetical Progressions 4, 3, 2, 1, 0, and 2, 1, 0, —1 —2, and there arises $8y^4 = 2aaty$. See the Operation. * *Art.* 201.

$$\begin{array}{llllll} y^4 & * - 2asyy & + 2aaty & + aass & = 0. \\ & + aa & & - aarr & \\ & & & + aatt & \\ 4, & 3, & 2, & 1, & 0. \\ 2, & 1, & 0, & -1, & -2. \\ \hline 8y^4 & * & * & -2aaty & * & = 0. \end{array}$$

Whence KC or PE $(t) = \frac{4y^3}{aa}$.

If it be required to find an Equation expressing the Nature of the Curve passing thro' all the Points (C), we must still multiply y^4, &c. by 0, 3, 4, 3 0, the Product of two arithmetical Progressions 4, 3, 2, 1, 0, and 0, 1, 2, 3, 4; and then will $8asy - 4aay = 6aat$: Whence (making, for Brevity's sake, $s - \frac{1}{2}a = u$) there comes out $y = \frac{3at}{4u}$, and $4y^3 = \frac{27a^3t^3}{16u^3} = aat$; and therefore $16u^3 = 27att$. Consequently the Curve passing thro' all the Points (C) is a second cubick Parabola, the Parameter to the Axis being $= \frac{27a}{16}$, and the Vertex is distant from the Vertex of the given Parabola by $\frac{1}{2}a$, since $u = s - \frac{1}{2}a$.

When the Position of the Parts of the Curve adjoining to the given Point M, is alike on each side that Point, as it happens when the Curvature there is a *Maximum* or *Minimum*: Then one of the Intersections of the touching Circle cannot coincide with the Point of Contact, except the other does so likewise; so that the Equation must have four equal Roots. Now if y^4, &c. be multiplied by 24, 6, 0, 0, 0, the Product of three arithmetical Progressions 4, 3, 2, 1, 0, and 3, 2, 1, 0, −1, and 2, 1, 0, −1, −2; we shall get $24y^4 = 0$. Therefore the Point *M* must fall in *A* the Vertex of the Parabola, in order for the Parts of the Curve adjoining to it on each Side to be similar or alike.

ANOTHER SOLUTION.

205. IF you call to mind what has been demonstrated in *Art.* 76, *viz.* that but one Line *CM* can be drawn from the Point *C* sought perpendicular to the Curve *AMD*; whereas from any other Point *G* in the said Perpendicular, there can be drawn two Perpendiculars *MG*, *GD*, to the Curve. From this Consideration we can solve the Problem thus: Suppose the Point *G* to be given, and seek * the Value of *x* or *y* with regard to *s* and *t* which are given: Then it is plain that the Equation must have two unequal Roots, *viz.* *AP*, *AB*, or *PM*, *BD*, which become equal when the Point *G* coincides with the Point *C* sought. Wherefore multiply that Equation by any arithmetical Progression, &c. FIG. 154. * *Art.* 200.

EXAMPLE.

206. LET, as before, $ax = yy$, then will * $4y^3$, &c. See the Operation. * *Art.* 101.

$$\begin{array}{cccc} 4y^3 & * & -4asy & +2aat = 0. \\ & & +2aa & \\ 2, & 1, & 0, & -1. \\ \hline 8y^3. & * & * & -2aat = 0. \end{array}$$

Which being multiplied by the arithmetical Progression 2, 1, 0, −1, and there comes out, as * before, $t = \frac{4y^3}{aa}$. * *Art.* 204.

Corol.

Fig. 153, 154. 207. It is manifest that the Point wherein the Radius of the Curvature touches the Curve, may be consider'd * as the Place where the Point wherein the Circle of the same Curvature with the Curve touches it, coincides with the Point of Intersection of the said Circle; or else as * the Point wherein two Points of Contact of different concentrick Circles coincide. Just as a Point of Inflection is looked upon *, as that wherein a Points of Contact coincides with the Intersection of the same right Line; or * as the Coincision of two Points of Contact of two right Lines issuing from the same Point.

* Art. 203.

* Art. 205.

* Art. 196.

* Art. 198.

Prop. VI.

Fig. 155. 208. *To find an Equation expressing the Nature of the Caustick* AFGK, *generated in the Quadrant* CAMNB, *by the reflected Rays* MH, NL, *&c. the incident Rays* PM, QN, *&c. being all parallel to* CB.

We may observe, 1°, That if the reflected Rays *MF*, *NG*, touching the Caustick in *F*, *G*, be continued out to meet the Radius *CB* in the Points *H*, *L*; then will *MH* be $=CH$, and $NL=CL$. For the Angle $CMH=CMP=MCH$; and in like manner the Angle $CNL=CNQ=NCL$.

2°, That from a given Point *F* in the Caustick *AFK*, there can be drawn but one right Line *MH* equal to *CH*; whereas from a given Point *D* between the Quadrant *AMB*, and the

the Caustick *AFK*, two Lines *MH*, *NL*, may be drawn so, that $MH=CH$, and $NL=CL$. For but one Tangent *MH* can be drawn from the Point *F*; but from *D* two Tangents *MH*, *NL*, can be drawn. This being well understood:

It is required to draw the right Line *MH* from a given Point *D*, in such manner, that it be equal to the Part *CH* determined thereby in the Radius *CB*.

Draw *MP*, *DO*, parallel to *CB*, and *MS* parallel to *CA*; call what is given, *viz.* *CO* or *RS*, u; *OD*, z; *AC* or *CB*, a; and the unknown Quantities *CP* or *MS*, x; *PM* or *CS*, y; *CH* or *MH*, r. Then because of the right-angled Triangle *MSH*, $rr=rr-2ry+yy+xx$; and so $CH(r)=\frac{xx+yy}{2y}$. Moreover, because of the similar Triangles *MRD*, *MSH*, $MR\,(x-u) : MS\,(x) :: RD\,(z-y) : SH=\frac{zx-xy}{x-u}$. And therefore $CS+SH$ or $CH=\frac{zx-uy}{x-u}=\frac{xx+yy}{2y}=\frac{aa}{2y}$, by putting aa for $xx+yy$. Whence (multiplying crosswise) and there arises $aax-aau=2zxy-2uyy$; and putting $aa-xx$ for yy, there comes out $2zxy=aax+aau-2uxx$. Then squaring both Sides, to get rid of the Surds, and again substituting $aa-xx$ for yy, and at length we have

$$\begin{array}{l} 4uux^4-4aaux^3-4aauuxx+2a^4ux+a^4uu=0. \\ 4zz \qquad\qquad\quad -4aazz \\ \qquad\qquad\qquad\quad +a^4 \end{array}$$

Now when u expresses *CO*, and z *OD*, it is manifest that this Equation must have two unequal Roots, *viz.* CP, CQ; and on the contrary, when u expresses CE; and z, EF; CQ will become equal to CP; so that then it will have two equal Roots. Therefore if the Terms thereof be multiplied by the Terms of the Arithmetical Progressions, 4, 3, 2, 1, 0, and 0, 1, 2, 3, 4, there will be two new Equations formed, from whence there will arise this Equation, after the unknown Quantity x is gotten out.

$$\begin{array}{llll} 64z^6 - 48aaz^4 & + 12a^4zz & - a^6 & = 0, \\ + 192uu & - 96aauu & - 15a^4uu & \\ & + 192u^4 & - 48aau^4 & \\ & & + 64u^6 & \end{array}$$

expressing the Relation between the Abscis GE (u) and the Ordinate EF (z). Which was to be found.

The touching Point F may be determined by what is explained in the eighth Section. For if another incident Ray pm be conceived infinitely near PM, it is plain that the reflected Ray mh shall cut MH in the Point F sought; from which having drawn FE parallel to PM, and making $CE = u$, $EF = z$, $CP = x$, $PM = y$, $CM = a$, you will find as before $\frac{aax + aau - 2uxx}{xy} = 2z$. Now it is manifest, that CM, CE, EF continue the same while CP and PM vary. Therefore with a, u, z, as standing Quantities, and x and y variable ones, throw that Equation into Fluxions, which will be $2uyxx\dot{x} + aauy\dot{x} - aaxx\dot{y} - aaux\dot{y} + 2ux^3\dot{y} = 0$. Wherein substitute $-\frac{y\dot{y}}{x}$ for its Equal $\dot{x}$, (because they are such by throwing $yy = aa - xx$ into Fluxions) and then $aa - xx$ for yy, and

and at length there comes out $CE\ (u) = \frac{x^3}{aa}$.

If the Curve *AMB* be not a Quadrant, but any other Curve, whereof the right Line *MC* is the Radius of Evolution in the Point *M*; it is manifest, * that the Portion *Mm* thereof may be esteemed as an Arch of a Circle described about the Centre *C*. Therefore if from that Centre the right Line *CP* be drawn perpendicular to the incident Ray *PM*, and assuming $CE = \frac{x^3}{aa}$ ($CP = x$, $CM = a$), you draw *EF* parallel to *PM*; it shall cut the reflected Ray *MH* in the Point *F*, wherein it touches the Caustick *AFK*.

* *Art.* 76.

If thro' all the Points *M*, *m* in any Curve *AMB* be drawn straight Lines *MC*, *mC* to a given Point *C* in the Axis *AC* of it, and other right Lines *MH*, *mb* terminated by *CB* perpendicular to the Axis; in such manner that the Angle $CMH = MCH$, and $Cmb = mCb$; and it be requir'd to find the Point *F*, in every Line *MH* wherein it touches the Curve *AFK* formed by the continual Interfection of the said right Lines *MH*, *mb*, we shall find as before $CH = \frac{xx+yy}{2y} = \frac{zx-uy}{x-u}$: From whence we get $\frac{x^3+uyy+xyy-uxx}{xy} = 2z$, which thrown into Fluxions (with *u* and *z* as standing Quantities, and *x* and *y* as variable ones) will be $2x^3y\dot{x} - uxxy\dot{x} - uxxy\dot{x} - x^4\dot{y} + ux^3\dot{y} + xxyy\dot{y} + uxyy\dot{y} - uy^3\dot{x} = 0$; and therefore $CE\ (u) = \frac{2x^3y\dot{x} - x^4\dot{y} + xxyy\dot{y}}{2xxy\dot{x} - x^3\dot{y} + y^3\dot{x} - xyy\dot{y}}$. Now the Nature of the

the Line AMB being given, we shall have a Value of $\dot{y}$ in $\dot{x}$, which being substituted in the Expression of CE, and the same will be freed from Fluxions.

Prop. VII.

Fig. 156. 209. *Let* AO *be an indefinite right Line, the Beginning whereof is the stable Point* A; *and let there be an infinite Number of Parabola's* BFD, CDG, *having the right Line* AQ *as a common Axis, to which the right Lines* AB, AC *intercepted between the stable Point* A *and their Vertices* B *and* C, *are the Parameters. It is requir'd to find the Nature of the Line* AFG *touching all those Parabola's.*

We may observe, 1°, That any two of these Parabola's BFD, CDG intersect one another in the Point D, situate between the Line AFG and the Axis AO; and that when $AC = AB$, the Point of Intersection D coincides with the Point of Contact F. This being well understood,

It is required to draw a Parabola thro' the given Point D having the denoted Property. Draw the Ordinate DO, and call the given Quantities AO, u; OD, z; and the unknown one AB, x; then by the Nature of the Parabola given $AB \times BO$ $(ux - xx) = \overline{DO}^2$ (zz); and ordering the Equation $xx - ux + zz = 0$. Now when u expresses AO, and z OD; the said Equation will have two unequal Roots, *viz.* AB, CA: and on the contrary when u expresses AE; and z, EF; AC becomes equal to AB, that is, the Equation then has two equal Roots. Therefore it must be multiply'd by by the Arithmetical Progreſſion 1, 0, —1: and ſo $x = z$, and ſubſtituting z for x, there ariſes $u = 2z$, which muſt expreſs the Nature of the Line *AFG*. Whence it follows, that *AFG* is a right Line making the Angle *FAO* with *AO*, being ſuch that *AE* is the Double of *EF*.

If the Problem is requir'd to be ſolved generally, *viz.* let the Parabola's *BFD*, *CDG* be of what nature you pleaſe. Recourſe is to be had to the Method explain'd in the Eighth Section, and the ſame muſt be uſed thus. Call *AE*, u; *EF*, z; *AB*, x; then will $\overline{u-x}^{m} \times x^{n} = z^{m+n}$ expreſs generally the Nature of the Parabola *BF*. This Equation thrown into Fluxions (making x and z invariable, and x variable) and we have $-m \times \overline{u-x}^{m-1} \dot{x} \times x^{n} + n x^{n-1} \dot{x} \times \overline{u-x}^{m} = 0$; and dividing by $\overline{u-x}^{m-1} \dot{x} \times x^{n-1}$, there comes out $x = \frac{n}{m+n} u$; and therefore $u - x = \frac{m}{m+n} u$. Now ſubſtituting theſe Values for $u - x$, and x in the general Equation; and making (for brevity's ſake) $\frac{m}{m+n} = p$, $\frac{n}{m+n} = q$, $m + n = r$, and then will z be $= \sqrt[r]{p^{m} q^{n}}$. Whence it appears that *AFG* is always a right Line, be the Parabola's of what Nature ſoever, the Ratio of *AE* to *EF* only varying.

From what has been explained in this Section, it evidently appears how Deſcartes *and* Hudde's *Method muſt be uſed in the Solution of Problems of theſe Kinds when the Curves are geometrical.*

But

But it is not comparable to the Method of Fluxions, which furnishes us with general Solutions, and extends to all Kinds of Curves, without any Necessity of clearing Equations of Surds. Whereas, by the former Method, we only get particular Solutions, and are necessitated to throw out the Surds: which very often cannot be done.

F I N I S.

PART II.

Being an

APPENDIX,

Containing the

Inverſe Method of FLUXIONS,

WITH THE

Application thereof in the Inveſtigation of the Areas of Superficies, Lengths of Curve Lines, Contents of Solids, and the Determination of their Centres of Gravity and Percuſſion.

Wherein are Examples of Solutions, according to the excellent compendious Way of the late Learned Mr. ROGER COTES, by the Meaſures of Ratios and Angles, or Tables of Logarithms, and natural Sines and Tangents.

APPENDIX.

OF THE

Inverse Method of FLUXIONS.

SECTION I.

Of the Reduction of Fractional Expressions and Surd Quantities to infinite Series.

THE Fluents or flowing Quantities of Fluxions express'd fractionwise, or of such wherein there are Surds or radical Quantities, in general cannot be found till the said fractional Expressions are freed from their compound Denominator, and brought to simple ones, and the radical Quantities from their Surds, by throwing such Expressions into infinite Series. Which may be done by the two following Problems.

PROB. I.

1. *TO throw* $\frac{b}{a+x}$ *(a and b being standing, and x a variable Quantity) into an infinite Series, whereby it will be freed from its binomial Denominator.*

Divide the Numerator b by the Denominator $a+x$, after the very same manner as you do decimal Fractions, by adding o to the Remainder, and repeating the Operation till you have gotten 4, 5 or 6 Terms in the Quotient; after which, in many Cases, you may find as many Terms as you please, by considering the Law of the Progression of those Terms already found. And an infinite Number or Series of Terms so found, will be the exact Quotient of the Division; but usually a few of the first Terms are sufficiently near the Truth for any Purpose.

EXAMPLE I.

$$a+x)\ b+o\left(\frac{b}{a}-\frac{bx}{a^2}+\frac{bx^2}{a^3}-\frac{bx^3}{a^4}, \&c.\right.$$

$$b+\frac{bx}{a}$$

$$o-\frac{bx}{a}+o$$

$$-\frac{bx}{a}-\frac{bx^2}{a^2}$$

$$o+\frac{bx^2}{a^2}+o$$

$$+\frac{bx^2}{a^2}+\frac{bx^3}{a^3}$$

$$o-\frac{bx^3}{a^3}+o$$

$$-\frac{bx^3}{a^4}-\frac{bx^4}{a^4}$$

$$o+\frac{bx^4}{a^4}, \&c.$$

For

For dividing b by a, the Quotient is $\frac{b}{a}$. The Product of $\frac{b}{a}$ by $a+x$ is $\frac{ba}{a}+\frac{bx}{a}$ or $b+\frac{bx}{a}$. Which substracted from the Dividend b, and there remains $o-\frac{bx}{a}$. Again, if $o-\frac{bx}{a}$ be divided by a, the Quotient will be $-\frac{bx}{a^2}$. Therefore the Product of $a+x$ into $-\frac{bx}{a^2}$, that is, $-\frac{abx}{a^2}-\frac{bx^2}{a^2}$, or $-\frac{bx}{a}-\frac{bx^2}{a^2}$ substracted from the Dividend $-\frac{bx}{a}$ leaves $o+\frac{bx^2}{a^2}$. Whence the Law of the Continuance of the Division is evident. Now the Quotient consists of an infinite Series of Terms, whose Numerators are the Powers of x, less by 1 than the Number of the Order multiply'd by b, and denominates the Powers of a, whose Exponents are equal to the Number of the Order of the Terms. For Example: In the third Term, the Exponent of the Power of x in the Numerator is 2, and of a in the Denominator is 3.

In like manner, if you put x the first Letter in the Divisor, and then divide b by $x+a$, as before, the Quotient will be $\frac{b}{x}-\frac{ba}{x^2}+\frac{ba^2}{x^3}-\frac{ba^3}{x^4}$, &c. So that it's plain there will be as many Quotients or infinite Series gotten by this Division, as there are Terms in the Divisor; and those Terms of the Divisor which are greatest must stand first, as well as those in the Dividend, in order to have a true Series.

For Example: Let $b=1$, $x=1$, and $a=2$. Then if the Division be perform'd with a as the first Letter of the Divisor, you will find $\frac{1}{3}=\frac{1}{2+1}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}$, &c. which we know to be true from other Principles. But if x be the first Letter, then will $\frac{1}{3}=\frac{1}{1+2}$ be $=1-2+4-8+16$, &c. which is false. For this Series diverges, and differs so much the more from the true Quotient, as the Number of Terms is greater. For Example: One Term 1 exceeds $\frac{1}{3}$ by $\frac{2}{3}$; two Terms are deficient by $\frac{4}{3}$; three Terms exceeds it by $\frac{8}{3}$; four Terms are deficient by $\frac{16}{3}$, and so on.

EXAMPLE II.

$$b+x\,(aa+0)\,\frac{aa}{b}-\frac{aax}{b^2}+\frac{aax^2}{b^3}-\frac{aax^3}{b^4}, \text{ \&c.}$$

$$aa+\frac{aax}{b}$$

$$0-\frac{aax}{b}+0$$

$$-\frac{aax}{b}-\frac{aax^2}{b^2}$$

$$0+\frac{aax^2}{b^2}+0$$

$$+\frac{aa^2}{b^2}+\frac{aax^3}{b^3}$$

$$0-\frac{aax^3}{b^3}+0$$

$$-\frac{aax^3}{b^3}-\frac{aax^4}{b^4}$$

$$0+\frac{aax^4}{b^4}, \text{ \&c.}$$

If

If b be put first, then will the Quotient be $\frac{aa}{x} - \frac{aab}{x^2} + \frac{aab^2}{x^3} - \frac{aab^3}{x^4}$, &c.

Again; the Quotient of $\frac{1}{1+xx}$ will be found $1 - x^2 + x^4 - x^6 + x^8$, &c. or (making xx the first Term in the Divisor) $x^{-2} - x^{-4} + x^{-6} - x^{-8}$, &c.

Moreover, $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x}$ freed from its compound Denominator by Division will be brought to $2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$, &c.

And lastly, This Fraction

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8, \text{ \&c.}}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6 - \frac{5}{128}b^4x^8, \text{ \&c.}}$$

having both Numerator and Denominator infinite Series may be freed from its compound Denominator, or brought into an infinite Series by dividing the Numerator by the Denominator, as before: the Operation being alike to that whereby one interminate decimal Fraction is divided by another.

This Quotient or infinite Series will be

$$\begin{array}{lllll} 1 & + \frac{1}{2}bx^2 & + \frac{3}{8}b^2x^4 & + \frac{5}{16}b^3x^6 & + \frac{35}{128}b^4x^8, \text{ \&c.} \\ & + \frac{1}{2}a & + \frac{1}{4}ab^2 & + \frac{3}{16}ab^2 & + \frac{5}{32}ab^3 \\ & & - \frac{1}{8}a^2 & - \frac{1}{16}a^2b & - \frac{3}{64}a^2b^2 \\ & & & + \frac{1}{16}a^3 & + \frac{1}{32}a^3b \\ & & & & - \frac{5}{128}a^4. \end{array}$$

PROB. II.

2. *TO free a compound Expression from Surds, by throwing it into an infinite Series.* Suppose $\sqrt{aa+xx}$.

Extract

Extract the Root thereof, as you do the Root of a decimal Fraction by the Addition of Cyphers, and bringing out so many Terms, till you may discover the Law of the Progression; that from those already found you may continue on the Terms at pleasure, and the thing is done.

EXAMPLE I.

$$aa+xx\ \left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7},\ \&c.\right.$$

$$aa$$

$$0+xx$$

$$xx+\frac{x^4}{4a^2}$$

$$0-\frac{x^4}{4a^2}$$

$$-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$$

$$0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$$

$$\frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}}$$

$$0-\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}},\ \&c.$$

For the square Root of aa is $+a$, for the first Letter or Term of the Root. Which squar'd and substracted from $aa+xx$, there remains xx. Which being divided by $2a$ (as in the Extraction of the square Root) and the Quotient will be $+\frac{x^2}{2a}$ the second Term of the Root. Which added to $2a$, and the whole multiplied by $\frac{x^2}{2a}$, will be $xx+\frac{x^4}{4a^2}$. This taken

ken

ken from xx, and there remains $-\frac{x^4}{4a^2}$. Which divided by $2a-\frac{x^2}{a}$ the Double of the two first Terms, and the Quotient $-\frac{x^4}{8a^3}$ will be the third Term of the Root. This added to $2a-\frac{x^2}{a}$, and the whole multiplied by $-\frac{x^4}{8a^3}$, will be $-\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6}$. Which taken from $-\frac{x^4}{4a^2}$, and there remains $0+\frac{x^6}{8a^4}-\frac{x^8}{64a^6}$ And after this manner you may find any Number of Terms. And so continue the Series on at Pleasure.

If xx be the first Term of the Surd, then will the Root be $x+\frac{aa}{2x}-\frac{a^4}{8x^3}+\frac{a^6}{16x^5}-\frac{5a^8}{128x^7}$, &c.

And here, as well as in the Division aforegoing, the greatest Term of the Surd must stand first, otherwise the Root or Series will not be true.

EXAMPLE II.

$$x-xx\ (x^{\frac{1}{2}}-\tfrac{1}{2}x^{\frac{3}{2}}-\tfrac{1}{8}x^{\frac{5}{2}}-\tfrac{1}{16}x^{\frac{7}{2}}-\tfrac{5}{128}x^{\frac{9}{2}},\ \&c.$$

$$x$$

$$0-x^2$$

$$-x^2+\tfrac{1}{4}x^3$$

$$0-\tfrac{1}{4}x^3$$

$$-\tfrac{1}{4}x^3+\tfrac{1}{8}x^4+\tfrac{1}{64}x^5$$

$$0-\tfrac{1}{8}x^4-\tfrac{1}{64}x^5$$

$$-\tfrac{1}{8}x^4+\tfrac{1}{16}x^5+\tfrac{1}{64}x^6+\tfrac{1}{256}x^7$$

$$0-\tfrac{5}{64}x^5-\tfrac{1}{64}x^6-\tfrac{1}{256}x^7$$

$$-\tfrac{5}{64}x^5+\tfrac{5}{128}x^6+\tfrac{5}{512}x^7+\tfrac{5}{1024}x^8+\tfrac{5}{16384}x^9$$

$$0-\tfrac{7}{128}x^6-\tfrac{7}{512}x^7-\tfrac{5}{1024}x^8-\tfrac{5}{16384}x^9,$$

&c.

EXAMPLE III.

$$1-2x+3x^2-4x^3+5x^4, \&c. \;(1-x+x^2, \&c.$$
$$1$$
$$0-2x+3x^2-4x^3+5x^4, \&c.$$
$$-2x+x^2$$
$$0+2x^2-4x^3+5x^4, \&c.$$
$$2x^2-2x^3+x^4$$
$$0-2x^3-4x^4, \&c.$$

After the same way you may extract the Cube, Biquadrate, *&c.* Root of a Surd, even if it be an infinite Series.

But these Extractions, as well as the Divisions aforegoing, will be very much shorten'd by a Theorem invented for that Purpose by Sir *Isaac Newton*, which is this: $\overline{P+PQ}^{\frac{m}{n}}$

$$=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{4n}CQ$$
$$+\frac{m-3n}{4n}DQ+\&c.$$

Where $P+PQ$ is the Quantity whose Root or any Dimension, or Root of the Dimension, is to be found; that is, express'd by an infinite Series. P is the first Term of that Quantity: Q the rest of the Terms divided by first; and $\frac{m}{n}$ is the numeral Index of the Dimension of $P+PQ$. Moreover, A, B, C, D, &c. are used for the Terms found in the Quotient, *viz.* A for the first Term $P^{\frac{m}{n}}$; B for the second $\frac{m}{n}AQ$, and so on.

A few

A few Examples will shew this wonderful Theorem's Use.

EXAMPLE I.

$\sqrt{aa+xx}$ or $\overline{aa+xx}^{\frac{1}{2}}$ is $=a+\frac{xx}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c. for in this Case, P is $=aa$, $Q=\frac{xx}{aa}$, $m=1$, $n=2$, $A\,(=P^{\frac{m}{n}}=aa^{\frac{1}{2}})=a$. $B=\frac{m}{n}AQ)=\frac{xx}{2c}$. $C\,(=\frac{m-n}{2n}BQ)=-\frac{x^4}{8a^3}$, &c.

EXAMPLE II.

$\sqrt[5]{a^5+a^4x-x^5}$, or $\overline{a^5+a^4x-x^5}^{\frac{1}{5}}$ is $=a+\frac{a^4x-x^5}{5a^4}-\frac{2a^8xx+4a^4x^6-2x^{10}}{25a^9}+$&c. for here $m=1$, $n=5$, $P=a^5$, and $Q=\frac{a^4x-x^5}{a^5}$.

EXAMPLE III.

$\frac{b}{\sqrt[2]{y^3-a^2y}}$, or $b\times\overline{y^3-a^2y}^{-\frac{1}{2}}$ is $=b\times\frac{1}{y}+\frac{aa}{3y^3}+\frac{2a^4}{9y^5}+\frac{14a^6}{81y^7}$, &c. for $P=y^3$. $Q=-\frac{aa}{yy}$. $m=-1$, $n=3$. $A\,(P\frac{m}{n}=y^3\times-\frac{1}{3})=y^{-1}$, that is, $\frac{1}{y}$. $B\,(=\frac{m}{n}AQ=-\frac{1}{3}\times\frac{1}{y}\times-\frac{aa}{yy})=\frac{aa}{3y^3}$, &c.

EXAMPLE IV.

$\sqrt[3]{\overline{a+x}^4}$ or $\overline{a+x}^{\frac{4}{3}}$ is $= a^{\frac{4}{3}} + \frac{4xa^{\frac{1}{3}}}{3} + \frac{2xx}{9a^{\frac{2}{3}}} - \frac{4x^3}{81a^{\frac{5}{3}}} +$ &c. for $P = a$. $Q = \frac{x}{a}$. $m = 4$, $n = 3$. $A\left(= P^{\frac{m}{n}}\right) = a^{\frac{4}{3}}$, &c.

EXAMPLE V.

$\overline{a+x}^5$ or $\overline{a+x}^{\frac{5}{1}}$ is $= a^5 + 5a^4x + 10a^3xx + 10a^2x^3 + 5ax^4 + x^5$. For $P = a$. $Q = \frac{x}{a}$. $m = 5$, and $n = 1$. $A\left(= P^{\frac{m}{n}}\right) = a^5$. $B\left(= \frac{m}{n}AQ\right) = 5a^4x$. and so $C = 10a^3xx$. $D = 10aax^3$. $E = 5ax^4$. $F = x^5$. and $G\left(= \frac{m-5n}{6n}FQ\right) = 0$.

EXAMPLE VI.

$\frac{1}{a+x}$ or $\overline{a+x}^{-1}$ or $\overline{a+x}^{-\frac{1}{1}}$ is $= \frac{1}{a} - \frac{x}{aa} + \frac{xx}{a^3} - \frac{x^3}{a^4}$, &c. for in this Case $P = a$, $Q = \frac{x}{a}$, $m = -1$, $n = 1$. and $A\left(= P^{\frac{m}{n}} = a^{-\frac{1}{1}}\right) = a^{-1}$ or $\frac{1}{a}$. $B\left(= \frac{m}{n}AQ = -1 \times \frac{1}{a} \times \frac{x}{a}\right) = -\frac{x}{aa}$. and $C = \frac{xx}{a^3}$. $D = -$

$\frac{x^3}{a^4}$, &c. So that the said wonderful Theorem likewise frees Fractions from their Denominators, as well as extracts Roots.

EXAMPLE VII.

$\frac{3}{a+x} = 3 \times \overline{a+x}^{-1}$ is $= \frac{1}{a^3} - \frac{3x}{a^4} + \frac{6xx}{a^5} - \frac{10x^3}{a^6}$, &c.

EXAMPLE VIII.

And $\frac{b}{\sqrt[3]{a+x}} = b \times \overline{a+x}^{-\frac{1}{3}}$ is $= b \times \frac{1}{a^{\frac{1}{3}}} - \frac{x}{3a^{\frac{4}{3}}} + \frac{2xx}{9a^{\frac{7}{3}}} - \frac{14x^3}{81a^{\frac{10}{3}}}$, &c.

EXAMPLE IX.

And $\frac{b}{\sqrt[5]{\overline{a+x}^3}} = b \times \overline{a+x}^{-\frac{3}{5}}$ is $= b \times \frac{1}{a^{\frac{3}{5}}} - \frac{3x}{5a^{\frac{8}{5}}} + \frac{12xx}{25a^{\frac{13}{5}}} - \frac{52x^3}{125a^{\frac{18}{5}}}$, &c.

In the Philosophical Transactions, N°. 230. Mr. *De Moivre* has given us the following Theorem for raising an infinite Series to a given Power *m*, or extracting the Root thereof, *viz.*

$\overline{az+bz^2+cz^3+dz^4+ez^5+fz^6, \&c.}^{m}$ will be $=$

$$= a^m z^m +$$

$$+ \frac{m}{1} a^{m-1} b z^{m+1}$$

$$\left.\begin{array}{l} + \frac{m}{1} \times \frac{m-1}{2} a^{m-2} b^2 \\ + \frac{m}{1} a^{m-1} c \end{array}\right\} z^{m+2}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3 \\ +\frac{m}{1}\times\frac{m-1}{1}a^{m-2}bc \\ +\frac{m}{1}a^{m-1}d \end{array}\right\} z^{m+3}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}a^{m-4}b^4 \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^2c \\ +\frac{m}{1}\times\frac{m-1}{1}a^{m-2}bd \\ +\frac{m}{1}\times\frac{m-1}{2}a^{m-2}c^2 \\ +\frac{m}{1}a^{m-1}e \end{array}\right\} z^{m+4}$$

$$\left.\begin{array}{r} +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}a^{m-5}b^5 \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}a^{m-4}b^3c \\ +\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^2d \\ +\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{2}a^{m-3}bc^2 \\ +\frac{m}{1}\times\frac{m-1}{1}a^{m-2}be \\ \frac{m}{1}\times\frac{m-1}{1}a^{m-2}cd \\ +\frac{m}{1}a^{m-1}f \end{array}\right\} z^{m+5}$$

+

$$\left.\begin{array}{r}
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}\times\frac{m-5}{6}a^{m-6}b^{6} \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{1}a^{m-5}b^{4}c \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{1}a^{m-4}b^{3}d \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}\times\frac{m-3}{2}a^{m-4}b^{2}c^{2} \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{1}a^{m-3}b^{2}e \\
+\frac{m}{1}\times\frac{m-1}{1}\times\frac{m-2}{1}a^{m-3}bcd \\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-3}bf \\
+\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}c^{3} \\
+\frac{m}{1}\times\frac{m-1}{1}a^{m-2}ce \\
+\frac{m}{1}\times\frac{m-1}{2}a^{m-2}d^{2} \\
\frac{m}{1}a^{m-1}g
\end{array}\right\} z^{m+6}, \text{ \&c.}$$

For underſtanding of which, it is only neceſſary to conſider all the Terms by which the ſame Power of z is multiply'd: In order to which two things in each of theſe Terms muſt be conſider'd, 1°, The Product of certain Powers of the given Quantities or Coefficients a, b, c, d, &c. And, 2°, The *Unciæ* or Products of $\frac{m}{1}\times\frac{m-1}{2}$, *&c.* prefixed to them.

Now to find all the Products belonging to the ſame Power of z. For Example; to find that

that Product whose Index is $m+r$ (r being any whole Number) the said Products must be distinguished into several Classes. Those which immediately after some certain Power of a (by which all these Products begin) are Products of the first Class: as $a^{m-4}b^3e$ is a Product of the first Class, because b immediately follows a^{m-4}. Those which immediately after some Power of a have c; are Products of the second Class. So $a^{m-3}ccd$ is a Product of the second Class: Those which immediately after some Power of a have d, are Products of the third Class, and so of the rest.

This being understood, 1°, Multiply all the Products belonging to z^{m+r-1} (which immediately precedes z^{m+r}) by b, and divide them all by a. 2°, Multiply by c, and divide by a, all the Products belonging to z^{m+r-2} except those of the first Class. 3°, Multiply by d, and divide by a all the Products belonging to z^{m+r-3} except those of the first and second Class. 4°, Multiply by e, and divide by a, all the Terms belonging to z^{m+r-4}, except those of the first, second, and third Class; and so on, till you meet twice with the same Term. Lastly, Add the Product of a^{m-r} into the Letter whose Exponent is $r+1$ to all these Terms.

Note, The Exponent of a Letter is the Number expressing what Place that Letter has in the Alphabet, as 3 is the Exponent of the Letter c.

By this Rule it is manifest that it is easy to find all the Products belonging to the several Powers of z, if you have but the Product belonging to z^m, viz. a^m.

Now

Now to find the *Unciæ* prefix'd to every Product, you must consider the Sum of the Units contained in the Exponents of the Letters that compose it (the Index of a excepted); then I write as many Terms of the Series $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}$, &c. as there are Units in the Sum of these Indexes; this Series is to be the Numerator of a Fraction, whose Denominator is the Product of the several Series $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3$ $4 \times 5 \times 6$, &c. the first of which contains as many Terms as there are Units in the Index of b; the second as many as there are Units in the Index of c; the third as many as there are Units in the Index of d, &c.

The Demonstration of this see in the above cited Transaction.

Here follows an Example or two of the Use of this Theorem.

EXAMPLE. I.

To raise this infinite Series $\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}$, &c. to the second Power, or to square it.

In this Case in the Theorem $m = 2$, $z = x$, $a = \frac{1}{xx}$, $b = \frac{1}{x^4}$, $c = \frac{1}{x^6}$, $d = \frac{1}{x^8}$, &c. therefore $\overline{\frac{1}{x} + \frac{1}{xx} + \frac{1}{x^3} + \frac{1}{x^4}, \&c.}^{\,2}$ will be $\frac{1}{xx} + \frac{2}{x^3} + \frac{3}{x^4} + \frac{4}{x^5}$, &c. for the first Term $a^m z^m \left(= \frac{1}{x^4} \times xx\right)$ is $= \frac{1}{xx}$. The second Term $\frac{m}{1} a^{m-1} b z^{m-1}$ $\left(= \frac{2}{xx} \times \frac{1}{x^4} \times x^3\right)$ is $= \frac{2}{x^3}$.

The

The Third $\left.\begin{matrix}\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^2\\ +\ \frac{m}{1}a^{m-1}c\end{matrix}\right\}z^{m+2}\left\{=2\times\frac{1}{2}\times\frac{1}{x^8}\times x^4\right.$

$=\frac{1}{x^4}+2\times\frac{1}{xx}\times\frac{1}{x^6}\times x^4=\frac{2}{x^4}\Big\}$ is $=\frac{3}{x^4}$.

The Fourth Term $\left.\begin{matrix}\frac{m}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}b^3\\ \frac{m}{1}+\frac{m-1}{1}a^{m-2}bc\\ \frac{m}{1}a^{m-1}d\end{matrix}\right\}z^{m+3}=\left\{\begin{matrix}2\times\frac{1}{2}\times\frac{0}{3}\times xxx\times\frac{1}{x^{12}}\times x^5=0\\ 2\times 1\times 1\times\frac{1}{x^4}\times\frac{1}{x^0}\times x^5=\frac{2}{x^5}\\ 2\times\frac{1}{xx}\times\frac{1}{x^8}\times x^5=\frac{2}{x^5}\end{matrix}\right\}$ is $=\frac{4}{x^5}$.

EXAMPLE II.

To square this infinite Series $1-x+x^2+x^3+x^4$, &c.

In this Case in the Theorem $m=2$, $z=x$. $a=\frac{1}{x}-1$. $b=1$. $c=-1$. $d=1$, &c. and so $\overline{1-x+x^2-x^3+x^4}^2$, &c. will be $=1-2x+3x^2-4x^3+5x^4$, &c. for $a^m z^m$ $\left(=\overline{\frac{1}{x}-1}^2\times x^2\right)=1-2x+xx$. $\frac{m}{1}a^{m-1}$ $bz^{m+1}\left(=\frac{2}{1}\times\overline{\frac{1}{x}-}^1\times x^3\right)=2xx-2x^3$.

$\frac{m}{1}$

$$\left.\begin{matrix}\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^2\\ \frac{m}{1}\,a^{m-1}c\end{matrix}\right\}z^{m+2}=\left\{\right.$$

$$\left.\begin{matrix}\frac{2}{1}\times\frac{1}{2}\times\overline{\frac{1}{x}-1}^{\,2-2}\times 1\times x^4\\ \frac{2}{1}\times\overline{\frac{1}{x}-1}^{\,1}\times -1\times x^4\end{matrix}\right\}=-x^3+2x^4,\ \&c.$$

EXAMPLE III.

To raise $1-x+x^3-x^5+x^7$, &c. to the third Power, or to cube it.

Here $m=3$. $z=x$. $a=\frac{1}{x}-1$, $b=0$, $c=1$, $d=0$. and so the third Power will be $1-3x+3x^2+2x^3-6x^4$, &c. for $a^m z^m$ $\left(=\overline{\frac{1}{x}-1}^{\,3}\times x^3\right)=1-3x+3xx-x^3$.

$\frac{m}{1}a^{m-1}bz^{m+1}\left(=3\times\overline{\frac{1}{x}-1}^{\,3-1}\times 0\times x^4\right)=0$.

$$\left.\begin{matrix}\frac{m}{1}\times\frac{m-1}{2}a^{m-2}b^2\\ \frac{m}{1}a^{m-1}c\end{matrix}\right\}z^{m+2}=\left\{\right.$$

$$\left.\begin{matrix}3\times 1\times\overline{\frac{1}{x}-1}^{\,1}\times 0^2\times x^5=0\\ 3\times\overline{\frac{1}{x}-1}^{\,2}\times 1\times x^5=3x^3-6x^4\times 3x^5\end{matrix}\right\}=$$

$3x^3-6x^4+3x^5$.

And because $b=0$, and also d, therefore the next Term of the general Theorem will be o. And thus you may proceed on.

EXAMPLE IV.

To extract the Root of an infinite Series; that is, if z be $= ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. to find the Value of x in an infinite Series of Terms affected with z, and free from x.

First, Let us suppose $x = fz + hz^2 + kz^3 + lz^4 + mz^5 + nz^6$, &c. Then by the Theorem

$$x^2 = f^2z^2 + 2fhz^3 + h^2z^4 + 2hkz^5 + k^2z^6$$
$$+ 2fkz^4 + 2flz^5 + 2hlz^6$$
$$+ 2fmz^6$$
&c.

$$x^3 = f^3z^3 + 3f^2hz^4 + 3fh^2z^5 + h^3z^6, \text{ \&c.}$$
$$+ 3f^2kz^5 + 3f^2lz^6$$
$$+ 6fhkz^6$$
$$x^4 = f^4z^4 + 4f^3hz^5 + 6f^2h^2z^6, \text{ \&c.}$$
$$x^5 = f^5z^5 + 5f^4hz^6$$
$$x^6 = f^6z^6, \text{ \&c.}$$

Now substitute these Values in the Equation $0 = -z + ax + bx^2 + cx^3 + dx^4 + ex^5$, &c. and then will $-z = -z$.

$$+ax = +afz + ahz^2 + akz^3 + alz^4 + amz^5 + anz^6, \text{ \&c.}$$
$$+bx^2 = * \; bf^2z^2 + 2bfhz^3 + bh^2z^4 + 2bflz^5 + bk^2z^6,$$
$$+ 2bhkz^4 + \;\; ^5$$
$$+ 2bhkz^5 + 2bfmz^6$$
$$+cx^3 = * \; * \; + cf^3z^3 + 3cf^2hz^4 + cfh^2z^5 + 3cf^2lz^6$$
$$+ 3cf^2kz^5 + \;\; \text{\&c.}$$
$$3cf^2lz^6$$
$$+ 6fhkz^6$$
$$dx^4 = * \; * \; * \; + dfz^4 + 4df^3z^5 + 6df^2h^2z^6$$
$$+ 4df^3kz^6$$
$$ex^5 = * \; * \; * \; * \; + ef^5z^5 + 5ef^4hz^6.$$

Now if the Sum of the Coefficients of every Term in this Equation be made equal to no-

nothing, we may get the Values of the Coefficients f, h, k, l, m, n, thus. The Sum of the Coefficients of the first Term $\left.\begin{matrix}-1\\ af\end{matrix}\right\} z$ will be $af-1$. Whence if $af-1=0$, then will f be $=\frac{1}{a}$. In like manner the Sum of the Coefficients of the second Term $\left.\begin{matrix}ah\\ bf^2\end{matrix}\right\} z^2$ will be $ah+bf^2$. Whence if $ah+bf^2=0$, there will arise $h=\frac{-bf^2}{a}=\frac{-b}{a^3}$. In like manner the Sum of the Coefficients of the third Term made equal to 0 will be $ak+2bfh+cf^3=0$. Whence $k=\frac{-2bfh-cf^3}{a}=\frac{2b^2-ac}{a^5}$. Again, $al+bh+2bfk+3cf^2h+df^4=0$. Whence $l=\frac{-bh-2bfk+3cfh-df^4}{a}=\frac{-b^3}{a^7-4b^3}+\frac{a^7+2ba}{a^6+3bc}+\frac{a^6-d}{a^5}=\frac{5abc-5b^3-a^2d}{a^7}$. So likewise m is $=\frac{14b^4+6abd-21ab^2c+3a^2c^2-a^3e}{a^9}$. And $n=$

$$\frac{-42b^5+84ab^3c-28a^2bc-28a^2bd+7a^3cd+7a^3cd+7a^3bea^4f}{a^{11}}$$

Whence at length substituting these Values of the Coefficients f, h, k, l, m, n, in the assumed Equation $x=fz+hz^2+kz^3+lz^4+mz^5+nz^6$, &c. and the Root sought will be $x=\frac{z}{a}-\frac{bz^2}{a^3}+\frac{2b^2-ac}{a^5}z^3+\frac{5abc-5b^3-a^2d}{a^7}z^4+\frac{14b^4+6a^2bd-21ab^2c+3a^2c^2-a^3e}{a^9}z^5$, &c.

Note, If there are any Terms wanting in the proposed Equation, it is plain that they will likewise be wanting in the Root. For Example: If z be $=ax+cx^3+ex^5$, &c. then will $x=\frac{z}{a}-\frac{ac}{a^5}z^3+\frac{3a^2c^2-a^3e}{a^9}z^5$, &c. In like manner, if z be $=ax+bx^3+cx^5+dx^7+ex^9$, &c. then on the contrary will x be $=\frac{z}{a}-\frac{b}{a^4}z^3+\frac{3bb-ac}{a^7}z^5+\frac{8abc-aad-12b^3}{a^{10}}z^7+\frac{55b^4-55abbc+10aabd+5aacc-a^3e}{a^{13}}z^9$, &c.

SCHOLIUM.

3. These two Expressions for the Root x being thus found, will now serve as Canons for finding the Root of a proposed infinite Equation by Substitution.

For Example: If you would extract the Root of this Equation $z=x-\frac{xx}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}$, &c. Then substituting in the first Expression 1 for a, $-\frac{1}{2}$ for b, $\frac{1}{3}$ for c, $-\frac{1}{4}$ for d, and $\frac{1}{5}$ for e; there will come out $x=z+\frac{1}{2}zz+\frac{1}{6}z^3+\frac{1}{24}z^4$, &c. Moreover, the Root of this Equation $z=x-\frac{x^3}{3r}+\frac{x^5}{5r^4}-\frac{x^7}{7r^6}+\frac{x^9}{9r^7}$, &c. will be $x=z+\frac{z^3}{3r^2}+\frac{2z^5}{15r^4}+\frac{17z^7}{315r^6}+\frac{62z^9}{2835r^8}$, &c. by putting in the Expression 1 for a, $-\frac{1}{3r}$ for b, $\frac{1}{5r^4}$ for c, $-\frac{1}{7r^6}$ for d, $\frac{1}{9r^7}$ for e, &c.

SECTION II.

Of finding the Fluents or flowing Quantities of given fluxionary Expressions.

DEFINITION.

THE *fluent* or *flowing Quantity* of a given fluxionary Expression, is that Quantity whereof the given fluxionary Expression is the Fluxion. As the Fluent of $\dot{x}$ is x, the Fluent of $\dot{x}+\dot{y}$ is $x+y$, the Fluent of $x\dot{y}+y\dot{x}$ is xy, the Fluent of $mx^{m-1}\dot{x}$ is x^m, the Fluent of $ax^{\frac{m}{n}}\dot{x}$ is $\frac{an}{m+n}x^{\frac{m+n}{n}}$.

COROL. I.

4. HENCE if the Ordinate $PM(y)$ of a Curve (or strait Line) AM at right Angles to the Abscis AP (x) drawn into Pp $(\dot{x})$ represents any given fluxionary Expression; then the Area of the Space APM will be the fluent or flowing Quantity of the given fluxionary Expression; and, *vice versâ*, the Rectangle under the Ordinate PM, and $\dot{x}$ the Fluxion of the Abscis AP, will be the Fluxion of the Area or Space APM. For this Rectangle may be taken for the Trapezium $PMmp$, which FIG. 1.

is

is the real Fluxion of that Area: because their Difference is only the small Triangle *Mmn*, infinitely less than *PMmp*. And so it may be rejected. (*Axiom* 1. *Part* 1.)

COROL. II.

5. HENCE likewise it appears that the inverse Method of Fluxions is a kind of a general Way of summing up of Series.

SCHOLIUM.

6. HERE we may observe, that a Fluent can have but one Fluxion; but on the contrary, a Fluxion may have an infinite Number of Fluents. For Example: The Fluent ax will have but only this Fluxion $a\dot{x}$. And if b, c, d, f, g, &c. be constant Quantities; then will $ax \pm b$, $ax \pm c$, $ax \pm d$, $ax \pm f$, $ax \pm g$: or $ax \pm$, an infinite Variety of other constant Quantities, be each the Fluent of $a\dot{x}$. Whence accurately speaking, the Fluent or flowing Quantity of $a\dot{x}$ is not ax, but $ax \pm p$. p being any given Quantity, which may be equal to any other Expression whatsoever, consisting of constant Quantities. The same may be understood of the Fluents of other fluxionary Expressions.

As it is easy to raise a given Quantity to any given Power; but on the contrary, any Root thereof cannot be had in finite Terms; so likewise in the Business of Fluxions, it is easy to find the Fluxion of any variable Quantity, or variable and constant Quantities any how compounded together. But on the contrary, the Fluent of any given Fluxion cannot be had in finite

finite Terms. For as in Algebra we have recourse to Approximations in the Extraction of Surd Roots, where they cannot be exactly expreſs'd; ſo in the Inverſe Method of Fluxions we make uſe of infinite Series, where Fluents cannot be had exactly.

PROB. I.

7. TO *find the Fluent of a given fluxionary Expreſſion.*

Caſe I. 1. When fluxionary Expreſſions conſiſt of no Powers of flowing or variable Quantities, but Products of flowing Quantities multiply'd by Fluxions, aud alſo the Fluxion of every flowing Quantity that is in the Expreſſion, as $\dot{y}x+\dot{x}y$, or $xy\dot{z}+zx\dot{y}+zy\dot{x}$. The Fluents are had by this Rule, which is the Reverſe of the direct Operation, *viz.*

Inſtead of each Fluxion ſubſtitute its reſpective variable Quantity; and adding all the Terms together, divide that Sum by the Number of Terms.

So the Fluent of $\dot{y}x+\dot{x}y$ will be xy, and of $xy\dot{z}+zx\dot{y}+zyx$ will be xyz.

2. When ſimple fluxionary Expreſſions, involving ſome Power of the variable Quantity, occur, *viz.* multiplied into ſome ſtanding Quantity, as $2x\dot{x}$, or $3xx\dot{x}$, or $mx^{m-1}\dot{x}$, or $\frac{n}{m}x^{\frac{n-m}{m}}\dot{x}$, or $ax^{\frac{m}{n}}\dot{x}$, which is the moſt general Expreſſion of all of this Nature: The Fluent will be had by the Reverſe of the direct Operation. For as in the direct Operation, any of the foregoing Expreſſions is found by leſſening the Index of the Power of the variable Quantity by 1, putting in the fluxionary

nary Letter $\dot{x}$, and multiplying the whole by the Index of the Power of the variable Quantity; so we come back again to the Fluent by adding 1 to the Index of the Power of the variable Quantity, striking out the fluxionary Letter $\dot{x}$, and dividing all by the Exponent thus increas'd by 1: therefore in all such Cases this is the Rule.

Strike out the fluxionary Letter, add Unity to the Exponent of the variable Quantity in the Expression, and divide it by that Exponent thus increas'd by Unity.

Hence the Fluent of $2x\dot{x}$, or $2x^1\dot{x}$ is x^2. For striking out $\dot{x}$, and adding 1 to 1 (the Exponent of x in the given Expression) there will be had $2x^2$; which divided by 2 (= the Exponent increased by 1) the Quotient x^2 will be the Fluent of $2x\dot{x}$.

In like manner the Fluent of $3x^2\dot{x}$ will be x^3; of $mx^{m-1}\dot{x}$ will be x^m; of $\frac{n}{m}x^{\frac{n-m}{m}}\dot{x}$ will be $x^{\frac{n}{m}}$, of $-nx^{-n-1}\dot{x}$ will be x^{-n} or $\frac{1}{x^n}$; of $ax^{\frac{m}{n}}\dot{x}$ will be $\frac{an}{m+n}x^{\frac{m+n}{n}}$. For in this latter Case by adding 1 to $\frac{m}{n}$ the Index of the Power of the variable Quantity, and striking out the fluxionary Letter $\dot{x}$, we have $ax^{\frac{m+n}{n}}$; which divided by $\frac{m+n}{n}$ (the new Index) and the Quotient is $\frac{an}{m+n}x^{\frac{m+n}{n}}$, *viz.* the Fluent of $ax^{\frac{m}{n}}\dot{x}$.

Other-

Otherwise:

Because this latter Fluxion and its Fluent are the most general of any of those of the above-named Condition: They may serve as a Canon for finding of Fluents of such simple fluxionary Expressions as abovesaid, by bringing them under the same Form, and afterwards substituting. For Example: To find the Fluent of $x^2\dot{x}$. Now this brought to the same Form with the Fluxion $ax^{\frac{m}{n}}\dot{x}$ will be $1x^{\frac{2}{1}}\dot{x}$; so that a is $=1$, $n=1$, and $m=2$. Whence putting 1 for a, and 1 for n, and 2 for m in the Fluent $\frac{an}{m+n}x^{\frac{m+n}{n}}$, and then we shall have $\frac{1}{3}x^3$ for the Fluent of $1x^2\dot{x}$.

In like manner the Fluent of $4\sqrt{x}\,\dot{x}\ (=4x^{\frac{1}{2}}\dot{x})$ will be $\frac{8}{3}x^{\frac{3}{2}}\ (=\frac{8}{3}\sqrt[2]{x^3})$.

The Fluent of $\sqrt[3]{x^5}\,\dot{x}\ (=x^{\frac{5}{3}}\dot{x})$ will be $\frac{3}{8}x^{\frac{8}{3}}$ $=\frac{3}{8}\sqrt[3]{x^8}$.

The Fluent of $\frac{1}{x^2}\dot{x}\ (=1x^{-2}\dot{x})$ will be $\left(\frac{1}{-1}x^{\frac{-1}{1}}=\right)-x^{-1}$. For here $a=1$, $n=1$, and $m=-2$.

The Fluent of $\frac{1}{\sqrt{x^3}}\dot{x}\left(=x^{\frac{-3}{2}}\dot{x}\right)$ will be $\left(\frac{2}{-1}x^{\frac{-1}{2}}=\right)\frac{2}{-\sqrt{x}}$.

Lastly, The Fluent of $\frac{1}{x}\dot{x}\left(=1x^{-1}\dot{x}\right)$

will be $\frac{1}{0}x^{\frac{0}{1}} = \frac{1}{0}x^{0} = \frac{1}{0}\times 1 = \frac{1}{0}$ = Infinity. For here $a = 1$, $m = -1$, and $n = 1$.

Case II. If a fluxionary Expression consists of any Number of simple Terms, such as those in *N.* 2. *Case* I. the Fluent thereof will likewise consist of the Fluents of the several Terms of the Expression connected together with the Signs + and —. For Example:

The Fluent of $x^2\dot{x} + x^{\frac{3}{2}}\dot{x}$ or $\overline{x^2 + x^{\frac{3}{2}}} \times \dot{x}$, will be $\frac{1}{3}x^3 + \frac{2}{5}x^{\frac{5}{2}}$. For (by *N.* 2. *Case* I.) the Fluent of $x^2\dot{x}$ will be $\frac{1}{3}x^3$, and that of $x^{\frac{3}{2}}\dot{x}$ will be $\frac{2}{5}x^{\frac{5}{2}}$. And so the Sum of these Fluents will be the Fluent of the Sum of those Fluxions.

So likewise the Fluent of $x^2\dot{x} - x^{\frac{3}{2}}\dot{x}$ will be $\frac{1}{3}x^3 - \frac{2}{5}x^{\frac{5}{2}}$.

And that of $3x\dot{x} - 2x^2\dot{x} + x^3\dot{x} - 5x^4\dot{x}$ will be $\frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4 - x^5$.

Moreover, that of $x^{-2}\dot{x} + x^{-\frac{3}{2}}\dot{x}$, or that of $x^{-2}\dot{x} - x^{-\frac{3}{2}}\dot{x}$, will be $-x^{-1} - 2x^{-\frac{1}{2}}$, or $-x^{-1} + 2x^{-\frac{1}{2}}$; and by changing the Signs we shall have the affirmative Values, *viz.* $x^{-1} + 2x^{-\frac{1}{2}}$, or $x^{-1} - 2x^{-\frac{1}{2}}$.

That of $x^2\dot{x} + x^{-2}\dot{x}$, will be $\frac{1}{3}x^3 - x^{-1}$.

Case III. When the Term or Terms of a Fluxion are more compounded than any of those aforegoing, they must first be reduc'd to simple Terms like some of those of the foregoing *Cases*, by throwing the Expression, or some

some Part of it, into an infinite Series, according to the Rules of the first Section: then the Fluent of the Expression, thus brought into an infinite Series of simple Terms, may be had by the Rules before laid down. For Example: Let $\frac{b}{a+x}\dot{x}$ be a given Fluxion.

Now this thrown into an infinite Series (by *Prob.* 1. *Sect.* 1.) will be $\frac{b}{a}\dot{x} - \frac{bx}{a^2}\dot{x} + \frac{bx^2}{a^3}\dot{x} - \frac{bx^3}{a^4}\dot{x}$, &c. the Fluent of which by *Case* 2. will be $\frac{b}{a}x - \frac{b}{2a^2}x^2 + \frac{b}{3a^3}x^3 - \frac{bx^4}{4a^4}$, &c.

In like manner the Fluent of $\frac{1}{1+xx}\dot{x}$ = * $1\dot{x} - x^2\dot{x} + x^4\dot{x} - x^6\dot{x}$, &c. will be * $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$, &c. * Art. 1. * Art. 7.

The Fluent of $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1+x^{\frac{1}{2}}-3x}\dot{x}$ = * $2x^{\frac{1}{2}}\dot{x} - 2x\dot{x} + 7x^{\frac{3}{2}}\dot{x} - 13x^2\dot{x} + 34x^{\frac{5}{2}}\dot{x}$, &c. will be * $\frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{3}x^3$, &c. * Art. 1. * Art. 7.

Also the Fluent of $\sqrt{aa+xx}\times\dot{x}$ = * $a\dot{x} + \frac{x^2}{2a}\dot{x} - \frac{x^4}{8a^3}\dot{x} + \frac{x^6}{16a^5}\dot{x} - \frac{5x^8}{128a^7}$, &c. will be * $ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, &c. * Art. 2. * Art. 7.

The Fluent of $\frac{\sqrt{1+ax^2}}{\sqrt{1-bx^2}}$ will be

$$x \left.\begin{matrix} +\frac{1}{6}b \\ +\frac{1}{6}a \end{matrix}\right\} x^3 \left.\begin{matrix} +\frac{3}{40}b^2 \\ +\frac{1}{20}ab \\ -\frac{1}{40}a^2 \end{matrix}\right\} x^5, \text{ \&c.}$$

By the same Rules the Fluent of $dx^m \times \overline{e+fx^n}^p \times \dot{x}$ may be found: Where d, e, f are any given Quantities, and m, n, p the Indices of the Powers of the Quantities to whom they are affixed. For make $\frac{m+1}{n} = r$, $p+r=s$, $\frac{d}{nf} \times \overline{e+fx^n}^{p+1} = Q$, and $rn-n=t$: then will the Fluent be $Q \times \frac{x^t}{s} - \frac{r-1}{s-1} \times \frac{eA}{fx^n} + \frac{r-2}{s-2} \times \frac{eB}{fx^n} - \frac{r-3}{s-3} \times \frac{eC}{fx^n} + \frac{r-4}{s-4} \times \frac{eD}{fx^n}$, &c. The Letters A, B, C, D, &c. expressing the nearest preceding Terms, *viz.* A the Term $\frac{x^t}{s}$, B the Term $-\frac{r-1}{s-1} \times \frac{eA}{fx^n}$, &c. This Series when r is a Fraction or negative Number will not terminate, that is, the Fluent will consist of an infinite Series of Terms. But when r is an affirmative whole Number, the Fluent will consist of a finite Number of Terms, *viz.* so many as there are Units in r.

Otherwise.

This last Fluxion and its Fluent will serve as a Canon for finding the Fluents of fluxionary Expressions any how compounded, not exceeding Binomials, by bringing them to the same Form with the fluxionary Expression, and afterwards by Substitution, as you may see in the following Examples.

EXAMPLE I.

To find the Fluent of $\sqrt{ax} \times \dot{x}$. This brought to the same Form with the fluxionary Expreſſion above, and then $1x^0 \times \overline{0+ax^1}|^{\frac{1}{2}} \dot{x} = dx^m \times \overline{e+fx^n}|^p \dot{x}$. Whence $d=1$, $m=0$, $e=0$, $f=a$, $n=1$, $p=\frac{1}{2}$, $Q=\frac{1}{a}\times ax^{\frac{3}{2}}$, $t=0$, $r=1$, $s=1\frac{1}{2}$. And ſo by ſubſtituting theſe Values for their Equals in the general Fluent above, and we ſhall have $\frac{1}{a}\times ax^{\frac{3}{2}}\times\frac{1}{1\frac{1}{2}}=\frac{2}{3}x\sqrt{ax}$ the Fluent ſought. And generally the Fluent of $cx^n\dot{x}$ will be $\frac{c}{n+1}x^{n+1}$.

EXAMPLE II.

$\frac{a^4\dot{x}}{c^4-2ccxx+x^4}$ brought to the Form of the general Binomial fluxionary Expreſſion will be $a^4\dot{x}\times\overline{cc-xx}|^{-2}$, or $a^4x^{-3}\dot{x}\times\overline{-1+ccx^{-2}}|^{-2} = dx^m\times\overline{e+fx^n}|^p$. In the firſt Caſe, $d=a^4$, $m=1$, $e=cc$, $f=-1$, $n=2$, $p=-2$. Whence $r=1$, $s=-1$, $Q=-\frac{a^4}{2}\times\overline{cc-xx}|^{-1}$, that is, $-\frac{a^4}{2cc-2xx}$, $t=0$. And the Fluent will be $Q\times-\frac{x^0}{1}$, that is, $=\frac{a^4}{2cc-2xx}$. In the ſecond Caſe, d is $=a^4$, $m=-3$, $e=-1$, $f=cc$, $n=-2$, $p=-2$, $r=1$, $s=-1$, $Q=-\frac{a^4}{2cc}\times\overline{-1+ccx^{-2}}|^{-1}$, that is, $-\frac{a^4xx}{2c^4-2ccxx}$, $t=0$. And the Fluent will be $Q\times-\frac{x^0}{1}$, that is, $\frac{a^4xx}{2c^4-2ccxx}$.

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EXAMPLE III.

$\frac{a^5}{x^4}\sqrt{bx+xx}\,\dot{x}$ brought to the general Form will be $a^5x^{-\frac{9}{2}}\times\overline{b+x}|^{\frac{1}{2}}\times\dot{x}$, or $a^5x^{-4}\times\overline{1+bx^{-1}}|^{\frac{1}{2}}\times\dot{x}$. In the first Case d will be $=a^5$, $m=-\frac{9}{2}$, $e=b$, $f=1$, $n=p=\frac{1}{2}$. And so $r=-\frac{7}{2}$, &c. Now since r is negative, I try the other Case. Here d is $=a^5$, $m=-4$, $e=1$, $f=b$, $n=-1$, $p=\frac{1}{2}$. And so $r=3$, $s=3\frac{1}{2}$, $Q=-\frac{a^5}{b}\times\overline{1+bx^{-1}}|^{\frac{3}{2}}$ or $-\frac{a^5x+a^5b}{bxx}\sqrt{xx+bx}$, and $t=-2$. Whence the Fluent will be $Q\times\frac{x^{-2}}{3\frac{1}{2}}-\frac{2}{2\frac{1}{2}}\times\frac{x^{-1}}{3\frac{1}{2}b}+\frac{1}{1\frac{1}{2}}\times\frac{2}{2\frac{1}{2}}\times\frac{x^0}{3\frac{1}{2}bb}$, that is, $-\frac{30bb+24bx-16xx}{105bbxx}\times\frac{a^5x+a^5b}{bxx}\sqrt{xx+bx}$.

EXAMPLE IV.

$\frac{bx^{\frac{1}{3}}}{\sqrt[5]{c^3-3accx^{\frac{2}{3}}+3aacx^{\frac{4}{3}}-a^3xx}}\,\dot{x}$, brought to the Form as above, will be $bx^{\frac{1}{3}}\times\overline{c-ax^{\frac{2}{3}}}|^{-\frac{3}{5}}\times\dot{x}$. And so $d=b$, $m=\frac{1}{3}$, $e=c$, $f=-a$, $n=\frac{2}{3}$, $p=-\frac{3}{5}$, $r=2$, $s=\frac{7}{5}$, $Q=-\frac{2b}{2a}\times\overline{c-ax^{\frac{2}{3}}}|^{\frac{2}{5}}$, $t=\frac{2}{3}$. Then the Fluent will be $Q\times\frac{5x^{\frac{2}{3}}}{7}-\frac{5}{2}\times-\frac{5c}{7a}$, that is, $-\frac{30abx^{\frac{2}{3}}+75bc}{28aa}\times\overline{c-ax^{\frac{2}{3}}}|^{\frac{2}{5}}$.

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SCHOLIUM I.

8. GENERAL Forms may be found for the Fluents of Trinomial, or other more compounded fluxionary Expressions, after the same manner as the general Form above; and they used for finding the Fluents of fluxionary Expressions proposed, which are less compounded, or can be brought to the same Form with them. But the Work this way being most commonly extremely tedious, it will be best to get the Fluents by (the first way of *Case* 3. as foregoing) bringing those compound Terms to infinite Series of single Terms.

But compound Expressions must not be thrown into infinite Series before we have tried to reduce them by augmenting, lessening, multiplying, dividing, &c. the variable Quantities: For by this means they may often be brought down to such simple Forms as come under *Case* 2. aforegoing, and the Fluents of them be had in finite Terms.

Here it will be of Use to observe likewise, that if in radical fluxionary Expressions, the rational Part of the Expression, or that without the *Vinculum*, multiplied into the Fluxion of the variable Quantity, be the Fluxion of the Part under the *Vinculum*, or in some given Ratio to it; the Fluent will always be had in finite Terms, by *Case* 2. aforegoing, and by Substitution.

EXAMPLE. I.

THE Fluent of $a\dot{x}\sqrt{ax-aa}$ or $\overline{ax-aa}^{\frac{1}{2}} \times a\dot{x}$, where $a\dot{x}$ is the Fluxion of $\sqrt{ax-aa}$ or $\overline{ax-aa}^{\frac{1}{2}}$ will (by *Case* 2.) be $\frac{2}{3}\overline{ax-aa}^{\frac{3}{2}} = \frac{2}{3}\overline{ax-aa}\sqrt{ax-aa}$.

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EXAMPLE II.

THE Fluent of $2x\dot{x}\sqrt{xx+aa}$ or $\overline{xx+aa}^{\frac{1}{2}} \times 2x\dot{x}$, where $2x\dot{x}$ is the Fluxion of $\overline{xx+aa}^{\frac{1}{1}}$, will be $\frac{2}{3}\overline{xx+aa}^{\frac{3}{2}} = \frac{2xx+2aa}{3}\sqrt{xx+aa}$.

EXAMPLE III.

THE Fluent of $\overline{a+x}^{\frac{m}{n}} \times \dot{x}$ will be $\frac{n}{m+n}\overline{a+x}^{\frac{m+n}{n}}$.

EXAMPLE IV.

THE Fluent of $x\dot{x}\sqrt{xx+aa}$. Where the Fluxion $x\dot{x}$ without the *Vinculum* is to the Fluxion of the Quantity under it, *viz.* $2x\dot{x}$ as 1 to 2, will be $\frac{1}{3}xx+\frac{1}{3}a \times \sqrt{xx+aa}$. For make $\sqrt{xx+aa}=z$: then $2z\dot{z}=2x\dot{x}$; and so $\sqrt{xx+aa} \times x\dot{x}=z^2\dot{z}$. Whence the Fluent (by *Case* 2.) $=\frac{1}{3}z^3=$ by Substitution to the aforesaid Fluent. After this manner might the Fluent of the second Example have been found by putting $xx+aa=z^2$.

EXAMPLE V.

THE Fluent of $\overline{x^m+a^q}^n \times x^{m-1}\dot{x}$. Where the Fluxion $x^{m-1}\dot{x}$ without the *Vinculum* is to the Fluxion of the Quantity contained under it, *viz.* $mx^{m-1}\dot{x}$, as 1 to m, will be $\frac{1}{mn+m}\overline{x^m+a^q}^{n+1}$. For put $x^m+a^q=z$:

then

then will $\overline{x^m + aq}^n = z^n$, and $\overline{x^m + aq}^{n-1} = z^{n-1}$. Also $nz^{n-1}\dot{z} = nmx^{m-1}\dot{x} \times \overline{x^m + aq}^{n-1} = nmx^{m-1}\dot{x} \times z^{n-1}$. And dividing by nz^{n-1}, there comes out $\dot{z} = mx^{m-1}\dot{x}$, or $x^{m-1}\dot{x} = \frac{\dot{x}}{m}$. Whence $x^{m-1}\dot{x} \times \overline{x^m + aq}^n = \frac{z^n\dot{x}}{m}$; and the Fluent $= \frac{1}{nm + m} z^{n+1} = \frac{1}{nm + m} \overline{x^m + aq}^{n+1}$.

Here it may not be amiss to give the following Table of simple Fluxions, (from Sir ISAAC NEWTON's *Quadrature of Curves*) whose Fluents standing against them are expressed in finite Terms. By which Means the Fluent of a Fluxion coming under any of the Forms of the Fluxions in the said Table may be had by Inspection, or else a very easy Substitution. Here z is the variable Quantity, aud d, e, f, g, h, n are invariable or given Quantities.

	Forms of Fluxions.	Fluents.
I.	$d\dot{z}z^{n-1}$	Flu. $= \frac{d}{n} z^n$.
II.	$\frac{d\dot{z}z^{n-1}}{e^2+2efz^n+f^2z^{2n}}$	Flu. $= \frac{dz^n}{ne^2+nefz^n}$, or $\frac{-d}{nef+nf^2z^n}$.
III. 1.	$d\dot{z}z^{n-1}\sqrt{e+fz^n}$	Flu. $= \frac{2d}{3nf}R^3$, R being $= \sqrt{e+fz^n}$.
III. 2.	$d\dot{z}z^{2n-1}\sqrt{e+fz}$	Flu. $= \frac{-4e+6fz^n}{15nf^2} dR^3$.
III. 3.	$d\dot{z}z^{3n-1}\sqrt{e+fz^n}$	Flu. $= \frac{16e^2-24efz^n+30f^2z^{2n}}{105nf^3} dR^3$.
III. 4.	$d\dot{z}z^{4n-1}\sqrt{e+fz^n}$	Flu. $= \frac{-96e^3+144e^2fz^n-180ef^2z^{2n}+210f^3z^{3n}}{945nf^4} dR^3$.
IV. 1.	$\frac{d\dot{z}z^{n-1}}{\sqrt{e+fz^n}}$	Flu. $= \frac{2d}{nf} R$.
IV. 2.	$\frac{d\dot{z}z^{2n-1}}{\sqrt{e+fz^n}}$	Flu. $= \frac{-4e+2fz^n}{3nf^2} dR$.
IV. 3.	$\frac{d\dot{z}z^{3n-1}}{\sqrt{e+fz^n}}$	Flu. $= \frac{16e^2-8efz^n+6f^2z^{2n}}{15nf^3} dR$.
IV. 4.	$\frac{d\dot{z}z^{4n-1}}{\sqrt{e+fz^n}}$	Flu. $= \frac{-96e^3+48e^2fz^n-36ef^2z^{2n}+30f^3z^{3n}}{105nf^3} dR$.

SCHOLIUM II.

BEFORE I conclude this Section, I will take the Liberty of adding a Word or two concerning the excellent compendious Method of expressing the Fluents of given Fluxions by the Measures of Ratio's and Angles

gles written by the late Mr. *Cotes*, Professor of Astronomy and Experimental Philosophy in the University of *Cambridge*; and publish'd after his Death by his Successor Dr. *Smith*, under the Title of *Harmonia Mensurarum*.

Here the Labour of throwing Quantities into infinite Series, which in many Cases is very troublesome, and on account of their converging sometimes too slowly are not fit for Use, is entirely avoided, and elegant Constructions of the Fluents of Fluxions are had geometrically with the Assistance of ample Tables of Logarithms of *Briggs*'s *Form*, for finding the Measures of Ratio's, and of large Tables of natural Sines and Tangents for finding the Measures of Angles. And from hence may be deduced wonderful neat and compendious Solutions of all difficult Problems; such as the Quadrature of Curve-lin'd Spaces, Rectification of Curves, Cubation of Solids, &c. wherein the Fluents of given Fluxions are concerned. Several Examples of which I shall give hereafter.

In the Treatise before us, you have two Series of Tables of several Forms of Fluxions at the Head of each Page, with their Fluents underneath them expressed in the Measures of Ratio's or Angles; the one Series composed by Mr. *Cotes* himself, and the other by Dr. *Smith*. In Mr. *Cotes*'s Tables, which I shall confine myself to, they being sufficient for Purposes that usually occur, z is the variable Quantity, d, e, f standing Quantities, n any Index of the Power of z, θ any affirmative or negative Number; and the Quantities R, S, T, always being the three Sides of a right-angled Triangle, whose Values are set down at the Bottoms of each Page, express the Ratio or Angle, by the Measure of which the Fluent of the given Fluxion is had: and if R be the square Root of an affirmative Quantity, they express a Ratio; always being that of $R+T$ to S. But if R be the square Root of a negative Quantity they denote an Angle, which Angle will be always that whose Tangent and Secant are to the Radius as T and S to R, where the Sign of that negative Quantity is changed into an affirmative one. The Figures in the Column of each Page of the Tables, with θ at the top, are some of the affirmative and negative Values of θ, against which the Fluents of the fluxionary Forms a top stand. As in the second Form $\frac{dzz^{\theta n+\frac{1}{2}n-1}}{e+fz^n}$.

When θ is 2, the Fluent will be $\frac{2dz^{\frac{3}{2}n}}{3nf}$ $-\frac{2dez^{\frac{1}{2}n}}{nff}+\frac{2e}{nff}dR\left|\frac{R+T}{S}\right.$. When θ is o, the Fluent will be $\frac{2}{ne}dR\left|\frac{R+T}{S}\right.$. And when θ is -1, the Fluent will be $\frac{-2d}{nez^{\frac{1}{2}n}}$ $+\frac{2}{ne}dr\left|\frac{r+t}{s}\right.$. And so of others.

But these Fluents cannot be said to be entirely known till the Quantity $R\left|\frac{R+T}{S}\right.$ be found: Which is called the Measure of the Ratio of $R+T$ and S to the *Module* R, when R is affirmative, or till the Quantity $\frac{2e}{nff}dR$ $\left|\frac{R+T}{S}\right.$, or $\frac{2}{ne}dR\left|\frac{R+T}{S}\right.$, or $\frac{2}{ne}dr\left|\frac{r+t}{s}\right.$.

in

in the ſeveral Fluents be found; the firſt of which is the Meaſure of the Ratio of $R+T$ and S to the *Module* $\frac{2e}{nff}dR$; the ſecond the Meaſure of the Ratio of $R+T$, and S to the *Module* $\frac{2d}{ne}R$; and the third of $R+T$ and S to the *Module* $\frac{2}{ne}dR$. Or when R is negative, the Quantity $R\left|\frac{R+T}{S}\right.$ is $=$ Meaſure of an Angle, whoſe Radius, Tangent and Secant are the reſpective Values of R, T and S, to the Radius R as a *Module*. And the Way to find this Meaſure of a Ratio or Angle to a given *Module* I ſhall ſhew preſently; but firſt take the following Definitions, or Deſcriptions of Terms, in order to clear this ſo far, as that a Perſon may in ſome meaſure underſtand the Uſe of theſe excellent Tables, without being at the pains of reading the Propoſitions in the firſt Part of the *Harmonia Menſurarum:* which are too generally handled to be perceived by a moderate Capacity without much Application.

DEFIN. I.

THE *Meaſure* of a Ratio is any Quantity proportional to that Ratio; that is, if M be the Meaſure of the Ratio of A to B, or of $\frac{A}{B}$, and m the Meaſure of the Ratio of a to b, or $\frac{a}{b}$; then will $M:\frac{A}{B}::m:\frac{a}{b}$. Therefore equal Ratio's have the ſame Meaſure. If one Ratio be the Double of the other, the Meaſure

Measure of the former will be the Double of the Measure of the latter: if the former be the Triple of the latter, the Measure of the former will be the Triple of the Measure of the latter: if the Half, the Half, &c. So that if the Ratio be never so much increased or lessen'd by Composition or Resolution, the Measure thereof will be likewise increased or lessen'd proportionably.

Moreover the Measure of a Ratio of Equality is 0; and if the Measure of the Ratio of a greater Quantity to a less be supposed positive, then the Measure of the Ratio of a lesser Quantity to a greater will be negative.

DEFIN. II.

THE *Numerical Measure of a Ratio* is the Excess of the Logarithm of a Number expressing the Antecedent above the Logarithm of the Number expressing the Consequent; that is, the Logarithm of the Quotient of the Division of the Antecedent by the Consequent, is the numerical Measure of a numerical Ratio.

DEFIN. III.

THE *Trigonometrical Measure* of an Angle is the Quantity of Degrees, Minutes, Seconds, &c. contained in that Angle.

DEFIN. IV.

THE *Module* of *Brigge*'s, *Vlaque*'s, &c. Logarithms is 0,434294481903, &c. by which if you divide 1, the Quotient 2,302585092994, &c. will be the reciprocal *Module*

Module of the ſaid Logarithms; that is, the Quotient of the Diviſion of any Quantity by the firſt *Module*, is equal to the Product of that Quantity multiplied by the reciprocal *Module*.

This Definition, or rather Account of the Quantity of the *Module* of the Logarithms, is ſufficient for my Purpoſe. Thoſe who are not ſatisfy'd with it may conſult *Prop.* 1. and its *Corollaries*, and *Scholia* in Part 1. *Harmonia Menſurarum.* The ſame may be ſaid likewiſe of the following Definition, which is a Conſequence of the *Prop.* in the Notes of the ingenious Dr. *Smith* contained in *p.* 94. at the latter End of *Harmonia Menſurarum.*

DEFIN. V.

THE *Module* of the Trigonometrical Canon, or the Number of Degrees contained in the Arch of a Circle equal to the Radius, which is to 180 Degrees, as the Radius of a Circle to ½ the Circumference, is 57° 17′ 44″, or 57,2957795130; and dividing 1 by it, the reciprocal *Module* of the ſaid Canon will be 0,0174532925.

PROP. I.

TO find the Meaſure of a given Ratio to a given Module, or to find the Quantity of the Expreſſion $R \left| \frac{R+T}{S} \right.$ *when* R *is the ſquare Root of an affirmative Quantity, and* R, T, S, *the three Sides of a right-angled Triangle.*

Rule. As the *Module* of the Logarithms 0,434294481903, &c. is to the *Module R* of the

the Ratio $R+T$ to S; so is the Logarithm of this Ratio to the Measure of it, having R as a *Module*, which will be equal to $R\left|\frac{R+T}{S}\right.$ or multiply the Product of the Logarithm of the proposed Ratio of $R+T$ to S into the Quantity R, as a *Module*, by the reciprocal *Module* 2.302585092994, &c. and this second Product will be the Measure of the Ratio of $R+T$ to S with R, being the *Module* thereof; and it is equal to the Value of the Quantity $R\left|\frac{R+T}{S}\right.$ sought.

Take this Numerical Example. Let $R=8$, $T=6$, $S=10$.

Module of the Logarithm 0,434294481903 : 8 :: Logarithm $\left(\text{of } \frac{14}{10}\right)$ 0,1461280 : 2,6916777 = Measure of the Ratio of 14, and 10 to the *Module* 8, equal to $8\left|\frac{8+6}{10}\right.$.

Or shorter thus: 1 : recip. Mod. Logarithm 2.302585092994 :: Logarithm $\left(\text{of } \frac{14}{10}\right)$ 0,1461280×8 the given *Module* : 2,6916777 the Measure of the Ratio as before.

This Problem may be solved without Computation by means of the Sector of an Hyperbola, after the following manner. Let AG be an Hyperbola, CA the Semi-transverse Diameter, and CB the Semi-conjugate, and CE an Asymptote: and draw AQ parallel to CB. Then make $R : T :: AQ : AD$. And if $CA \times CB$ be $= 2R$, the Sector CAM will be $= R\left|\frac{R+T}{S}\right.$. The Triangle CAD being equal to T, when T is less than R; and the Triangle

FIG. 2.

gle CBE equal to T, when T is greater than R.

COROL. I.

10. HENCE if m be the standing *Module* of the Logarithms, and l the Logarithm of the Ratio $\frac{R+T}{S}$, then will $R\left|\frac{R+T}{S}\right. = R \times \frac{l}{m}$.

COROL. II.

11. HENCE $\frac{nR}{m}\left|\frac{T^{\frac{m}{n}}}{S^{\frac{m}{n}}}\right. = R\left|\frac{T}{S}\right.$. m being any whole Number, and n another. This follows from the Nature of the Logarithms.

PROB. II.

12. *TO find the Measure of an Angle, whose Radius is as* R, *Tangent as* T, *and Secant as* S, *to the Quantity* R *as a Module: Or to find the Value of this Expression* $R\left|\frac{R+T}{S}\right.$. *When* R *is the square Root of a negative Quantity, and so impossible; all of them being given Quantities.*

Rule. First say, As the Value of R to the Value of T, or as the Value of R to the Value of S; so is the Radius of the Tables of the Tangents and Secants to the Tabular Tangent or Secant of the Angle to be measur'd. Against which stands the Quantity of that Angle in Degrees, Minutes, &c. being the Trigonometrical Meaſure of it. Then as the *Module* of the Trigonometrical Canon 57°. 17′. 44″. or 57,2957795130, is to the Trigonometrical Meaſure of the Angle juſt now found, ſo is the Module of that Angle, *viz.* *R*, to the Meaſure of the ſaid Angle, having *R* as a Module; which is equal to the Quantity $R \left| \frac{R+T}{S} \right.$. Or multiply the Product of the Trigonometrical Meaſure of the Angle into the *Module R* by the reciprocal *Module* of the Canon 0,0174532925. And this ſecond Product will be the Meaſure of the Angle to the given *Module R*.

Here follows a numerical Example. Let *R* be = 16, *T* = 12, and *S* = 20.

As 16 : 12 :: Radius 10000000 : 7500000 = Tangent.

And as 16 : 20 :: Radius 10000000 : 12500000 = Secant.

Againſt both which in the Tables you have 36°. 52′. 6″. for the Trigonometrical Meaſure of the Angle.

Again: *Module* Trigonometrical Can. 57.2957795130 : Trigonometrical Meaſure 36,868333331 :: *Module* 16 : 10.4330985 = Meaſure of the Angle, whoſe Radius is as 16, Tangent as 12, and Secant as 20, to the *Module* 16; or equal to $16 \left| \frac{16+12}{20} \right.$.

Note, By uſing the reciprocal *Module*, 0174532925 will be ſhorter than by uſing the direct *Module*: And the Operations will be ſtill very much abbreviated, by uſing the Logarithms to find the fourth Terms of the Proportion, in the aforegoing Problems as well as this.

This

This Problem may be solv'd likewise by means of the Sector of a Circle or Ellipsis, after the following manner. Let CA, CB, the Semiaxes of the Quadrant of a Circle or Ellipsis AB, drawn into each other, be equal to $2R$. Draw AD parallel to CB, and BG parallel to CA; then make $R:T::CB:AD$, and draw the right Line $CMDG$; then the Sector CAM will be equal to $R\,\Big|\,\frac{R+T}{S}$, when R is greater than T = Triangle CAD, and the Sector $CBM = R\,\Big|\,\frac{R+T}{S}$. When R is less than T = Triangle CBG. FIG. 3.

Note, When S happens to be the Square Root of a negative Quantity, and so impossible, you may change the Sign, and every thing will succeed right.

COROL.

13. If a be the Angle, whose Radius, Tangent and Secant are as R, T, S, and m be the standing Module of the Canon; then $R\,\Big|\,\frac{R+T}{S}$ will be $= \frac{aR}{m}$.

SCHOLIUM.

14. BECAUSE R, T, S are always the three Sides of a right-angled Triangle; therefore $R\,\Big|\,\frac{R+T}{S}$ is $= R\,\Big|\,\frac{S}{T-R}$; supposing T to be the Hypothenuse. For from the Nature of a right-angled Triangle $T = \sqrt{RR+SS}$: and so $R\,\Big|\,\frac{R+T}{S}$ is $= R\,\Big|\,\frac{R+\sqrt{RR+SS}}{S}$. And

$R \left| \frac{S}{T-R} \right.$ is $= R \left| \frac{S}{\sqrt{RR+SS}-R} \right.$. And multiplying crosswise, you will find that $\overline{\sqrt{RR+SS}+R} \times \overline{\sqrt{RR+SS}-R}$ is $= SS$. The Sum of two Numbers multiplied by their Difference being equal to the Difference of their Squares; therefore $\frac{R+\sqrt{RR+SS}}{S}$ is $= \frac{S}{-R+\sqrt{RR+SS}}$; that is, $\frac{R+T}{S}$ is $= \frac{S}{T-R}$. Consequently $R \left| \frac{R+T}{S} \right. = R \left| \frac{S}{T-R} \right.$.

This evidently appears also in *Fig.* 83. For let the Hypothenuse $AB = T$, the Perpendicular $CB = R$, and the Base $AC = S$. Continue out AB till $BD = BC = R$, and make $BE = BC$; then $AD = R+T$, and $AE = T-R$. If a Circle be described about B with the Radius BC, it will pass through E and D, and AC will touch it in C; therefore (by *Prop.* 36. *lib.* 3. *Eucl.*) $\overline{AC}^2$ (SS) is $= AD \times AE = \overline{R+T} \times \overline{T-R}$; and so $\frac{R+T}{S}$ is $= \frac{S}{T-R}$.

SECT.

SECTION III.

Use of the inverse Method of Fluxions in the Quadrature of Curve-lined Spaces.

PROB.

15. TO *square a given Curve-lined Space.*

Having gotten the Equation expressing the Relation of an Abscifs *AP* (x) to its correspondent Ordinate *PM* (y) at right Angles to each other, find the Value of y, which multiply by $\dot{x}$, and the Fluent of the Expression thus arising will express the Quadrature of the indeterminate mixed-line Space contained under the Abscifs *AP*, Ordinate *PM*, and Curve *AM*. And if the Abscifs *AP* be determinate, *viz.* = to a given Quantity a; and accordingly the Curve itself so likewise: Then by substituting a for x in the Fluent aforesaid, there will come out an Expression for the Quadrature of the determinate mixed-lined Space. This will be plainer by the following Examples. FIG. 1.

But if the Area *CDEF* contained under two Curves or right Lines *DE*, *CF*, the right Line *CD*, and the Part *EF* of any right Line *AE* drawn from a given Point *A* in the right Line *DC* be sought, draw *Afe* infinitely near *AFE*, and from the Centre *A* describe the small FIG. 4.

ſmall Arches Fp, Eq: then from the Nature of the Curves find the Area of the Quadrilateral Figure $FEqp$; which is equal to the Difference of the little Sectors AFp, AEq, or equal to $\frac{1}{2}\,AE \times Eq - \frac{1}{2}\,AF \times Fp$: and this will be = Space $FEef$, being the Fluxion of the Area $CDEF$; the Fluent of which will be = ſaid Area. Examples of this will be ſhewn hereafter.

EXAMPLE I.

16. *To find the Area of a Triangle* ABC.

FIG. 5. Draw AD perpendicular to one of the Sides, as BC; in which aſſume any where between A and D the Point P. Thro' which draw the Line MN perpendicular to AD; and let mn be infinitely near and parallel to MN; and draw Mp, Nq perpendicular to MN. Now the Rectangle $MNqp$, viz. under Mq, or Nq, or Pp, and the Ordinate MN will be the Fluxion of the indeterminate Area AMN. Which muſt firſt be found thus:

Call the variable Quantities AP, x; PM, y; and the given Quantities AD, a; CB, b. Then becauſe MN is parallel to AB. $AD\ (a) : CB\ (b) :: AP\ (x) : MN\ (y)$. Whence $y = \frac{bx}{a}$. But $Pp\ (= Mp = Nq) = \dot{x}$. Therefore the Fluxion of the indefinite Area AMN will be $\frac{bx}{a}\dot{x}$. The Fluent of which (by *Caſe* 2.) will be $\frac{bx^2}{2a}$ = Area AMN. Now if for $AP\ (x)$ you ſubſtitute $AD\ (a)$, there will ariſe $\frac{ba^2}{2a}$ $(= \frac{1}{2}ab) = \frac{1}{2}AD \times CB$. And this will be the

Area of the whole Triangle *ACB*; which we know likewiſe to be true from the Elements of Geometry.

Here it may not be amiſs likewiſe to ſhew the way of finding the Area of a Trapezium, *GPCB*, having two Sides *PC*, *GB* parallel, and the Angles *B*, *C* right ones, by the Inverſe Method of Fluxions; tho' the thing is much ſhorter found out by the Elements of common Geometry. But it is delightful to perceive the ſame Truth ariſe from ſuch very different Principles. FIG. 6.

In order to this, continue out *CB* and *PG* to meet one another in *A*. From *A* draw *Amp* infinitely near *AGP*, and with the Diſtances *Am*, *Ap* deſcribe from *A* the ſmall Arches *mr*, *pn*. This being done, let $AB=a$, $BC=b$, $BG=x$, $AG=y$. Now the Triangles *ACP*, *pPn* are ſimilar; becauſe the Angles at *n* and *C* are right ones, and the Angle *P* is common to both. Whence $AG\ (y):AB\ (a)::Gm\ (\dot{x}):mr=\frac{a\dot{x}}{y}$. This multiplied by $\frac{1}{2}Ar$ $\left(=\frac{1}{2}AG=\frac{y}{2}\right)$, and the Product $\frac{a\dot{x}}{2}$ is equal to the Area of the little Triangle *Arm* or *AGm*: theſe differing only by the Triangle *Gmr* being infinitely leſs than either of them.

Again; Becauſe the Triangles *ABG*, *ACP* are ſimilar; therefore $AB\ (a):AG\ (y)::BC\ (b):GP=\frac{by}{a}$. Conſequently $AP=\frac{by}{a}+y$. And ſince *Am*, *AG*, and *AP*, *Ap* differ from one another by infinitely ſmall Quantities only: therefore *Am* may be taken for *AG* and *Ap* for *AP*. This being granted, and the Triangles *Amr*, *Apn*, taken as ſimilar; we ſhall

ſhall have $AG(y) : rm\left(\frac{a\dot{x}}{y}\right) : AP\left(\frac{by}{a}+y\right) :$ $pn = \frac{b\dot{x}}{y} + \frac{a\dot{x}}{y}$; which multiplied by $\frac{1}{2}$ AP $\left(\frac{by}{2a}+\frac{y}{2}\right)$; and the Product $\frac{bb\dot{x}}{2a} + b\dot{x} + \frac{1}{2}a\dot{x}$ is equal to the Area of the Triangle Apn or APp. From which if you ſubſtract the Area of the Triangle $AmG\left(\frac{a\dot{x}}{2}\right)$ before found, the Remainder $b\dot{x} + \frac{bb}{2a}\dot{x}$ will be the Area of the Trapezium $mpnr$, which may be taken for the Area of the Trapezium $mpPG$, being the Fluxion of the Trapezium $BCPG$. But the Fluent of this Fluxion is $bx + \frac{bbx}{2a} = \frac{2abx + bbx}{2a} = \overline{\frac{2ax+xb}{a}}$ $\times \frac{b}{2}$. But $\frac{ax+xb}{a}$ is $= PC$. Since from the Similarity of the Triangles ABG, ACP, AB $(a) : BG\ (x) :: AC\ (a+b) : CP = \frac{ax+xb}{a}$. Therefore $\overline{GB+PC} \times \frac{1}{2}BC =$ Area of the Trapezium $GPCB$. Which is a known Truth from the Elements of common Geometry.

EXAMPLE II.

17. *To find the Area of (or to ſquare) the common parabolical Space* ABD.

FIG. 1. Call the given Quantities AD, a; BD, b: and the invariable Abſciſs AP, x; and the correſpondent Ordinate PM, y: and let Pp be $= \dot{x}$. Now the Equation expreſſing the Relation between $AP\ (x)$ and $PM\ (y)$ is $px = yy$. Whence $y = \sqrt{px} = p^{\frac{1}{2}}x^{\frac{1}{2}}$. And ſo the ſmall Rectangle $PMnp$, equal to the Fluxion of the indefinite Space AMP will be $\sqrt{px} \times \dot{x}$; that is

is, $y\dot{x} = p^{\frac{1}{2}} \times x^{\frac{1}{2}}\dot{x}$. The Fluent of which (by *Case* 2. *Sect.* 2.) will be $= (\frac{2}{3} p^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{2}{3}\sqrt{px^3} = \frac{2}{3}\sqrt{x^2 y^2}) \frac{2}{3} xy$, equal to the indeterminate Space *AMP*; and by substituting *a* for *x*, and *b* for *y* in this Fluent; and then we shall have $\frac{2}{3}ab = \frac{2}{3} AP \times PM$; that is, the parabolick Space is to the Rectangle under the Semi-ordinate and Abscifs, as $\frac{2}{3}xy$ to xy, or as 2 to 3.

EXAMPLE III.

18. TO *square Parabola's of all Kinds.*

If *AP* (*x*) be the Abscifs, and *PM*(*y*) the correspondent Ordinate, then the Relation between the Ordinate and Abscifs of a Parabola of any kind will be expressed by this general Equation $p^m x^n = y^q$. Whence $p^{\frac{m}{q}} x^{\frac{n}{q}} = y$; and so the Fluxion of the Area will be $y\dot{x} = p^{\frac{m}{q}} x^{\frac{n}{q}} \dot{x}$. The Fluent of which (by *Case* 2. *Sect.* 2.) will be $\frac{q}{n+q} p^{\frac{m}{q}} \frac{n+q}{xq} = \frac{q}{n+q} xy$, because $p^{\frac{m}{q}} x^{\frac{n}{q}} = y$. And so any Paraboloid is to the Rectangle under the Ordinate and Abscifs, as $\frac{qxy}{n+q}$ to xy, or q to $n+q$. FIG. 1.

EXAMPLE IV.

19. TO *square the Segment of the Parabolick Space* PMNQ *contained under the Ordinates* PM, NQ, *the Part* PQ *of the Abscifs, and the Part* MN *of the Curve of the Parabola.* FIG. 7.

Here let $AP = a$ be invariable, and let P be the Beginning of the variable Quantity $PQ = x$. Alſo let QN be $= y$, and the Parameter $= p$; then AQ will be $= a + x$, draw nq parallel and infinitely near to NQ.

Now the Nature of the Curve is $\overline{AP + PQ} \times p = \overline{NQ}^2$; that is, $pa + px = yy$, and $\sqrt{pa + px} = y$. Therefore $QN \times Qq = y\dot{x} = \dot{x}\sqrt{pa + px} = 1\,x^0 \times \overline{pa + px}^{\frac{1}{2}}\dot{x}$ is the Fluxion of the Area, the Fluent of which may be found (by the latter way of *Caſe* 2. *Sect.* 2.) or more eaſily thus: Let $\sqrt{pa + px} = z$; then will $pa + px = zz$, and $p\dot{x} = 2z\dot{z}$, and $\dot{x} = \frac{2z\dot{z}}{p}$: therefore $y\dot{x}$ is $= \frac{2z^2\dot{z}}{p}$: the Fluent of which is $\frac{2}{3}\frac{z^3}{p} = \frac{2}{3}\frac{\overline{pa + px} \times \sqrt{pa + px}}{p} = \frac{2}{3}\overline{a + x} \times \sqrt{pa + px} = \frac{2}{3} AQ \times NQ$.

But ſince in the Point P x is $= 0$, the Space $PMNQ$ will vaniſh; and ſo making $x = 0$ in the Fluent juſt now found, and the Terms x and px will vaniſh; ſo that the Fluent will be now $\frac{2}{3}a\sqrt{pa}$. Which ſhews what is to be added to the Fluent, that ſo the Space $MPNQ$ may become 0 in P; and conſequently the Quadrature of the ſame had. In this Caſe $\frac{2}{3}a\sqrt{pa}$ is to be taken from it: therefore the Area of $MPNQ$ will be $= \frac{2}{3}\overline{a + x}\sqrt{pa + px} - \frac{2}{3}a\sqrt{pa} = \frac{2}{3}AQ \times NQ - \frac{2}{3}AP \times PM$.

Otherwiſe.

Draw mp infinitely near to MP, and let $AQ = a$ be invariable. Let the Beginning of x be in Q, and let QP be $= x$, $PM = y$; then will AP be $= a - x$.

Now

Now from the Nature of the Curve $\overline{AQ - QP} \times p = \overline{PM}^2$; that is, $pa - px = yy$. Whence $\sqrt{pa - px} = y$: and so $MP \times Pp$ the Fluxion of the Area will be $y\dot{x} = \dot{x}\sqrt{pa - px}$. The Fluent of which may be found as before thus: Make $pa - px = zz$. Then will $-p\dot{x}$ be $= 2z\dot{z}$, and consequently $\dot{x} = -\frac{2z^2\dot{z}}{a}$. Therefore $y\dot{x} = -\frac{2z^2\dot{z}}{a}$. The Fluent of which will be $-\frac{2z^3}{3a} = -\frac{\frac{2}{3}\overline{a - x} \times \sqrt{pa - px}}{a} = \frac{2}{3}\overline{x - a} \times \overline{pa - px}$.

Now to find what is to be added to the Fluent to give us the Quadrature of the Space $PMNQ$, making as before $x = 0$ in the Fluent, and we have $-\frac{2}{3}a\sqrt{pa}$. Whence it is manifest, that if $+\frac{2}{3}p\sqrt{pa}$ be added to the Fluent, the Space $PMNQ$ will be $= \frac{2}{3}a\sqrt{pa} + \frac{2}{3}\overline{x - a} \times \overline{pa - px}$.

COROL. I.

20. THE Space $PMNQ = ANQ - AMP$. But in the first way $ANQ = \frac{2}{3}AQ \times QN = \frac{2}{3}\overline{a + x} \times \overline{pa + px}$; and $AMP = \frac{2}{3}AP \times PM = \frac{2}{3}a\sqrt{pa}$: therefore $PMNQ = \frac{2}{3}AQ \times QN - \frac{2}{3}AP \times PM$. Moreover in the latter way: $ANQ = \frac{2}{3}AQ \times QN = \frac{2}{3}a\sqrt{ap}$, and $AMP = \frac{2}{3}AP \times PM = \frac{2}{3}\overline{a - x} \times \sqrt{pa - px}$; therefore $QNMP$ is $= \frac{2}{3}AQ \times QN - \frac{2}{3}AP \times PM$ here also.

COROL. II.

21. If the Curve be not described, and the Equation expressing the Nature of it be only given, and so it is uncertain where x is to begin; it is evident from the Solution above, that o must be substituted for x in the Fluent; and striking out all the Terms affected with x, what is left must be added to the Fluent with the Sign changed, and the whole will be the Quadrature sought.

EXAMPLE V.

22. *To square a Curve expressed by this Equation* $x^5 + ax^4 + a^2x^3 + a^3x^2 + a^5 = a^4y$.

Since $y = \frac{x^5}{a^4} + \frac{x^4}{a^3} + \frac{x^3}{a^2} + \frac{x^2}{a} + a$; the Fluxion of the Area will be $y\dot{x} = \overline{\frac{x^5}{a^4} + \frac{x^4}{a^3} + \frac{x^3}{a^2} + \frac{x^2}{a} + a} \times \dot{x}$, and the Fluent will be $\frac{x^6}{6a^4} + \frac{x^5}{5a^3} + \frac{x^4}{4a^2} + \frac{x^3}{3a} + ax$.

EXAMPLE VI.

23. *To square any Curve, whose Nature is expressed by this general Equation* $y = x\sqrt[m]{x+a}$.

Because y is $= x \times \overline{x+a}^{\frac{1}{m}}$, therefore the Fluxion of the Space sought will be $y\dot{x} = x\dot{x} \times \overline{x+a}^{\frac{1}{m}}$; the Fluent of which will be easily found by making $\overline{x+a}^{\frac{1}{m}} = z$. For then

$x+a=z^m$; and both Sides of this last Equation thrown into Fluxions will be $\dot{x}=mz^{m-1}\dot{z}$. Whence $y\dot{x}=mz^m\dot{z}$, the Fluent of which will be $=\frac{mz^{m+1}}{m+1}=\frac{m}{m+1}\overline{x+a}\times\sqrt[m]{x+a}$. Now in order to know whether this be the true Fluent, suppose $x=0$; then this last Expression will become $\frac{m}{m+1}a\sqrt[m]{a}$. Which must therefore be taken from the Fluent, (by *Cor.* 2. *Example* 3.) so that the true Fluent or Quadrature of the Curve will be $\frac{m}{m+1}\overline{x+a}\times\sqrt[m]{x+a}-\frac{m}{m+1}a\sqrt[m]{a}$.

EXAMPLE VII.

23. *TO square Hyperbola's between the Asymptotes of all Kinds; or, which is the same, to find the Area of the interminate Space* HMPAS, *or* hMPs. *The former contained under the Abscis* AP, *Ordinate* PM, *Asymptote* AS, *and the Part* MH *of the Curve of the Hyperbola; and the latter under the Ordinate* PM, *the remaining Part* Ps *of the Asymptote, and the remaining Part* Mh *of the Curve of the Hyperbola.* FIG. 8.

In these Curves the Relation between *AP*(x) and *PM* (y) is expressed by the following Equation $a^{m+n}=y^m x^n$.

Whence $a^{m+n}x^{-n}=y^m$, and $a^{\frac{m+n}{m}}x^{-\frac{n}{m}}=y$. Consequently $PM\times Pp=y\dot{x}$ is $=a^{\frac{m+n}{m}}x^{-\frac{n}{m}}\dot{x}$. The Fluent of which will be $\left(\frac{m}{m-n}a^{\frac{m+n}{m}}\right.$

$\times x^{-}$

$\times x^{\frac{-n+m}{m}} = \frac{m}{m-n}\sqrt[m]{a^{m+n}x^{m+n}} = \frac{m}{m-n}\sqrt[m]{y^m x^n}$) $= \frac{m}{m-n}xy$. If m be greater than n, then we have always the Quadrature of the Interminate Space *HMPAS*: the same being $= \frac{m}{m-n} AP \times PM$: If m be less than n, then will $\frac{m}{m-n}xy$ be negative; and so we have the Quadrature of the Interminate Space *hMPs* lying on the other Side of the Ordinate *PM*. But when $m = n$, neither of these Spaces can be squared, they being in this Case both infinite. For if $xy^2 = a^3$, then $m = 2$, $n = 1$; and so $HMPAS = \sqrt{a^3 x} = (\sqrt{x^2 y^2}) = xy$. If $xy^4 = a^5$, then $m = 4$, $n = 1$; and so $HMPAS = \frac{4}{3}xy$. If $x^2 y = a^5$, then $m = 1$, $n = 2$; and so $-xy$ will be the Quadrature of the Space *hMPs*. If $x^4 y = a^5$, then will $m = 1$, $n = 4$, and so $-\frac{1}{3}xy$, that is, $\frac{1}{3}xy = hMSs$. But when $m = n$, then $\frac{m}{m-n}$ is $= \frac{1}{0}$: therefore the Numerator is infinite in respect of the Denominator.

EXAMPLE VIII.

25. *To square the common Hyperbola between its Asymptotes; or, which is the same thing, to find the Area of the Space* CcMP
FIG. 8. *contained under the Ordinates* Cc, PM, *the Part* CP *of the Asymptote, and the Part* cM *of the Curve of the Hyperbola.*

Let the given Quantity *AC* be $= b$, and let the Beginning of x be at *C*.

Then

Then the Equation expressing the Relation of AP $(b+x)$ to PM (y) will be $a^2=by+xy$. And so $\frac{a^2}{b+x}=y$; and $\dot{x}y$ the Fluxion of the Area will be $\frac{a^2}{b+x}\dot{x}$.

Now the Fluent of this (by *Case* 3. *Sect.* 1.) will be $\frac{a^2}{b}x-\frac{a^2}{2b^2}x^2+\frac{a^2x^3}{3b^3}-\frac{a^2x^4}{4b^4}$, &c. = to the Area $CcMP$. And if you suppose $a=b=1$, then will $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$, &c. be = Area aforesaid.

Otherwise by the Measure of a Ratio or Angle.

The Fluxion $\dot{x}\frac{a^2}{b+x}$ may be referred to the 1st Form in Mr. *Cotes*'s Tables. For making $z=x$, $n=1$, $\theta=1$, $d=a^2$, $e=b$, $f=1$; then will $\frac{d\dot{z}z^{\theta n-1}}{e+fz^n}=\frac{a^2\dot{x}}{b+x}$. And the Fluent standing against $\theta=1$, is $\frac{d}{nf}\Big|\frac{e+fz^n}{e}=a^2\Big|\frac{b+x}{b}=\overline{AC}^2\Big|\frac{AP}{AC}=$ to the Measure of the Ratio of AP and AC to $\overline{AC}^2$ for a Module, which you may find by *Art.* 9. = Area of the Space $CcMP$; and if the Asymptote AS be not perpendicular to As, and so the Ordinate PM parallel to it, not perpendicular to the Absciss AP; the Measure of the Ratio of AP to PM, with the Parallelogram ACc as a Module, will be still the Area of the Space $CcMP$.

This is demonstrated synthetically *Page* 12. *Part* 1. *Harmonia Mensurarum*.

EXAM-

EXAMPLE IX.

FIG. 9. 26. *TO find the Area of the Space* ACMP *contained under the Part* AP(x) *of an infinite right Line perpendicular to the Axis of the Hyperbola, the Line* AC (a) *being the Continuation of the same Axis, and any Line or Ordinate* PM (y) *parallel to* AC.

Here $aa+xx=yy$; and so $y=\sqrt{aa+xx}$. Consequently $\dot{x}y=\dot{x}\sqrt{aa+xx}$. The Fluent of which (by *Case* 3. *Sect.* 1.) will be $=ax+\frac{x^3}{6a}-\frac{x^5}{40a^3}+\frac{x^7}{112a^5}-\frac{5x^9}{1152a^7}$, &c. $=$ Area *ACMP* sought. Which gives the Quadrature of the Segment *DCM* of the Hyperbola, by substracting the same from the Rectangle *ADMP* $(\dot{x}y)$; and if a be $=1$, then will the Series be $x+\frac{x^3}{6}-\frac{x^5}{40}+\frac{x^7}{112}-\frac{5x^9}{1152a^7}$, &c.

Otherwise by the Measure of a Ratio or Angle.

The Fluxion $\dot{x}\sqrt{aa+xx}$ comes under the fourth Form of the Tables of *Cotes*. For if $z=x$, $n=2$, $\theta=0$, $d=1$, $e=aa$, $f=1$, then will $d\dot{z}z^{\theta n+\frac{1}{2}n-1}\sqrt{e+fz^n}$ be $=\dot{x}\sqrt{aa+xx}$. The Fluent of which is $\frac{z^n}{n}dP+\frac{e}{nf}dR\left|\frac{R+T}{S}\right.$; and making $P\left(=\sqrt{\frac{e+fz^n}{z^n}}\right)=\frac{1}{x}\sqrt{aa+xx}$, $R(=\sqrt{f})=1$, $T\left(=\sqrt{\frac{e+fz^n}{z^n}}\right)=\frac{1}{x}\sqrt{aa+xx}$, $S\left(=\sqrt{\frac{e}{z^n}}\right)=\frac{a}{x}$. The said Fluent will become

come $\frac{x}{2}\sqrt{aa+xx}+\frac{aa}{2}\Big|\frac{x+\sqrt{aa+xx}}{a}=\frac{1}{2}AP$ $\times PC+\frac{1}{2}\overline{AC}^2\Big|\frac{AP+PC}{AC}=$ Area $ACPM$; that is, $\frac{1}{2}AP\times PC$ *plus* the Measure of the Ratio of $AP+PC$, and AC to $\frac{1}{2}AC$, as a Module will be the said Area.

The Quadrature of the Hyperbolick Space AMP may be had thus, as laid down by Mr. *Cotes* in *Harmonia Mensurarum*.

Make the Semi-conjugate Diameter $CB=b$, the Semi-transverse $CA=a$, $CP=x$, $PM=y$; then from the Nature of the Curve we have $\frac{b}{a}\sqrt{xx-aa}=y$; and so $\frac{b}{a}\dot{x}\sqrt{xx-aa}=$ Fluxion of the Space AMP. FIG. 10.

Now making $d=\frac{b}{a}$, $z=x$, $\theta=0$, $n=2$, $e=-aa$, $f=1$; the Fluxion of the 4th Form in the Tables of Mr. *Cotes*, viz. $d\dot{z}z^{\theta n+\frac{1}{2}n-1}\sqrt{e+fz^n}$ will become $\frac{b}{a}\dot{x}\sqrt{xx-aa}$. And the Fluent of it against $\theta=0$, is $\frac{z^n}{n}dP+\frac{e}{nf}dR\Big|\frac{R+T}{S}$. And making $P\left(=\sqrt{\frac{e+fz^n}{z^n}}\right)=\frac{1}{x}\sqrt{xx-aa}$, $R(=f)=1$, $T\left(=\sqrt{\frac{e+fz^n}{z^n}}\right)=\frac{1}{x}\sqrt{xx-aa}$, $S\left(=\sqrt{\frac{e}{z^n}}\right)=\frac{a}{x}$; the same will become $\frac{bx}{2a}\sqrt{xx-aa}-\frac{ab}{2}\Big|\frac{x+\sqrt{xx-aa}}{a}=$ $\frac{xy}{2}-\frac{ab}{2}\Big|\frac{a}{x-\frac{ay}{b}}$, by substituting y for its Equal

$\frac{b}{a}\sqrt{xx-aa}$. And *Art.* 14. where the Ratio of $x+\sqrt{xx-aa}$ to a, is shewn to be equal to the Ratio of a to $x-\sqrt{xx-aa}=x-\frac{ay}{b}$. Therefore $\frac{xy}{2}-\frac{ab}{2}\left|\frac{a}{x-\frac{ay}{b}}\right.$ is the Fluent of the Fluxion $\frac{b}{a}\dot{x}\sqrt{xx-aa}$. Which may be thus constructed.

Assume $CF:CA(a)::PM(y):CB(b)$; that is, make $CF=\frac{ay}{b}$. And assume $CG:CA(a)::CB(b):PM(y)$; that is, make $CG=\frac{ab}{y}$. Then if CH be taken equal to the Measure of the Ratio between $CA(a)$ and $FP\ (=x-\frac{ay}{b})$ to the Module $\frac{ab}{y}$; that is, if PH be taken $=\frac{ab}{y}\left|\frac{a}{x-\frac{ab}{y}}\right.$: And you draw the right Line MH; the right-lined Triangle HMP is = Space AMP. For it is $=\frac{y}{2}\times\overline{x-\frac{ab}{y}}\left|\frac{a}{x-\frac{ab}{y}}\right.=\frac{yx}{2}-\frac{ab}{2}\left|\frac{a}{x-\frac{ab}{y}}\right.$. And to find the Quadrature of the external Hyperbolick Space *CAMP* after the same manner, we may proceed thus.

FIG. 11. Let AC, CB be Semi-conjugate Diameters, $AC=a$, $CB=b$, $PM=x$, $CP=y$; then

$\frac{b}{a}$

$\frac{b}{a}\sqrt{xx-aa}$ is $=y$, from the Nature of the Curve. This Equation thrown into Fluxions, and there will arise $\frac{bbx\dot{x}}{aay}=\dot{y}=\frac{bbx\dot{x}}{ab\sqrt{xx-aa}}=\frac{bx\dot{x}}{a\sqrt{xx-aa}}$. But $\dot{y}\times x$ is $=\frac{bxx\dot{x}}{a\sqrt{xx-aa}}=$ Fluxion of the Hyperbolick Space $CAMP$.

Now making $d=\frac{b}{a}$, $z=x$, $n=2$, $\theta=1$, $e=-aa$, $f=1$; the Fluxion $\frac{d\dot{z}z^{\theta n+\frac{1}{2}n-1}}{\sqrt{e+fz^n}}$ of the sixth Form in the Tables will become $\frac{bx^2\dot{x}}{a\sqrt{xx-aa}}$. And the Fluent of it against $\theta=1$, will be $\frac{z^n}{nf}dP-\frac{e}{nff}dR\left|\frac{R+T}{S}\right.$. Whence making $P\left(=\sqrt{\frac{e+fz^n}{z^n}}\right)=\frac{1}{x}\sqrt{xx-aa}$, $R(=\sqrt{f})=1$, $T\left(=\sqrt{\frac{e+fz^n}{z^n}}\right)=\frac{1}{x}\sqrt{xx-aa}$, $S\left(=\sqrt{\frac{e}{z^n}}\right)=\frac{a}{x}$; the said Fluent will become $\frac{bx}{2a}\sqrt{xx-aa}+\frac{ab}{2}\left|\frac{x+\sqrt{xx-aa}}{a}\right.=\frac{xy}{2}+\frac{ab}{2}\left|\frac{a}{x-\frac{ay}{b}}\right.$, (by substituting y for $\frac{b}{a}\sqrt{xx-aa}$, and *Art.* 14.) $=$ Fluent of the Fluxion $\frac{bx^2\dot{x}}{a\sqrt{xx-aa}}$. The Construction of which may be thus.

Assume $CF:CB\,(b)::PM\,(x):AC\,(a)$; that is, make $CF=\frac{bx}{a}$; and $CG:CB\,(b)::AC$

(a) : PM (x); that is, make $CG = \frac{ba}{x}$. Then if CH be taken equal to the Meaſure of the Ratio between BC (b) and PF $\left(\frac{bx}{a} - y\right)$. (Which is equal to the Ratio of a to $x - \frac{ay}{b}$) to the Module CG $\left(\frac{ba}{x}\right)$: the ſaid CH will be $= \frac{ba}{x} \bigg| \frac{a}{x - \frac{ay}{b}}$. And you draw the right Line MH, the right-lined Triangle is equal to the Space $CAMP$. For it is $= \frac{x}{2} \times y + \frac{ba}{x} \bigg| \frac{a}{x - \frac{ay}{b}} = \frac{xy}{2} + \frac{ba}{2} \bigg| \frac{a}{x - \frac{ay}{b}}$, the Fluent of the given Fluxion.

EXAMPLE X.

27. *TO ſquare the Circle; or, which is the ſame thing, to find the Area of any Semi-ſegment* APM *thereof.*

Make $AB = 1$, $AP = x$, $PM = y$. Then
FIG. 12. from the Nature of the Circle $AP \times PB = \overline{PM}^2$; that is, $x - xx = yy$; and ſo $y = \sqrt{x - xx}$. Conſequently $\dot{x}y = \dot{x}\sqrt{x - xx} = PM \times Pp$ is the Fluxion of the Area AMP. The Fluent of which (by *Caſe* 3. *Sect.* 1.) will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. $=$ Area AMP, or $x^{\frac{1}{2}}$ into $\frac{2}{3}x - \frac{1}{5}x^2 - \frac{1}{28}x^3 - \frac{1}{72}x^4$, &c.

Otherwise.

If the Radius CM be $= a$, and CP be supposed $= x$, $PM = y$; then from the Nature of the Circle $\overline{CM}^2 = \overline{CP}^2 + \overline{PM}^2$; that is, $aa = xx + yy$, $yy = aa - xx$, and $y = \sqrt{aa - xx}$. Therefore $\dot{x}y = \dot{x}\sqrt{aa - xx}$ is the Fluxion of the Indeterminate Space $PMDC$, and the Fluent of this, *viz.* $ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, &c. = Area $PMDC$. Now if a be $= 1 = x$, then this Series will become $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152}$, &c. = equal to the Area of the Quadrant ADC. And four times this will be the Quadrature of the whole Circle $ADBA$; or if the Diameter be 1, the said Series will express the Area of the whole Circle.

Otherwise:

Let AE the Tangent of $\frac{1}{2}$ the Arch AM be $= x$, the Radius $AC = 1$. Let AB be the Tangent of the Arch AM. Draw the Secants CE, CB, and the Sine MP of the Arch AM. Let pm be infinitely near PM, and from the Centre C draw the Secant Cb thro' the Point m; also from the Point M draw Mt perpendicular to pm, and from the Point B draw Bs perpendicular to Cb. FIG. 13.

Now we propose here to find the Area of the indefinite Sector ACM; the Fluxion of which, being the little Sector MCm, must first be found thus:

First,

First, the Tangent *AB* of the Arch *BM* will be $=\frac{2x}{1-xx}$. Now because the Angle *ACB* is bisected by the right Line *CE*, therefore $AE(x):AC(1)::EB\left(\frac{x+x^3}{1-xx}\right):CB=\frac{1+x^2}{1-x^2}$. Also because of the similar Triangles *ACB*, *PCM*, $CB\left(\frac{1+x^2}{1-x^2}\right)::AB\left(\frac{2x}{1-x^2}\right)::AC(1):PM=\frac{2x}{1+x^2}$, and $CB\left(\frac{1+x^2}{1-x^2}\right):AC(1)::CM(1):CP=\frac{1-x^2}{1+x^2}$. Whence $AP\ \frac{2x^2}{1+x^2}$; the Fluxion of which will be $\frac{4x\dot{x}}{\overline{1+xx}^2}=Pp$ or Mt.

Again, the little Triangle *Mmt* right-angled at *t*, will be similar to the right-angled Triangle *CMP*, the Angle *tMm* being = Angle *PMC*, and the Angle *tmM* = Angle *PCM*, as is easy to prove. Therefore $MP\left(\frac{2x}{1+xx}\right):AC(1)::Mt\left(\frac{4x\dot{x}}{\overline{1+xx}^2}\right):Mm=\frac{2\dot{x}}{1+xx}$. And so $\frac{1}{2}MC(1)\times Mm=\frac{\dot{x}}{1+xx}$ = Area of the little Sector *MCm*, being the Fluxion of the Sector *AMC*. The Fluent of which will be $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{9}x^9$, &c. = Area of the Indeterminate Sector *AMC*. And when the Tangent *AE* (x) of half the Arch *AM* becomes $=1=$ Radius, then the Sector *ACM* will become a Quadrant; and the Series aforesaid, expressing the Area of the same, will be $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}$, &c.

&c. and if the Diameter of the Circle be $=1$, the whole Area will be expressed by that Series.

The said Series may be found shorter thus: If AB be $=x$, then will $Bb=\dot{x}$, and $CB=\sqrt{1+xx}$. Now the little Triangle BSb right-angled at S, is similar to the Triangle ABC, the Angle ABC differing from the Angle b only an infinitely small Quantity, and so they may be taken for Equals; therefore $CB\ (\sqrt{1+xx}) : AC\ (1) :: Bb\ (\dot{x}) : Bs = \frac{\dot{x}}{\sqrt{1+xx}}$. Moreover, since Bs is infinitely small, CB and Cs differ from one another only by an infinitely small Quantity: therefore $CB\ (\sqrt{1+xx}) : Bs \left(\frac{\dot{x}}{\sqrt{1+xx}}\right) :: MC\ (1) : Mm = \frac{\dot{x}}{1+xx}$. Whence the little Sector $MCm = \frac{\dot{x}}{2+2xx}$ = Fluxion of the Area of the Sector AMC. The Fluent of which will be $\frac{1}{2}x - \frac{1}{6}x^3 + \frac{1}{10}x^5 - \frac{1}{14}x^7 + \frac{1}{18}x^9$, &c. = Area of the said Sector: so that when the Sector ACM is an eighth Part of the Circle, *viz.* when the Tangent $AB\ (x)$ = Radius AC is $=1$; then the aforesaid Series will become $\frac{1}{2} - \frac{1}{6} + \frac{1}{10} - \frac{1}{14} + \frac{1}{18}$, &c. which doubled will be $1 - \frac{1}{3} + \frac{1}{5}$, &c. = Area of the Quadrant as before.

EXAMPLE XI.

28. *TO square the Elliptic Space: or, which is the same thing, to find the Area of any Indeterminate Elliptic Segment* ACMP *contain'd under the Semi-conjugate Diameter* AC, *the Ordinate* FIG. 14.

dinate PM, *Part of the Abscifs* AP, *and* CM *Part of the Curve of the Ellipsis.*

Call AC, a, AB or AD, b, AP, x, PM, y; then from the Nature of the Ellipsis $\overline{MP}^2 = \frac{\overline{AD}^2 \times \overline{\overline{AB}^2 - \overline{AP}^2}}{\overline{AC}^2}$, that is, $yy = \frac{bb \times \overline{aa - xx}}{aa}$, and so $y = \frac{b}{a}\sqrt{aa - xx}$. Consequently $\frac{b}{a}\dot{x}\sqrt{aa - xx}$ will be the Fluxion of the Space $ACPM$, and the Fluent of this will be $bx - \frac{bx^3}{6a^2} - \frac{bx^5}{40a^4} - \frac{bx^7}{112a^6}$, &c. = Area $ACPM$.

Now if you put a for x in this Series, it will become $ab - \frac{1}{6}ab - \frac{1}{40}ab - \frac{1}{112}ab$, &c. = Area of the Quadrant ACD of the Ellipsis. And if a be = Axis BD, then this last Series will express the Area of the whole Ellipsis. And if $\sqrt{ab}$ be = 1, then the Area of the Ellipsis will be $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{1}{1152}$, &c. and therefore an Ellipsis is equal to a Circle, whose Diameter is a mean Proportional between the Transverse and conjugate Diameters of the Ellipsis. And so an Ellipsis is to a Circle of the same Diameter with the transverse Axis, as ab to a^2, or b to a, *viz.* as the conjugate Axis is to the transverse Axis.

Otherwise.

To find the Area of any Sector CMA *of the Ellipsis.*

FIG. 15. 24. Let CB be the Semi-conjugate, and CA the Semi-transverse Diameter, MP a Semi-Ordinate. Now draw mp infinitely near MP,

MP, join the Points *C* and *M*, and *m* by the right Lines *CM*, *Cm*; and from *m* draw the short right Line *mH* perpendicular to *MP*, cutting *MP* in *I*, and *CM* in *H*; as also the little Line *mK* perpendicular to *CM* continued out.

This done, let $AC=a$, $BC=1$, $AP=x$, PM (y), $CM=u$, and $CP=z$. Now the first thing to be found must be the Area of the small Triangle *CMm*. Thus: Because the Triangles *CPM*, *HIM* are similar; therefore $PM(y):CP(z)::MI(\dot{y}):IH=\frac{z\dot{y}}{y}$. Whence (since $Pp=Im=\dot{x}=-\dot{z}$) $Hm=\frac{z\dot{y}}{y}-\dot{z}$. Again; because of the similar Triangles *CMP*, *HKM*. $CM(u):PM(y)::Hm\left(\frac{z\dot{y}}{y}-\dot{z}\right)$: $mK=\frac{z\dot{y}-y\dot{z}}{u}$. Which drawn into half the Base $CM(u)$; and then the Area of the fluxionary Triangle *CMm* will be $=\frac{z\dot{y}-y\dot{z}}{2}$; and throwing the Equation of the Curve $\frac{aa-zz}{aa}=yy$ into Fluxions we have $-\frac{z\dot{z}}{aay}=\dot{y}$. Which substitute in $\frac{z\dot{y}-y\dot{z}}{2}$ for $\dot{y}$, and there arises $\frac{-zz\dot{z}}{2aay}-\frac{y\dot{z}}{2}=\frac{-zz\dot{z}-aayy\dot{z}}{2aay}$. In which substituting $aa-zz$ for its Equal $aayy$, and it will be $\frac{-zz\dot{z}-aa\dot{z}+zz\dot{z}}{2aay}=\frac{-a\dot{z}}{2ay}=\frac{a\dot{x}}{2ay}$; since $\dot{x}$ is $=-\dot{z}$. And again substituting $\sqrt{2ax-xx}$ for ay; and the Triangle *CMm* will be

be

FIG. 16. be $= \frac{a\dot{z}}{2\sqrt{2ax-xx}} =$ Fluxion of the Sector ACM of the Ellipsis. And if you make $\sqrt{2ax-xx} = \frac{x}{n}$ then will x be found $= \frac{2ann}{1+nn}$, and $\dot{x} = \frac{4an\dot{n}}{1+nn}$; and by due Substitution $\frac{a\dot{x}}{2\sqrt{2ax-xx}} = \frac{a\dot{n}}{1+nn}$. The Fluent of which is $an - \frac{an^3}{3} + \frac{an^5}{5} - \frac{an^7}{7}$, &c. $=$ Sector of the Ellipsis.

And after the same manner you will find the Sector of the Hyperbola (*Fig.* 16.) to be $an + \frac{an^3}{3} + \frac{an^5}{5} + \frac{an^7}{7}$, &c. and the Sector of the Circle will be $= n - \frac{n^3}{3} + \frac{n^5}{5} - \frac{n^7}{7}$, &c. and this becomes $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$, &c. by making $n = 1$.

This may be done something shorter, thus:

FIG. 17, 18. Draw the Tangent AD, and continue out CM to cut the same in D. Draw Cd infinitely near CD, and from C describe the small Arches Mn, De. Make $CB = b$, $CA = a$, $AD = x$, $CD = y$, $CP = z$; because the Triangles CAD, Ded are similar, the Angle A differing from the Angle D only by an infinitely small Angle, which may be rejected, and the Angles at e, and A right ones. Therefore $CD\ (y) : AC\ (a) :: Dd\ (\dot{x}) : De = \frac{a\dot{x}}{y}$. And because of the similar Triangles CPM, CAD, $AC\ (a) : CD\ (y) :: CP\ (z) : CM \left(\frac{yz}{a}\right)$. Likewise since the Sectors CDe, CMn are similar, $CD\ (y) :$

De

$De\left(\frac{a\dot{x}}{y}\right)::CM\left(\frac{yz}{a}\right):Mn=\frac{z\dot{x}}{y}$. Now $\frac{1}{2}CM\times Mn$, that is, $\frac{z\dot{x}}{2y}\times\frac{yz}{a}=\frac{zz\dot{x}}{2a}$ is = Area of the Triangle CMm, being the Fluxion of the Sector CAM. Again, from the Nature of the Curve, and the Similarity of the Triangles CPM, CAD, in the Ellipsis we have $PM\left(\frac{b}{a}\sqrt{aa-zz}\right):CP\ (z)::AD\ (x):AC\ (a)$; and in the Hyperb. $PM\left(\frac{b}{a}\sqrt{-aa+zz}\right):CP(z)::AD(x):AC(a)$. And multiplying the Means and Extremes, we have $zx=b\sqrt{aa+zz}$ in the Ellipsis, and $zx=b\sqrt{-aa+zz}$ in the Hyperbola. Consequently $zz=\frac{aabb}{bb+xx}$ in the Ellipsis, and $zz=\frac{aabb}{bb-xx}$ in the Hyperbola. Which being substituted in $\frac{zz\dot{x}}{2a}$, we have $\frac{abb\dot{x}}{2bb+2xx}$ in the Ellipsis, and $\frac{abb\dot{x}}{2bb-2xx}$ in the Hyperbola, = Fluxion of the Area of the Sector CAM, the Fluent of which will be $\frac{ax}{2}-\frac{ax^3}{6bb}+\frac{ax^5}{10b^4}-\frac{ax^7}{14b^6}$, &c. in the Ellipsis, and $\frac{ax}{2}-\frac{ax^3}{6bb}-\frac{ax^5}{10b^4}-\frac{ax^7}{14b^6}$, &c. in the Hyperbola = Area of the Sector CAM; and making $CB\ (b)=1$, the said Fluent will be $\frac{1}{2}ax-\frac{1}{6}ax^3+\frac{1}{10}ax^5-\frac{1}{14}ax^7$, &c. in the Ellipsis, and $\frac{1}{2}ax-\frac{1}{6}ax^3-\frac{1}{10}ax^5-\frac{1}{14}ax^7$, &c. in the Hyperbola. And in the Ellipsis, if x be $=b$, the Area of the Sector CAM will be $\frac{1}{2}ab-\frac{1}{6}ab+\frac{1}{10}ab-\frac{1}{14}ab$, &c. and when x

 $=a$,

$=a$, the Area of the Sector CBM will be also $\frac{1}{2}ab - \frac{1}{6}ab + \frac{1}{10}ab - \frac{1}{14}ab$, &c. the Sum of which, *viz.* $ab - \frac{1}{3}ab + \frac{1}{5}ab - \frac{1}{7}ab$, &c. will be the Area of the Quadrant of the Ellipsis ABC.

COROL.

HENCE if $a=b$, *viz.* when the Ellipsis is a Circle; the Area of the Quadrant of a Circle, whose Radius is a, will be $aa - \frac{1}{3}aa + \frac{1}{5}aa - \frac{1}{7}aa$, &c. and if a be $=1$, the Area of the Quadrant will be $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$, &c.

The Fluent of $\frac{abb\dot{x}}{2bb-2xx}$, the Fluxion of the Sector of the Hyperbola, may be found in the Measure of a Ratio; for it may be referr'd to the second Form in the Tables of Mr. *Cotes*; since making $z=x$, $d=abb$, $\theta=0$, $n=2$, $e=2bb$, $f=-2$, the Fluxion in the second Form $\frac{d\dot{z}z^{\theta n+\frac{1}{2}n-1}}{e+fz^n} = \frac{abb\dot{x}}{2bb-2xx}$, and the Fluent of it against $\theta=0$, is $\frac{2}{ne}dR\left|\frac{R+T}{S}\right.$. Now $R\left(=\sqrt{\frac{-e}{f}}\right)=b$, $T\left(=z^{\frac{1}{2}n}\right)=x$, and $S\left(=\sqrt{\frac{e+fz^n}{f}}\right)$ $=\sqrt{bb-xx}$. Whence $\frac{2}{ne}dR\left|\frac{R+T}{S}\right. = \frac{1}{2}ab$ $\left|\frac{b+x}{\sqrt{bb-xx}}\right. = \frac{1}{2}AC \times CB\left|\frac{CB+AD}{\sqrt{\overline{CB}^2-\overline{AD}^2}}\right.$; that is, the Area of the Hyperbolical Sector CAM is equal to the Measure of the Ratio of $CB+AD$ to $\sqrt{\overline{CB}^2-\overline{AD}^2}$, the Triangle ACB being the Module.

EXAM-

EXAMPLE XII.

30. TO square the Space AMp, called the Figure of the Tangents.

The Nature of this Figure is, that any Abscifs AP is equal to any Arch (AP) of a Circle, and the correspondent Ordinate PM at right Angles to it, is equal to the correspondent Tangent AT of the Arch. FIG. 19, 20.

Draw the Secant CT, and the Secant Ct infinitely near to it; and from C with the Distance CT describe the small Arch Tc. Make the Radius $AC = a$, the Arch AP (= Abscifs AP) $= x$, and the correspondent Tangenr AT (= correspondent Ordinate pM) $= y$. Now because the Triangles ATC, Ttc, are similar, the Angles at T, t, differing from each other only by an infinitely small Angle, and the Angles at A and C right Angles; therefore $TC\ (\sqrt{aa+yy}) : AC\ (a) :: Tt\ (\dot{y}) : Tc = \frac{a\dot{y}}{\sqrt{aa+yy}}$. Again, because of the similar Sectors CPp, CTc, therefore $CT\ (\sqrt{aa+yy}) : Tc\ \left(\frac{a\dot{y}}{\sqrt{aa+yy}}\right) :: CP\ (a) : Pp\ (=\dot{x})$. And multiplying the Means and Extremes, we get $\dot{x}\sqrt{aa+yy} = \frac{aa\dot{y}}{\sqrt{aa+yy}}$; and dividing by $\sqrt{aa+xx}$, there will be had $\dot{x} = \frac{aa\dot{y}}{aa+yy}$ = Fluxion of the Abscifs Ap; and multiplying by y, there will arise $\dot{x}y = \frac{aay\dot{y}}{aa+yy}$ = Fluxion of the Space

 AMp;

AMp; and the Fluent of it will be $yy - \frac{y^4}{4aa} + \frac{y^6}{6a^4} - \frac{y^8}{8a^6} +$, &c. = Area sought.

Otherwise by the Measure of a Ratio.

This Fluxion $\frac{aay\dot{y}}{aa+yy}$ may be compared with the first Form in Mr. *Cotes*'s Tables. For if $z=y$, $n=2$, $\theta=1$, $d=aa$, $e=aa$, $f=1$; the first Form $\frac{d\dot{z}z^{\theta n-1}}{e+fz^n}$ will be $= \frac{aay\dot{y}}{aa+yy}$. And against θ $(=1)$, the Fluent will be $\frac{d}{nf}\left|\frac{e+fz^n}{e}\right. = \frac{1}{2}aa\left|\frac{aa+yy}{aa}\right. = \frac{1}{2}\overline{AC}^2\left|\frac{\overline{CT}^2}{\overline{AC}^2}\right.$, which is equal to $\overline{AC}^2\left|\frac{CT}{AC}\right.$. For putting $2l$ for the Logarithm of the Ratio $\frac{\overline{CT}^2}{\overline{AC}^2}$, and m for the Module of the Logarithms; then will $\frac{1}{2}\overline{AC}^2$

* Art. 11. $\left|\frac{\overline{CT}^2}{\overline{AC}^2}\right.$ be * $= \frac{1}{2}\overline{AC}^2 \times \frac{2l}{m} = \overline{AC}^2 \times \frac{l}{m}$. But from the Nature of the Logarithms l is equal to the Logarithm of the Ratio $\frac{CT}{AC}$; therefore $\overline{AC}^2$

* Art. 11. $\times \frac{l}{m}$ is * $= \overline{AC}^2\left|\frac{CT}{AC}\right. = \frac{1}{2}\overline{AC}^2\left|\frac{\overline{CT}^2}{\overline{AC}^2}\right.$; therefore the Area *ApM* is equal to the Measure of the Ratio between the Secant *CT* of the Arch $AP = Ap$, and the Radius *AC*, having the Square of the Radius for a Module.

EXAM-

EXAMPLE XIII.

31. TO square the Space ABpM, called the Figure of the Secants.

Here the Abscifs Ap is equal to the Arch AP, as in the last Example; but the correspondent Ordinate pM is = Secant CT, and the Ordinate AB = Radius AC, every thing else being as in the last Example; only let CT be $= y$. Then from the Similarity of the Triangles ACT, Ttc, we have $AT\,(\sqrt{yy-aa})$ $::AC\,(a):tc\,(\dot{y}):Tc = \frac{a\dot{y}}{\sqrt{yy-aa}}$; and because of the similar Sectors CTc, CPp, therefore $CT\,(y):Tc\,\left(\frac{a\dot{y}}{\sqrt{yy-aa}}\right)::CP\,(a):Pp\,(\dot{x})$; and multiplying the Means and Extremes, there arises $y\dot{x} = \frac{aa\dot{y}}{\sqrt{yy-aa}}$ = Fluxion of the Space $ABMp$, and the Fluent thereof will be very easily found in the Measure of a Ratio, it coming under the sixth Form of Mr. *Cotes*'s Tables. For making $z=y$, $n=2$, $\theta=0$, $d=aa$, $e=-aa$, $f=1$, we shall have $\frac{d\dot{z}z^{\theta n+\frac{1}{2}n-1}}{\sqrt{e+fz^n}} = \frac{aa\dot{y}}{\sqrt{-aa+yy}}$. Also in this Form $R\,(=\sqrt{f})=1$, $T\left(=\frac{fz^{\frac{1}{2}n}}{\sqrt{e+fz^n}}\right) = \frac{y}{\sqrt{-aa+yy}}$, and $S\left(=\sqrt{\frac{-ef}{e+fz^n}}\right) = \frac{a}{\sqrt{-aa+yy}}$. Whence the

FIG. 21.

the Fluent (θ being $= 0$) $\frac{nf}{2} dR \left| \frac{R+T}{S} \right.$ will be $aa \left| \frac{y+\sqrt{-aa+yy}}{a} \right. = \overline{AC}^2 \left| \frac{CT+AT}{AC} \right.$.

Consequently the Area of the Space $ABMp$ will be equal to the Measure of the Ratio between the Sum of the Secant and Tangent of the same Arch, and the Radius to the Square of the Radius for a Module.

COROL.

HENCE the Meridional Parts in *Mercator's* Chart may be computed for any given Latitude AP. It is well known *, that the Meridional Parts of any given Latitude AP, are to the Length of the Arch AP, as the Sum of the Secants in them to the Sum of as many Semidiameters; that is, as the Area of the Curve-lin'd Space $ABMp$ to the Rectangle $BA \times pA$; or as $\overline{AC}^2 \left| \frac{CT+AT}{AC} \right.$ to $AC \times AP$, or as $AC \left| \frac{CT+AT}{AC} \right.$ to AP. Whence the Meridional Parts are equal to $AC \left| \frac{CT+AT}{AC} \right.$ or* $AC \left| \frac{AC}{CT-AT} \right.$.

* *Art.* 14.

EXAMPLE XIV.

32. TO *find the Area of the Space* CPQc *contained under the Parts of the Conchoids* CPD, cqd *of* Nicomedes, *the Part* Cc *of the right Line* AC *drawn from the Pole* A *perpendicular to the common Asymptote* BG, *and the*

* See *Philosophical Transf.* N°. 176.

the Part QP *of any right Line drawn from the said Pole* A, *and intercepted between the Curves.*

Let $Ac = a$, $cB = CB = b$, $cC = 2b$, $AB = c$, $AG = y$, $BG = x$. Draw Ap infinitely near AP; and from A, as a Centre, describe the little Concentrick Arches qs, mr, pn. Now the Nature of these Curves is, that BC or Bc is $= GP$ (pm) or GQ (mq). Because the Triangles AGB, rGm are similar, the Angle at G being common, and the Angles at r and B right ones (the Arch rm being taken for a straight Line) therefore AG $(y) : AB$ (c) $:: GM(\dot{x}) : rm = \frac{c\dot{x}}{y}$. Again, since the Sectors Arm, Asq are similar, therefore $Am = AG$ $(y) : rm$ $\left(\frac{c\dot{x}}{y}\right) :: Aq = AQ = AG - AQ$ $(y - b) : sq = \frac{cy\dot{x} - cb\dot{x}}{yy}$. And again, from the Similarity of the Sectors Arm, Anp; therefore AG $(y) : mr$ $\left(\frac{c\dot{x}}{y}\right) : AP$ $(y + b) : pn = \frac{cy\dot{x} + cb\dot{x}}{yy}$. Consequently $pn + qs$ is $= \frac{2c\dot{x}}{y}$. And so $\frac{1}{2}\overline{pn + qs} \times PQ = \frac{2cb\dot{x}}{y}$ is $=$ Area of the little Space $sPpq =$ Space $QPpq =$ Fluxion of the Space $CPQc$. And substituting $\sqrt{cc + xx}$ for y, the Fluxion will be $\frac{2cb\dot{x}}{\sqrt{cc + xx}}$. Which comes under the sixth Form in the Tables of Mr. *Cotes*. For making $d = 2cb$, $z = x$, $\theta = 0$, $n = 2$, $e = cc$, $f = 1$, the said sixth Form $\frac{d\dot{z}z^{\theta n + \frac{1}{2}n - 1}}{\sqrt{e + fz^n}}$ will become $\frac{2cb\dot{x}}{\sqrt{cc + xx}}$. The Flu-

FIG. 22.

ent of which ſtanding againſt $\theta=0$, is $\frac{2}{nf}dR$ $\left|\frac{R+T}{S}\right.$. And making $R\,(=\sqrt{f})=1$, $T\,(=\sqrt{\frac{e+fz^n}{z^n}})=\frac{1}{x}\sqrt{cc+xx}$, and $S\,(=\sqrt{\frac{e}{z^n}})=\frac{c}{x}$, the ſaid Fluent will become $2bc\left|\frac{x+\sqrt{cc+xx}}{c}\right.$ $=$ Fluent of the given Fluxion $=$ Area of the Space $CPQc$, $=Bc\times AB\left|\frac{BG+AG}{AB}\right.$. That is, the Area of the ſaid Space is equal to a Rectangle under Bc, and the Meaſure of the Ratio between $BG+AG$, and AB to the Module AB.

FIG. 23. *But to find the Area of the Space* CPGB *contained under the Part* PC *of only one of the Conchoides of* Nicomedes, BE *being the Aſymptote*, A *the Pole, and* ABC *at right Angles to* BE.

Call AB, a, BC, b, AG, y, AB, x. Draw Ap infinitely near to AP, and from A deſcribe the ſmall Arches mr, pn. Then by reaſoning as before, we ſhall have $\frac{abb\dot{x}}{2aa+2xx}+\frac{ab\dot{x}}{\sqrt{aa+xx}}=$ Area of the little Space $GPpm=$ Fluxion of the Space $CPGB$. The firſt Part of which Fluxion comes under the ſecond Form in the Tables of Mr. *Cotes*: for making $d=abb$, $z=x$, $\theta=0$, $n=2$, $e=2aa$, $f=2$, the Fluxion of the ſecond Form, *viz.* $\frac{d\dot{z}z^{\theta n+\frac{1}{2}n-1}}{e+fz^n}$ will become $\frac{abb\dot{x}}{2aa+2xx}$; the Fluent of which is $\frac{2}{ne}dR$ $\left|\frac{R+T}{S}\right.$.

$\left|\frac{R+T}{S}\right.$. In which substituting for $R\ (=\sqrt{\frac{-e}{f}})$ $\sqrt{-aa}$, for $T\ (=z^{\frac{1}{2}n})\ x$, and for $S\ (=\sqrt{\frac{e+fz^n}{f}})$ $\sqrt{aa+xx}$, and we shall have $\frac{1}{2}bb\left|\frac{a+x}{\sqrt{aa+xx}}\right.$ for the Fluent of the first Part $\frac{abb\dot{x}}{2aa+2xx}$.

Again, the latter Part $\frac{ab\dot{x}}{\sqrt{aa+xx}}$ of the Fluxion may be compared with the sixth Form in the Tables of Mr. *Cotes* by making $d=ab$, $z=x$, $\theta=0$, $n=2$, $e=aa$, $f=1$; and the Fluent thereof against $\theta=0$, *viz.* $\frac{2}{nf}dR\left|\frac{R+T}{S}\right.$, by making $R=1$, $T=\frac{1}{x}\sqrt{aa+xx}$, and $S=\frac{a}{x}$, will be $ab\left|\frac{x+\sqrt{aa+xx}}{a}\right.$. Therefore the Sum of these Fluents $\frac{1}{2}bb\left|\frac{a+x}{\sqrt{aa+xx}}\right.+ab$ $\left|\frac{x+\sqrt{aa+xx}}{a}\right.$ will be the Fluent of the whole Fluxion. Consisting likewise of two Parts, the former of which is the Measure of an Angle, because R is $=\sqrt{-aa}$; and the latter of a Ratio, because R is $=\sqrt{1}=1$.

The Construction of which Fluent may be thus: Draw CF perpendicular to AC, assume BH equal to the Measure of the Ratio between $BG+AG$, and AB to AB taken as a Module, that is, make $BH=a\left|\frac{x+\sqrt{aa+xx}}{a}\right.$. FIG. 24.

Moreover, assume BI equal to the Measure of the Angle GAB, to the same Module AB, that is, $=a\left|\frac{a+x}{\sqrt{aa+xx}}\right.$. The Radius, Tangent, and Secant of which being as a, x, and $\sqrt{aa+xx}$, or AB, BG, AG. This done, draw AIE and HD parallel to the same; then will the Trapezium $BHDC$ be equal to the whole Fluent; and consequently equal to the Space $CPGB$. For since the Triangles ABH, ACD are similar, therefore AB (a) : BI $\left(a\left|\frac{a+x}{\sqrt{aa+xx}}\right.\right)$: : AC $(a+b)$: CE = $a\left|\frac{a+x}{\sqrt{aa+xx}}\right.+b\left|\frac{a+x}{\sqrt{aa+xx}}\right.$. And $\frac{\overline{CE+IB}}{2}\times BC=\frac{ab}{2}\left|\frac{a+x}{\sqrt{aa+xx}}\right.+\frac{bb}{2}\left|\frac{a+x}{\sqrt{aa+xx}}\right.=$ Trapezium $BIEC$. Again, HI $(=BH-BI)$ is $=a\left|\frac{x+\sqrt{aa+xx}}{a}\right.-a\left|\frac{a+x}{\sqrt{aa+xx}}\right.$; which multiplied by BC (b), and the Product $ab\left|\frac{x+\sqrt{aa+xx}}{a}\right.-ab\left|\frac{a+x}{\sqrt{aa+xx}}\right.$ (= Parallelogram $HDEI$) added to the Trapezium $BIEC$ will be $\frac{ab}{2}\left|\frac{a+x}{\sqrt{aa+xx}}\right.+\frac{bb}{2}\left|\frac{a+x}{\sqrt{aa+xx}}\right.-\frac{ab}{2}\left|\frac{a+x}{\sqrt{aa+xx}}\right.+ab\left|\frac{x+\sqrt{aa+xx}}{a}\right.=\frac{bb}{2}\left|\frac{a+x}{\sqrt{aa+xx}}\right.$ $+ab\left|\frac{x+\sqrt{aa+xx}}{a}\right.$ = Fluent to be constructed, = Trapezium $BHDC$.

If

If the Conchoid CPD be of such a Nature, that $AB \times BC$ (ab) be $= AG \times GP$ (yb); by reasoning as above, we shall have $\frac{aaby^{-1}\dot{y}}{\sqrt{yy-aa}} + \frac{a^3bb^{-3}\dot{y}}{2\sqrt{yy-aa}}$ for the Area of the little Space $GPpm$ = Fluxion of the Space $GPCB$; and the Fluents of each Part of this Fluxion may be had from those of the Fluxion of the fifth Form in the Tables of Mr. *Cotes*. For making $d = aab$, $z = y$, $\theta = 0$, $n = 2$, $e = -aa$, $f = 1$, the Fluxion of the Form, *viz.* $\frac{d\dot{z}z^{\theta n - 1}}{\sqrt{e + fz^n}}$ will become $\frac{aaby^{-1}\dot{y}}{\sqrt{yy-aa}}$, the first Part of the Fluxion. And making $d = \frac{a^3bb}{2}$, $z = y$, $\theta = -1$, $n = 2$, $e = -aa$, $f = 1$, the said Fluxion in the Form will become $\frac{a^3bby^{-3}\dot{y}}{2\sqrt{yy-aa}}$, the latter Part of the Fluxion; and the Fluent in the former Case corresponding to $\theta = 0$, is $\frac{-2}{ne} dR \left| \frac{R+T}{S} \right.$; and in the latter, the Fluent against $\theta = -1$, is $\frac{-1}{nez^n} dP + \frac{f}{nee} dR \left| \frac{R+T}{S} \right.$. Consequently writing for P ($= \sqrt{e + fz^n}$) $\sqrt{yy-aa}$, for R ($= \sqrt{e}$) $\sqrt{-aa}$, for T ($= \sqrt{e + fz^n}$) $\sqrt{yy-aa}$, and for S ($= \sqrt{fz^n}$) y; the former Fluent will become $ab \left| \frac{a + \sqrt{yy-aa}}{y} \right.$, and the latter $\frac{abb}{4yy} \sqrt{yy-aa} + \frac{bb}{4} \left| \frac{a - \sqrt{yy-aa}}{y} \right.$; and the Sum of these Fluents

Fluents $\frac{abb}{4yy}\sqrt{yy-aa}+\overline{ab+\frac{bb}{4}}\left|\frac{a+\sqrt{yy-aa}}{y}\right.$ is = Fluent of the Fluxion above. Which may be constructed thus:

FIG. 24. Make $AG\ (y):\frac{1}{2}AB\ (\frac{1}{2}a)::\frac{1}{2}BC\ (\frac{1}{2}b):BN=\frac{ab}{4y}$. And $AG(y):GB\ (\sqrt{yy-aa})::BN\ (\frac{ab}{4y}):BM=\frac{ab}{4yy}\sqrt{yy-aa}$; then if to BM you add MO = to the Measure of the Angle BAG (whose Radius, Tangent and Secant are as $AB\ (a)$, $BG\ (\sqrt{yy-aa})$, and $AG\ (y)$ to the Module $AB+\frac{BC}{4}\ (a+\frac{b}{4})$, the Rectangle $CB\times BO$ will be equal to the Fluent $\frac{abb}{4yy}\sqrt{yy-aa}+\overline{ab+\frac{bb}{4}}\left|\frac{a+\sqrt{yy-aa}}{y}\right.$ = Area of the Space $GPCB$.

EXAMPLE XV.

33. *LET* DPEBQD *be half the Lune of* Hippocrates, A *being the Centre of the Arch* DQB, *and* C *the Centre of* DPE; *it is required to find the Area of the Space* QPEB, *contained under* BE, *Part of a Line* ACEB *joining the Centres* A, C; *the Parts* QB, PE *of the Arches forming the Semi-Lune; and the Part* QP *of any right Line* AQP *drawn from the Centre* A *of the larger Arch of Formation, lying between the said Arches.*

FIG. 25. Let DCF be at right Angles to ACE. Draw $Agqp$ infinitely near $AGQP$, and from the Centre A describe the small Arches Gr, ps, and join the Points P and E.

This done, let $AC = a$, $CG = x$, $AG = y$. Because the Triangles AGC, rgG are similar; therefore $AG\ (y) : AC\ (a) :: Gg\ (\dot{x}) : Gr = \frac{a\dot{x}}{y}$. And since the Triangles AGC, AEP, are likewise similar, the Angle A being common to both, and the Angle APE in a Semicircle equal to the right Angle ACG; therefore $AG\ (y) : AC\ (a) :: AE\ (2a) : AP = \frac{2aa}{y}$. Moreover, from the Similarity of the Sectors AGr, Asp, we have $AG\ (y) : Gr\ \left(\frac{a\dot{x}}{y}\right) :: AP\ \left(\frac{2aa}{y}\right) : sp = \frac{2a^3\dot{x}}{y^3}$. Which drawn into $\frac{1}{2} AP\left(\frac{aa}{y}\right)$, and the Product $\frac{2a^5\dot{x}}{y^4}$ will be $=$ Area of the little Sector Aps $=$ little Triangle ApP. For these differ only by the Triangle pPs, which is infinitely less than Aps. Again, because the Sectors AGr, Asp are similar, we have $AG(y) : Gr\ \left(\frac{a\dot{x}}{y}\right) :: AQ\ (a\sqrt{2}) : Qq = \frac{aa\dot{x}\sqrt{2}}{yy}$. Which drawn into $\frac{1}{2} A\left(\frac{a}{2}\sqrt{2}\right)$, and the Product $\frac{a^3\dot{x}}{yy}$, will be equal to the Area of the little Sector or Triangle AQq; which taken from $\frac{2a^5\dot{x}}{y^4}$, the Area of the Triangle ApP, and the Remainder $\frac{2a^5\dot{x}}{y^4} - \frac{a^3\dot{x}}{yy}$ will be $=$ Area of the Trapezium $QPpq$. Which is the Fluxion of the lunar Space $EPQB$.

Again,

Again, becauſe $yy = aa + xx$; therefore $xx = yy - aa$: and throwing this Equation into Fluxions, we get $x\dot{x} = y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{x} = \frac{y\dot{y}}{\sqrt{yy-aa}}$, by putting for x its equal $\sqrt{yy-aa}$; then if this laſt Value be put for $\dot{x}$ in the Fluxion of the ſought Lunar Space, the ſame will become $\frac{2a^5\dot{y}}{y^3\sqrt{yy-aa}} - \frac{a^3\dot{y}}{y\sqrt{yy-aa}} = \frac{2a^5y^{-3}\dot{y}}{\sqrt{yy-aa}} - \frac{a^3y^{-1}\dot{y}}{\sqrt{yy-aa}}$. Both Parts of which come under the fifth Form in the Tables of Mr. *Cotes*. For in the firſt Part, making $d = 2a^5$, $z =$, $\theta = -1$, $n = 2$, $e = -aa$, $f = 1$, we have the Fluxion of the Form $\frac{d\dot{z}z^{\theta n - 1}}{\sqrt{e+fz^n}} = \frac{2a^5y^{-3}\dot{y}}{\sqrt{yy-aa}}$. In like manner, in the ſecond Part, making $d = -a^3$, $z = 0$, $\theta = 0$, $n = 2$, $e = -aa$, $f = 1$, we have $\frac{d\dot{z}z^{\theta n - 1}}{\sqrt{e+fz^n}} = \frac{-a^3y^{-1}\dot{y}}{\sqrt{yy-aa}}$. The Fluent againſt $\theta = -1$ is $\frac{-1}{nez^n}dP + \frac{f}{nee}dR \left| \frac{R+T}{S} \right. = \frac{a^3}{yy}\sqrt{yy-aa} + aa \left| \frac{a+\sqrt{yy-aa}}{y} \right.$, by ſubſtituting $\sqrt{yy-aa}$ for P ($\sqrt{e+fz^n}$), a for R ($\sqrt{e}$), $\sqrt{yy-aa}$ for T ($\sqrt{yy-aa}$), and y for S ($\sqrt{fz^n}$). And that againſt $\theta = 0$, is $\frac{-2}{ne}dR \left| \frac{R+T}{S} \right. = -aa \left| \frac{a+\sqrt{yy-aa}}{y} \right.$, by ſubſtituting the ſame Values. Therefore the Sum of theſe Fluents

$$\frac{a^3}{yy}\sqrt{yy-aa} + aa \left| \frac{a+\sqrt{yy-aa}}{y} \right. - aa \left| \frac{a+\sqrt{yy-aa}}{y} \right.$$ is $= \frac{a^3}{yy}\sqrt{yy-aa}$ (ſince the

Meaſures

Measures of the same Angle to the Module aa are, the one affirmative, and the other negative, and consequently destroy each other.) = Fluent of the Fluxion of the Lunar Space $EPQB$, $= EPQB$.

Now this Fluent may be easily constructed; for if a right Line be drawn from the Centre C to P, the Isoscceles right-lined Triangle CPE will be = Fluent $\frac{a^3}{yy}\sqrt{yy-aa} = \frac{a^3}{yy}x$ (since $x = \sqrt{yy-aa}$) = Lunar Space $EPQB$, as may be shewn thus:

Draw PH parallel to GC; then the Triangles AGC, APH are similar: therefore AG $(y) : AP \left(\frac{2aa}{y}\right) : GC\ (x) : PH = \frac{2aax}{yy}$.

And so $\frac{1}{2}PH\left(\frac{aax}{yy}\right) \times CE\ (a)$ is $= \frac{a^3x}{yy} =$ Fluent to be constructed = Area of the Triangle CPE. Hence the Area of the Semi-Lune is $= CE \times \frac{1}{2}DC = \frac{1}{2}aa$. And thus you have the Quadrature of the aforesaid Space by the Method of Fluxions; tho' indeed it may be shewn much shorter by common Geometry.

EXAMPLE XVI.

34. *TO square the Cycloidal Space, or to find the Area of any Segment* AMG *of it.*

Let APB be the generating Circle. Let AP be any Ordinate, PM the correspondent Absciss; let mp be infinitely near MP; let PM touch the generating Circle in P, and MT the Cycloid in M. Then from the Nature of the Cycloid, the Subtangent $PT = PM =$ Arch AP. Draw AG perpendicular FIG. 26.

to AB, and from the Points M, m, draw MG, mg perpendicular to AG.

Now let $AQ = x$, $AB = 1$. Because $TP = PM$, the Angle $MTP = PMT$; and therefore the Angle $TPQ = 2TMP$. But the Measure of the Angle APQ is $\frac{1}{2}$ the Arch AP, which is also the Measure of the Angle TPA; therefore $APQ = TMP = Mms$. Consequently the Triangles APQ, MmS are similar, therefore $AQ\ (x) : QP\ (\sqrt{x - xx}) :: MS\ (\dot{x}) : mS = \dot{x}\frac{\sqrt{x - xx}}{x}$: But $\dot{x}\frac{\sqrt{x - xx}}{x}$ thrown into an infinite Series, will be $x^{-\frac{1}{2}}\dot{x} - \frac{1}{2}x^{\frac{1}{2}}\dot{x} - \frac{1}{8}x^{\frac{3}{2}}\dot{x} - \frac{1}{16}x^{\frac{5}{2}}\dot{x}$, &c. = the Fluxion of the Ordinate QM to the Axis AB of the Cycloid, and the Fluent of this, *viz.* $2x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}} - \frac{1}{20}x^{\frac{5}{2}} - \frac{1}{56}x^{\frac{7}{2}}$, &c. will be the said Ordinate MQ. Whence $QM \times \dot{x} =$ the Fluxion of the Cycloidal Space AMQ is $= 2x^{\frac{1}{2}}\dot{x} - \frac{1}{3}x^{\frac{3}{2}}\dot{x} - \frac{1}{20}x^{\frac{5}{2}}\dot{x} - \frac{1}{56}x^{\frac{7}{2}}\dot{x}$, &c. the Fluent of which will be $\frac{4}{3}x^{\frac{3}{2}} - \frac{2}{15}x^{\frac{5}{2}} - \frac{1}{70}x^{\frac{7}{2}} - \frac{1}{252}x^{\frac{9}{2}}$, &c. = Area of the indefinite Cycloidal Space AMQ.

Now if $mS = gG = \frac{\dot{x}\sqrt{x - xx}}{x}$ be drawn into $GM = AQ = x$; then will the Fluxion $GMSG$ of the Area AMG be equal to $\dot{x}\sqrt{x - xx}$. Therefore since this is the same as the Fluxion of the Segment of the Circle APQ, the Space AMG will be equal to the Segment APQ of the Circle, and so the Area $ADBC$ of the whole Cycloidal Space ADB is equal to the Area of the Semicircle APB.

COROL.

COROL.

BECAUSE DB is equal to $\frac{1}{2}$ the Circumference of the Circle; if you call the same p, and AB, a; then the Rectangle $ABDE = ap =$ Area Semicircle APB. Whence the external Cycloidal Space $AEDMA = \frac{1}{4}ap$. Therefore the Area of the Semicycloid $ADB = \frac{3}{4}ap$ $AMDPA = \frac{1}{2}ap$. Consequently the Area of the Cycloid is the triple of the generating Circle.

EXAMPLE XVII.

35. *TO square the Cissoid of* Diocles, *or to find the Area of any Segment* APM *of it.* FIG. 27.

Let ADB be the generating Circle, BH the Asymptote to the Curve (AI) of the Cissoid, at right Angles to the Diameter AB; let the Diameter AB be $=1$, the Absciss $AP = x$, the correspondent Ordinate PM to the Cissoid $= y$.

Now the Equation expressing the Nature of this Curve will be $PB \times \overline{PM}^2 = \overline{AP}^3$; that is, $y^2 - xy^2 = x^3$, and so $y^2 = \frac{x^3}{1-x}$. Whence $y = \sqrt{\frac{x^3}{1-x}} = \frac{x\sqrt{x}}{\sqrt{1-x}} = x^{\frac{3}{2}} \times \overline{1-x}^{-\frac{1}{2}}$; and $\dot{x}y = x^{\frac{3}{2}} \times \overline{1-x}^{-\frac{1}{2}}\dot{x} =$ Fluxion of the Area APM. The Fluent of which will be $\frac{2}{5}x^{\frac{5}{2}} + \frac{2x^{\frac{7}{2}}}{2.7} + \frac{1.3x^{\frac{9}{2}}}{4.9} + \frac{1.3.5x^{\frac{11}{2}}}{4.6.11} + \frac{1.3.5.7x^{\frac{13}{2}}}{4.6.8.13}$, &c. $= \sqrt{x}$ into $\frac{2}{5}x^4 + \frac{1}{7}x^6 + \frac{1.3x^8}{4.9} + \frac{1.3.5x^{10}}{4.6.11} + \frac{1.3.5.7x^{12}}{4.6.8.13}$ &c. = Space *APM*. And when $x = 1$, then will this Series become $\frac{2}{5} + \frac{1}{7} + \frac{1.3}{4.9} + \frac{1.3.5}{4.6.11} + \frac{1.3.5.7}{4.6.8.13}$, &c. = Area of the infinite Space *ABHIA*.

Otherwise:

Let $AB = a$, and $PN = z$. Now because from the Nature of the Curve $ay^2 - xy^2 = x^3$; this thrown into Fluxions, and there will arise $2ay\dot{y} - 2xy\dot{y} - y^2\dot{x} = 3x^2\dot{x}$, and $\overline{2a - 2x} \times \dot{y} = y\dot{x} = \frac{3x^2\dot{x}}{y}$. But since from another Property of the Curve $x^2 = zy$; therefore $\frac{x^2}{y} = z$. Now make $a - x = PB = u$; then we shall have $2u\dot{y} - y\dot{x} = 3z\dot{x}$; and so the Fluent of the one will be equal to the Fluent of the other. But $z\dot{x}$ is the Fluxion $PNnp$ of the Segment *ANP* of the Circle; and because $u = PB = OM$ and $\dot{y} = mR = Oo$, $mMOo$, will be the Fluxion of the Area *AMOB*, and $y\dot{x}$ the Fluxion of the Area *AMP*. Now when the Fluent of $u\dot{y}$ expresses the Area of the whole Cycloidal Space *ABHIA*, the Fluent of $x\dot{y}$ will be the same Area; and so the Fluent of $2u\dot{y} - y\dot{x} =$ Fluent $u\dot{y}$. Therefore since in the same Case the Fluent of $z\dot{x}$ is the Area of the Semicircle *ANB*; and because the Fluent of $u\dot{y} = 3z\dot{x}$, the whole Cissoidal Space *ABHIA* will be the triple of the generating Semicircle *ANB*.

EXAM-

EXAMPLE XVIII.

36. *TO square any Interminate Space* HRMI *contained under the Asymptote* PH, *Ordinate* MP, *and Part* MSI *of the Logarithmick Curve* MI. FIG. 28.

Call the Subtangent PT, a (because from the Nature of this Curve it is a standing Quantity) and the Ordinate PM, y. Draw the infinitely near Ordinate pm, and from M draw Mq perpendicular to mq. Now because of the similar Triangles MTP, mMq, $MP\ (y) : TP\ (a) :: mq\ (\dot{y}) : Mq = \frac{a\dot{y}}{y}$. Whence $MP \times Mq = y \times \frac{a\dot{y}}{y} = a\dot{y}$, is the Fluxion of the Area $IMPH$, and the Fluent thereof will be ya; that is, the Area of the infinitely extended Space $IMPH$ is $= MP \times PT$.

COROL.

IF the Ordinate QS be $= z$; then the Interminate Space $ISQH = az$, and consequently $SMPQ = ay - az = a \times \overline{y - z}$; that is, the Space contained under any two Semi-ordinates MP, SQ, the Part of the Abscis PQ, and the Curve MS, is equal to $TP \times \overline{MP - SQ}$; and so the Space $BAPM$ is to the Space $BMSQ$, as the Difference of the Ordinates AB, PM to the Difference of the Ordinates PM, QS.

EXAM-

EXAMPLE XIX.

37. TO square Spiral Spaces.

FIG. 29. Let the Semi-diameter of the Circle, *viz.* $AC=a$, the Periphery $=b$, any Arch $AB=x$, as an Abscifs, and the correspondent Ordinate $CM=y$. Conceive the Radius Cb infinitely near CB, and draw the small Arch ME.

Now the Nature of *Archimedes*'s Spiral is $AC\times AB=$ Periphery b drawn into CM; that is, $ax=by$. This being granted, the small Arch $ME=\frac{y\dot{x}}{a}$, since $CB:Bb::CM:ME$. Therefore $\frac{1}{2}CM\times ME=$ Area of the little Sector $MCE=$ to the little trilineal Space CMm, which is the Fluxion of the Spiral Space $=\frac{y^2\dot{x}}{2a}$; but from the Nature of the Curve $ax=by$; therefore $\frac{a^2x^2}{b^2}=yy$. Consequently substituting $\frac{a^2x^2}{b^2}$ for yy in the Fluxionary Expression, and then it will be $\frac{ax^2\dot{x}}{2b^2}$; the Fluent of which will be $\frac{ax^3}{6b^2}$, the Area of the Segment of the Spiral Space; and if for x be put b, the whole Circumference, the whole Spiral Space will be $=\frac{1}{6}ab$.

Again; The Nature of all Kinds of circular Spirals will be expressed by $a^m x^n=b^n y^m$; therefore $\frac{a^m x^n}{b^n}=y^m$, and $\frac{ax^{\frac{n}{m}}}{b^{\frac{n}{m}}}=y$. Consequently $\frac{y^2}{2a}\dot{x}=\frac{ax^{\frac{2n}{m}}\dot{x}}{2b^{\frac{2n}{m}}}$. The Fluent of which will be $\frac{m a \dot{x}^{\frac{2n+m}{m}}}{4n+2m \times b^{\frac{2n}{m}}}$. Therefore putting b for x, and the whole Spiral Spaces will be $\frac{mab}{4n+2m}$.

Moreover, if the Arch AB be to BM as the Abſciſs to the Ordinate in any Algebraical Curve, the Spiral Space may be ſquared after the ſame manner as above.

For Example: Let AB be to BM as the Abſciſs of a Parabola to the Ordinate; then aſſuming p for the Parameter $px = a^2 - 2ay + yy$, and $\dot{x} = \frac{2y\dot{y} - 2a\dot{y}}{p}$. Whence $\frac{y^2\dot{x}}{2a} = \frac{y^3\dot{y} - ay^2\dot{y}}{ap}$, the Fluent of which will be $\frac{\frac{y^4}{4ap} - y^3}{3r}$

Much after the ſame manner you may find the Area of the Space contained under the Arch AB, and the Spiral AM; whoſe Fluxion is the Trapezium $BMmb = \overline{Bb + Mm} \times \frac{1}{2}mb$. But $Bb = \dot{x}$, $Mm = \frac{y\dot{x}}{a}$, $mb = a - y$; therefore $BMmb \left(\dot{x} + \frac{y\dot{x}}{a} \times \frac{1}{2}\overline{a - y} \right) = \frac{a^2\dot{x} - y^2\dot{x}}{2a}$.

Now let the Curve be a Parabolical Spiral; ſubſtitute $\frac{2y\dot{y} - 2a\dot{y}}{p}$ for its Equal $\dot{x}$: then will $\frac{ay^2\dot{y} + a^2y\dot{y} - y^3\dot{y} - a^3\dot{y}}{ap}$ be the Fluxion of the Space ABM, and the Fluent thereof

y^3

or mn ($\dot{y}$) to a fourth Proportional, which will be $= Mn$ the Fluxion of the Curve AM. For the little right-angled Triangle Mmn is similar to the right-angled Triangle TMP; and the Fluent of that Fluxion will be the Arch sought. Some Examples will make this evident.

COROL.

HENCE if PQ be drawn from P perpendicular to the Tangent TM; PQ or QM : PM (y) : : Mn ($\dot{x}$), or nm ($\dot{y}$) : Mm the Fluxion of the Curve AM.

EXAMPLE I.

40. *TO find the Length of any Arch* AM *of the Curve of the common Parabola.*

FIG. 31. Here $AP \times a = \overline{PM}^2$; that is, $ax = yy$; and both Parts thrown into Fluxions is $a\dot{x} = 2y\dot{y}$. Whence $aa\dot{x}^2 = 4y^2\dot{y}^2$, and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{aa}$. And adding $\dot{y}^2$ to this last Expression, we shall have $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{y}^2 + \frac{4y^2\dot{y}^2}{aa}} = \frac{\dot{y}}{a}\sqrt{aa + 4yy} = Mm$ the Fluxion of the Curve AM. The Fluent of which will be $y + \frac{2y^3}{3a^2} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6} - \frac{10y^9}{9a^8}$ &c. $=$ Arch AM.

Otherwise by the Measure of a Ratio.

The said Fluxion of the Curve of the Parabola $\frac{\dot{y}}{a}\sqrt{aa + 4yy}$ may be referred to the fourth Form in the Tables of Mr. *Cotes*. For if you make $z = \frac{1}{2}y$, $n = 2$, $\theta = 0$, $d = \frac{1}{a}$, $e = aa$, $f = 4$. Then will $d\dot{z}z^{\theta n + \frac{1}{2}n - 1}\sqrt{e + fz^n}$ be $= \frac{\dot{y}}{a}\sqrt{aa + 4yy}$. And the Fluent of this Form (against $\theta = 0$) is $\frac{z^n}{n}dP + \frac{e}{nf}dR \left| \frac{R+T}{S} \right.$. And making $P\left(= \sqrt{\frac{e+fz^n}{z^n}}\right) = \frac{1}{y}\sqrt{aa + 4yy}$, $R\ (= \sqrt{f}) = 2$, $T\left(= \sqrt{\frac{e+fz^n}{z^n}}\right) = \frac{1}{y}\sqrt{aa + 4yy}$, $S\left(= \sqrt{\frac{e}{z^n}}\right) = \frac{a}{y}$. The said Fluent will be $\frac{y}{2a}\sqrt{aa + 4yy} + \frac{a}{4}\left| \frac{\sqrt{aa + 4yy} + 2y}{a} \right.$. Which may be thus constructed: From the Vertex A draw AB bisecting the Ordinate PM (y) in B; then will $AB\ (= \sqrt{\overline{AP}^2 + \overline{PB}^2}) = \frac{y}{2a}\sqrt{aa + 4yy}$. FIG. 32. And multiplying the Numerator and Denominator of the Ratio $\frac{\sqrt{aa + 4yy} + 2y}{a}$ by $\frac{y}{2a}$; the same will become $\frac{\frac{y}{2a}\sqrt{aa + 4yy} + \frac{yy}{a}}{\frac{1}{2}y}$. That is, $\frac{a}{4}\left| \frac{\sqrt{aa + 4yy} + 2y}{a} \right.$ is $= \frac{a}{4}\left| \frac{\frac{y}{2a}\sqrt{aa + 4yy} + \frac{yy}{a}}{\frac{1}{2}y} \right.$. But $\frac{y}{2a}\sqrt{aa + 4yy} + \frac{yy}{a}$ is $= AB + BP$, and $\frac{1}{2}y$ is $= PB$; also $\frac{a}{4}$ is $=$ focal Distance from the Nature of the Parabola. Therefore

$\frac{a}{4} \Big| \frac{\sqrt{aa+4yy}+2y}{a}$ is $= AF \Big| \frac{AB+AP}{PB} =$ to the Measure of the Ratio between $AB+AP$, and PB to the focal Distance AF, as a Module. This added to AB, and the whole will be $\frac{y}{2a}\sqrt{aa+4y^2} + \frac{a}{4}\Big|\frac{\sqrt{aa+4yy}+2y}{a} =$ Fluent of the Fluxion $\frac{\dot{y}}{a}\sqrt{aa+4yy} =$ Length of the Arch AM of the Parabola. Hence the Rule for finding the Length of the Curve of the common Parabola is this.

Let A be the Vertex, F the Focus, AP the Axis, and PM an Ordinate to the same. Draw AB bisecting the Ordinate PM in the Point B; to which continued out, add BC the Measure of the Ratio between $AB+AP$, and PB to the Module AF; and then AC will be the Length of the Arch AM of the Parabola. This is the Construction given by Mr. *Cotes* in *Harmonia Mensurarum*, p. 12.

In this Curve the Subtangent $TP = 2AP = 2x$. Whence $TM = \sqrt{4xx+ax}$. Therefore $TP\ (2x) : TM\ (\sqrt{4xx+ax}) :: Mn\ (\dot{x}) : Mm = \frac{\sqrt{4xx+ax}}{2x} \times \dot{x} =$ Fluxion of the Curve. Or since $ax = yy$; therefore $x = \frac{yy}{a}$, and $2x = \frac{2yy}{a}$, and $4xx = \frac{4y^4}{aa}$. Consequently $\sqrt{\frac{4y^4}{aa}+yy} = TM$. Now $PM\ (y) : TM\left(\sqrt{\frac{4y^4}{aa}+yy}\right) :: nm\ (\dot{y}) : Mm = \dot{y}\,\frac{\sqrt{\frac{4y^4}{aa}+yy}}{y} = \dot{y}\,\frac{\sqrt{aa+4yy}}{a}$

as

as before; and the Fluent of $\dot{x}\frac{\sqrt{4xx+ax}}{2x}$, supposing $a=1$, will be $x^{\frac{1}{2}}+\frac{2}{3}\sqrt{x^3}-\frac{2}{5}\sqrt{x^5}+\frac{4}{7}\sqrt{x^7}-\frac{10}{9}\sqrt{x^9}$, &c.

COROL.

If AC, DC be the Semi-conjugate Axes of FIG. 32.
an Equilateral Hyperbola, and AC $(=DC)$ be supposed $=a$, the *Latus Rectum* of the Parabola; and the Ordinate PM be $=2y$, and the Abscifs $QM=x$: then will AP be $=x-a$. Consequently because $PC\times AP=\overline{PM}^2$; that is $xx-aa=4yy$, and so $xx=4yy+aa$. Therefore $x=\sqrt{4yy+aa}$. And if qm be drawn infinitely near qm, then $Qq=\dot{y}$; and so the Fluxion $QqmM$ of the Area of the Hyperbolick Space $CQMA$ will be $=\dot{y}\sqrt{aa+4yy}$ $=$ Fluxion of the Curve of the Parabola. Therefore the ~~Quadrature~~ of the Curve of the Parabola depends upon the Quadrature of the said Hyperbolick Space. And so the Rectification of any Curve may be brought to the Quadrature of a Curve, by supposing the Fluxion of the Curve to be rectify'd (found as above) as an Ordinate, and the variable Quantity in that Fluxion as an Abscifs to that Ordinate. Consequently the Business of rectifying Curves sometimes may be shorten'd, from a Pre-knowledge of the Quadrature of that Curve it may be reduced too.

EXAM-

EXAMPLE II.

41. *To rectify a Parabola of the second Kind; where $ax^2 = y^3$, or making $a = 1$, $x^2 = y^3$.*

Because $x^2 = y^3$; therefore $2x\dot{x} = 3y^2\dot{y}$, and $4x^2\dot{x}^2 = 9y^4\dot{y}^2$. Whence $\dot{x}^2 = \frac{9y^4\dot{y}^2}{4x^2} = \frac{9y^4\dot{y}^2}{4y^3}$ (by substituting y^3 for ax^2) $= \frac{9}{4}y\dot{y}^2$; therefore $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{9}{4}y\dot{y}^2 + \dot{y}^2}$ (by adding $\dot{y}^2$ to each Side of the Equation $\dot{x}^2 = \frac{9}{4}y\dot{y}^2$) $= \frac{1}{2}\sqrt{9y\dot{y}^2 + 4\dot{y}^2} = \frac{1}{2}\dot{y}\sqrt{9y+4}$, = Fluxion of the Curve, and the Fluent thereof will be $\frac{1}{27} \times \overline{9y+4} \times \sqrt{9y+4}$. But in order to find whether any thing is to be added or taken from this, make $y = 0$, and then the Remainder will be $\frac{4}{27}\sqrt{4} = \frac{8}{27}$; therefore the Length of the Curve will be * $\frac{1}{27} \times \overline{9y+4} \times \sqrt{9y+4} - \frac{8}{27}$.

* *Art.* 19.

COROL.

FIG. 31. Let the Parameter of the common Parabola be 1, $AP = 1$, $PQ = \frac{9}{4}y$; then will $AQ = \frac{9}{4}y + 1$, and because the Parameter is 1, $\overline{QN}^2$ $(\frac{9}{4}y + 1) = \frac{9y+4}{4}$. Consequently $QN = \frac{1}{2}\sqrt{9y+4}$; therefore the Fluxion $QNnq$ of the Parabolick Space $PMNQ$ is $= \frac{1}{2}\dot{y}\sqrt{9y+4}$. And so the Length of the Curve of a Parabola of the second Kind expressed by $ax^2 = y^3$ depends on, or is the Quadrature of the common Parabola, and may be had in finite Terms.

EXAM-

EXAMPLE III.

42. *To rectify Parabola's of all Kinds.*

Let the Parameter be $=1$; then the Nature of infinite Numbers of Parabola's of different Kinds will be expressed by this Equation $y^m=x$. The Fluxion of which will be $my^{m-1}\dot{y}=\dot{x}$; and squaring both Sides $m^2y^{2m-2}\dot{y}^2=\dot{x}^2$. And if for Brevity's sake you make $2m-2=r$, then will $m^2y^r\dot{y}^2=\dot{x}^2$; and adding $\dot{y}^2$ to each Side, and afterwards extracting the square Root, $\sqrt{\dot{x}^2+\dot{y}^2}=\sqrt{m^2y^r\dot{y}^2+\dot{y}^2}=\dot{y}\sqrt{m^2y^r+1}$ $=$ Fluxion of the Arch of a Parabola of any Kind soever. The Fluent of which, *viz.*

$$y+\frac{m^2y^{r+1}}{2.\overline{r+1}}-\frac{m^4y^{2r+1}}{2.4.\overline{2r+1}}+\frac{1.3m^6y^{3r+1}}{2.4.6.\overline{3r+1}}$$

$$-\frac{1.3.5m^8y^{4r+1}}{2.4.6.8.\overline{4r+1}}, \text{ \&c.}$$ will be the Length of the Curve thereof. Now substituting $2m-2$ for r, and then the same Arch will be $=$

$$y+\frac{m^2y^{2m-1}}{2.\overline{2m-1}}-\frac{m^4y^{4m-3}}{2.4.\overline{4m-3}}+\frac{1.3m^6y^{6m-5}}{2.4.6.\overline{6m-5}}$$

$$-\frac{1.3.5m^8y^{8m-7}}{2.4.6.8.\overline{8m-7}}, \text{ \&c.}$$

EXAMPLE IV.

43. *TO find the Length of any Part* BC *of the Equiangular or Logarithmical Spiral.*

FIG. 33. Let the Radius $AB=a$, $AC=b$: Let BD (c) touch the Spiral in B, and AD be perpendicular to BD. Call BD, p. Also let PE touch the Spiral in P, and let AE be perpendicular to it. Moreover let Ap be infinitely near AP; and from A describe the small Arch Pm. Call $AP-AB=FP$, y.

Now since from the Nature of this Curve any Radius AB cuts it in the same Angle, or makes the same Angle ABD with the Tangent BD; therefore any Triangle, as APE, will be similar to the given Triangle ABD. But the small Triangle mpP is also similar to the Triangle APE, the Angle at m being a right Angle $=E$, and the Angle at p common to both Triangles: whence the fluxionary Triangle mpP is similar to the Triangle ABD; and so $BD(c):AB(a)::mp(\dot{y}):Pp\left(\frac{a\dot{y}}{c}\right)=$ Fluxion of the Part BP of the Curve; and the Fluent $\frac{ay}{c}$ will be the Length of the Part BP; and putting $b-a$ for y, we shall have $\frac{ba-aa}{c}=$ Length of BC; that is, as BD $(c):AB(a)::AC-AB:$ Length BC.

EXAM-

EXAMPLE V.

44. TO *find the Length of any Arch* AP *of the Spiral* APC *of* Archimedes.

Let CAM be the generating Circle. Call AC, a, the Circumference b, the Arch CM, x, the Ordinate AP of the Spiral y, and let PT touch the Spiral in P; intersecting the Line AT drawn from A the Centre of the Spiral perpendicular to AP, in the Point T. Moreover, let $Mm = \dot{x}$, draw Am, and with the Distance An, from A describe the little Arch np. Now $ax = by$ expresses the Nature of this Curve. Because of the similar Sectors AMm, Apn, $AM (a) : Mm (\dot{x}) :: Ap = AP (y) : Pn = \frac{y\dot{x}}{a}$. And since the Triangles Ppn, PAT are similar also; therefore $pP (\dot{y}) : pn \left(\frac{y\dot{x}}{a}\right) :: AP (y) : AT$ the Subtangent $= \frac{yy\dot{x}}{a\dot{y}}$. But $a\dot{x} = b\dot{y}$; whence $\dot{x} = \frac{b\dot{y}}{a}$. Which substituted for $\dot{x}$ in $\frac{yy\dot{x}}{a\dot{y}}$, and we have $AT = \frac{byy}{aa}$. Again, since TAP is a right Angle, $\overline{AT}^2 \left(\frac{bby^4}{a^4}\right) + \overline{AP}^2 (yy) = \overline{TP}^2 = \frac{a^4yy + bby^4}{a^4}$; and so $TP = \frac{y}{aa}\sqrt{a^4 + bbyy}$; and $AP (y) : TP \left(\frac{y}{aa}\sqrt{a^4 + bbyy}\right) :: pP (\dot{y}) : Pn = \frac{1}{aa}\dot{y}\sqrt{a^4 + bbyy}$, = Fluxion of the Part AP of the Spiral. Which comes under the fourth Form of Mr. *Cotes*'s Tables. FIG. 34.

For making $z=y$, $n=2$, $\theta=0$, $d=\frac{1}{aa}$, $e=a^4$, $f=bb$, we have $d\dot{z}z^{\theta n+\frac{1}{2}n-1}\sqrt{e+fz^n} = \frac{1}{aa}\dot{y}\sqrt{a^4+bbyy}$. The Fluent of which is $\frac{z^n}{n}dP + \frac{e}{nf}dR\left|\frac{R+T}{S}\right.$. And writing for P, R, T, S, or $\sqrt{\frac{e+fz^n}{z^n}}$, $\sqrt{f}$, $\sqrt{\frac{e+fz^n}{z^n}}$, $\sqrt{\frac{e}{z^n}}$, their Values $\frac{1}{y}\sqrt{a^4+bbyy}$, b, $\frac{1}{y}\sqrt{a^4+bbyy}$, $\frac{a^2}{y}$, there arises $\frac{y}{2aa}\sqrt{a^4+bbyy} + \frac{aa}{2b}\left|\frac{\sqrt{a^4+bbyy}+yb}{aa}\right. =$ Fluent of the Fluxion aforesaid.

This Fluent may be constructed thus: Bisect AP (y) in the Point L, and draw DL parallel to the Tangent PT cutting the Subtangent in D; then will DL be $=\frac{y}{2aa}\sqrt{a^4+bbyy}$, since it is $\frac{1}{2}PT = \frac{y}{aa}\sqrt{a^4+bbyy}$. Again, the Module $\frac{aa}{2b}$ is had by making $AD\left(\frac{byy}{2aa}\right) : AL\left(\frac{y}{2}\right) :: AL\left(\frac{y}{2}\right) : AF = \frac{aa}{2b}$. And the Ratio $\frac{\sqrt{a^4+bbyy}+yb}{aa}$ is $=$

$$\frac{\frac{y}{2aa}\sqrt{a^4+bbyy}+\frac{by^2}{2aa}}{\frac{y}{2}},$$

as you will find by dividing the Numerator and Denominator of this last by $\frac{y}{2}$; therefore that Ratio is $= \frac{LD+AD}{AL}$. Consequently if to DL you add

LQ,

LQ, which is the Meaſure of the Ratio of $LD+AD$ to AL, the Line DF being the Module; the whole Line DQ will be equal to the Length AP of the Spiral; and thus you may get the Length of the whole Spiral by making $y=a$, the Radius of the generating Circle.

EXAMPLE VI.

45. *To find the Length of any Arch* CP *of the reciprocal Spiral* APC. FIG. 35.

About the Centre A, with the Diſtance AC, deſcribe the Quadrantal Arch $BCMH$, and continue out AP to M; then the Nature of this Curve is, that any Radius AC of it is reciprocally as the Angle BAC it makes with the firſt Radius AB, or as the Arch BC; that is, $AP : AC :: BC : BM$; and ſo $AP \times BM = AC \times BC$.

Now make AB or $AC=a$, the Arch $BC=b$, the Arch $BM=x$, and $AP=y$; then will ab be $=xy$. Draw PG to touch the Curve in P, and at right Angles to AP draw AG. Let Ap be infinitely near AP, and continue it out to interſect the Arch in m, and with the Diſtance Ap from A deſcribe the ſmall Arch pn. Becauſe the Sectors Anp, AMm are ſimilar, $AM(a) : Mm(\dot{x}) :: AP(y) : pn = \frac{\dot{x}y}{a}$. And becauſe of the ſimilar Triangles APG, nPp, therefore $Pn(\dot{y}) : pn\left(\frac{\dot{x}y}{a}\right) :: AP(y) : AG = \frac{\dot{x}y^2}{a\dot{y}}$. And throwing the Equation of the Curve $ab=xy$ into Fluxions, we have $\dot{x} = \frac{ab\dot{y}}{yy}$, which put for $\dot{x}$ in $\frac{\dot{x}y^2}{a\dot{y}}$, and we have $AG = b$; that is, the Line drawn from the Centre A perpendicular to any Radius AP of the Spiral, to interfect the Tangent to the Spiral at the Extremity of that Radius, will be a ftanding Quantity, *viz.* = Arch BC, which Arch will become a ftraight Line, perpendicular to the firft Radius AB, when the Point C is at an infinite Diftance, and the Angle BAC infinitely fmall; that is, it will be equal to AD the Diftance of the Afymptote DE from the Centre. Hence $AP\ (y) : GP$ $(\sqrt{bb+yy}) :: nP\ (\dot{y}) : Pp = \frac{\dot{y}}{y}\sqrt{bb+yy}$ = Fluxion of the Curve AP.

This may be referred to the third Form in the Tables of Mr. *Cotes*. For making $z = y$, $n = 2$, $\theta = 0$, $d = 1$, $e = bb$, $f = 1$, we have $d\dot{z}z^{\theta n - 1}\sqrt{e+fz^n} = \frac{\dot{y}}{y}\sqrt{bb+yy}$. And the Fluent of this $\frac{2}{n}dP - \frac{2}{n}dR\left|\frac{R+T}{S}\right.$, by writing for $P\ (=\sqrt{e+fz^n})\ \sqrt{bb+yy}$, for $R\ (=\sqrt{e})$ b, for $T\ (=\sqrt{e+fz^n})\ \sqrt{bb+yy}$, and for S $(=\sqrt{fz^n})\ y$, will be $= \sqrt{bb+yy} - b\left|\frac{b+\sqrt{bb+yy}}{y}\right.$. And if AC be fuppofed invariable $= z$, then the Fluent $\sqrt{bb+zz} - b\left|\frac{b+\sqrt{bb+zz}}{z}\right.$ will be = Arch APC. And the Difference of thefe Fluents, *viz.* $\sqrt{bb+zz} - \sqrt{bb+yy} + b\left|\frac{b+\sqrt{bb+zz}}{z}\right. - b\left|\frac{b+\sqrt{bb+yy}}{y}\right. = CF - PG + AF\left|\frac{AF+PG}{AP}\right. - AF\left|\frac{AF+CF}{AC}\right.$ = Length of the Part PC of the Curve; that is, if to the

the Difference $CF - PG$ of the Tangents, you add the Difference of the Measures of the Ratio between $AF + PG$ and AP to the Module AF, and of the Ratio between $AF + CF$ and AC to the same Module AF, the whole will be the Length of the Part PC of the Curve; or because $AF \Big| \frac{AF + PG}{AP}$ is $=$ to $AF \Big| \frac{AP}{PG - AF}$, and $AF \Big| \frac{AF + CF}{AC}$ equal $AF \Big| \frac{AC}{CF - AF}$ (by *Schol. Art.* 14.) therefore if LM be the Measure of the Ratio between AC and $CF - AF$, to the Module AF, and in like manner lm the Measure of the Ratio between AP and $PG - AF$ to the Module AF; then the Aggregate of the Difference of the Tangents $CF - PG$, and the Difference of the Measures $lm - LM$ will be $=$ Length of the Part PC of the Spiral.

EXAMPLE VII.

46. TO *find the Length of the Arch* CP *of the Logarithmick Curve* CPG. FIG. 36.

Let AC, aP be Ordinates perpendicular to the Asymptote Af, and let CF, Pf be Tangents. Also let n be infinitely near P, and draw np parallel to af.

Now call AC, z, aP, y, and the invariable Subtangent AF or af, b. Then because of the similar Triangles aPf, nPp, we have aP (y) $: fP$ $(\sqrt{bb + yy}) :: nP$ $(\dot{y}) : Pp = \frac{\dot{y}}{y}\sqrt{bb + yy} =$ Fluxion of the infinite Part PG of the Curve.

Which compared with the third Form of Mr. *Cotes*'s Tables, and the Fluent (as in the last Problem) will be $\sqrt{bb+yy}-b \Big| \frac{b+\sqrt{bb+yy}}{y}$. In like manner the Fluxion of the infinite Part CPG will be $\frac{\dot{z}}{z}\sqrt{bb+zz}$; and the Fluent thereof will be $\sqrt{bb+zz}-b \Big| \frac{b+\sqrt{bb+zz}}{z}$; and the Difference of these Fluents, *viz.* $\sqrt{bb+zz}-\sqrt{bb+yy}+b \Big| \frac{b+\sqrt{bb+zz}}{z} - b \Big| \frac{b+\sqrt{bb+yy}}{y} = CF - Pf + AF \Big| \frac{AF+Pf}{AP}$

* Art. 14. $-AF \Big| \frac{AF+CF}{AC}$, or * $= CF - Pf + AF \Big| \frac{aP}{Pf-AF} - AF \Big| \frac{AC}{CF-AF}$. Now if AL be taken equal to $CF-AF$, and al equal to $Pf-AF$, and LM, lm be drawn parallel to Af; then from the Nature of the Curve, LM (= Abscifs An) will be the Logarithm of the Ratio of the Ordinate AC to the Ordinate Mn; that is, of AC to AL: and consequently LM is the Logarithm of $\frac{AC}{CF-AF}$. So likewise lm is the Logarithm of $\frac{aP}{Pf-AF}$. And suppofing the invariable Subtangent AF, or af (b) equal to the Module of the Logarithms,

* Art. 14. the Line LM will be * $= AF \Big| \frac{AF+CF}{AC} = AF \Big| \frac{AC}{CF-AF}$, and $lm = AF \Big| \frac{AF+Pf}{aP} =$

AF

$AF \Big| \frac{aP}{Pf - Af}$; therefore if to $Cf - Pf$, the Difference of the Tangents, you add $lm - LM$, the Difference of the Parallels; the whole will be equal to the Length CP of the Curve.

EXAMPLE VIII.

47. *THE Sine* PM *of any Arch* PM *of a Circle being given, together with the Radius: to find the Length of that Arch.*

Let AP be $= x$, the Radius $AC = 1$, and the Sine or Ordinate $PM = y$. Then from the Nature of the Curve, $AP \times PB = \overline{PM}^2$, that is, $2x - xx = yy$. And throwing this Equation into Fluxions, we have $2\dot{x} - 2x\dot{x} = 2y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{1 - x}$. Which squared will be $\dot{x}^2 = \frac{yy\dot{y}^2}{1 - 2x + xx} = \frac{y^2\dot{y}^2}{1 - y^2}$, because from the Equation of the Curve $2x - xx = yy$; therefore $\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{y^2\dot{y}^2}{1 - y^2} + \dot{y}^2} = \sqrt{\frac{y^2\dot{y}^2 + \dot{y}^2 - y^2\dot{y}^2}{1 - y^2}} = \sqrt{\frac{\dot{y}^2}{1 - y^2}} = \sqrt{\frac{\dot{y}}{1 - y^2}} = \overline{1 - y^2}^{-\frac{1}{2}}\dot{y}$; which will be the Fluxion of the Arch AM. And the Fluent thereof will be $y + \frac{y^3}{2.3} + \frac{1.3.}{2.4.5}y^5 + \frac{1.3.5.}{2.4.6.7}y^7 + \frac{1.3.5.7}{2.4.6.8.9}y^9$, &c. = Length of the Arch AM. FIG. 12.

Now if the first Term be called A, the second B, the third C, the fourth D, &c. and the second Term be multiplied by $\frac{1}{1}$, the third by $\frac{1}{3}$, the fourth by $\frac{1}{5}$, *&c.* the Series aforesaid will

will become this $y + \frac{1}{2.3} Ay^{2} + \frac{3}{4.5} 3 By^{2} + \frac{5}{6.7} 5 Cy^{2} + \frac{7}{8.9} 7 Dy^{2}$, &c.

Otherwise:

Draw the Radius MC, compleat the infinitely small Rectangle $PMnp$, make $AB = 1$; and as before, $AP = x$, $PM = y$; then $x - xx = yy$, and because the small right-angled Triangle Mmn is similar to the Triangle PMC. Therefore PM $(\sqrt{x - xx}) : MC (\frac{1}{2}) :: Pp$ or Mn $(\dot{x}) : Mm = \frac{1}{2\sqrt{x - xx}} \dot{x} = \frac{\sqrt{x - xx}}{2x - 2xx} \times \dot{x} =$ Fluxion of the Arch AM; and the Fluent thereof will be $x^{\frac{1}{2}} + \frac{1}{6} x^{\frac{3}{2}} + \frac{3}{40} x^{\frac{5}{2}} + \frac{5}{112} x^{\frac{7}{2}} + \frac{35}{1152} x^{\frac{9}{2}}$ &c. $= x^{\frac{1}{2}}$ into $1 + \frac{1}{6} x + \frac{3}{40} x^{2} + \frac{5}{112} x^{3} + \frac{35}{1152} x^{4}$ &c.

If the Cosine PC be made x, and the Radius AC be $= 1$; then the Fluxion Mm of the Arch CD, being the Complement of AM to a Quadrant, will be $\frac{1}{\sqrt{x - xx}} \dot{x}$ or $\frac{\sqrt{x - xx}}{x - xx} \dot{x}$; for since the Nature of the Circle is $1 - yy = xx$, this thrown into Fluxions is $-2y\dot{y} = 2x\dot{x}$, viz. $y\dot{y} = x\dot{x}$, and $\dot{y} = \frac{x\dot{x}}{y}$, and $\dot{y}^{2} = \frac{x^{2}\dot{x}^{2}}{yy} = \frac{x^{2}\dot{x}^{2}}{1 - xx}$, and thence $\sqrt{\dot{x}^{2} + \dot{y}^{2}} = \sqrt{\frac{x^{2}\dot{x}^{2}}{1 - xx} + \dot{x}^{2}} = \dot{x}\sqrt{\frac{1}{1 - xx}}$. The Fluent of which will be $x + \frac{1}{6} x^{3} + \frac{3}{40} x^{5} + \frac{5}{112} x^{7}$, &c. $=$ Length of the Arch CD.

SCHO-

SCHOLIUM I.

If the Arch AM (suppose z) be given in the Series found above, and you want the versed Sine or Base AP (x), or the Sine MP; then you must extract the Root of this Equation * $z = x^{\frac{1}{2}} + \frac{1}{6}x^{\frac{3}{2}} + \frac{3}{40}x^{\frac{5}{2}} + \frac{5}{112}x^{\frac{7}{2}}$, &c. AP being $= x$, for the former; or of this, $z = x + \frac{1}{6}z^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$, &c. where $PM = x$. * Art. 3.

SCHOLIUM II.

From this Example we have the Investigation of the Theorem of *Hugen*'s for finding the Length of any Arch AM of a Circle by having the Chord AM thereof, and the Chord Am of half the same given. The Theorem is $\frac{8Am - AM}{3} =$ Arch AM nearly. FIG. 37.

Call the Radius AC, a, the Arch AM, z, the Chord AM thereof A, and the Chord Am of half the same B. Then will A ($= mp =$ twice the Sine of $Am = \frac{1}{2}z$) be * $= z - \frac{z^3}{4.6aa} + \frac{z^5}{4.4.120a^4} -$, &c. and $B = \frac{1}{2}z - \frac{z^3}{2.16.6aa} + \frac{z^5}{2.16.16.120a^4} -$, &c. Now multiply B by any supposed Number n, and from the Product substract A, and that the second Term of the Remainder $- \frac{nz^3}{2.16.6a} + \frac{z^8}{4.6aa}$ may vanish, make it $= 0$. Then there comes out $n = 8$; and so $8B - A = 3z * - \frac{3z^5}{64.120a^4} +$, &c. that is, $\frac{8B - A}{3} = z$; the Error being only $\frac{z^5}{7680a^4} -$ &c. too much. * Art. 3.

SCHOLIUM III.

FIG. 38. IF the Sagitta *AP* of any Arch *MAm* be continued out, and it be requir'd to find the Point *F* in the same, from which if the right Lines *FME* and *Fme* be drawn, they may include the Part *Ee* of the Tangent to the Circle in *A* very nearly equal to the Length of the Arch *Mm*.

Let *C* be the Centre, and $AG=a$ the Diameter, and the Saggita $AP=x$. Then will

$$PM\ (=\sqrt{ax-xx})\ \text{be} = a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{3}} - \frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}} - \&c.$$

and $AE\ (=\text{Arch } AM) = a^{\frac{1}{2}}x^{\frac{1}{2}}$

$$+\frac{x^{\frac{3}{2}}}{6a^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40a^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112a^{\frac{5}{2}}} + \&c.$$

But because the Triangles *FMP*, *FEA* are similar, therefore

$$AE-PM\left(\frac{1x^{\frac{3}{2}}}{4a^{\frac{1}{2}}} + \frac{1x^{\frac{5}{2}}}{12a^{\frac{3}{2}}} + \frac{3x^{\frac{7}{2}}}{64a^{\frac{5}{2}}}\ \&c.\right)$$

$$:AP\ (x)::AE\left(a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6a^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40a^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112a^{\frac{5}{2}}} + \&c.\right)$$

$$AF = \tfrac{3}{2}a - \tfrac{1}{5}x - \frac{12xx}{175a} - \text{or} + \&c.$$

Now let us suppose $AF=\frac{3}{2}a-\frac{1}{5}x$. Then if *AH* be taken $=\frac{1}{5}\ AP\ (x)$, and *GF* be taken equal to *HC*, a right Line drawn from *F* thro' *M*, and another thro' *m*, will cut off *Ee* nearly equal to the Length of the Arch *MAm*; the Error being only $\frac{2\times16x^{3}}{525a^{3}}\sqrt{ax}+$ or $-$ &c.

SCHO-

SCHOLIUM IV.

AND if the Area of any Segment MAm of a Circle be wanted, nearly true, reduce the same to an infinite Series, *viz.* let the Segment MAm be $= \frac{4}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14a^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36a^{\frac{5}{2}}}$, &c. to which $\frac{2}{3} \overline{AM + PM} \times \frac{4}{5} AP$ will be nearly equal, the Error being only $\frac{x^3}{70a^2}\sqrt{ax} +$ &c. too little. FIG. 39.

EXAMPLE IX.

48. *TO rectify the Ellipsis, or find the Length of any Arch* AM *thereof.*

Let the Semi-transverse Axis be AC, a, and the Semi-conjugate $Cc = b$; and let $AP = x$, $PM = y$. Now the Nature of the Curve will be expressed thus $\frac{AP \times Pa \times \overline{Cc}^2}{\overline{AC}^2} = \overline{PM}^2$; that is $\frac{bb}{aa} \times \overline{2ax - xx} = yy$. And throwing both Sides of this Equation into Fluxions we shall have $2a\dot{x} - 2x\dot{x} = \frac{2a^2}{b^2} \times y\dot{y}$, or $a\dot{x} - x\dot{x} = \frac{a^2}{b^2} \times y\dot{y}$, and so $\dot{x} = \frac{\frac{a^2}{b^2} y\dot{y}}{\overline{a - x}}$, and squaring both Sides $\dot{x}^2 = \frac{a^4 y^2 \dot{y}^2}{a^2 b^4 - 2ab^4 x + b^4 xx}$. And since from the Na- FIG. 40.

ture of the Curve $bb \times \overline{2ax - xx} = aayy$, and $b^4 \times \overline{2ax - xx} = aabbyy$; if $aabbyy$ be put for its Equal in the Denominator, we shall have $\dot{x}^2 = \frac{a^2 y^2 \dot{y}^2}{a^2 b^4 - a^2 b^2 y^2} = \frac{a^2 y^2 \dot{y}^2}{b^4 - b^2 y^2}$, and adding $\dot{y}^2$ to both Sides; $\dot{x}^2 + \dot{y}^2 = \frac{a^2 y^2 \dot{y}^2}{b^4 - b^2 y^2} + \dot{y}^2 =$ $\frac{a^2 y^2 \dot{y}^2 + b^4 \dot{y}^2 - b^2 y^2 \dot{y}^2}{b^4 - b^2 y^2} = \frac{b^4 + \overline{a^2 - b^2} \times y^2}{b^4 - b^2 y^2} \times \dot{y}\dot{y}$; and consequently $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{b}\sqrt{\frac{b^4 + \overline{a^2 - b^2}yy}{b^2 - yy}}$ = Fluxion Mm of the Arch AM of the Ellipsis. And supposing $b = 1$, and $a^2 - bb$ $(= a^2 - 1) = c$; then will Mm the Fluxion of the Arch be $\dot{y}\sqrt{\frac{1 + cyy}{1 - yy}}$; and the Fluent of the same will be $y + \frac{ny^3}{3.2} + \frac{ny^5}{5.2} - \frac{n^2 y^5}{5.2.2.2.} + \frac{ny^7}{7.2}$ $- \frac{n^2 y^7}{7.2.2.} + \frac{n^3 y^7}{7.2.2.2.2.2}$, &c. supposing $n = 1 + c$.

But to find the Length of the Arch Mc of the Ellipsis, you may proceed thus: Here let PC be $= x$; then will $\overline{AC^2 - PC^2} \times \frac{Cc^2}{AC^2} = \overline{PM}^2$ that is, $\overline{aa - xx} \times \frac{bb}{aa} = yy$; and so $\overline{aa - xx} \times bb$ $= aayy$. And throwing both Parts of the Equation into Fluxions, we shall have $-2bbx\dot{x}$ $= 2aay\dot{y}$, or $-bbx\dot{x} = aay\dot{y}$, and $\dot{x} = \frac{aay\dot{y}}{-bbx}$, and squaring both Sides $\dot{x}^2 = \frac{a^4 y^2 \dot{y}^2}{b^4 x^2}$. Now from the Equation of the Curve we shall get $x^2 = aa - \frac{aa}{bb}y^2$, and so $b^4 x^2 = aab^4 - b^2 y^2$; and

so

so substituting $aab^4-b^2y^2$ for b^4x^2 in the Denominator, and $\dot{x}^2=\frac{a^4y^2\dot{y}^2}{aab^4-a^2b^2y^2}=\frac{a^2y^2\dot{y}^2}{b^4-b^2y^2}$ and then adding $\dot{y}^2$ to both Sides, there will be had $\dot{x}^2+\dot{y}^2=\left(\frac{a^2y^2\dot{y}^2}{b^4-b^2y^2}+\dot{y}^2=\frac{b^4+a^2y^2-b^2y^2}{b^4-b^2y^2}\times\dot{y}^2\right)=\frac{b^4+\overline{a^2-b^2}\times yy}{b^4-b^2y^2}\times\dot{y}^2$; therefore $\sqrt{\dot{x}^2+\dot{y}^2}$ $=\sqrt{\frac{b^4+\overline{a^2-b^2}yy}{b^4-b^2yy}}\,\dot{y}=$ Fluxion of the Arch *Mc*. Which is the same as the Fluxion of the Arch *AM*, when *AP* was equal to x; and so the Fluent of this will be the same Expression as the Fluent of the Arch *AM*.

If *Aa* be $=a$, and $CP=x$, we shall have the Length of the Arch $Mc=$

$$x$$
$$+\frac{bb}{2aa}\times\frac{x^3}{3a}$$
$$+\overline{\frac{b^2}{2aa}-\frac{b^4}{8a^4}}\times\frac{x^5}{5a^4}$$
$$+\overline{\frac{bb}{2aa}-\frac{b^4}{4a^4}+\frac{b^6}{16a^6}}\times\frac{x^7}{7a^6}$$
$$+\frac{bb}{2aa}-\frac{b^4}{8a^4}+\frac{3b^6}{16a^6}-\frac{5b^8}{128a^8}\times\frac{x^9}{9a^8}, \&c.$$

for $\frac{bb}{aa}\times\overline{aa-xx}=yy$; and $\frac{b}{a}\times\overline{aa-xx}^{\frac{1}{2}}=y$. Whence $\frac{b}{a}\times\frac{-x\dot{x}}{\overline{aa-xx}^{\frac{1}{2}}}=\dot{y}$; and so $\frac{bb}{aa}\times\frac{xx}{aa-xx}$ $\times\dot{x}\dot{x}=\dot{y}\dot{y}$. Consequently $\overline{1+\frac{bb}{aa}\times\frac{xx}{aa-xx}}$ $\times\dot{x}\dot{x}=\dot{y}\dot{y}+\dot{x}\dot{x}$. And the Fluxion of the Arch

Arch Mc will be $\overline{1+\frac{bb}{aa}\times\frac{xx}{aa-xx}}\Big|^{\frac{1}{2}}\times\dot{x}$. Which thrown into an infinite Series will be $= 1 + \frac{1}{2}\times\frac{bb}{aa}\times\frac{xx}{aa-xx} - \frac{1}{8}\times\frac{b^4}{a^4}\times\overline{\frac{xx}{aa-xx}}\Big|^2\times\frac{1}{16}\times\frac{b^6}{a^6}\times\overline{\frac{xx}{aa-xx}}\Big|^3 - \frac{5}{128}\times\frac{b^8}{a^8}\times\overline{\frac{xx}{aa-xx}}\Big|^4$ &c. $\times\dot{x} =$

$$\dot{x}$$
$$+\tfrac{1}{2}\dot{x}\times\frac{bb}{aa}\times\overline{\frac{x^2}{a^2}+\frac{x^4}{a^4}+\frac{x^6}{a^6}+\frac{x^8}{a^8}}, \text{ \&c.}$$
$$-\tfrac{1}{8}\dot{x}\times\frac{b^4}{a^4}\times\overline{\frac{x^4}{a^4}+\frac{2x^6}{a^6}+\frac{x^8}{a^8}}, \text{ \&c.}$$
$$+\tfrac{1}{16}\dot{x}\times\frac{b^6}{a^6}\times\overline{\frac{x^6}{a^6}+\frac{3x^8}{a^8}}, \text{ \&c.}$$
$$-\tfrac{5}{128}\dot{x}\times\frac{b^6}{a^8}\times\frac{x^8}{a^8}, \text{ \&c.}$$

And the Fluent of this will be

$$x$$
$$+\tfrac{1}{2}\times\frac{bb}{aa}\times\frac{x^3}{3aa}+\frac{x^5}{5a^4}+\frac{x^7}{7a^6}+\frac{x^9}{9a^8}, \text{ \&c.}$$
$$-\tfrac{1}{8}\times\frac{b^4}{a^4}\times\qquad\frac{x^5}{5a^4}+\frac{2x^7}{7a^6}+\frac{x^9}{9a^8}, \text{ \&c.}$$
$$+\tfrac{1}{16}\times\frac{b^6}{a^6}\times\qquad\frac{x^7}{7a^6}+\frac{3x^9}{9a^8}, \text{ \&c.}$$
$$-\tfrac{5}{128}\times\frac{b^8}{a^8}\qquad+\frac{x^9}{9a^8}, \text{ \&c.}$$

being = to the Series first proposed.

Fig. 41.

EXAMPLE X.

49. *TO rectify the Hyperbola, or find the Length of any Arch* AM *thereof.*

Let

Let C be the Centre, $aA (2a)$ the transverse Axis, $Cc (b)$ the Semi-conjugate, $AP (x)$ any Abscifs, and $PM (y)$ the correspondent Ordinate. Now the Nature of this Curve is $\frac{aP \times AP \times \overline{Cc}^2}{\overline{aA}^2} = \overline{PM}^2$; that is, $\frac{bb}{aa} \times \overline{2ax + xx} = yy$. And throwing both Parts of the Equation into Fluxions we have $2a\dot{x} + 2x\dot{x} = \frac{2a^2}{b^2} \times y\dot{y}$, or $a\dot{x} + x\dot{x} = \frac{a^2}{b^2} \times y\dot{y}$; and so $\dot{x} = \frac{a^2 y\dot{y}}{b^2 \times \overline{a + x}}$; and squaring both Sides $\dot{x}^2 = \frac{a^4 y^2 \dot{y}^2}{a^2 b^4 + 2ab^4 x + b^4 xx}$. And since (from the Nature of the Curve) $bb \times \overline{2ax + xx} = aayy$, and $b^4 \times 2ax + xx = a^2 b^2 yy$; if $a^2 b^2 yy$ be put for its Equal $2ab^4 x + b^4 xx$, we shall have $\dot{x}^2 = \frac{a^4 y^2 \dot{y}^2}{a^2 b^4 + a^2 b^2 yy} = \frac{a^2 y^2 \dot{y}^2}{b^4 + b^2 yy}$. Then adding $\dot{y}^2$ to each Side, and $\dot{x}^2 + \dot{y}^2$ will be $= \frac{a^2 y^2 \dot{y}}{b^4 + b^2 yy} + \dot{y}^2 = \frac{b^4 + \overline{a^2 + b^2} \times yy}{b^4 + b^2 yy} \times \dot{y}^2$. Consequently $\sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}\sqrt{b^4 + \overline{a^2 + b^2} \times yy}}{b^4 \quad b^2 + yy} = Mm$, the Fluxion of the Arch AM; being the very same Expression as that of the Ellipsis in the last Example, only with this Difference, that in the Ellipsis the Sign of the Term $b^2 yy$ is negative, and here it is affirmative. Consequently the Fluent here will be the same as the Fluent for the Ellipsis; only with the Alteration of the Signs, *viz.* here $a^2 + 1 = c$, and $c - 1 = n$.

Note,

Note, If the Hyperbola be an Equilateral one, then $2ax + xx = yy$ expresses the Nature of the same. This thrown into Fluxions will be $2a\dot{x} + 2x\dot{x} = 2y\dot{y}$, or $a\dot{x} + x\dot{x} = y\dot{y}$, and $\dot{x} = \frac{y\dot{y}}{a+x}$. Which squared, and then $\dot{x}^2 = \frac{y^2\dot{y}^2}{a^2 + 2ax + xx}$; and substituting $yy (= 2ax + xx)$ in the Denominator, $\dot{x}^2$ will be $= \frac{y^2\dot{y}^2}{a^2 + yy}$. Consequently $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{y}\sqrt{\frac{a^2 + 2y}{a^2 + y^2}} =$ Fluxion of the Arch AM; and making $a = 1$, the the same will be $\dot{y}\sqrt{\frac{1 + 2y^2}{1 + y^2}}$. The Fluent of which will be $y + \frac{y^3}{3.2} - \frac{y^5}{5.2} + \frac{y^7}{7.2.2} - \frac{y^7}{7.2.2.2.2}$, &c.

SCHOLIUM.

FIG. 38. If in the Ellipsis MAm, AG be = transverse Axis, and AQ = *Latus Rectum*, and QF be assumed $= \frac{1}{2} AQ \frac{19AG + 21AQ}{10AG} \times AP$; and in the Hyperbola, $QF = \frac{1}{2} AQ + \frac{19AC + 21AQ}{10AG} \times AP$, and the Secant FME be drawn; then the Tangent AE will be nearly equal to the Arch AM of the Ellipsis or Hyperbola, if the same be not very large.

EXAMPLE XVI.

FIG. 26. 50. *To rectify the Cycloid, or find the Length of any Arch* AM *thereof.*

 Let

Let $AQ=x$, $\overline{AB=1}$; then $Qq=MS=\dot{x}$, and $PQ=\sqrt{x-xx}$ from the Nature of the Circle: therefore $AP=\sqrt{x}=x^{\frac{1}{2}}$. Consequently because the Triangles APQ, MmS are similar $AQ\,(x) : AP\,(x^{\frac{1}{2}}) :: MS\,(\dot{x}) : Mm = x^{-\frac{1}{2}}\dot{x}$ = Fluxion of the Arch AM; the Fluent of which will be $2x^{\frac{1}{2}}=2AP$=Arch AM. So the Length AD of $\frac{1}{2}$ the Cycloid is = to twice the Diameter AB of the generating Circle, which we know to be true from other Principles.

EXAMPLE XVII.

51. *TO find the Length of any Arch* AM *of the Cissoid of* Diocles AMI.

Let AB the Diameter of the generating Circle be a, and call PB, x, and the Ordinate PM, y; then $AP=a-x$. Now from the Nature of the Curve $PM\,(y)$ is $=AP\times\sqrt{\frac{AP}{PB}}$ FIG. 42.

that is, $y=\overline{a-x}\sqrt{\frac{a-x}{x}}=x^{-\frac{1}{2}}\times\overline{a-x}|^{\frac{3}{2}}$.

And throwing this Equation into Fluxions, we get $\dot{y}=\dot{x}\times-\frac{1}{2}x^{-\frac{3}{2}}\times\overline{a-x}|^{\frac{3}{2}}-\frac{3}{2}x^{-\frac{1}{2}}\times\overline{a-x}|^{\frac{1}{2}}$ $=\frac{-a-2x}{2x}\sqrt{\frac{a-x}{x}}\times\dot{x}$. Which squar'd, and

$$\dot{y}^2=\frac{\overline{aa+4ax+4xx}\times\overline{a-x}}{4x^3}\times\dot{x}^2=$$

$\frac{a^3+3aax-4x^3}{4x^3}\times\dot{x}^2$. Whence $\dot{y}^2+\dot{x}^2=$ $\frac{a^3+3aax}{4x^3}\times\dot{x}^2$, and so $\sqrt{\dot{y}^2+\dot{x}^2}=\frac{a\dot{x}}{2x}\sqrt{\frac{a+3x}{x}}$ $=\frac{1}{2}ax^{-\frac{3}{2}}\dot{x}\times\sqrt{a+3x}$.

 Now

Now this must be compar'd with Mr. *Cotes*'s 4th Form $d\dot{z}z^{\theta n+\frac{1}{2}n-1}\sqrt{e+fz^n}$. And making $z=x$, $d=\frac{1}{2}a$, $\theta=-1$, $n=1$, $e=a$, $f=3$, we get $P\left(\sqrt{\frac{e+fz^n}{z^n}}\right)=\sqrt{\frac{a+3x}{x}}$. $R(\sqrt{f})$ $=\sqrt{3}$. $T\left(\sqrt{\frac{e+fz^n}{z^n}}\right)=\sqrt{\frac{a+3x}{x}}$, and $S\left(\sqrt{\frac{e}{z^n}}\right)$ $=\sqrt{\frac{a}{x}}$. And so the Fluent $\frac{-2}{n}dP+\frac{2}{n}dR$ $\left|\frac{R+T}{S}\right.$ becomes the Fluent of the given Fluxion $-\sqrt{\frac{a+3x}{x}}+a\sqrt{3}\left|\frac{\sqrt{3}+\sqrt{\frac{a+3x}{x}}}{\sqrt{\frac{a}{x}}}\right.$ or $=$ $-a\sqrt{\frac{a+3x}{x}}+3\sqrt{\frac{1}{3}aa}\left|\frac{\sqrt{ax}+\sqrt{\frac{1}{3}aa+3ax}}{\sqrt{\frac{1}{3}aa}}\right.$. But this Fluent must be alter'd before it can express the Arch AM, ~~because x begins at B and not at A.~~

In order to this, make $x=a$, and the said Fluent becomes $=-2a+3\sqrt{\frac{1}{3}aa}\left|\frac{a+\sqrt{\frac{4}{3}aa}}{\sqrt{\frac{1}{3}aa}}\right.$; from which if the other Fluent be taken, we shall get $a\sqrt{\frac{a+3x}{x}}-2a+$ $3\sqrt{\frac{1}{3}aa}\left|\frac{a+\sqrt{\frac{4}{3}aa}}{\sqrt{ax}+\sqrt{\frac{1}{3}aa+ax}}\right.$, (remembring that the Substraction of the Ratio $\frac{\sqrt{ax}+\sqrt{\frac{1}{3}aa+3ax}}{\sqrt{\frac{1}{3}aa}}$ from the Ratio $\frac{\sqrt{a+\frac{4}{3}aa}}{\sqrt{\frac{1}{3}aa}}$ is the Division of the latter by the former) expressing the Length of the Arch AM.

Now

Now for the Construction of the Fluent. Draw AC $(\sqrt{\frac{4}{3}aa})$ cutting the Asymptote in C so, that CAB be $\frac{1}{3}$ of a right Angle; and let BD $(\sqrt{ax})$ be a mean Proportional between $AB\,(a)$ and $PB\,(x)$; and draw $CD\,(\sqrt{\frac{1}{3}aa+ax})$. Also draw AE bisecting PM, which will be $=\frac{1}{2}a\sqrt{\frac{a+3x}{x}}$. Then the Arch $A\overset{M}{P}$ of the Cissoid will be $=2AE-2AB+3BC\left|\frac{AB+AC}{BD+DC}\right.$ $=a\sqrt{\frac{a+3x}{x}}-2a+3\sqrt{\frac{1}{3}aa}\left|\frac{a+\sqrt{\frac{4}{3}aa}}{\sqrt{ax}+\sqrt{\frac{1}{3}aa+ax}}\right.$.

SECTION V.

Of the Use of Fluxions in the Cubation of Solids, and in the Quadrature of their Surfaces.

PROB.

FIG. 7. 52. *To cube or find the Solidity of any Solid generated by the Rotation of a plain Figure* AMN *about the Axis* AQ.

Draw the Ordinate *pm* infinitely near *PM*; then the Parallelogram *PMrp* may be taken for the Trapezium *PMmp*; and consequently the Cylinder described by the said little Parallelogram *PMrp*, while the Figure *ANQ* revolves about the Axis *AQ*, may be taken for the Increment or Fluxion of the Portion of the Solid, generated by the Rotation of the Portion *AMP* of the plain Figure; and the Fluent of that Fluxion will be equal to the Solidity of the said Portion, conceived as made up of an infinite Number of Cylinders of infinitely small equal Altitudes.

Now let $AP=x$, $PM=y$, and the Ratio of the Radius of a Circle to the Circumference, be expressed by $\frac{r}{p}$. Then will the Circumference of the Circle described by the Radius *PM*

PM be $= \frac{py}{r}$, and the Area of the said Circle will be $\frac{p}{2r} y^2$. Which multiplied by $Pp\,(\dot{x})$ is $\frac{p}{2r} \dot{x} y^2 =$ Solidity of the Cylinder aforesaid, or Fluxion of the Portion of the Solid. And if for y^2 in this Expression you substitute its Value gained from the Equation of the Curve *AMN*, you will have a fluxionary Expression affected with only one unknown Quantity x; and the Fluent thereof will express the Solidity required.

EXAMPLE I.

53. *TO cube a right Cone.*

A right Cone may be described by the Rotation of a right-angled Triangle *ABC* about the Side or Axis *AB*. Now let $AB = a$, $BC = r$, $AP = x$, $PM = y$. Then because the Triangles *APM*, *ABC* are similar, $AP\,(x) : PM\,(y) :: AB\,(a) : BC\,(r)$; therefore $\frac{rx}{a} = y$; and squaring both Sides $\frac{r^2 x^2}{a^2} = y^2$. Therefore $\frac{py^2 \dot{x}}{2r}$ (= Fluxion * of the Solid) $= \frac{pr^2 x^2 \dot{x}}{2a^2 r}$ $= \frac{prx^2 \dot{x}}{2a^2}$ (by substituting $\frac{rx}{a}$ for y) = the Fluxion of the Solidity of the Part of the Cone generated by the Triangle *APM*. The Fluent of which will be $= \frac{prx^3}{6a^2} =$ Solidity of the Part of the Cone; and if you substitute a for x,

FIG. 43.

* Art. 52.

x, the Solidity of the whole Cone will be $\frac{pra^3}{6a^2} = \frac{1}{6}apr \times \frac{1}{3}a = \frac{1}{2}pr \times \frac{1}{3}a$; that is, the Base is to be multiplied into $\frac{1}{3}$ of the Altitude.

EXAMPLE II.

54. *TO cube a Sphere, or any Segment thereof.*

FIG. 44. A Sphere is generated by the Rotation of a Semicircle ABa about the Axis or Diameter ACa. Let the Radius $AC = r$, $AP = x$, $PM = y$; then from the Nature of the Circle generating it, $yy = 2rx - xx$. Whence $\frac{py^2\dot{x}}{2r} = \frac{2prx\dot{x} - px^2\dot{x}}{2r}$ = Fluxion of the Segment of the Sphere generated by the Rotation of the Semi-segment of the Circle AMP; and the Fluent thereof will be $\frac{3prx^2 - px^3}{6r}$ = Solidity of the Segment AMm. And if the whole Diameter $2r$ be put for x, the Solidity of the whole Sphere will be $= \frac{12pr^2 - 8pr^3}{6r} = 2pr^2 - \frac{4}{3}pr^2 = \frac{2}{3}pr^2 = 2rp \times \frac{1}{3}r$; that is, the Rectangle under the Diameter $2r$, and Circumference p, is to be multiplied by $(\frac{1}{3}r)$ a third Part of the Radius, or sixth Part of the Diameter. And if the Diameter $2r$ be $= 1$, then the Solidity of the Sphere will be $\frac{1}{6}p$.

COROL. I.

HENCE a Sphere is equal to a Quadrangular Pyramid, whose Base is the Rectangle under the Diameter of the Sphere $2r$, and the

Periphery described by the same, and Altitude equal to the Semidiameter of the Sphere.

COROL. II.

AND because the Solidity of a Cylinder circumscribing the Sphere is pr^3; therefore it is to the Sphere as pr^3 to $\frac{2}{3}pr^3$, or as $3pr^3$ to $2pr^3$, or as 3 to 2.

EXAMPLE III.

55. *IF* EDBF *be a right Cylinder, and the Part* DBMAF *be cut off from the same by a Plane* DFA *passing thro'* C *the Centre of the lower Base, and* F *the Extremity of the Diameter* FD *of the upper Base; it is required to cube the said Part* DBMAF: *which is called the* Ungula *or* Hoof.

Let AD be at right Angles to $BE=2r$, draw CF, any where in AD take the Point P, draw PM parallel to CB, and PN parallel to CF; join the Points M,N, and call the Altitude FB, a. Then any Triangle PMN (formed thus) right-angled at M, will be similar to the right-angled Triangle CBF. This being allow'd, call CP, x; then will $\overline{PM}^2=rr-xx$, and $CB\,(r):BF\,(a)::PM\,(\sqrt{rr-xx}):MN=\frac{a}{r}\sqrt{rr-xx}$; and the Area of the Triangle PMN will be $\frac{1}{2}PM\times MN=\frac{a}{2r}\times\overline{rr-xx}$. Which multiplied by Pp $(\dot{x})$ will be $\frac{arr\dot{x}-axx\dot{x}}{2r}$ = the Fluxion of $APMN$ the Part of the *Ungula*; and the Fluent will be $\frac{arx}{2}-\frac{ax^3}{6r}=$ FIG. 45.

Soli-

Solidity of the ſaid Part *APMN*, and when $=x$ becomes $=r$, then will this laſt Expreſſion be $\frac{1}{3}ar^2$, = Solidity of $\frac{1}{2}$ the *Ungula*, and ſo the Solidity of the whole will be $\frac{2}{3}ar^2$.

EXAMPLE IV.

FIG. 46. 56. *TO cube a Solid generated by the Rotation of the Part* F m M *of the* Lunule F A D *of* Hypocrates *about the Radius* E D *as an Axis.*

Draw the Tangent *Aa*. Call the Radius *CE*, 1; then the Radius *CA* will be $\sqrt{2}$. Alſo let $EP=x$, the Ordinate $PM=y$, and $Pm=z$. Now $\overline{CA}^2-\overline{CP}^2=\overline{PM}^2$ from the Nature of the Circle; that is, $2-1-2x-xx$ $(=1-2x-xx)=yy$. And $\overline{ED}^2-\overline{EP}^2=\overline{Pm}^2$; that is, $1-xx=zz$. Whence the Area of the Circle deſcribed by *PM* will be $\frac{p-2px-pxx}{2r}$; and the Area of the Circle deſcribed by *Pm* will be $\frac{p-pxx}{2r}$; and the Difference of theſe Areas will be $\frac{px}{r}$ = *Anulus* deſcribed by *Mm*. And this drawn into the Fluxion $\dot{x}$ will be $\frac{px}{r}\dot{x}$ = to the Fluxion of the Part of the Solid deſcribed by the Part *FmM* of the *Lunule*; and the Fluent thereof will be $\frac{pxx}{2r}$ = Solidity of the ſaid Part.

EXAM-

EXAMPLE V.

57. *TO cube a Parabolical Conoid generated by the Rotation of a Parabolick Space of any Kind about its Axis.*

Let the Parameter be $= 1$; let AD be $=$ FIG. 1.
a, $BD = r$, $AP = x$, $PM = y$. Then the Nature of all Parabola's will be expressed by $y^m = x$. Whence $y = x^{\frac{1}{m}}$; and so $yy = x^{\frac{2}{m}}$; therefore $\frac{py^2\dot{x}}{2r} = \frac{px^{\frac{2}{m}}\dot{x}}{2r} =$ Fluxion of the Solid generated by the Rotation of the Portion APM of the Parabola. And the Fluent thereof will be $\frac{mp}{\overline{2m+4} \times r} x^{\frac{m+2}{m}}$; and substituting $y^2 x$ for $x^{\frac{m+2}{m}}$ its Equal, there will come out $\frac{mpy^2x}{\overline{4+2m} \times r}$ $=$ Fluent of the said Solid.

And if a the Altitude of the whole Conoid be put for x, and $2r$ the Diameter of the Base for $2y$; then the Solidity of the whole Conoid will be $\frac{mpr^2a}{\overline{4+2m} \times r} = \frac{m}{4+2m} \times apr = \frac{1}{2} pr \times \frac{ma}{2+m}$.

Hence if the generating Parabola be the common one, m will be $= 2$; and so $\frac{m}{2+m}$ is $= \frac{2}{2+2} = \frac{1}{2}$. Whence the Base is to be drawn into $\frac{1}{2}$ the Altitude; and consequently the generated Conoid will be $\frac{1}{2}$ a Cylinder of the same Base and Altitude.

EXAMPLE VI.

58. *TO cube a Spheroid generated by the Rotation of a Semi-elliptick Space about its transverse or conjugate Axes.*

FIG. 40. If you make $AP=x$, $PM=y$, and $Aa=a$, and the Parameter be $=b$; then will $\overline{PM}^2$ be $=AP\times b-\frac{\overline{AP}^2\times b}{Aa}$; that is, $y^2=bx-\frac{bx^2}{a}$ will express the Nature of the Ellipsis. Therefore * $\frac{py^2}{2r}\dot{x}=\frac{pbx}{2r}-\frac{pbx^2}{2ra}\times\dot{x}=\frac{pbx\dot{x}}{2r}-\frac{pbx^2\dot{x}}{2ra}$ will be the Fluxion of the Solid generated by the Rotation of the Part of the Ellipsis APM; and the Fluent of the said Fluxion will be $\frac{pbx^2}{4r}-\frac{pbx^3}{6ar}=$ the said Solid.

* Art. 52.

And if the whole Axis a be substituted for x, the whole Spheroid will be $\frac{pba^2}{4r}-\frac{pba^2}{6r}=\frac{6pba^2-4pba^2}{24r}=\frac{pba^2}{12r}$.

Hence if the conjugate Axis Cc be $=2r$. Then will $4r^2$ be $=ab$; and so the Solidity of the whole Spheroid will become $\frac{1}{3}par$; that is, the Elliptick Spheroid is equal to a Cone of the same Height with a the transverse Axis of the Ellipsis, and Diameter of the Base equal to four times the conjugate Axis of the Ellipsis, *viz.* $4r$. And because the Altitude of a Cylinder circumscribing the Spheroid being a, and Diameter $=r$, the Solidity of that Cylinder is $\frac{1}{2}apr$; therefore a Spheroid, as well as a Sphere, is $\frac{2}{3}$ of the circumscribing Cylinder.

Other-

Otherwise:

This Example may be effected something shorter thus: Let $r=1$, in the general Expression * $\frac{py^2}{2r}\dot{x}$; then will the same become $\frac{py^2}{2}\dot{x}$. * *Art.* 52.

Make $Cc=1$, $AC=a$, and $PC=x$, and the Nature of the Ellipsis will be $1-\frac{xx}{aa}=yy$; and substituting $1-\frac{xx}{aa}$ for yy in the said general Expression; and $\frac{py^2\dot{x}}{2}$ will be $=\frac{p\dot{x}}{2}-\frac{px^2\dot{x}}{2aa}$ = Fluxion of the Part of the Spheroid generated by the Rotation of the Part $MPcC$ of the Ellipsis, and the Fluent thereof will be $\frac{px}{2}-\frac{p}{6aa}x^3=\frac{p}{2}\times\overline{x-\frac{x^3}{3aa}}$ = Soldity of the said Part of the Spheroid. And substituting a for x, we shall have $\frac{p}{2}\times\overline{a-\frac{a}{3}}=\frac{p}{2}\times\overline{\frac{3a-a}{3}}=\frac{2}{3}a\times\frac{p}{2}$; but $\frac{p}{2}$ = Circle described by the Rotation of Cc; therefore the Solidity of the Spheroid is $=\frac{2}{3}$ of a Cylinder of the same Base and Altitude.

EXAMPLE VII.

59. *TO cube an Hyperbolical Conoid generated by the Rotation of a Semi-Hyperbola about the transverse Axis.* FIG. 41.

Let $AP=x$, $PM=y$, the Parameter $=b$, and the transverse Axis $=a$. Then the Nature of the Curve will be $AP \times b + \frac{\overline{AP}^2 \times b}{Aa}$; that is, $y^2 = bx + \frac{bx^2}{a}$. Therefore* $\frac{py^2}{2r}\dot{x} = \frac{pbx\dot{x}}{2r} + \frac{pbx^2\dot{x}}{2ra}$ is = Fluxion of the Conoid generated by the Rotation of the Part AMP of the Hyperbola about the Axis Aa; the Fluent of which will be $\frac{pbx^2}{4r} + \frac{pbx^3}{6ra}$ = Solidity of the said Conoid. And when the Altitude of the Conoid is equal to a; that is, when x is $= a$, the Solidity of the Conoid will become $\frac{6pba^2 + 4pba^2}{24r} = \frac{10pba^2}{24} = \frac{5}{12}pba^2$. And if $2r$ be = conjugate Axis, then will $2r = \sqrt{ab}$, and $4r^2 = ab$; which Value being substituted in the last Expression, and the Conoid will be $\frac{5}{3}par$.

If the Hyperbola be an Equilateral one, then the Nature of it will be expressed thus, $y^2 = ax + xx$; whence $\frac{py^2\dot{x}}{2r} = \frac{apx\dot{x} + px^2\dot{x}}{2r}$; and the Fluent thereof will be $\frac{apx^2}{4r} + \frac{px^3}{6r}$. And so since here $2r = a$, and $b = a$, the Solidity will be $\frac{5}{6}pa^2$.

Hence the Solidity of the circumscribing Cylinder will be $\frac{1}{2}par$; which therefore will be to the Solidity of the Conoid as $\frac{1}{2}par$ to $\frac{5}{3}par$, or as 3 to 10; and in the Conoid generated by the Equilateral Hyperbola, the circumscribing Cylinder is $\frac{1}{4}pa^2$. Whence the same will be to the Conoid as $\frac{1}{4}pa^2$ to $\frac{5}{6}pa^2$, or as 3 to 10.

By making $PC = x$, and $Cc = 1 = r$, and $AC = a$, as before in the Ellipsis; then the Nature of the Hyper-

Hyperbola will be $yy = \frac{xx}{aa} - 1$. Whence $\frac{py^2\dot{x}}{2} = \frac{px^2\dot{x}}{2aa} - \frac{p\dot{x}}{2}$; and the Fluent thereof will be $\frac{1px^3}{6aa} - \frac{px}{2}$. And since the Beginning of x is not at A, you must make $x = a$; and then this Fluent will become $= \frac{pa}{6} - \frac{pa}{2}$. Which being taken from the said Fluent, and we shall have $\frac{px^3}{6aa} - \frac{px}{2} + \frac{pa}{3} =$ Solidity of the Solid generated by the Part AMP of the Hyperbola.

EXAMPLE VIII.

60. *TO cube a Solid generated by the Rotation of the interminate hyperbolical Space* CABED *about one of the Asymptotes* CD. FIG. 47.

Let $AB = a$, $AC = b$, $CP = x$, $PM = y$; and let $Pp = \dot{x}$; then let the Circumference of the Circle described by the Radius AC be $= p$, and the Circumference described by the Radius PC will be $\frac{px}{a}$. Which drawn into PM (y) will give us $\frac{pxy}{a}$, the Superficies of a Cylinder described by the Parallelogram $CPMR$. This again drawn into Pp ($\dot{x}$) will give us the Solidity of the little Cylinder $PpqM$, viz. $\frac{pxy\dot{x}}{a} =$ Fluxion of the Solid; but the Nature of the Hyperbola with regard to the Asymptotes is $xy = ab$. Whence $y = \frac{ab}{x}$, and $\frac{pxy\dot{x}}{a} = \frac{pabx\dot{x}}{ax} = pb\dot{x}$; the Fluent of which is $pbx =$ Solidity of the interminate Space $CABED$; and if a be put for x, the whole Solid will be pba.

EXAMPLE IX.

61. *TO cube a Conoid generated by the Rotation of the hyperbolick Space* AMBDC *about*
FIG. 48. CD *the half of the conjugate Axis of the Hyperbola* AMB.

Call CA, a, CD, b, CP, x, PM, y, BD, r. Now $\frac{pr}{y}$. will be the Periphery described by the Point M, and $\frac{pyy}{2r}$ = the whole Circle, having PM for a Radius; which multiplied by $\dot{x}$, and $\frac{pyy\dot{x}}{2r}$ will be the Fluxion of the Solid. But $yy = \frac{aaxx + aabb}{bb}$, from the Nature of the Curve; and substituting this Value in $\frac{pyy\dot{x}}{2r}$, we have $\frac{aapxx\dot{x} + aabbp\dot{x}}{2bbr}$ for the Fluxion of the Conoid, the Fluent of which is $= \frac{aapx^3}{6bbr} + \frac{aapx}{2r}$; and substituting $\frac{bbyy}{xx+bb}$ for aa, there arises $\frac{px^3yy}{6rxx+6bbr} + \frac{bbpyyx}{2rxx+2bbr} =$ Solidity of the Conoid formed by the Space $AMPC$; and when x becomes $= b$, and y to r, then the whole Conoid will be $= \frac{bpr}{3}$.

COROL.

A Cylinder generated by the Rotation of the Parallelogram $ACSB$ about the Axis CS is $\frac{1}{2}pba$; and so it is to the said hyperbolick Solid

Solid as $\frac{1}{2}pba$ to pba, that is, as $\frac{1}{2}$ to 1, or 1 to 2.

EXAMPLE X.

62. *TO cube a Solid generated by the Rotation of a Parabola about a Semi-ordinate* CB.

Let $AB=r$, $BC=b$, $AP=x$, $PM=y$; then if the Parameter be 1, this Equation will express the Nature of the Parabola $y^2=x$. But it is manifest *, that the Fluxion of the Solid is the Circle described with the Radius MD drawn into $Dd=\dot{y}$. Let the Ratio of the Radius to the Circumference be as r to p; then $MD=BD=AB-AP=r-x$. And the Circumference of the Circle described by MD $=p-\frac{px}{r}$, and the Area of the said Circle will be $\frac{pr}{2}-px+\frac{px^2}{2r}$. Whence the Fluxion of the Solid will be $\frac{pr}{2}\dot{y}-px\dot{y}+\frac{px^2\dot{y}}{2r}$. Now if in this Expression for x and x^2 you substitute y^2 and y^4, their Equals (from the Equation of the Curve) we shall have $\frac{p\dot{y}}{2}-\frac{py^2\dot{y}}{2r}$ $+\frac{py^4\dot{y}}{2r}$ equal to the Fluxion of the indefinite Part of the Solid generated by the Rotation of the Portion MCD about the Axis BC; the Fluent of which will be $\frac{1}{2}py-\frac{py^3}{3or}+$ $\frac{py^5}{10r}=$ said Part of the Solid. FIG. 49. * Art. 52.

But if for y^2 you put x in the general Expression *, then the said Solid will be $\frac{1}{2}py-\frac{pxy}{3or}$ * Art. 52.

$\frac{pxy}{30r}+\frac{px^2y}{10r}=p\times\frac{1}{20}y-\frac{xy}{30r}+\frac{x^2y}{10r}$. And if for y you put b, and for x you put r, the whole Solid will be $p\times\frac{1}{2}br-\frac{1}{3}br+\frac{1}{10}br=30-20+6\times\frac{pbr}{60}=\frac{8}{30}pbr=\frac{1}{2}pr\times\frac{8}{15}b$; that is, the Base or Circle described by the Radius AB is to be drawn into $\frac{8}{15}$ of the Altitude BC.

Because a Cylinder of the same Base and Altitude is $\frac{1}{2}pbr$; therefore it will be to this parabolical Solid as $\frac{1}{2}pbr$ to $\frac{1}{2}pbr\times\frac{8}{15}b$; that is, as 1 to $\frac{8}{15}$, or as 15 to 8.

EXAMPLE XI.

FIG. 28. 63. *TO cube a Solid generated from the Rotation of the Logarithmetical Curve about the Asymptote* AH.

Here the Subtangent being always $=a$, $y\dot{x}$ is $=a\dot{y}$; and so $\dot{x}=\frac{a\dot{y}}{y}$. Whence $\frac{py^2\dot{x}}{2r}=\frac{pay\dot{y}}{4r}$ = Fluxion of Part of the Solid generated by the Rotation aforesaid; and the Fluent thereof will be $\frac{pay^2}{4r}$. And putting $r=AB$ for y, then will the whole Solid be $\frac{par^2}{4r}=\frac{1}{4}apr$.

Hence because a Cylinder, whose Altitude is $=a$, and Radius of the Base $=r$, is $\frac{1}{2}apr$: therefore it is to the Solid as $\frac{1}{2}a$ to $\frac{1}{4}a$, or as 2 to 1.

EXAMPLE XII.

FIG. 27. 64. *TO cube a Solid generated by the Rotation of the Cissoid about the Line* AB *as an Axis.*

Let

Let $AB=1$, $AP=x$, $PM=y$; then the Nature of the Curve will be $y^2=\frac{x^3}{1-x}$; and so $\frac{py^2\dot{x}}{2r}=\frac{px^3\dot{x}}{2r\times\overline{1-x}}$; that is, (making $2r=AB=1$) $=\frac{px^3\dot{x}}{1-x}$. The Fluent of which will be $=\frac{1}{4}px^4+\frac{1}{5}px^5+\frac{1}{6}px^6+\frac{1}{7}px^7$, &c. = Portion of the Solid described by APM. And putting $AB=1$, for x, we shall get $\frac{1}{4}p+\frac{1}{5}p+\frac{1}{6}p+\frac{1}{7}p$, &c. or $p\times\overline{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}$, *&c.* But this Series is infinite, as may be easily demonstrated from the Hyperbola; therefore the said Solid is infinite also.

Otherwise by the Measure of a Ratio.

If the Area APM of the Cissoid of *Diocles* *revolves about the Base* AB *as an Axis, it is requir'd to cube the Solid generated thereby.* FIG. 50.

From the Nature of the Curve, calling AP, x, and PM, y, we have $x\sqrt{\frac{x}{a-x}}=y$. Whence $\frac{pyy\dot{x}}{2r}=\frac{p}{2r}\dot{x}\,\frac{x^3}{a-x}$ = Fluxion of the Solid required. Which compared with the first Form in Mr. *Cotes*'s Tables, by making $d=\frac{p}{2r}$, $\theta=4$, $n=1$, $e=a$, $f=-1$; and we get $-\frac{px^3}{6r}-\frac{paxx}{4r}-\frac{paax}{2r}-\frac{pa^3}{2r}\Big|\frac{a-x}{a}$. The Fluent of the given Fluxion; or $-\frac{px^3}{6r}-\frac{paxx}{4r}-\frac{paax}{2r}+\frac{pa^3}{2r}\Big|\frac{a}{a-x}$. Since the Logarithm of the Ratio a to $a-x$ with an affirmative Sign, is the same as the Loga-

Logarithm of the Ratio $a-x$ to a with a negative Sign.

Now this Fluent may be thus constructed: Let $AP\,(x)$, $AB\,(a)$, $AR\left(\frac{aa}{x}\right)$, $AS\left(\frac{a^3}{xx}\right)$ $AT\left(\frac{a^4}{x^3}\right)$ be continual Proportionals. Then to the Module $TS\left(\frac{a^4-a^3x}{x}\right)$ assume PX equal to the Measure of the Ratio between $AB\,(a)$ and $PB\,(a-x)$; that is, make $PX = \frac{a^4-a^3x}{x}\Big|\frac{a}{a-x}$; and from X towards B, lay off XZ equal to $SR+\frac{RB}{2}+\frac{BQ}{3}=\frac{a^3-a^2x}{xx}+\frac{aa-ax}{2x}+\frac{a-x}{3}$; and the Solid will be equal to a Cylinder, whose Base is PM, the Value of which is $\frac{px^3}{2r\times a-x}$; and Altitude PZ, the Value of which is $-\frac{a^3-a^2x}{xx}-\frac{aa-ax}{2x}-\frac{a-x}{3}+\frac{a^4-a^3x}{x^3}\Big|\frac{a}{a-x}$; therefore $\frac{p}{2r}\times\frac{x^3}{a-x}\times=\frac{a^3-a^2x}{xx}-\frac{aa-ax}{2x}-\frac{a-x}{3}+\frac{a^4-a^3x}{x^3}\Big|\frac{a}{a-x}$ is $=-\frac{px^3}{6r}-\frac{paxx}{4r}-\frac{paax}{2r}+\frac{pa^3}{2r}\Big|\frac{a}{a-x}$; the Fluent to be constructed.

SCHOLIUM.

Fig. 50. 65. THE Value of the infinite Solid generated by the Rotation of the Cissoid AMI about the Asymptote BOH may be had thus:

Let

Let *ANB* be the generating Semicircle Circle, *AP*, x, *AB*, a; then all the Ordinates *PM* do describe cylindrical Surfaces. Whence $r:p::$ $PB \times PM\ (x\sqrt{ax-xx})::\frac{px}{r}\sqrt{aa-xx}$ = Surface described by *PM*; which multiplied by $\dot{x}$, and $\frac{p}{r}x\dot{x}\sqrt{ax-xx}$ is the Fluxion of the Solid generated as above.

Now to get the Fluent of that Fluxion, let us conceive the Generation Semicircle to revolve about an Axis parallel to the Asymptote *BOH*, and passing thro' the Point *A*; then all the Ordinates *PN* will also describe cylindrical Surfaces; and so $\frac{px\dot{x}}{r}\sqrt{ax-xx}$ will be the Fluxion of the Solid generated by this Motion, which is equal to the Fluxion of the former Solid to be cubed. Whence the infinite Cissoïdal Solid formed by the Revolution aforesaid, is equal to this latter Solid generated by the Revolution of the generating Semicircle about a right Line parallel to the Asymptote, and passing thro' the Point *A*.

PROP. II.

66. *To measure the Superficies of a Solid generated by the Rotation of a Figure* AMNQ *about its Axis* AQ. FIG. 7.

Let the Ratio of the Circumference to the Diameter of any Circle be $\frac{r}{p}$, let $AP = x$, $PM = y$, then will $Pp = Mq = \dot{x}$, $qm = \dot{y}$; and so $Mm = \sqrt{\dot{x}^2 + \dot{y}^2}$. the Circumference described by the Radius

$PM = \frac{py}{r}$. Which multiply'd by, or drawn into Mm, will be $= \frac{py}{r}\sqrt{\dot{x}^2 + \dot{y}^2}$ = Fluxion of Part of the Superficies of the Solid generated by the Rotation of the Part AMP of the Figure $AMNQ$; and finding the Value of $\dot{x}^2$ from the Equation of the Curve thrown into Fluxions, and substituting the same in the general Expression; the Fluent thereof being afterwards found, will be the Superficies sought.

SCHOLIUM.

67. If $\frac{p}{r}\sqrt{\dot{x}^2 + \dot{y}^2}$ be any Fluxion of a Superficies generated by the Rotation of a plain Figure about the Line x as an Axis; and if $\frac{p}{r}$ be the Ratio of the Radius to the Circumference of a Circle, and y represents any perpendicular Ordinate to the Abscis describing a Circle during the Generation of the Superficies; and if $\frac{p}{2r}zz$ be supposed the Fluent of that Fluxion, *viz* equal to a Circle whose Radius is z; then will the Fluent of $2y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ be $= zz$, the Square of the Radius of the said Circle. Consequently if instead of finding the Fluent of the Fluxion $\frac{p}{r}y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ of any Superficies generated, as above, you find the Fluent of $2y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$; this Fluent will be equal to the Square of the Radius of a Circle equal to the Fluent of the Fluxion $\frac{py}{r}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, or to a Circle

cle equal to the Superficies, whereof this last Expression is the Fluxion.

EXAMPLE I.

68. TO *find the Superficies of a right Cone.*

Because a right Cone is generated by the Rotation of the right-angled Triangle ABC about the Axis AB, the Value of $\dot{x}^2$ must first be gotten from the Equation of the Triangle thus: Let $AB=a$, $BC=r$, $AP=x$, $PM=y$. Now since $AP\ (x) : PM\ (y) :: AB\ (a) : BC\ (r)$: therefore $rx=ay$. Which thrown into Fluxions will be $r\dot{x}=a\dot{y}$. Whence $\dot{x}=\frac{a\dot{y}}{r}$, and FIG. 43.

$\dot{x}^2=\frac{a^2\dot{y}^2}{r^2}$; and substituting $\frac{a^2\dot{y}^2}{r^2}$ for $\dot{x}^2$ in the general Expression $\frac{py}{r}\sqrt{\dot{x}^2+\dot{y}^2}$ we shall get $\frac{py}{r}\sqrt{\dot{x}^2+\dot{y}^2}=\frac{py}{r^2}\sqrt{a^2\dot{y}^2+r^2\dot{y}^2}=\frac{py}{r^2}\times\dot{y}\sqrt{a^2+r^2}$ = Fluxion of Part of the Superficies of the Cone generated by the Triangle APM. The Fluent of which will be $\frac{py^2}{2r^2}\sqrt{a^2+r^2}$ = said Part of the Superficies; and substituting y for r, the Superficies of the whole Cone will be $=\frac{1}{2}p\sqrt{a^2+r^2}=\frac{1}{2}p\times AC$; that is, equal to the Rectangle under $\frac{1}{2}$ the Circumference of the Base into the Side AC of the Cone.

EXAMPLE II.

69. TO *find the Superficies of a Sphere, or of any Segment of it.* FIG. 45.

If

If the Diameter of the generating Circle be $=1$, then the Fluxion of the Arch *Mm* will be $=\frac{\dot{x}}{2\sqrt{x-xx}}$. Which drawn into the Circumference described by the Radius *PM*, will give $p\dot{x}$ the Fluxion of Part of the Superficies of the Sphere generated by the Semi-segment *AMP*; and the Fluent px of it will be the Superficies of the Segment of the Sphere, having p for the Periphery of the circular Base, and x for the Height; and if you put the Diameter 1 for x, the Superficies of the whole Sphere will be equal to p, or (making $1=a$) $= ap$.

Whence any Segment of the Superficies of a Sphere is to the whole Superficies of the Sphere as px to p or x to 1; that is, as the Altitude of the Segment to the Diameter of the Sphere.

EXAMPLE III.

70. *TO find the Superficies of a Parabolical Conoid.*

The Equation expressing the Nature of the Parabola is $AP\times a=\overline{PM}^2$ (yy), or $ax=yy$. Which thrown into Fluxions, and $a\dot{x}=2y\dot{y}$; whence $\dot{x}^2=\frac{4y^2\dot{y}^2}{a^2}$. Consequently $\frac{py}{r}\sqrt{\dot{x}^2+\dot{y}^2}$ $=\frac{py}{ar}\times\sqrt{4y^2\dot{y}^2+a^2\dot{y}^2}=\frac{py\dot{y}}{ar}\sqrt{4y^2+a^2}=$ Fluxion of the Superficies of the Part of the Conoid generated by the Portion *APM* of the Parabola. Which comes under the first Case of the third Form in the little *Table of simple Curves that may be squared*, Art. 8. or under the

the third Form in the Tables of Mr. *Cotes*. For if in the little Table you make $d = \frac{p}{ar}$, $z = y$, $n = 2$, $e = aa$, $f = 4$, we shall have the Fluxion $d\dot{z}z^{n-1}\sqrt{e+fz} = \frac{py\dot{y}}{ar}\sqrt{4y^2+a^2}$; and the Fluent $\frac{2d}{3nf}R^3$ of it, by making $R\ (=\sqrt{e+fz}) = \sqrt{4y^2+a^2}$ will become $\frac{p}{12ar} \times \overline{4y^2+aa}^{\frac{3}{2}} =$ Fluent of the Fluxion of the Superficies of the Conoid. In like manner, in Mr. *Cotes*'s 3d Form, the Fluent answerable to $\theta = 1$, *viz.* $\frac{2e+2fz^n}{3nf}dP$, making $P\ (=\sqrt{e+fz^n}) = \sqrt{4y^2+a^2}$; and substituting the same Values for d, n, e, f, as before, will become $\frac{2aa+8yy}{24}\ \frac{p}{ar}\sqrt{4y^2+a^2} = \frac{p}{12ar}\overline{4y^2+a^2}^{\frac{3}{2}} =$ to that found by the little Table of Quadratures.

This Fluent may be constructed thus: If r be made $= y$, and you draw $PC = 2y$, and make $PB = a$, and join BC; then will BC be $= \sqrt{4yy+aa}$. Call this n, and the Fluent to be constructed will be $\frac{p}{12ay}n^3$. Let z be the Diameter of the Circle equal to this Fluent; the Area of this will be $= \frac{pz^2}{8y}$. Therefore $\frac{p}{12ay}n^3$ must be $= \frac{pz^2}{8y}$; that is, $\frac{n^3}{12a} = \frac{z^2}{8}$; and so $\frac{8n^3}{12a} = z^2$. Whence $z = \sqrt{\frac{2n^3}{3a}} =$ * Fluent of $2y\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}$. FIG. 51. * *Art.* 67.

Now

Now make $PH=BC$, and $PE=\frac{1}{3}BC$, join B, E. From H draw HF parallel to BE. Make $GP=PF$, bisect GH in D; from which, as a Centre, describe the Semicircle GKH, cutting PM continued out in the Point K; and then the right Line PK will be the Diameter of a Circle equal to the Superficies of the Conoid,

EXAMPLE IV.

FIG. 52, 53. 70. *TO find the Superficies of a Spheroid generated by the Rotation of any Part* AM *of an Ellipsis about the Part* DC *of the Semi-axis* BC.

Call the Semi-~~conjugate~~ transverse Axis AC, a; and the Semi-conjugate BC, b; the Abscis PA, x, and the correspondent Semi-ordinate MP, y. Now $\frac{bb}{aa}\times\overline{2ax-xx}=yy$, and $2a\dot{x}-2x\dot{x}=\frac{2a^2}{b^2}\times y\dot{y}$, or $a\dot{x}-x\dot{x}=\frac{a^2}{b^2}\times y\dot{y}$. Whence $\dot{x}=\frac{\frac{a^2}{b^2}y\dot{y}}{a-x}$, and $\dot{x}^2=\frac{a^2y^2\dot{y}^2}{b^4-b^2y^2}$. Consequently the Fluxion of the Arch AM ($=\sqrt{\dot{x}^2+\dot{y}^2}$) will be $=\frac{\dot{y}}{b}\sqrt{\frac{b^4+\overline{a^2-b^2}\,yy}{b^2-y^2}}=\frac{\dot{y}}{b}\sqrt{\frac{b^4+ccyy}{b^2-y^2}}$, (when AC (a) is greater than BC (b), and you put cc for a^2-b^2) or $=\frac{\dot{y}}{b}\sqrt{\frac{b^4-ccyy}{b^2-y^2}}$ when AC (a) is less than BC (b), and cc be substituted for b^2-a^2.

Again, DM is $=\frac{a}{b}\sqrt{bb-yy}$, from the Nature of the Curve. Whence the Periphery of the Circle described by DM will be $\frac{pa}{rb}\sqrt{bb-yy}$.

Which drawn into the Fluxion of the Arch before found, and there will arise $\frac{pa\dot{y}}{rbb}\sqrt{\frac{b^4 \pm ccyy \times bb-yy}{bb-yy}} = \frac{pa\dot{y}}{rbb}\sqrt{b^4+ccyy}$, when AC is greater than BC; and $= \frac{pa\dot{y}}{rbb}\sqrt{b^4-ccyy}$; when AC is less than BC; each being the Fluxion of the Superficies of a Spheroid, generated by the Rotation of the Part AM of the Ellipsis about the Part CD of the Axis; and may be referred to the fourth Form in the Tables of Mr. *Cotes*. For making $d = \frac{pa}{rbb}$, $z=y$, $n=2$, $\theta=0$, $e=b^4$, $f=\pm cc$, the Fluxion $d\dot{z}z^{\theta n + \frac{1}{2}n - 1}\sqrt{e+fz^n}$ will become $\frac{pa\dot{y}}{rbb}\sqrt{b^4 \pm ccyy}$. And the Fluent $\frac{z^n}{n}dP + \frac{e}{nf}dR\left|\frac{R+T}{S}\right.$, by making $P\left(=\sqrt{\frac{e+fz^n}{z^n}}\right) = \frac{1}{y}\sqrt{b^4 \pm ccyy}$. $R\ (=\sqrt{f}) = c$, or $\sqrt{-cc}$, $T\left(=\sqrt{\frac{e+fz^n}{z^n}}\right) = \frac{1}{y}\sqrt{b^4 \pm ccyy}$, $S\left(=\sqrt{\frac{e}{z^n}}\right) = \frac{b^2}{y}$, will become $\frac{pay}{2rbb}\sqrt{b^4 \pm ccyy} + \frac{pab^2}{2rc}\left|\frac{yc+\sqrt{b^4 \pm ccyy}}{b^2}\right.$. It being $\sqrt{b^4+ccyy}$, and the Measure of a Ratio, when $AC\ (a)$ is greater than $BC\ (b)$; and $\sqrt{b^4-ccyy}$, and the Measure of an Angle, when $AC\ (a)$ is less than $BC\ (b)$.

FIG. 52. Now to construct the Fluent in the first Case, we may proceed thus: Let F be one of the Foci. Make CF ($\sqrt{aa-bb} = \sqrt{cc} = c$, from the Nature of the Ellipsis) : CB (b) : : $CB(b) : CE = \frac{bb}{c}$, and draw the right Line DE. Also make $CE\left(\frac{bb}{c}\right) : DE\left(\frac{1}{c}\sqrt{b^4+ccyy}\right)$: : $CD(y) : KL = \frac{y}{bb}\sqrt{b^4+ccyy}$. To which if you add LM, made equal to the Measure of the Ratio between $DE\left(\frac{1}{c}\sqrt{b^4+ccyy}\right) + DC$ (y) and $CE\left(\frac{bb}{c}\right)$ to the Module $CE\left(\frac{bb}{c}\right)$; that is, if you assume $LM = \frac{b^2}{c}\left|\frac{cy+\sqrt{b^4+ccyy}}{b^2}\right.$, then will CA (a) be to $KM = KL + LM$ $\left(= \frac{y}{bb}\sqrt{b^4+ccyy} + \frac{b^2}{c}\left|\frac{cy+\sqrt{b^4+ccyy}}{b^2}\right.\right)$ as a Circle described with CA (a), as a Radius, (which Circle is $= \frac{paa}{2r}$) is to the Superficies of the Spheroid generated by the Rotation of the Part AM of the Ellipsis $= \frac{pay}{2rbb}\sqrt{b^4+ccyy}$ $+ \frac{pab^2}{2rc}\left|\frac{cy+\sqrt{b^4+ccyy}}{b^2}\right.$; which is the Fluent to be constructed.

Now let the Radius of a Circle, equal to the Superficies of a Spheroid, be z. Then we shall have this Proportion, CA (a) : KM : : Circle described by CA, viz. $\frac{paa}{2r}$: Superficies

of

of the Spheroid ſuppoſed equal to a Circle $\frac{pzz}{2r}$ whoſe Radius is z. And multiplying the Extremes and Means, there ariſes $\frac{pazz}{2r} = KM \times \frac{paa}{2r}$; and ſo $zz = KM \times a$; that is, a mean Proportional between KM and AC (a) will be the Radius of a Circle equal to the Superficies of the Spheroid generated by the Rotation of the Part AM of the Ellipſis.

When DC (y) becomes equal to BC (b), then will DE become BE; and ſo if you make $CE\left(\frac{bb}{c}\right) : BE\left(\frac{b}{c}\sqrt{bb+cc}\right) :: BC(b) : KL = \sqrt{bb+cc} = AC = a$, and LM be = Meaſure of the Ratio between AC $(a) + BC$ (b), and $CE\left(\frac{bb}{c}\right)$ to the Module $CE\left(\frac{bb}{c}\right)$, we ſhall have CA to KM, as a Circle having CA for a Radius is to $\frac{1}{2}$ the Superficies of the whole Spheroid.

The Conſtruction of the Fluent in the latter Caſe is thus: In the Ordinate DM aſſume the Point E in ſuch manner, that the Line CE being drawn be equal to $\frac{bb}{c}$, or a third Proportional to CF (c) and $CB(b)$; and make $CE\left(\frac{bb}{c}\right) : DE\left(\frac{1}{c}\sqrt{b^4-ccyy}\right) :: DC(y) : KL = \frac{y}{bb}\sqrt{b^4-ccyy}$. Then if to KL you add LM equal to the Meaſure of the Angle DCE to the Module $CE\left(\frac{bb}{c}\right)$; $CA(a)$ will be to KM, FIG. 53.

as a Circle, whose Diameter is CA (a), is to the Superficies of the Spheroid generated by the Rotation of the Part AM of the Ellipsis about the Part DC of the Axis; and so a mean Proportional between KM and CA, will be the Radius of a Circle equal to the said Superficies.

When CD (y) becomes equal to CB (b), draw BG perpendicular to CB. In which assume the Point e such, that Ce being drawn, be $= EC \left(\frac{bb}{c}\right)$. Then if you make CE or $Ce \left(\frac{bb}{c}\right) : Be \left(\frac{b}{c}\sqrt{bb-cc}\right) :: AC\,(b) : KL = \sqrt{bb-cc} = a$. And to the same you add LM equal to the Measure of the Angle eCB to to the Module $eC \left(\frac{bb}{c}\right)$, we shall have CA to KM as a Circle having CA for a Radius is to ½ the Superficies of the whole Spheroid.

EXAMPLE V.

Fig. 54. 71. *TO find the Superficies of a Hyperbolical Conoid generated by the Rotation of any Part* AM *of an Hyperbola about the transverse Axis* AP.

Let C be the Centre, Cc an Asymptote, $AC = b$ one of the Semi-axes, and $Ac = a$ the other. Let CP be $= y$, and $PM = x$. Now $\frac{aa}{bb} \times \overline{yy - bb} = xx$, from the Nature of the Curve, and $2aay\dot{y} = 2bbx\dot{x}$; and so $\dot{x} = \frac{aay\dot{y}}{bbx}$. Consequently $\dot{x}\dot{x} = \frac{a^4yy\dot{y}\dot{y}}{b^4xx}$. And the Fluxion of the

the Arch AM $(=\dot{y}\dot{y}+\dot{x}\dot{x})$ will be $\frac{\dot{y}}{b}\sqrt{\frac{a^2yy+b^2yy-b^4}{yy-bb}}=\frac{\dot{y}}{b}\sqrt{\frac{ccyy-b^4}{yy-bb}}$, by putting cc for a^2+b^2. Again, PM is $=\frac{a}{b}\sqrt{yy-aa}$ from the Nature of the Curve; and therefore the Periphery described by PM will be $\frac{pa}{rb}\sqrt{yy-aa}$. This drawn into the Fluxion of the Arch before found, and there will arise $\frac{pa\dot{y}}{rbb}\sqrt{\frac{ccyy-b^4\times yy-bb}{yy-bb}}=\frac{pa\dot{y}}{rbb}\sqrt{ccyy-b^4}$, being the Fluxion of the Superficies of an hyperbolical Conoid generated by the Rotation of the Part AM of the Hyperbola about the Axis AP. Which comes under the same Form in the Tables of Mr. *Cotes*, as the Fluxion of the Superficies of the Spheroid in the last Example. And making $d=\frac{pa}{rbb}$, $z=y$, $\theta=0$, $n=2$, $e=-b^4$, $f=cc$, $P=\frac{1}{y}\sqrt{ccyy-b^4}$, $R=c$, $T=\frac{1}{y}\sqrt{ccyy-b^4}$, $S=\frac{b^2}{y}$, we shall have the Fluent of that Fluxion thus expressed; $\frac{pay}{2rbb}\sqrt{ccyy-b^4}-\frac{pab^2}{2rc}\Big|\frac{cy+\sqrt{ccyy-b^4}}{b^2}$. ~~But because y begins at the Centre C, and not at A the Vertex,~~ we must make $b=y$. Then the first Part $\frac{y}{bb}\sqrt{ccyy-b^4}$ of the Fluent will be $=\sqrt{cc-bb}=a$; which therefore must be taken from the said first Part. Moreover, the Logarithmical Part $\frac{bb}{c}\Big|\frac{cy+\sqrt{ccyy-b^4}}{bb}$, by making

king $b=y$, will become $\frac{bb}{c}\left|\frac{ab+bc}{bb}\right.$. Which must be taken from $\frac{bb}{c}\left|\frac{cy+\sqrt{ccyy-b^4}}{bb}\right.$, to have the true Logarithmical Part. Now because the Logarithm of the Ratio of bb to $ab+bc$ having an affirmative Sign, is the same as the Logarithm of the Ratio of $ab+bc$ to bb with a negative Sign (by *Def.* 1. *Sect.* 2. *Schol.* 2.) therefore the Sum of $\frac{bb}{c}\left|\frac{cy+\sqrt{ccyy-b^4}}{bb}\right.$, and of $\frac{bb}{c}\left|\frac{bb}{ab+bc}\right.$, will be equal to the Difference sought. But the Ratio of $cy+\sqrt{ccyy-b^4}$ to bb, and the Ratio of bb to $ab+bc$, do compound the Ratio of $cy+\sqrt{ccyy-b^4}$ to $ab+bc$. Whence the Sum now mention'd will be $\frac{bb}{c}\left|\frac{cy+\sqrt{ccyy-b^4}}{ab+bc}\right.$. Therefore the true Fluent to be constructed will be $\frac{pay}{2rbb}\sqrt{ccyy-b^4}-\frac{pa^2}{2r}-\frac{pab^2}{2rc}\left|\frac{cy+\sqrt{ccyy-b^4}}{ab+bc}\right.$. And this may be done thus:

Let F be the Focus of the Hyperbola. Make CF ($\sqrt{aa+bb}=c$, (from the Nature of the Hyperbola) : CA (b) : : CA (b) : $CE=\frac{bb}{c}$. Draw EG perpendicular to CA meeting the Asymptote in G. In the Angle CEG, inscribe the right Line $CH=CP$ (y), which continue out to meet PM also continued out in the Point I. Then assume KL equal to $PI-Ac$ $=\frac{y}{bb}\sqrt{ccyy-b^4}-a$, since from the similar Tri-

 angles

angles CEH, CPI, we have $CE\left(\frac{bb}{c}\right):EH$ $\left(\frac{1}{c}\sqrt{ccyy-b^4}\right)::CP(y):PI=\frac{y}{bb}\sqrt{ccyy-b^4}$. Again, assume LM equal to the Measure of the Ratio between $CH(y)+EH$ $\left(\frac{1}{c}\sqrt{ccyy-b^4}\right)$ and $CG(b)+EG\left(\frac{ab}{c}\right)$ to the Module $CE\left(\frac{bb}{c}\right)$. Then the Superficies generated by the Rotation of the Arch AM about the Axis AP, will be to a Circle described with the Semidiameter Ac (a), viz. $\frac{paa}{2r}$, as KM to Ac.

EXAMPLE VI.

72. *TO find the Superficies of a hyperbolical Conoid generated by the Rotation of any Part* AM *of an Hyperbola about the conjugate Axis* CPB.

Let C be the Centre, $CA=a$ the Semi-transverse Axis, $CB=b$ the Semi-conjugate, F the Focus, $PM=x$, any Ordinate to CA, and $CP=y$, the correspondent Abscis. FIG. 55.

Now $\frac{bb}{aa}\times\overline{xx-aa}=yy$, from the Nature of the Curve. Whence $\frac{2bb}{aa}\times x\dot{x}=2y\dot{y}$, that is, $\frac{bb}{aa}\times x\dot{x}=y\dot{y}$, and $bbx\dot{x}=aay\dot{y}$, and $\dot{x}=\frac{aay\dot{y}}{bbx}$, and $\dot{x}\dot{x}=\frac{a^4yy\dot{y}\dot{y}}{b^4xx}$; and substituting $\frac{aayy+aabb}{bb}$ for xx we shall get $\dot{x}\dot{x}=\frac{a^4bbyy\dot{y}\dot{y}}{aab^4\times\overline{bb+yy}}=\frac{a^2yy\dot{y}\dot{y}}{b^2\times\overline{bb+yy}}$.

Whence

Whence $\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}=\frac{\dot{y}}{b}\sqrt{\frac{\overline{a^2+b^2}\times yy+b^4}{bb+yy}}$. But x is $=\frac{a}{b}\sqrt{bb+yy}$, which multiplied by $\frac{p}{r}$, and $\frac{pa}{rb}\sqrt{bb+yy}$ will be the Periphery described by the Point M; and if the Fluxion of the Arch be drawn into this, we shall get $\frac{pa\dot{y}}{rbb}\sqrt{\frac{\overline{a^2+b^2}\times yy+b^4\times bb+yy}{bb+yy}}=\frac{pa\dot{y}}{rbb}\sqrt{\overline{a^2+b^2}yy+b^4}$. And substituting cc for a^2+b^2, the same will become $\frac{pa\dot{y}}{rbb}\sqrt{ccyy+b^4}$. Which is the Fluxion of the Superficies generated by the Rotation of the Arch AM about the Semi-conjugate Diameter CP.

Now this Fluxion being the same as that in the first Case of Example 4. aforegoing, the Fluent thereof must be the same as the Fluent of that, *viz.* $\frac{pay}{2rbb}\sqrt{ccyy+b^4}+\frac{pab^2}{2rc}\left|\frac{cy+\sqrt{ccyyb^4}}{b^2}\right.$. Which may be thus constructed.

Make $CF\ (\sqrt{aa+bb}=c) : CB\ (b) :: CB\ (b) : CE=\frac{bb}{c}$, and draw the right Line PE $(\frac{1}{c}\sqrt{ccyy+b^4}.)$ Then make $CE\ (\frac{bb}{c}) : PE\ (\frac{1}{c}\sqrt{ccyy+b^4}) :: PC\ (y) : KL=\frac{y}{bb}\sqrt{ccyy+b^4}$. And assume LM equal to the Measure of the Ratio between $PE\ (\frac{1}{c}\sqrt{ccyy+b^4})+PC\ (y)$; that is, between $\frac{cy+\sqrt{ccyy+b^4}}{c}$, and CE

$(\frac{bb}{c})$

$\left(\frac{bb}{c}\right)$ to the Module CE $\left(\frac{bb}{c}\right)$. And the Superficies generated by the Rotation of the Arch AM of the Hyperbola about the Semiconjugate Axis CB, will be to a Circle described with the Semidiameter CA (a), (which Circle is $\frac{paa}{2r}$) as the Sum of the Lines KL and LM to the said Semidiameter CA.

EXAMPLE VII.

73. *TO find the Superficies generated by the Motion of the Arch* CM *of an equilateral Hyperbola about the Asymptote* ABP.

Let A be the Centre, and let AB be equal to BC, drawn parallel to the other Asymptote; assume AP, and draw PM parallel to the other Asymptote. Now call AB or BC, a, AP, x, and PM, y. Then from the Nature of the Curve, $aa = xy$. Whence $x = \frac{aa}{y}$, and $\frac{aa}{x} = y$. Which thrown into Fluxions, and $\dot{y} = \frac{-aa\dot{x}}{xx}$. Consequently $\dot{y}\dot{y} = \frac{a^4\dot{x}\dot{x}}{x^4}$. Again, $x = \frac{aa}{y}$ thrown into Fluxions, and $\frac{aa\dot{y}}{yy} = \dot{x}$, and $\frac{a^4\dot{y}\dot{y}}{y^4} = \dot{x}\dot{x}$. Whence $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \sqrt{\frac{a^4\dot{x}\dot{x}}{x^4} + \frac{a^4\dot{y}\dot{y}}{y^4}} = \frac{\dot{y}}{yy}\sqrt{a^4 + y^4} = Mm$ the Fluxion of the Arch CM. Which drawn into $\frac{py}{r}$, and the Product $\frac{p\dot{y}}{ry}\sqrt{a^4 + y^4} = \frac{p}{r}\,\dot{y}\,y^{-1}\sqrt{a^4+y^4}$

FIG. 56.

will be the Fluxion of the Superficies generated by the Rotation of the Arch CM as aforesaid. This Fluxion can be compared to that of the third Form of Mr. *Cotes*'s Tables, and the Fluent will be had by making $d = \frac{p}{r}$, $z=y$, $\theta=0$, $n=4$, $e=a^4$, $f=y^4$, $P=\sqrt{a^4+y^4}$. $R=aa$, $T=\sqrt{a^4+y^4}$, and $S=yy$: for then the same will be $\frac{1}{2}\frac{p}{r}\sqrt{a^4+y^4} - \frac{1}{2}\frac{p}{r}aa \left| \frac{aa+\sqrt{a^4+y^4}}{yy} \right.$. But this Fluent must be corrected, because the Abscis AP (x) increases, while the correspondent Ordinate PM (y) decreases; therefore the Signs must be changed; that is, it will be $-\frac{1}{2}\frac{p}{r}\sqrt{a^4+y^4} + \frac{1}{2}\frac{p}{r}aa \left| \frac{aa+\sqrt{a^4+y^4}}{yy} \right.$. Moreover, ~~since the said Abscis AP (x) begins at A, and not at B,~~ the said Fluent must undergo another Correction, as to Magnitude. And this is done thus: Make $y=a$, and then the first Part of the Fluent will become $\frac{1}{2}\frac{p}{r}aa\sqrt{2}$. And this must be added to $-\frac{1}{2}\frac{p}{r}\sqrt{a^4+y^4}$, and the Sum will be $\frac{1}{2}\frac{paa}{r}\sqrt{2} - \frac{1}{2}\frac{p}{r}\sqrt{a^4+y^4}$, the true first Part of the Fluent. Again, the Logarithmick Part $\frac{1}{2}\frac{p}{r}aa\left| \frac{aa+\sqrt{a^4+y^4}}{yy} \right.$ must be alter'd; which may be done by making $a=y$; for then it will become $\frac{1}{2}\frac{p}{r}aa\left| \frac{\sqrt{2}+1}{1} \right.$. Which being

ing ſubſtracted from $\frac{1}{2}\frac{p}{r}aa \left| \frac{aa+\sqrt{a^4+y^4}}{yy} \right.$,

and then we ſhall have $\frac{1}{2}\frac{p}{r}aa \left| \frac{\frac{aa}{yy}+\frac{\sqrt{a^4+y^4}}{yy}}{\sqrt{2}+1} \right.$,

which is the true Logarithmical Part. And conſequently the true Fluent will be $\frac{paa}{2r}\sqrt{2}-\frac{p}{2r}\sqrt{a^4+y^4}+\frac{1}{2}\frac{p}{r}aa \left| \frac{\frac{aa}{yy}+\frac{\sqrt{a^4+y^4}}{yy}}{\sqrt{2}+1} \right.$, which may be thus conſtructed.

Draw *AM* and *AC*, and from the Point *C* draw *CG* parallel to *AM*, meeting the Aſymptote *AP* continued out in *G*. Now becauſe of the ſimilar Triangles *APM*, *ABF*, $AP\left(\frac{aa}{yy}\right):PM(y)::AB(a):BF=\frac{yy}{a}$. Whence $\frac{1}{a}\sqrt{a^4+y^4}=AF$. Again, becauſe *GC* is parallel to *AM*, the Triangles *APM*, *GBC* are ſimilar; therefore $PM(y):AP\left(\frac{aa}{y}\right)::CB(a):BG=\frac{a^3}{yy}$. Whence $CG=\frac{a}{yy}\sqrt{a^4+y^4}$. Then if you make $AH=AC\,(a\sqrt{2})-AF\left(\frac{1}{a}\sqrt{a^4+y^4}\right)+$ the Meaſure of the Ratio between $BG\left(\frac{a^3}{yy}\right)+GC\left(\frac{a}{yy}\sqrt{a^4+y^4}\right)$, and $AC\,(a\sqrt{2})+AB\,(a)$ (which Ratio is $=\frac{\frac{aa}{yy}+\frac{\sqrt{a^4+y^4}}{yy}}{\sqrt{2}+1}$) to the Module *AB* (*a*); and the Superficies generated by the Rotation of the Arch *CM* about the Asymptote *AP*, will be to a Circle whose Semidiameter is *AB* (a), viz. $=\frac{paa}{2r}$, as *AH* ($a\sqrt{2} - \frac{1}{a}\sqrt{a^4+y^4} + a \left| \frac{\frac{a^3}{yy} + \frac{a\sqrt{a^4+y^4}}{yy}}{a\sqrt{2}+a} \right)$ to *BC* (a); and therefore the Value of *AH* drawn into $\frac{pa}{2r}$ will be equal to the said Superficies.

EXAMPLE VIII.

74. *TO find the Superficies generated by the Rotation of the infinite Arch* PZ *of the Logarithmical Curve about its Asymptote* AX.

FIG. 57. Let *AP* be an Ordinate at right Angles to the Asymptote, which call y; let *TP* touch the Curve in *P*, and let the invariable Subtangent *AT* be $= a$. Draw *pm* parallel to *AT*, and infinitely near the Point *P*.

Then because the Triangles *ATP*, *mpP* are similar, $AP\,(y) : TP\,(\sqrt{aa+yy}) :: mP\,(\dot{y}) : pP = \frac{\dot{y}}{y}\sqrt{aa+yy} =$ Fluxion of the Arch *PZ*. This multiplied by $2y$, and the Product $2\dot{y}\sqrt{aa+yy}$ will be the Fluxion of a Square, whose Side is equal to the Diameter of a Circle equal to the Superficies sought by *Scholium*, *Art.* 67. which Square let $= AOq$.

Now this Fluxion may be referr'd to the fourth Form of Mr. *Cotes*'s, by writing y, 2, 0, 2, aa, 1, for z, n, θ, d, e, f. And again, writing for P, R, T, S, $\frac{1}{y}\sqrt{aa+yy}$, 1, $\frac{1}{y}\sqrt{aa+yy}$, and

$\frac{a}{y}$,

$\frac{a}{y}$, we ſhall have the Fluent of the Fluxion $2\dot{y}\sqrt{xa+yy}$ equal to $y\sqrt{aa+yy}+aa\left|\frac{y+\sqrt{aa+yy}}{a}\right.$.

In order to conſtruct this Fluent, we may obſerve that the right Line *AO* is a mean Proportional between a and $\frac{y}{a}\sqrt{aa+yy}+a\left|\frac{a+\sqrt{aa+yy}}{a}\right.$; and the Quantity $\frac{y}{a}\sqrt{aa+yy}$ is equal to the right Line *EP* at right Angles to the Curve in *P*, and bounded by the Aſymptote in *E*. For becauſe of the ſimilar Triangles *TPA*, *APE*, *TA* (a) : *TP* ($\sqrt{aa+yy}$) :: *AP* (y) : *PE* $=\frac{y}{a}\sqrt{aa+yy}$. And the other Quantity $a\left|\frac{y+\sqrt{aa+yy}}{a}\right.$ is the Meaſure of the Ratio between *AP* + *TP*, and *AT* to the Module *AT*. Or (becauſe of the ſimilar Triangles *APF*, *APE*) the Meaſure of the Ratio between *AE* + *EP*, and *AP* to the ſame Module *AT*.

Hence the Fluent is thus conſtructed. Draw *PE* ($\frac{y}{a}\sqrt{aa+yy}$) perpendicular to the Curve in *P* terminating at the Aſymptote at *E*. Continue out the Ordinate *AP* to *L*, ſo that *AL* be $=$ *AE* + *EP* ($\frac{yy}{a}+\frac{y}{a}\sqrt{aa+yy}$), and draw *LK* the ſame way as the Tendency of the Curve, *viz.* towards *Z*; which make $=$ *EP* ($\frac{y}{a}\sqrt{aa+yy}$); and let the ſame cut the Curve in *M*. Laſtly, between *KM* ($\frac{y}{a}\sqrt{aa+yy}+$

$a\left|\frac{y+\sqrt{aa+yy}}{a}\right)$ and AT, find a mean Proportional AO, (LM from the Nature of the Curve being the Logarithmick Part;) which will be the Semidiameter of a Circle equal to the Superficies generated by the Rotation of the infinite Arch PZ.

EXAMPLE IX.

75. *TO find the Superficies generated by the Rotation of the infinite Arch* MZ *or* AZ *of the Cissoid of* Diocles *about its Asymptote* AG.

FIG. 42. Draw the Ordinate MP at right Angles to the Asymptote, and mp infinitely near to it. Then AB being $=a$, and $AN=x$, MN being at right Angles to AB, the Fluxion Mm of the Arch MZ will be $\frac{a\dot{x}\sqrt{4a-3x}}{2\times\overline{a-x}\sqrt{a-x}}$; which multiplied by $\frac{p}{r}\times MP=\frac{p}{r}\times\overline{a-x}$, and the Fluxion of the Superficies generated by the abovesaid Rotation of the Arch MZ will be had, *viz.* $\frac{pa\dot{x}}{2r}\sqrt{\frac{4a-3x}{a-x}}$.

Now this may be brought under the eleventh Form of Mr. *Cotes*'s. For making $z=x$, $\theta=1$, $n=1$, $d=\frac{pa}{2r}$, $e=4a$, $f=-3$, $g=a$, and $h=-1$, the Fluxion in the eleventh Form $d\dot{z}z^{\theta n-1}\sqrt{\frac{e+fz^n}{g+hz^n}}$ will become $\frac{pa\dot{x}}{2r}\sqrt{\frac{4a-3x}{a-x}}$, and the Fluent $\frac{1}{nh}dPQ+\frac{eh-fg}{nfh}dR\left|\frac{R+T}{S}\right.$, by making $P\,(\sqrt{e+fz^n})=\sqrt{4a-3x}$, $Q\,(\sqrt{g+hz^n})$ =

$=\sqrt{a-x}$, $R\,(\sqrt{\frac{f}{h}}) = \sqrt{3}$, $T\,(\sqrt{\frac{e+fz^n}{g+hz^n}}) = \sqrt{\frac{4a-3x}{a-x}}$, $S\,(\sqrt{\frac{eh-fg}{h,g+hz^n}}) = \sqrt{\frac{-a}{-a+x}}$ or $\sqrt{\frac{a}{a-x}}$, will become $-\frac{pa}{2r}\times\sqrt{4a-3x}\times\sqrt{a-x}-\frac{pa}{2r}\times\frac{a}{\sqrt{3}}\,\big|\,\frac{\sqrt{3a-3x}+\sqrt{4a-3x}}{\sqrt{a}}$, being the Fluent of the given Fluxion $\frac{pa\dot{x}}{2r}\sqrt{\frac{4a-3x}{a-x}}$; wherein the negative Signs of the two Parts may be alter'd.

Now this Fluent may be conſtructed thus: Biſect *MN* in *F*, and draw *AFE* meeting the Aſymptote in *E*. Make *DB* a mean Proportional between *AB* and *NB*. Alſo let the Angle $CAB = \frac{1}{3}$ of a right Angle, and continue out the Aſymptote towards *B* meeting *AC* in the Point *C*. Then $BC = \frac{a}{\sqrt{3}}$ for $\sqrt{\overline{BC}^2+\overline{AB}^2} = 2\overline{BC}^2$. Whence $\overline{BC}^2+\overline{AB}^2 = 4\overline{BC}^2$, and ſo $\frac{\overline{AB}^2}{3} = \overline{BC}^2$. Conſequently $BC = \frac{AB}{\sqrt{3}} = \frac{a}{\sqrt{3}}$. Again, $AF = \frac{x}{2}\sqrt{\frac{4a-3x}{a-x}}$. And becauſe of the ſimilar Triangles *ANF*, *ABE*, $AN(x) : AF\left(\frac{x}{2}\sqrt{\frac{4a-3x}{a-x}}\right) :: NB\ (a-x) : FE = \frac{1}{2}\sqrt{4a-3x}\times\sqrt{a-x}$. And $DC = \sqrt{\frac{4aa-3ax}{3}}$. Now the Superficies generated by the Rotation of the infinite Arch *MZ*, will be to a Circle $\left(=\frac{paa}{2r}\right)$ whoſe Semidiameter is

AB (a), as $2EF(\sqrt{4a-3x}\times\sqrt{a-x})$ + the Measure of the Ratio between BD $(\sqrt{aa-ax})$ + $DC\left(\sqrt{\frac{4aa-3ax}{3}}\right)$, and $BC\left(\frac{a}{\sqrt{3}}\right)$ to the Module $BC\left(\frac{a}{\sqrt{3}}\right)$.

Now making $x=0$ in the aforesaid Fluent, and it will become $\frac{pa}{2r}\times 2a+\frac{pa}{2r}\times\frac{a}{\sqrt{3}}\left|\frac{\sqrt{3a}+2\sqrt{a}}{\sqrt{a}}\right.$. Which is the Fluent expressing the Quantity of the infinite Arch AZ. Being to a Circle whose Semidiameter is AB (a), as $2AB$ + the Measure of the Ratio between BA (a) + $AC\left(\frac{2a}{\sqrt{3}}\right)$ and $BC\left(\frac{a}{\sqrt{3}}\right)$ to the Module BC. And the Difference of these Fluents, omitting $\frac{pa}{2r}$, viz. $2AB$ $(2a)$ — EF $(\sqrt{4a-3x}\times\sqrt{a-x})$ + Measure of the Ratio BA (a) + $AC\left(\frac{2a}{\sqrt{3}}\right)$ to BD $(\sqrt{aa-ax})$ + $DC\left(\sqrt{\frac{4aa-3ax}{3}}\right)$, the Module being BC $\left(\frac{a}{\sqrt{3}}\right)$, will give us the Quantity of the finite Arch AM. For it will be to a Circle having BA for a Semidiameter, as that Difference is to the same Semidiameter BA.

When the Curve revolves about the Base AB, Mr. *Cotes* in *Harmonia Mensurarum* gives the Quantity of the Superficies generated. But the Operation is long and troublesome, on account of the Fluxion's not directly coming under any of his Forms.

SECT.

SECTION VI.

Of the Use of Fluxions in finding the Centres of Gravity of Figures.

PROB.

76. *TO find the Centre of Gravity of a plain Figure.*

Let AB be the Axis, and MN an Ordinate to it; and let mn be another infinitely near MN, and suppose C to be the Centre of Gravity. FIG. 58.

Now if the infinitely small Parts, as $MmnN$ of the Figure be conceived as so many Weights hung on the Axis AB, at the several Points P, P, P, &c. and the Point of Suspension be in A the Vertex of the Figure; the Distance of the Centre of Gravity C from A the Point of Suspension, will be equal to the Quotient of the Division of the Sum of the *Momentums* of all the said infinitely small Parts or Weights $MmnN$ by the Sum of all those little Parts or Weights; that is, by the Area of the whole Figure. This is plain from common Books of Mechanicks.

Therefore calling AP, x, MP, y, $Pp, \dot{x}$, the infinitely small Part or Weight $MmnN$ will be $=2y\dot{x}$, and the Sum of them $=$ Fluent of $2y\dot{x}$. And the *Momentum* of one of those infinitely

finitely ſmall Parts or Weights will be $2y\dot{x}$ multiplied by AP $(x) = 2yx\dot{x}$. The Fluent of which divided by the Fluent of $2y\dot{x}$ will be $= AC$, the Diſtance of the Centre of Gravity from A, the Vertex of the Figure.

EXAMPLE I.

77. *TO find the Centre of Gravity of a Triangle* ADE.

FIG. 59. Draw the right Line AB biſecting the Baſe DE in B. Then becauſe the Triangle $ABD = ABE$, each of them may be reſolved into an infinite Number of little Parts or Weights $MmpP$, $PNnp$, each equal to one another on either Side the Line AB taken as an Axis; and conſequently the Centre of Gravity C muſt be ſomewhere in the ſaid Line AB.

Now call AB, a, DE, b; AP, x, MN, y; and draw AF perpendicular to DE, which call c. Then ſince the Triangles AMN, ADE are ſimilar, $AB\ (a) : DE\ (b) :: AP\ (x) : MN = \frac{bx}{a}$. Alſo becauſe the Triangles APQ, ABF are ſimilar; therefore $AB\ (a) : AF\ (c) :: AP\ (x) : AQ = \frac{cx}{a}$. And as $AP\ (x) : AQ\ \left(\frac{cx}{a}\right) :: Pp\ (\dot{x}) : Qq = \frac{c\dot{x}}{a}$. Whence the *Momentum* $yx\dot{x}$ will be $= \frac{cbx^2\dot{x}}{a^2}$; the Fluent of which is $= \frac{cbx^3}{3a^2}$. Which being divided by $\frac{cbx^2}{2a^2}$, the Area of the Triangle ADE, and the Quotient will be $\frac{2}{3}x =$ Diſtance of the Centre of Gravity

vity of the Part AMN of the Triangle ADE from the Vertex; and ſubſtituting a for x, the Diſtance of the Centre of Gravity of the whole Triangle ADE from the Vertex A will be $= \frac{2}{3} AB (a)$.

EXAMPLE II.

78. *To determine the Centre of Gravity* C *in the common Parabola.*

Calling AP, x, MN, y, and AB, a, and the Parameter p; then will px be $= yy$; and ſo $y = \sqrt{px}$. Whence $y\dot{x} = \dot{x}\sqrt{px}$; and ſo the *Momentum* $yx\dot{x} = x\dot{x}\sqrt{px} = x\dot{x}p^{\frac{1}{2}}x^{\frac{1}{2}}$. The Fluent of which is $\frac{2}{5}p^{\frac{1}{2}}x^{\frac{5}{2}}$; but the Fluent of $y\dot{x}$ is $= \frac{2}{3}p^{\frac{1}{2}}x^{\frac{3}{2}} =$ Area of the Portion AMN of the Parabola; therefore dividing $\frac{2}{5}p^{\frac{1}{2}}x^{\frac{5}{2}}$ by $\frac{2}{3}p^{\frac{1}{2}}x^{\frac{3}{2}}$, and the Quotient will be $= \frac{3}{5}x = AC$. And ſubſtituting a for x, the Diſtance of the Centre of Gravity C from the Vertex A will be $= \frac{3}{5}AB$. FIG. 58.

EXAMPLE III.

79. *To determine the Centre of Gravity* C *in all Parabola's of any higher Kind.*

Here $p^n x^m = y^r$ expreſſes the Nature of all Curves of this Kind. Whence $y = p^{\frac{n}{r}} x^{\frac{m}{r}}$, and conſequently $y\dot{x} = p^{\frac{n}{r}} x^{\frac{m}{r}} \dot{x}$, and the *Momentum*

 $xy\dot{x}$

$xy\dot{x} = p^{\frac{n}{r}} x^{\frac{m+r}{r}} \dot{x}$. The Fluent of which is $= \frac{r}{m+2n} p^{\frac{n}{r}} x^{\frac{m+2r}{r}}$. Which divided by the Fluent of $y\dot{x} = \frac{r}{m+n} p^{\frac{n}{r}} x^{\frac{m+n}{r}}$, and the Quotient will be $= \frac{m+r}{m+2r} x =$ the Distance of the Centre of Gravity of the Portion *MAN* from the Vertex *A*; and substituting *a* for *x*, there comes out $\frac{m+r}{m+2r} a = AC$.

EXAMPLE IV.

FIG. 60. 80. *TO find the Centre of Gravity of the Space* ADE *contained under two equal Parabola's* AD, AE, *touching one another in their Vertex* A, *and the straight Line* DE *parallel to the common Axis of the Parabola's.*

Make $AP = x$, $PM = y$, and let the Parameter be $= 1$: then will $\overline{AP}^2$ (x^2) be $(PM \times 1)$ $= y$; therefore the *Momentum* $xy\dot{x} = x^3\dot{x}$. The Fluent of which is $\frac{1}{4}x^4$; but the Fluent of $y\dot{x}$ is $\frac{1}{3}x^3$. Whence $\frac{\frac{1}{4}x^4}{\frac{1}{3}x^3} = \frac{3}{4}x = AC$, the Distance of the Centre of Gravity from *A*.

If the Parabola's *AME*, *AMD* be of any Kind whatsoever, this Equation will express the Relation of *AP* to *PM*, viz. $x^m = y^n$. Therefore $y\dot{x} = x^{\frac{m}{n}}\dot{x}$, and the *Momentum* $xy\dot{x} = x^{\frac{m+n}{n}}\dot{x}$. The Fluent of which is $\frac{n x^{\frac{m+2n}{n}}}{m+2n}$.

but

but the Fluent of $y\dot{x}$ is $\frac{n}{m+n}x^{\frac{m+n}{n}}$. Whence the Quotient of the Division of the former of these Expressions by the latter, *viz.* $\frac{m+n}{m+2n}x$ will be $= AC$.

EXAMPLE V.

81. TO *determine the Centre of Gravity of any Arch of a Circle.*

Let BE the Chord of the given Arch BDE be parallel to the Diameter FG; which being consider'd as an Axis, to which the small Weights BM are suspended; and so the *Momentums* of them as $BM \times PB$. And since the Numbers and *Momentums* of the said little Weights on each Side the Radius AD bisecting the Arch BDE are equal, the Centre of Gravity will be in AD. FIG. 61.

Now let $AB = a$, $AP = \overline{HB} = x$, and $Pp = Bm = \dot{x}$. Then PB $(\sqrt{aa-xx}) : AB$ $(a) :: Bm$ $(\dot{x}) : BM = \frac{a\dot{x}}{\sqrt{aa-xx}}$, and $BM \times BP$ $= \sqrt{aa-xx} \times \frac{a\dot{x}}{\sqrt{aa-xx}} = a\dot{x} = AB \times BM$; and the Sum of the *Momentums*, or Fluent of this Fluxion, is $ax = AB \times BH$. Which divided by the Arch BD and $\frac{AB \times BH}{BD}$ is $=$ the Distance of the Centre of Gravity C from the Centre A of the Circle. And substituting the Quadrant FD for BD, and the Radius FH or AB for BH; then will $\frac{\overline{AB}^2}{FD}$ be the Distance

of

of the Centre of Gravity C of the Semicircle FDG from the Centre A.

EXAMPLE VI.

82. *TO determine the Centre of Gravity of a Sector of a Circle* ABE.

FIG. 62. The Centre of Gravity will be in the Radius AD, which bisects the Arch BE. Describe the Arch MPM with any Distance AP, and another Arch mpm infinitely near it; then the *Momentum* of the Arch MPM drawn into Mm or Pp, will be the *Momentum* of the annular Segment $mMPMm$, or the Fluxion of the *Momentum* of the Sector.

Now let AB be $=a$, $AF=b$, and $BD=c$; then the *Momentum* of the Arch BDE ($2ab$) is to the *Momentum* of the Arch MPM, as the Triangle ABE to the Triangle AMM, or as $\overline{AB}^2$ to $\overline{AM}^2$; since the Triangles ABE, AMM are similar. Whence the *Momentum* of the Arch $MPM=\frac{2abx^2}{a^2}=\frac{2bx^2}{a}$, and the *Momentum* of the annular Segment $mMPMm=\frac{2bx\dot{x}}{a}$. The Fluent (or Sum) of which $\frac{2bx^3}{3a}$ divided by the Sum of the Weights, or the Area ac of the Sector, and the Quotient $\frac{2bx^3}{3aac}$ is the Distance of the Centre of Gravity of the Sector of the Circle MPM.

EXAMPLE VII.

83. *TO find the Centre of Gravity of any Segment* MAm *of an Hyperbola.*

Making

Making AC, a, CB, b, AP, x, and PM, y, FIG. 63. we have from the Nature of the Curve $\frac{bb}{aa} \times \overline{2ax + xx} = yy$. Therefore the Fluxion of the *Momentums* will be $\frac{2b}{a} x^{\frac{1}{2}} \dot{x} \sqrt{2a + x}$, and the Fluxion of the Weights $\frac{2b}{a} x^{\frac{1}{2}} \dot{x} \sqrt{2a + x}$. Each of which may be compared with that of the fourth Form in the Tables of Mr. *Cotes*; and so making $\theta = 2$, $n = 1$, &c. we shall have the Sum of the *Momentums* $= \frac{\overline{-3aa + ax + 2xx}}{3} \times y + 2a^2 b \left| \frac{\sqrt{x} + \sqrt{2a + x}}{\sqrt{2a}} \right.$. And again, putting $\theta = 1$, &c. the Sum of the Weights will be $= \overline{a + x} \times y + 2ab \left| \frac{\sqrt{x} + \sqrt{2a + x}}{\sqrt{2a}} \right.$; and so dividing the former Fluent by the latter, we have

$$\frac{\overline{-3aa + ax + 2xx} \times y + 2a^2 b \left| \frac{\sqrt{x} + \sqrt{2a + x}}{\sqrt{2a}} \right.}{\overline{a + x} \times y + 2ab \left| \frac{\sqrt{x} + \sqrt{2a + x}}{\sqrt{2a}} \right.}$$

= Distance of the Centre of Gravity from the Vertex D.

Now to construct this Expression, make $CB(b) : CA(a) :: PM(y) : CF = \frac{ay}{b}$. And again, $PM(y) : CB(b) :: CA(a) : CG = \frac{ab}{y}$. Then take CH equal to the Measure of the Ratio between $CA(a)$ and $FP\ (a + x - \frac{ay}{b})$. Which Ratio is = duplicate Ratio of $\sqrt{2a}$

to $\sqrt{x}-\sqrt{2a+x}$. When this is done, if you make as $3\,PH\left(3a+3x-\frac{3ab}{y}\,\Big|\,\frac{a}{a+x-\frac{ay}{b}}\right)$ $:2\,CF\left(\frac{2ay}{b}\right)::CF\left(\frac{ay}{b}\right):CZ=$

$$\frac{4ax+2xx}{3a+3x-\frac{3ab}{y}\,\Big|\,\frac{a}{a+x-\frac{ay}{b}}}.$$

This fourth Proportional will be = Distance of the Centre of Gravity z from the Centre C. For if CA (a) be taken from it, we shall have $\frac{4ax+2xx}{3a+3x-\frac{3ab}{y}\,\Big|\,\frac{a}{a+x-\frac{ay}{b}}}-a=$

$$\frac{-\overline{3aa+ax+2xx}\times y+2a^2b\,\Big|\,\frac{\sqrt{x}+\sqrt{2a+x}}{\sqrt{2a}}}{\overline{a+x}\times y+2ab\,\Big|\,\frac{\sqrt{x}+\sqrt{2a+x}}{\sqrt{2a}}}$$

the Expression first found.

SCHOLIUM.

84. THE Distance of the Centre of Gravity of the Segment of an Ellipsis or Circle from the Vertex will be expressed thus:

$$\frac{-\overline{3aa-ax+2xx}\times y-2a^2b\,\Big|\,\frac{\sqrt{x}+\sqrt{2a-x}}{\sqrt{2a}}}{\overline{a-x}\times y-2ab\,\Big|\,\frac{\sqrt{x}+\sqrt{2a-x}}{\sqrt{2a}}}$$

it being the same as that for the Centre of Gravi-

Gravity of the Hyperbola only with the Alteration of some Signs, and the Measure of the Ratio there being here changed into that of an Angle, because $R(=\sqrt{-f})=\sqrt{-1}$ is negative.

EXAMPLE VIII.

85. TO *find the Centre of Gravity of the external hyperbolick Space* AMmaA. FIG. 64.

Let the Semi-conjugate Diameter $BC=b$, the Semi-transverse $AC=a$, the Abscis $CP=y$, and the Semi-ordinate $PM=x$.

Then from the Nature of the Curve, $\frac{b}{a}\sqrt{xx-aa}=yy$; and so $x=\frac{a}{b}\sqrt{bb+yy}$.

Whence $\frac{2a}{b}\dot{y}\sqrt{bb+yy}$ is the Fluxion of the Weights, and $\frac{2a}{b}y\dot{y}\sqrt{bb+yy}$ that of the *Momentums*. And from the Comparison of this Fluxion with the 3d Form of Mr. *Cotes*'s Tables, θ being $=1$, $n=2$, &c. the following Fluent will be had, viz. $\frac{2a}{3b}\overline{bb+yy}^{\frac{3}{2}}$; and making $y=0$, the same will become $2ab^2$; which must be taken from that Fluent, and the Remainder $\frac{2a}{3b}\overline{bb+yy}^{\frac{3}{2}}-2ab^2$ will be the Fluent of the Fluxion $\frac{2a}{b}y\dot{y}\sqrt{bb+yy}$.

Also the Fluent of $\frac{2a}{b}\dot{y}\sqrt{bb+yy}$, by comparing it with the fourth Form, θ being $=0$, $n=2$, &c. will be $\frac{ay}{b}\sqrt{bb+yy}+ab\left|\frac{y+\sqrt{bb+yy}}{b}\right.$. By which dividing the Fluent just now found,

 and

and the Quotient $\dfrac{\frac{2a}{3b}\times\overline{bb+yy}^{\frac{3}{2}}-2ab^2}{\frac{ay}{b}\sqrt{bb+yy}+ab\left|\frac{y+\sqrt{bb+yy}}{b}\right.}$

$=\dfrac{2\times\overline{bb+yy}^{\frac{3}{2}}-2b^3}{3y\sqrt{bb+yy}+3bb\left|\frac{y+\sqrt{bb+yy}}{b}\right.}$ will be $=$ Distance of the Centre of Gravity Z from the Centre C.

Now to construct the Expression: Make $AC(a):PM\left(\frac{a}{b}\sqrt{bb+yy}=x\right)::BC(b):CR=\sqrt{bb+yy}$. And again, $PM\left(\frac{a}{b}\sqrt{bb+yy}\right):AC(a)::BC(b):CS=\frac{b^2}{\sqrt{bb+yy}}$. And moreover, $PM\left(\frac{a}{b}\sqrt{bb+yy}\right):AC(a)::CS\left(\frac{b^2}{\sqrt{bb+yy}}\right):CT=\frac{b^3}{bb+yy}$. And let CR, CT, tend both the same way, *viz.* from C towards PM, but CS the contrary way. Then take CN equal to the Measure of the Ratio between $CB\ (b)$ and $ER\ (\sqrt{bb+yy}-y)$ to the Module CS $\left(\frac{b^2}{\sqrt{bb+yy}}\right)$. Which will be thus expressed, $\frac{bb}{\sqrt{bb+yy}}\left|\frac{b}{\sqrt{bb+yy}-y}\right.$. This being done, make $3PN\left(3y+\frac{3b^2}{\sqrt{bb+yy}}\left|\frac{b}{\sqrt{bb+yy}}\right.\right):2TR\left(2\sqrt{bb+yy}-\frac{2b^3}{bb+yy}\right)::CR\left(\sqrt{bb+yy}\right)$: to a fourth a fourth Proportional, which will be =

$$\frac{2\times\overline{bb+yy}^{\frac{3}{2}}-2b^3}{3y\sqrt{bb+yy}+3bb\,\Big|\,\frac{y+\sqrt{bb+yy}}{b}} = CZ$$ the Distance sought.

Note, The Construction of these two Examples are the same as Mr. *Cotes* has given in *Harmonia Mensurarum*, p. 25, 26. Part I.

EXAMPLE VIII.

86. *TO find the Centre of Gravity of a right Cone and Pyramid.*

The Centre of Gravity will be somewhere in the Axis *AB*. FIG. 43.

Now if *AP* be $= x$, and $BC = a$, and $AB = b$, we shall have $PM = \frac{ax}{b}$. And the Ratio of the Radius to the Periphery being that of *r* to *p*, the Fluxion of the Weight will be $\frac{paaxx}{2rbb}\dot{x}$, and the *Momentum* of it $\frac{paax^3}{2rbb}\dot{x}$. Consequently the Sum of the *Momentums* $\frac{paax^4}{8rbb}$, divided by the Sum of the Weights $\frac{paax^3}{6rbb}$, will give us $\frac{3}{4}x = AG$, the Distance of the Centre of Gravity of the Part *AMPm* of the Cone; and so $\frac{3}{4}AB$ is the Distance of the Centre of Gravity of the whole Cone from the Vertex *A*.

Much after the same way you will find the Distance of the Centre of Gravity of a Pyramid from the Vertex to be $\frac{3}{4}$ of the Axis from the Vertex.

EXAMPLE IX.

FIG. 44. 87. *TO find the Centre of Gravity of any Segment of a Sphere.*

Let $AC = r$, $AP = x$; then the Fluxion of the Weights will be $px\dot{x} - \frac{px^2\dot{x}}{2r}$, and that of the *Momentums* $px^2\dot{x} - \frac{px^3\dot{x}}{2r}$, and the Sum of the *Momentums* $\frac{px^3}{3} - \frac{px^4}{8r}$; which being divided by $\frac{pxx}{2} - \frac{px^3}{6r}$, the Sum of the Weights, and the Quotient $\frac{8rx - 3xx}{12r - 4x}$ will be the Distance of the Centre of Gravity from A, of the Segment of a Sphere generated by the Semi-segment AMP of a Semicircle about AP.

COROL.

88. HENCE the Centre of Gravity of $\frac{1}{2}$ the Sphere will be $\frac{5}{8}r$: for here x becomes $= r$.

EXAMPLE X.

89. *TO find the Centre of Gravity of a parabolick Conoid formed by the Revolution of a Parabola about its Axis.*

Here $\frac{px\dot{x}}{2}$ is the Fluxion of the Weights, and $\frac{px^2\dot{x}}{2}$ the Fluxion of the *Momentums*, r being

ing the *Semi-latus Rectum*. The Sum of the Weights is $\frac{px^2}{4}$, and the Sum of the *Momentums* $\frac{px^3}{6}$; therefore this divided by $\frac{px^2}{4}$, and the Quotient will be $\frac{2}{3}x$. Which is the Distance of the Centre of Gravity from the Vertex *A* of the common Parabola.

EXAMPLE XI.

90. *TO find the Centre of Gravity of a Solid formed by the Revolution of the parabolick Space* AMBD *about the Line* BT *parallel to the Axis.*

Let *DB* or *AT* be $=r$, $AP=x$, $PM=y$; FIG. 65. then will $\frac{pr}{2} - py + \frac{pyy}{2r}$ be $=$ Circle described by *MQ*; and consequently $px^{\frac{1}{2}}\dot{x} - \frac{px\dot{x}}{2r}$ will be the Fluxion of the Weights, and $px^{\frac{3}{2}}\dot{x} - \frac{px^2\dot{x}}{2r}$ that of the *Momentums*. The Fluent of the former Expression, or Sum of the Weights, is $\frac{2}{3}px^{\frac{3}{2}} - \frac{px^2}{4r}$, and the Sum of the *Momentums* $\frac{2}{5}px^{\frac{5}{2}} - \frac{px^3}{6r}$. Which divided by the Sum of the Weights, and the Quotient will be $\frac{24rx^{\frac{3}{2}} - 10xx}{40rx^{\frac{1}{2}} - 15x} = \frac{24rx - 10yx}{40r - 15y}$, being the Distance of the Centre of Gravity of the Part of the Solid generated by *APM*; and

and when x becomes $= a$, and so $y = r$, the said Distance will be $= \frac{14}{25}a$.

EXAMPLE XII.

91. *TO find the Centre of Gravity of an hyperbolick Conoid* ABG *generated by the Rotation of the hyperbolick Space* ABG *about the transverse Axis* aCAP.

FIG. 41. Now calling AC, $2b$, AB, a, and the Ordinate BG, r, the Fluxion of the Weights will be $\frac{prx\dot{x} + 2bpr\dot{x}}{2aa + 4ab}$, and the Fluxion of the *Momentums*. A being the Centre of Motion, will be $\frac{prx^3\dot{x} + 2bprxx\dot{x}}{2aa + 4ab}$; the Fluent of which is $\frac{prx^4}{8aa + 16ab} + \frac{bprx^3}{3aa + 6ab} = \frac{3prx^4 + 8bprx^3}{24aa + 48ab}$. Which being divided by $\frac{prx^3 + 3bprxx}{6aa + 12ab} = \frac{4prx^3 + 12bprxx}{24aa + 48ab}$ the Sum of the Weights, and the Quotient $\frac{3xx + 8bx}{4x + 12b}$ will be the Distance of the Centre of Gravity from the Point A in the Axis AP of the Conoid form'd by the Revolution of the Part AMP of the Hyperbola; and when $x = a$, $\frac{3aa + 8ab}{4a + 12b}$ will be the Distance of the Centre of Gravity of the whole Solid from the Vertex A. Whence $3a + 8b : 4a + 12b :: a : \frac{3aa + 8ab}{4a + 12b}$.

EXAMPLE XIII.

92. *To find the Centre of Gravity of an hyperbolick Conoid form'd by the Revolution of the hyperbolick Space* AMBDC *about* CD *the half of the conjugate Axis.*

The same being supposed as in *Art.* 61. the Fluxion of the Weights is $\frac{aapx^2\dot{x}+aabbp\dot{x}}{2bbr}$, and that of the *Momentums* with regard to *AC*, is $\frac{aapx^3\dot{x}+aabbpx\dot{x}}{2bbr}$; the Fluent of which is $\frac{aapx^4}{8bbr}+\frac{aabbpxx}{4bbr}=\frac{px^4yy+2bbpx^2y^2}{8rx^2+8bbr}$, by putting $\frac{bbyy}{xx+bb}$ for aa. Now this divided by the Sum of the Weights, and the Quotient $\frac{3x^3+6bbx}{4xx+12bb}$, will be the Distance of the Centre of Gravity of the Conoid generated by the Space *ACPM* from the Line or Axis *AC*; and when x becomes $=b$, we shall have $\frac{9}{16}b$ for the Distance of the said Centre. FIG. 48.

Therefore the Centre of Gravity of the whole Conoid is so situate in the Axis *AC*, that the Part from the Centre *C* to the Centre of Gravity is to *AC* as 9 to 16.

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EXAMPLE XIV.

93. *TO find the Centre of Gravity of a Semi-spheroid generated by the Rotation of ½ the Eliptick Space* AMca *about the Axis* Aa.

FIG. 40. Make $AC = a$, $Cc = r$, $AP = x$, $PM = y$. Now when PC is x, the Fluxion of the Weights will be $\frac{pyy\dot{x}}{2r}$; but from the Nature of the Ellipsis, (AP being $= x$) $yy : 2ax - xx :: rr : aa$. Whence $yy = \frac{2arrx - rrxx}{aa}$; and so substituting this Value in $\frac{pyy\dot{x}}{2r}$, there arises $\frac{2aprx\dot{x} - prx^2\dot{x}}{2aa}$, and multiplying by x; the Fluxion of the *Momentums* will be $\frac{2aprxx\dot{x} - prx^3\dot{x}}{2aa}$, the Fluent of which, *viz.* $\frac{prx^3}{3a} - \frac{prx^4}{8aa} = \frac{8aprx^3 - 3prx^4}{24aa}$, divided by the Sum of the Weights $\frac{12aprxx - 4prx^3}{24aa}$, and the Quotient $\frac{8ax - 3xx}{12a - 4x}$ will be the Distance of the Centre of Gravity of the partial Solid generated by the Space APM from the Point A; and when x becomes $= a$, the said Distance will be $\frac{5}{8}a$; that is, the Centre of Gravity of the Semi-spheroid will be distant from A by $\frac{5}{8}a$.

SECT.

SECTION VII.

Of the Use of Fluxions in finding the Centres of Percussion of Figures.

DEFINITION.

94. THE Centre of *Percussion*, or *Oscillation* of a Figure in Motion, is that Point in which all the Forces of the same are consider'd as united together in one; so that if the said Figure meets any Obstacle contrary to the Motion thereof, it strikes the Obstacle with a greater Force than any other Point of the Figure.

In order to this, it is necessary that the Parts of the Figure do constantly alter their Disposition to move; that they separate their Quantity of Motion, not as in the Centre of Gravity in the Ratio of the Spaces run thro'; but in a Ratio compounded of their Velocities, and the Distances of that Centre reciprocally proportional to the said Velocities; or, which is the same thing, into equal Quantities of Motion on each Side that Point. Therefore the Centre of Percussion is the same with regard to Velocities, as the Centre of Gravity is with respect to Weights: and as in finding the Centre of Gravity, we divide the Sum of the *Momentums* by the Sum of the Weight; so to find the Centre of Percussion, we must multiply the Sum of the *Momentums* by straight Lines equal or proportional to the Spaces moved thro', and divide the Product by the Sum of the *Momentums*. Whence the

General Rule for finding the Centre of Percussion of a Figure that revolves about a given Point or Axis, is to multiply all the small Parts of which the Figure consists, (that is, the Area or Solidity of it) looked upon as so many Weights, by the Squares of their Distances from the Point of Suspension, and divide the Product by the Product of the same Weights into the Distances from the Axis of Motion, and the Quotient will be the Distance of the Centre of Percussion from the Point or Axis of Motion.

FIG. 59. Hence if AP be $= x$, $MN = 2y$, $Pp = \dot{x}$, the *Momentum* of the whole small Weight $MNnm$ will be $= 2yx\dot{x}$. Consequently the Distance of the Centre of Percussion from the Point A is $=$ to the Fluent of $2yx^2\dot{x}$ divided by the Fluent of $2yx\dot{x}$. Consequently, if from the Equation of the Figure you get the Value of y, and put it in those Fluxions, and then find the Fluents of them, you will get the Distance of the Centre of Percussion from the Point A. This will be evident from the following Examples.

EXAMPLE I.

FIG. 65. 95. *TO find the Centre of Percussion of a right Line* AB, *moving about one End* A *thereof.*

Now if the said Line be conceived to be divided into an infinite Number of equal small Parts

Pp

Pp ($\dot{x}$) (AB being a, and AP, x), it is manifest that in equal times they will describe equal Arches of concentrick Circles, that will be to each other as the Distances from the Point A: But the Velocities wherewith the said Arches are moved through, are proportional to the said Arches; and so the Velocities are as the said Distances.

Now $x\dot{x}$ will be the Fluxion of the *Momentums*, which multiplied by x representing the Velocities, and there will arise $x^2\dot{x}$ for the Fluxion of Forces; the Fluent of which $\frac{x^3}{3}$ being divided by the Sum of the *Momentums* $\frac{x^2}{2}$, and the Quotient $\frac{2}{3}x$ will be the Distance of the Centre of Percussion of the Part AP of the Line from the Point A; and the Centre of Percussion of the whole Line will be $\frac{2}{3}a$, by making $x=a$.

EXAMPLE II.

96. *TO find the Centre of Percussion of a Rectangle* RISH *moving about one of the Sides* RI.

If $RI=SH$ be $=a$, $AP=x$; then will $Pp=\dot{x}$ be the Fluxion of the Area, and one of the small Weights will be $=a\dot{x}$, and the *Momentum* of it will be $ax\dot{x}$. Whence the Fluent of $ax^2\dot{x}$ divided by the Fluent of $ax\dot{x}$ (or $\frac{\frac{1}{3}ax^3}{\frac{1}{2}ax^2}$), and the Quotient $\frac{2}{3}x$ will be the Distance of the Centre of Percussion of the FIG. 66.

 Part

Part $RCDI$ of the Parallelogram from the Side RI; and if for x be put the Altitude $RS = b$ of the whole Rectangle, the Distance of the Centre of Percussion from the Side RI will be $= \frac{2}{3}b$.

EXAMPLE III.

97. *TO find the Centre of Percussion of an Isosceles Triangle* SAH *moving about the Line* RI *passing thro' the Vertex* A, *and parallel to the Base* SH.

FIG. 66. Let the Altitude AE be $= a$, $AP = x$; $EH = \frac{1}{2}b$, $PL = y$. Then $AP(x) : PL(y) :: AE(a) : EH = \frac{1}{2}b$. Whence $ay = \frac{1}{2}bx$, and $y = \frac{bx}{2a}$. Now the Fluent of $yx^2\dot{x}$ = Fluent of $\frac{bx^3\dot{x}}{2a}$ is $= \frac{bx^4}{8a}$; and the Fluent of $yx\dot{x}$ = Fluent of $\frac{bx^2\dot{x}}{2a}$ is $= \frac{bx^3}{6a}$. Whence the Fluent of $yx^2\dot{x}$, divided by that of $yx\dot{x}$, or $\frac{6abx^4}{8abx^3}$ is $= \frac{6}{8}x = \frac{3}{4}x$.

Now if for x you substitute the whole Altitude $AE = a$, the Distance of the Centre of Percussion of the whole Triangle ASH from the Vertex A will become $\frac{3}{4}a = \frac{3}{4}AE$.

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EXAMPLE IV.

98. *To find the Centre of Percussion of an Isosceles Triangle* SAH *moving about the Base* SH.

Let all things be as in the last Example, then will *PE* be $= a - x$. Whence the Fluent of $yx^2\dot{x}$ = Fluent of $\frac{bx\dot{x}}{2a} \times \overline{a-x}^2$ (= Fluent of $\frac{1}{2}abx\dot{x} - bx^2\dot{x} + \frac{bx^3\dot{x}}{2a}$) is $= \frac{1}{4}abx^2 - \frac{1}{3}bx^3 + \frac{bx^4}{8a}$, and the Fluent of $yx\dot{x}$ = Fluent of $\frac{bx\dot{x}}{2a} \times \overline{a-x}$ = (Fluent of $\frac{1}{2}bx\dot{x} - \frac{bx^2\dot{x}}{2a}$) $= \frac{1}{4}bx^2 - \frac{bx^3}{6a}$. Whence the Quotient of the Fluent of $yx^2\dot{x}$ divided by that of $yx\dot{x}$ FIG. 66.

$$\left(= \frac{\frac{1}{4}abx^2 - \frac{1}{3}bx^3 + \frac{bx^4}{8a}}{\frac{1}{4}bx^2 - \frac{bx^3}{6a}} \right)$$

$$\text{is} = \frac{\frac{24a^2bx^2 - 32abx^3 + 12bx^4}{96a}}{\frac{6abx^2 - 4bx^3}{24a}}$$

$$= \frac{6a^2bx^2 - 16abx^3 + 6bx^4}{12abx^2 - 8abx^3} = \frac{6a^2 - 8ax + 3x^2}{6a - 4x}$$

= Distance of the Centre of Percussion of the Segment *SZVH* from the Base *SH*.

Now if for x you substitute a, we shall have the Distance of the Centre of Percussion of of the whole Triangle $SAH = \left(\frac{6a^2 - 8a^2 + 3a^2}{6a - 4a}\right) = \frac{1}{2}a = \frac{1}{2}AE.$

EXAMPLE V.

99. *To find the Centre of Percussion of a parabolick Space, moving about a Line passing thro' the Vertex parallel to the Base.*

Now calling the Abscis x, and the whole Height a, and the Fluxion of the *Momentums* will be $x^{\frac{3}{2}}\dot{x}$, and that of the Forces $x^{\frac{5}{2}}\dot{x}$; the Fluent of which will be $\frac{2}{7}x^{\frac{7}{2}}$. Which divided by $\frac{2}{5}x^{\frac{5}{2}}$ the Sum of the *Momentums*; the Quotient will be $\frac{5}{7}x$ = Distance of the Centre of Percussion of the Part of the Parabola whose Height is x from the Vertex; and when x becomes equal to a, the Distance of the Centre of Percussion of the whole Parabola will be $\frac{5}{7}a$.

SCHOLIUM.

100. If the Centre of Percussion of a Parabola of any Kind be sought, you will have $\frac{2m+1}{3m+1}a$ = Distance of the Centre of Percussion from the Vertex; where m is the Exponent of the Power of the Ordinate of the Parabola. So that if m be $= 2$, the Distance will be $\frac{5}{7}a$, as in the common Parabola. If

m be

m be $=3$, as in the cubick Parabola, the Distance will be $\frac{7}{10}a$, &c.

EXAMPLE VI.

101. *TO find the Centre of Percussion of a parabolick Space moving about its Base* DD. Fig. 58.

Multiply the Fluxion of the Weights $x^{\frac{1}{2}}\dot{x}$ by $a-x$, and we get $ax^{\frac{1}{2}}\dot{x}-x^{\frac{1}{2}+1}\dot{x}=$ the Fluxion of the *Momentums*; which being again multiplied by $a-x$, and the Fluxion of the Forces will be $aax^{\frac{1}{2}}\dot{x}-2ax^{\frac{1}{2}+1}\dot{x}+x^{\frac{1}{2}+2}\dot{x}$, the Fluent of which, *viz.* $\frac{2}{3}aax^{\frac{1}{2}+1}-\frac{4}{5}ax^{\frac{1}{2}+2}+\frac{2}{7}x^{\frac{1}{2}+3}=\frac{70aax^{\frac{1}{2}+1}-84ax^{\frac{1}{2}+2}+30x^{\frac{1}{2}+3}}{105}$ being divided by the Sum of the *Momentums* $=\frac{10ax^{\frac{1}{2}+1}-6x^{\frac{1}{2}+2}}{15}$, and the Quotient will be $\frac{35aa-42ax+15xx}{35a-21x}$, the Distance of the Centre of Percussion of the Space $DMND$ from the Base DD; and putting x for a, the Distance of the Centre of Percussion of the whole Parabola from the Base DD will be $\frac{4}{7}a=\frac{4}{7}AB$.

EXAMPLE VII.

102. *TO find the Centre of Percussion of a Cylinder* AB *moving about the End* A *thereof.* Fig. 67.

Now

Now it is evident that the Velocities of all the small equal Parts of this Solid will be to each other, on account of the equal times, in the same Ratio as the Spaces *rm* thro' that is, as the Arches described in their Motion, or as the Radii or Distances from the Point of Suspension. Now let the Axis *AB* of the Cylinder be $= a$, any Part $AP = x$, the Periphery of the Base $= p$, and the Radius of it $= r$; then the Fluxion of the *Momentums* will be $\frac{prx\dot{x}}{2}$, and the Fluxion of the Forces $\frac{prx^2\dot{x}}{2}$, the Fluent of which is $\frac{prx^3}{6}$. Which being divided by the Sum of the *Momentums* $= \frac{prx^2}{4}$, and the Quotient $\frac{2}{3}x$ will be the Distance of the Centre of Percussion of the Part of the Cylinder, whose Altitude is *AP* from the Point *A*; and $\frac{2}{3}a$ will be the Distance of the Centre of Percussion of the whole Cylinder from the said Point *A*.

EXAMPLE VIII.

103. *To find the Centre of Percussion of a Cylinder moving about the Point R in the Axis continued out.*

FIG. 67. Make $RB = a$, $RA = b$, $AP = x$; then $RP = b + x$, and $AB = a - b$. Whence $\frac{bprx\dot{x} + prx\dot{x}x}{2}$ is the Fluxion of the *Momentums*: But the Velocities of the little equal Weights of the Solid are to one another as the Arches described from the Point *R* by

those Weights; so the Forces are to each other as the right Lines that fill up a Trapezium described by the Cylinder. Whence multiplying the Fluxion of the *Momentums* by $b+x$; and we get $\frac{bbpr\dot{x}+2bprx\dot{x}+prx^2\dot{x}}{2}$ = Fluxion of the Forces; the Fluent whereof is = $\frac{bbprx}{2}+\frac{bprx^2}{2}+\frac{prx^3}{6}$. Which divided by the Sum of the *Momentums* = $\frac{bprx}{2}+\frac{prx^2}{4}$, and the Quotient $\frac{6bb+6bx+2xx}{6b+3x}$ will be the Distance of the Centre of Percussion of the Part whose Altitude is *AP* from the Point *R*; and $\frac{2aa+2ab+2bb}{3a+3b}$ will be the Distance of the Centre of Percussion of the whole Cylinder from the Point *R*, since then x becomes = $a-b$.

Hence it is easy to find on what Part of a Cylindrical Stick a Man ought to strike in order to gain the greatest Blow possible, supposing *AR* represents the Man's Arm, and *AB* the Stick.

EXAMPLE IX.

104. *TO find the Centre of Percussion of a Cone moving about its Vertex* A.

The Fluxion of the *Momentums* will be $\frac{prx^3\dot{x}}{2aa}$, and the Fluxion of the Forces $\frac{prx^4\dot{x}}{2aa}$; the Fluent $\frac{prx^5}{10a^2}$ of which being divided by the Sum of the *Momentums* = $\frac{prx^4}{8aa}$, and the Quotient $\frac{4}{5}x$ will be the FIG. 43.

 Distance

Distance of the Centre of Percussion of the Part of the Cone, whose Altitude is AP from the Vertex A, and $\frac{4}{5}a$ will be the Distance of the Centre of Percussion of the whole Cone from the Vertex A.

Note, This Centre of Percussion is the same as the Centre of Gravity of the Complement of a cubick Parabola; because the Forces are as right Lines which fill up that Complement.

EXAMPLE X.

105. *TO find the Centre of Percussion of a Sphere moving about a Point* A *in the End of a Diameter* AD.

FIG. 44. Let the Radius be $=r$, the Periphery $=p$, and $AP=x$; then the Fluxion of the *Momentums* will be $px^2\dot{x}-\frac{px^3\dot{x}}{2r}$, and the Fluxion of the Forces $px^3\dot{x}-\frac{px^4\dot{x}}{2r}$; and the Fluent of this will be $=\frac{px^4}{4}-\frac{px^5}{10r}$; which being divided by the Sum of the *Momentums* $=\frac{px^3}{3}-\frac{px^4}{8r}$, and the Quotient $\frac{30rx-12xx}{40r-15x}$ will be the Distance of the Centre of Percussion of the Segment of the Sphere, whose Height is x, from the Point A; and $\frac{6}{5}r$ will be the Distance of the Centre of Percussion of the whole Sphere from that Point.

EXAM-

EXAMPLE XI.

106. *TO find the Centre of Percussion of a Parabolick Conoid moving about the Vertex.*

The Fluxion of the *Momentums* will be $\frac{px^2\dot{x}}{2r}$, and the Fluxion of the Forces $\frac{px^3\dot{x}}{2r}$. The Fluent of which is $=\frac{px^4}{8r}$; which being divided by $\frac{px^3}{6r}$, and $\frac{3}{4}x$ will be the Distance of the Centre of Percussion of the Part of the Conoid, whose Altitude is x from the Vertex, and $\frac{3}{4}a$ will be the Distance of the Centre of Percussion of the whole Conoid from the Vertex.

EXAMPLE XII.

107. *TO find the Centre of Percussion of a Spheroid moving about one End of the transverse Axis.*

The same things being supposed as in *Art.* 93. the Fluxion of the *Momentums* will be $\frac{2aprx^2\dot{x}-prx^3\dot{x}}{2aa}$, and the Fluxion of the Forces $\frac{2aprx^3\dot{x}-prx^4\dot{x}}{2aa}$; the Fluent of which, *viz.* $\frac{prx^4}{4a}-\frac{prx^5}{10aa}$, being divided by the Sum of the *Momentums* $=\frac{prx^3}{3a}-\frac{prx^4}{8aa}$, and the Quotient

$\frac{30ax - 12xx}{40a + 15x}$ will be the Distance of the Centre of Percussion of the Part of the Spheroid, whose Altitude is x from the Point of Motion, and $\frac{5}{6}a$ will be the Distance requir'd.

SECT. VIII.

Of the Resolution of some miscellaneous Problems by Fluxions.

PROB. I.

108. *To find a Line, wherein the Subtangent is equal to the Semi-ordinate.*

In all Cases it is plain that $\frac{y\dot{x}}{\dot{y}}$ is an Expression of the Subtangent; and so from the Condition of the Problem $\frac{y\dot{x}}{\dot{y}}=y$, and $y\dot{x}=\dot{y}y$; that is, $\dot{x}=\dot{y}$, and the Fluent of each Side will be $x=y$. Whence the Line sought is the Hypothenuse of a right-angled Equicrural Triangle; a Line bisecting the right Angle being looked upon as the Axis. But if x be the Arch of a Circle, then will the Line sought be a Cycloid.

PROB. II.

109. *To find a Curve, whose Subtangent is equal to twice the Square of the Semi-ordinate divided by a constant Quantity: suppose* a.

The

The Subtangent is $=\frac{y\dot{x}}{\dot{y}}$. Whence $\frac{y\dot{x}}{\dot{y}}$ must be $=\frac{2yy}{a}$. Consequently $ay\dot{x}=2y^2\dot{y}$, that is, $a\dot{x}=2y\dot{y}$; and finding the Fluents of each Side, we have $ax=yy$; therefore the Curve sought is the *Apollonian* Parabola.

PROB. III.

110. *To find a Curve, whose Subtangent is a third Proportional to some standing Quantity* a *lessen'd by the Abscifs, and the Semi-ordinate.*

Here $a-x:y::y:\frac{y\dot{x}}{\dot{y}}$; and multiplying the Means and Extremes we get $\frac{ay\dot{x}-yx\dot{x}}{\dot{y}}=yy$, and $ay\dot{x}-yx\dot{x}=y^2\dot{y}$; that is, $a\dot{x}-x\dot{x}=y\dot{y}$; and finding the Fluents of each Side, we have $ax-\frac{1}{2}xx=\frac{1}{2}yy$, or $2ax-xx=yy$; therefore the Curve sought is a Circle whose Radius is $=a$.

PROB. IV.

111. *To find a Curve, whose Subtangent is an invariable Line.*

Here $\frac{y\dot{x}}{\dot{y}}$ must be $=a$, and so $\dot{x}=\frac{a\dot{y}}{y}=ay^{-1}\dot{y}$. Whence the Fluent of $\dot{x}$, viz. x must be $=$ Fluent of $ay^{-1}\dot{y}$; and multiplying $ay^{-1}\dot{y}$ by a, we shall have $a^2y^{-1}\dot{y}$, which is the Fluxion of an Hyperbola between the Asymptotes. And so if y be taken for an Abscifs, the correspondent Semi-ordinate $x=$ Fluent of $ay^{-1}\dot{y}$ will be equal to the Asymptotical Hyperbolical Space divi-

divided by the invariable Quantity a, being the Side of the Power of the Hyperbola.

PROB. V.

112. *A Number being given: to find the Logarithm of it.*

Let the Ordinate of the Logarithmical Curve AB be $=1=$ Subtangent; then PM will represent a Number greater than 1, and QN a Number less than 1; AP the Logarithm of the Number greater than 1, and AQ the Logarithm of the Number less than 1. FIG. 68.

Now let the Difference between AB and PM be $=y$; then will PM be $=1+y$. Whence AP, the Logarithm of a Number greater than 1, will be the Fluent of $\frac{\dot{y}}{1+y}=\dot{y}-y\dot{y}+y^2\dot{y}-y^3\dot{y}+y^4\dot{y}$, &c. Which Fluent is $=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4$, &c. $=$ Logarithm of a Number greater than 1.

Again, if the Difference between AB and QN be y; then will $QN=1-y$. And so AQ, or the Logarithm of a Number less than 1, will be $=$ Fluent of $\frac{-\dot{y}}{1-y}=\dot{y}-y\dot{y}-y^2\dot{y}-y^3\dot{y}-y^4\dot{y}$, &c. the Fluent of which will be $=-y-\frac{1}{2}y^2-\frac{1}{3}y^3-\frac{1}{4}y^4$, &c. being the Logarithm of a Number less than 1.

SCHOLIUM.

113. If the Side AB or BC of the Power of an Hyperbola be $=1$, and $BP=y$; then will $AP=1+y$, and the Asymptotical Hyperbolical Space will be $=y-\frac{1}{2}y^2+\frac{1}{3}y^3$ FIG. 69.

$-\frac{1}{4}$

$-\frac{1}{4}y^4$, &c. And if *BQ* be $= y$, then will *AQ* be $= 1 - y$, and the Fluxion $\frac{-y\dot{y}}{1-y}$ will be that of the Hyperbolick Asymptotical Space. Whence the Space will be $= -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c.

Whence Logarithms may be represented also by the Hyperbola: for if the Side *AB* of the Power of the Hyperbola be $= 1$, the Abscis *AP* is a Number greater than 1, and the Asymptotical Space *BCMP* is the Logarithm of a Number greater than 1; in like manner the Abscis *AQ* is a Number less than 1, and the Asymptotical Hyperbolical Space *QNCB* is the Logarithm of a Number less than 1. Again, if $y = 1$, then will $1 + y = 2$; and so the Hyperbolical Logarithm of 2 will be $\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$, *&c.* But these Series converge very slowly; which may be remedied by substituting $\frac{x}{1+x}$ for y.

Note, These Hperbolical Logarithms are the same as *Napier*'s; and so are different from *Briggs*'s, which we commonly used; but they may be reduced to *Briggs*'s: being to his, as the Hyperbolical Logarithm of 10, *viz.* 2.302585092994, &c. to *Briggs*'s Logarithm of 10, that is, 1.000000000000, *&c.*

SCHOLIUM II.

114. If z be an odd Number, whose Logarithm is sought; the Numbers $z - 1$ and $z + 1$ will be even; and so their Logarithms and the Difference of their Logarithms will be given, which let be y. Likewise the Logarithm of a Number being a geometrical

Mean

Mean between the Numbers $z-1$ and $z+1$, *viz.* half the Sum of the Logarithms. The Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}$, *&c.* will be the Logarithm of the Ratio, which the geometrical Mean between the Numbers $z-1$ and $z+1$ has to an arithmetical Mean, *viz.* the Number z.

PROB. VI.

115. *IF a Body freely descends by its own Gravity from the Point* A *along two inclined Plains* AB, AC, *to the Points* B *and* C; *it is required to find the Proportion of the Times of Description.* FIG. 70.

Let ADE be a vertical Line, and BD, CE horizontal ones. Call AD, a, AE, b, DB, x, and EC, z. Now it is evident, that the Time of a Body's describing an infinitely small Part of any Line, as AB or AC may be taken for the Fluxion of the Time of its describing the whole Lines AB and AC. This being premised, $AB=\sqrt{aa+xx}$, and $AC=\sqrt{bb+zz}$. The Fluxion of the former will be $\frac{x\dot{x}}{\sqrt{aa+xx}}$, and that of the latter $\frac{z\dot{z}}{\sqrt{bb+zz}}$

Again, the Velocity of the Body describing the infinitely small Part of AB expressed by $\frac{x\dot{x}}{\sqrt{aa+xx}}=Bb$, and that of AC expressed by $\frac{z\dot{z}}{\sqrt{bb+zz}}=Cc$, will be equal to the respective

Velocities that the Body falling perpendicularly from A will have in the Points D and E; the Velocities of the Description of the very small Parts Bb, Cc being looked upon as equable: which are proportional to $\sqrt{AD}$ $(\sqrt{a})$, and $\sqrt{AE}$ $(\sqrt{b})$. Therefore the Times of describing Bb and Cc are as $\frac{\dot{x}x}{\sqrt{a^3+ax}}$ and $\frac{z\dot{z}}{\sqrt{b^2+bz^2}}$ (the Times of the Descriptions of any Spaces being as the Spaces directly, and the Velocities inversly.) Consequently the Time of describing AB to the Time of describing AC, will be as the Fluent of the former Fluxion to that of the latter. But these Fluents are easily had from the fifth Form of Mr. *Cotes*'s Tables, by making $\theta=1$, $n=2$, &c. that of the former Fluxion being $\sqrt{\frac{aa+xx}{a}}$, and that of the other $\sqrt{\frac{bb+zz}{b}}$. So that the Time of the Description of AB to that of the Description of AC will be as $\frac{AB}{\sqrt{AD}}\left(\sqrt{\frac{aa+xx}{a}}\right)$ to $\frac{AC}{\sqrt{CE}}\left(\sqrt{\frac{bb+zz}{b}}\right)$.

PROB. VII.

116. *To find the Nature of the Curve* BC *being such, that a Body freely falling*
FIG. 71. *by its own Gravity perpendicularly from the Point* A *to* B, *and thence continuing to move on along the said Curve, shall descend equal Spaces in equal Times.*

Let BD be the Axis; then let $AB=a$, the Abscis $BP=x$, and the Ordinate $PM=y$.

Now

Now the Fluxion of the Ordinate $\dot{x}$ is $=0$ in the Point B the Beginning of the Curve, because it is convex next to the Axis, and the Axis is a Tangent to it in B; therefore Mm ($\sqrt{\dot{x}^2+\dot{y}^2}$) is $=\dot{x}$ at the Point B; therefore the Time of the Bodies describing the Arch Bb is as $\frac{\dot{x}}{\sqrt{a}}$; since the Times are as the Spaces directly, and Velocities inversly: and the Velocities acquir'd in B and M being those acquir'd by the Fall from A to the Points B and M, (which Velocities are in the subduplicate Ratio of AB to AP), and the Time of the Description of the small Arch Mm is as $\sqrt{\frac{\dot{x}^2+\dot{y}^2}{a+x}}$. But since from the Condition of the Problem, during the Description of the Curve, the Body falls equal Spaces in equal Times; therefore $\frac{\dot{x}}{\sqrt{a}}$ must be $=\sqrt{\frac{\dot{x}^2+\dot{y}^2}{a+x}}$; and squaring both Sides, there arises $\frac{\dot{x}^2+\dot{y}^2}{a+x}=\frac{\dot{x}^2}{a}$. Whence $a\dot{x}^2+a\dot{y}^2=a\dot{x}^2+x\dot{x}^2$; that is, $a\dot{y}^2=x\dot{x}^2$; and extracting the Root of both Sides, we get $\sqrt{a}\times\dot{y}=\sqrt{x}\times\dot{x}$. Lastly, finding the Fluents, and $ay^2=\frac{4}{9}x^3$, or $\frac{9}{4}ay^2=x^3$. Which is the Equation of the Curve; and so it is a Semicubical Parabola.

Otherwise:

Let $AP=x$, $PM=y$, the Velocity at the End of the Fall from A to $M=v$, and $z=$ Time of the Fall to M. Now from the Principles of Mechanicks $\frac{Mm}{v}$ is as $\dot{z}$, the Time of Description of the small Arch Mm, that is, $\frac{\sqrt{\dot{x}^2+\dot{y}^2}}{v}$ is as $\dot{z}$. Whence if a be a proper standing Quantity, we shall have $a\sqrt{\dot{x}^2+\dot{y}^2}=\dot{v}\dot{z}$. But v is as $\sqrt{x}$, or as $\sqrt{ax}$; and so we may take $v=\sqrt{ax}$. Also z is as $x-a$; since from the Condition of the Problem, the Time is as the Altitudes from whence the Body falls. Whence $x-a$ may be taken for z, and $\dot{x}$ for $\dot{z}$; so that substituting $\dot{x}$ for $\dot{z}$, and $\sqrt{ax}$ for v in the Equation $a\sqrt{\dot{x}^2+\dot{y}^2}=v\dot{z}$, and there comes out $a\sqrt{\dot{x}^2+\dot{y}^2}=\dot{x}\sqrt{ax}$. Whence $a^2\dot{x}^2+a^2\dot{y}^2=ax\dot{x}^2$, and $a^2\dot{y}=\dot{x}\sqrt{x-a}$. Then finding the Fluent of each Side, (which may be easily done from the little Table of Curves that may be squared *page* 34, or from the third Table of Forms of Mr. *Cotes*) and a^2y will be $=\frac{2ax-2a^2}{3}\sqrt{ax-a^2}$, or $ay=\frac{2x-2a}{3}\sqrt{ax-aa}$. Now making $n=x-a$, and we have $\frac{2n}{3}\sqrt{an}$, or $a^2y^2=\frac{4an^3}{9}$, or $\frac{9}{4}ay^2=n^3$: therefore the Curve sought is a second cubical Parabola, AB being $=a$, and $BP=n$.

PROB. VIII.

FIG. 72. 117. *To find the Law of Refraction, admitting this Principle, viz. that Nature in all its Operations takes the shortest ways.*

Because Light cannot move in different Mediums with the same Velocity; let the Ratio of the Velocity of the Light during its Motion from A to B, where it begins to be refracted,

fracted, to its Velocity, while it is refracted in its Motion from *B* to *C*, be expressed by $\frac{m}{n}$; then the Times of the Descriptions of the Lines *AB*, *BC* will be as $n \times AB$ to $n \times BC$. Let fall the Perpendiculars *AQ*, *CP*, and make $AQ = a$, $CP = b$, $PQ = c$, $PB = x$; then will $BQ = c - x$, and consequently $BC = \sqrt{bb + xx}$, and $AB = \sqrt{aa + cc - 2cx + xx}$. Whence the Time in which $AB + BC$ is moved thro', is $= m\sqrt{bb + xx} + n\sqrt{aa + cc - 2cx + xx}$, which must be a *Minimum*; and so the Fluxion thereof, *viz.* $\frac{mx\dot{x}}{\sqrt{bb + xx}} + \frac{nx\dot{x}}{\sqrt{aa + cc - 2cx + xx}} = 0$. Whence $\frac{mx}{\sqrt{bb + xx}} = \frac{n \times \overline{c - x}}{\sqrt{aa + cc - 2cx + xx}}$, that is, $\frac{m \times PB}{BC} = \frac{n \times BQ}{AB}$. Make $BC = AB$, then will $m \times PB = n \times BQ$, and consequently $m : n :: BQ : PB$.

Whence if *BA* or *BC* be taken for the Radius, *BQ* will be the Sine of the Angle *A*, and *PB* the Sine of the Angle *C*; that is, since *AQ* and *PC* are parallel to *DE*, *PB* is the Sine of the Angle *CBE*, and *BQ* the Sine of *ABD*, viz. *PB* is the Sine of the refracted Angle, *BQ* the Sine of the Angle of Incidence. Whence the Sine of the Angle of Incidence is to the Sine of the refracted Angle in a constant Ratio, *viz.* that of the Velocity of Light before Refraction to the Velocity during its Refraction.

PROB.

PROB. IX.

FIG. 73. 118. *To find the Angle* BCD, *in which a Body from* A *obliquely striking a Plain in* C *may be reflected to a given Point* B, *so as to pass from the given Point* A *to the given Point* B *the shortest way.*

From A, B let fall the Perpendiculars AE, BD. Let $AE = a$, $BD = b$, $ED = c$, and $EC = x$; then $CD = c - x$, and $AC = \sqrt{aa + xx}$; also $CB = \sqrt{bb + cc - 2cx + xx}$; and so $AC + CB$ must be *Minimum*, that is, $\sqrt{aa + xx} + \sqrt{bb + cc - 2cx + xx}$. Whence the Fluxion of it must be made equal to o; therefore $\frac{\dot{x}}{\sqrt{aa + xx}} + \frac{x\dot{x} - c\dot{x}}{\sqrt{b^2 + c^2 - 2cx + x^2}} = 0$; and so $x\sqrt{b^2 + c^2 - 2cx + x^2} + \overline{x - c} \times \sqrt{a^2 + x^2} = 0$. Whence $x\sqrt{b^2 + c^2 - 2cx + xx} = \overline{c - x} \times \sqrt{a^2 + x^2}$; that is, $EC \times CB = CD \times AC$; and so $EC : AC :: CD : CB$. Consequently (by *Prop.* 7. *lib.* 6. *Eucl.*) the Triangles AEC, BDC are equiangular: Whence the Angle ACE must be $=$ Angle BCD.

PROB. X.

FIG. 74. 119. *If a thin Fluid consists of equal Particles freely disposed at equal Distances from each other; it is required to find that* Frustum *of a Cone, which of all others of the same Base* A a, *and Altitude* B C, *moving in that Fluid according to the Direction of the Axis, with the lesser Base* D d *foremost, that shall have the least Resistance.*

It

It is the same thing to consider the *Frustum* at rest, and the Particles to move against it with the same Velocity.

Draw DE parallel to BC. Let the given Altitude $BC=b$, the Radius of the given Base $=a$, and $AE=x$. Now it is well known that the Effect of any Particle of the Fluid striking the Surface AD of the *Frustum* obliquely in the Direction DE to move it according to the same Direction, is to the Effect of the same Particle striking directly against the Annulus generated by the Line AE (while AB revolves about) to move it in the Direction DE, as the Square of the Sine of the Angle of Incidence FEV or ADE to the Square of the Radius. And since here the Angle of Incidence is invariable, the Effect of all the Particles striking the Superficies of the *Frustum* generated by AD, will be to the Effect of all the Particles that can strike the *Annulus* aforesaid is in that Proportion, that is, the Resistance of the Superficies of the *Frustum* generated by AD, is to the Resistance of the *Annulus* generated by AE, as the Square of the Sine of the Angle of Incidence is to the Square of the Radius.

Whence if BC (b) be made the Radius, the Sine of the Angle of Incidence will be $\frac{bx}{\sqrt{b^2+x^2}}$, for AD ($\sqrt{b^2+x^2}$) $: AE\,(x) :: BC\,(b) : \frac{bx}{\sqrt{b^2+x^2}}$. Now if the aforesaid *Annulus* be made the Resistance of itself, then will the Circle described by BE be the Resistance of itself also; but since Circles are to each other as the Squares of their Radii, therefore the Resistance of the *Annulus* is to the Resistance of the Circle, as $\overline{AB}^2 - \overline{BE}^2$ $(2ax - x^2)$ is to $\overline{BE}^2$ $(a^2 - 2ax + x^2)$. Which Quantities now let represent the respective Resistances of the *Annulus* and Circle.

Then $\overline{BC}^2$ (b^2) : $\frac{b^2x^2}{b^2+x^2}$ square Sine : : $\overline{AB}^2 - \overline{BE}^2$ $(2ax - x^2)$: $\frac{2ax^3 - x^4}{b^2+x^2}$ = Resistance of the Superficies of the *Frustum* generated by *AD*. To which adding the Resistance of the lesser Base *Dd* (being $a^2 - 2ax + x^2$) and the Resistance of the whole *Frustum* will be

$$\frac{2ax^3 - x^4}{b^2+x^2} + a^2 - 2ax + x^2 = \frac{a^2b^2 - 2ab^2x + b^2x^2 + a^2x^2}{b^2+x^2}.$$

The Fluxion of which must be a *Minimum*; therefore

$$\frac{2ab^4x\dot{x} + 2ab^2x^2\dot{x} - 2ab^4\dot{x}}{\overline{b^2+x^2}|^2} = 0.$$

Whence $x^2 + \frac{b^2}{a}x = b^2$; and so $x = \frac{b}{2a}\sqrt{b^2+x^2} - \frac{b^2}{2a}$. But because the Triangles *AED*, *ABV* are similar, therefore $AE\,(x) = \frac{b}{2a}\sqrt{b^2+x^2} - \frac{b^2}{2a} : BC$ $(b) :: AB\,(a) : BV = \frac{a^2}{\frac{1}{2}\sqrt{4a^2+b^2} - \frac{1}{2}b}$, which is $= \frac{1}{2}\sqrt{4a^2+b^2} + \frac{1}{2}b$; since $\frac{1}{2}\sqrt{4a^2+b^2} - \frac{1}{2}b \times \frac{1}{2}\sqrt{4a^2+b^2} + \frac{1}{2}b$ is $= a^2$, as appears by bare Inspection: the Sum of any two Quantities drawn into their Difference being equal to the Difference of their Squares.

From hence arises the following Construction. Bisect *BC* (*b*) in *G*, and draw *AG*: in *BC*

continu-

continued out, make $GV = AG\ (\frac{1}{2}\sqrt{4a^2+b^2})$ and V will be the Vertex of the Cone.

PROB. XI.

120. *TO find the Duration of a Pendulum oscillating in the Curve of the Cycloid.*

Let the Diameter of the generating Circle, or the Altitude of the whole Cycloid be $=a$; and let HB, the Altitude of the Point Q from whence the Pendulum begins to fall, and describe the Arch QB, be $=b$. Also let $HP = z$; and so $PB = 2b - z$. Now let the Time of the Pendulum's describing QB be $=x$, and on HB describe the Semicircle HNB, and draw PM, pm infinitely near one another, and perpendicular to HB; then will PN be $= \sqrt{2bz - zz}$, $Pp = Nq = Rm = \dot{z}$, and the Velocity in P, and so in N and $Q = \sqrt{z}$. FIG. 75.

Consequently since the Particle of the Curve Mm is described by an uniform Motion, the Time of the Description of the same, *viz.* $\dot{x}$ is $= \frac{Mm}{\sqrt{z}}$. But from the Nature of the Cycloid $Mm : mR :: BS : BP$, and $AB : BS :: BS : BP$, from the Nature of the Circle. Whence $BS : BP :: \sqrt{AB} : \sqrt{PB}$, and so $Mm : mR :: \sqrt{AB} : \sqrt{PB}$; therefore $Mm = \frac{mR \times \sqrt{AB}}{\sqrt{PB}}$, and $\dot{x} = \frac{\dot{z}\sqrt{a}}{\sqrt{2bz - zz}} = \frac{2b\dot{z}\sqrt{a}}{2b\sqrt{2bz - zz}}$. But $\frac{b\dot{z}}{\sqrt{2bz - zz}} = Nn$; therefore $\dot{x} = \frac{2\sqrt{a} \times Nn}{2b}$. Now when the Fluent of $\dot{x}$, viz. x does express the Time of the Descent *BD* of the whole Arch of the Cycloid, the Fluent of *Nn* will be = the Periphery *HNB*; therefore as $2b$ the Diameter of the Circle to $\frac{1}{2}$ the Circumference thereof, so is $2\sqrt{a}$ to the Time of the Pendulums describing the Arch *BQ*. Consequently because $2\sqrt{a}=\frac{2a}{\sqrt{a}}$ denotes the Time of the perpendicular Descent thro' *AB*, we have the following Theorem, *viz.* The Time of an whole Oscillation thro' any Arch of the Cycloid, is to the Time of the perpendicular Descent thro' the Diameter *AB* of the generating Circle, as the Periphery of a Circle to the Diameter.

COROLL.

HENCE the Times of describing all Arches of a Cycloid are equal.

PROB. XII.

121. *THE Course and Difference of Latitude of two Places being given: to find the Difference of Longitude.*

FIG. 76. Let *P* be the Pole; the Circle *ABF*, the Equator; *PCB*, *PDA* Meridians; *ACG* the Rhumb Line passing thro' two given Places *A* and *C*. Draw *Pdcb* infinitely near *PCB*, and with the Distance *PC* describe the Arch *CD*. Now make the Radius $PA=a$; the Difference of Latitude of the two Places, *viz.* *AD* or $BC=y$; the Difference of Longitude sought $AB=x$; the Tangent of the given Course (or constant Angle that the Rhumb Line makes with

with any Meridian) *viz.* of the Angle $Ccd = m$; also make the Sine of the Latitude of the Place $C = r$, and the Cosine $= z$. All these are variable Quantities except AP (a), and the Tangent of the given Course $= m$.

Now since the Arches Bb, Cd are similar; therefore PB (a) : Cosine of Lat. of C (z) : : $Bb(\dot{x}) : Cd = \frac{z\dot{x}}{a}$. Again, from the Nature of the Circle, $\overline{PB}^2$ — Square of Sine Lat. of C, viz. $a^2 - r^2$ is $= z^2$, and $dc\,(\dot{y})$ is $= \sqrt{\dot{r}^2 + \dot{z}^2}$, as easily appears. Whence throwing the Equation $a^2 - r^2 = z^2$ into Fluxions, and we get $\dot{z}^2 = \frac{r^2\dot{r}^2}{a^2 - r^2}$. And so putting $\frac{r^2\dot{r}^2}{a^2 - r^2}$ for $\dot{z}^2$ in the Equation $\dot{y} = \sqrt{\dot{r}^2 + \dot{z}^2}$, and there arises $\dot{y} = \frac{a\dot{r}}{z}$. If cd $(\dot{y})$ be made the Radius, then will Cd be the Tangent of the Angle Ccd of the Course $= m$; therefore $PB(a)$ $: m : : cd\,(\dot{y}) : Cd = \frac{z\dot{x}}{a}$; and drawing the Means and Extremes into each other, there arises $m\dot{y} = z\dot{x}$; and so $\dot{x} = \frac{m\dot{y}}{z}$. Whence substituting $\frac{a\dot{r}}{z}$ for $\dot{y}$, and we have $\dot{x} = \frac{ma\dot{r}}{z^2} = \frac{ma\dot{r}}{a^2 - r^2}$; and finding the Fluents, there arises $BA\,(x) = m \times \frac{r}{a} + \frac{r^3}{3a^3} + \frac{r^5}{5a^5} + \frac{r^7}{7a^7}$, &c. = Difference of Longitude of the Places A and C.

The Fluent of $\frac{am\dot{r}}{a^2 - r^2}$ may be had likewise after Mr. *Cotes*'s way in the Measure of a Ra-

 tio;

tio; for it may be referr'd to the second Form of his Tables; whence making $\theta = 0$, $n = 2$, $d = am$, $e = a^2$, $f = -1$, $R\ (= \sqrt{\frac{-e}{f}}) = a$, $T\ (x^{\frac{1}{2}n}) = r$, and $S\ (= \sqrt{\frac{e+fz^n}{f}}) = \sqrt{aa-rr}$, the Fluxion of the Form $\frac{d\dot{z}z^{\theta n + \frac{1}{2}n - 1}}{e+fz^n}$ will become $\frac{am\dot{r}}{a^2-r^2}$, and the Fluent $\frac{2}{ne}dR\left|\frac{R+T}{S}\right.$ will be $m\left|\frac{a+r}{\sqrt{a^2-r^2}}\right. = m\left|\frac{a+r}{z}\right.$ being the Expression for the Difference of Longitude BA.

That is, in Words, the Difference of Longitude is equal to the Measure of the Ratio of the Radius added to the Sine of the Latitude of the Place C, and the Sine Complement of the same, the Tangent of the Course being the Module; and if E be some other Place in the same Rhumb Line, the Sine of whose Latitude is given; then by the same Rule we can get the Difference of Longitude AH, and so the Difference of Longitude BH of the Places C and E.

COROL.

If the Rhumb Line AC be $= u$, and cd (y) be made the Radius; then will Cc be the Secant of the Course. Whence $PB\ (a)$: $Cc :: cd\ (\dot{y}) : Cc\ (\dot{u})$; and so $a\dot{u} = Cc \times \dot{y}$, and taking the Fluents $au = Cc \times y$, therefore $u = \frac{Cc \times y}{a}$; that is, as the Radius is to the Secant of the Course, so is the Difference of Latitude AD to the Length $AC\ (y)$.

PROB.

PROB. XIII.

122. *TO cube the Solids generated by the Rotation of the Conchoidal Spaces* CPGB, *and* BGQc *about the Line* ABC *drawn from the Pole* A *at right Angles to the Asymptote* BG. FIG. 77.

Draw Ap infinitely near AP; from A describe the small Arches Qs, Gr, Pn, with the Distance BC or Bc, from A describe the Arch EF, and from F, Q, P draw FH, QK, PI perpendicular to AE. Call cB ($=AE=AF=BC=QG=GB$) a, AB, b, EH, x, AH, z, and HF, y. Now from the Similarity of the Triangles ABG, AHF, we have $AH(z):AF(a)::AB(b):AG=\frac{ab}{z}$. Whence $AQ=\frac{ab}{z}-a$, and for the same Reason $AP=\frac{ab}{z}+a$. Again, $AF(a):FH(y)::AQ\left(\frac{ab}{z}-a\right):QK=\frac{by}{z}-y$. So likewise $PI=\frac{by}{z}+y$. Now the Fluxion of the Arch EF, viz. Ff will be $\frac{a\dot{z}}{y}$; and since the Sectors AFf, AQs are similar, therefore $AF(a):Ff\left(\frac{a\dot{z}}{y}\right)::AQ\left(\frac{ab}{z}-a\right):Qs=\frac{ab\dot{z}}{zy}-a\dot{z}$. In like manner $Pn=\frac{ab\dot{z}}{zy}+a\dot{z}$.

Again, the Fluxion of the Solid generated by the Space AcQ during the Motion abovemention'd, will be $=\frac{1}{3}AQ\times Qs$ drawn into the

the Periphery described by the Point Q. So also will the Fluxion of the Solid generated by the Space ACP be $= \frac{1}{3} AP \times Pn$ into the Periphery described by the Point P, and the Fluxion of the Cone generated by the right-angled Triangle ABG during the aforesaid Motion will be $\frac{1}{3} AG \times Gr$ into the Periphery described by the Point G; therefore the Fluxion of the Solid generated from AcQ will be $\overline{\frac{ab}{3z} - \frac{a}{3}} \times \frac{p}{r} \times \overline{\frac{by}{z} - y} \times \frac{ab\dot{z}}{zy} - a\dot{z}$ ($\frac{p}{r}$ being the Ratio of the Radius to the Periphery of a Circle) $= \frac{p}{r} \times \overline{\frac{a^2b^3\dot{z}}{3z^3} - \frac{a^2b^2\dot{z}}{z^2} + \frac{a^2b\dot{z}}{z} - \frac{a^2\dot{z}}{3}}$; and the Fluxion of the Solid generated from ACP will be $\overline{\frac{ab}{3z} + \frac{a}{3}} \times \frac{p}{r} \times \overline{\frac{by}{z} + y} \times \overline{\frac{by}{z} + y} = \frac{p}{r} \times \frac{a^2b^3\dot{z}}{3z}$ $+ \frac{a^2b^2\dot{z}}{z^2} + \frac{a^2b\dot{z}}{z} + \frac{a^2\dot{z}}{3}$; and the Difference of these two last mentioned Fluxions will be $= \frac{p}{r} \times \overline{\frac{2a^2b^2\dot{z}}{z^2} + \frac{2}{3}a^2\dot{z}} =$ Fluxion of the Sum of the two Solids generated by the Spaces $cQGB$, $BGPC$: and so $\frac{p}{r} \times \frac{a^2b^2\dot{z}}{z^2} + \frac{1}{3}a^2\dot{z}$ is the Fluxion of $\frac{1}{2}$ the Sum; the Fluent of which will be $\frac{p}{r} \times \overline{\frac{a^2b^2}{z} - \frac{1}{3}a^2 z}$, $=$ (substituting $a - x$ for z)

$$\frac{p}{r} \times \overline{\frac{a^2b^2}{a-x} - \frac{a^3 - a^2x}{3}} = \frac{p}{r} \times \frac{\overline{a^2b^2 - \frac{a^4}{3} + \frac{2a^3x}{3} - \frac{a^2x^2}{3}}}{\times \overline{a-x}}.$$

and making $x = 0$, this Fluent becomes $\frac{p}{r} \times \overline{ab^2 - a^3}$ to be substracted: so that the true

Fluent

Fluent is $\frac{p}{r} \times \frac{\overline{a^2b^2 - \frac{a^4}{3} + \frac{2a^3x}{3} - \frac{a^2x^2}{3}}}{\overline{a-x}} - ab^2 + a^3$

$= \frac{p}{r} \times \frac{\overline{ab^2x + \frac{a^3x}{3} - \frac{a^2x^2}{3}}}{\overline{a-x}}$.

Much after the same way as the Fluxion of $\frac{1}{2}$ the Sum of the Solids was found, we may get the Fluxion of $\frac{1}{2}$ the Difference: for from the similar Triangles AHF, ABG, there arises $AH(z):AF(a)::AB(b):AG=\frac{ab}{z}$, and $AH(z):HF(y)::AB(b):BG=\frac{by}{z}$; and since the Sectors AFf, APr are also similar, therefore $AF(a):Ff\left(\frac{a\dot{z}}{y}\right)::AG\left(\frac{ab}{z}\right):Gr=\frac{ab\dot{z}}{yz}$; and consequently the Fluxion of the Cone described by the right-angled Triangle ABG will be $=\frac{p}{r}\times\frac{ab}{3z}\times\frac{by}{z}\times\frac{ab\dot{z}}{yz}=\frac{p}{r}\times\frac{a^2b^3\dot{z}}{3z^3}$, from which taking the Fluxion (before found) of the Solid described by the Space AcQ, and the Remainder $\frac{p}{r}\times\frac{a^2b^2\dot{z}}{z^2}-\frac{a^2b\dot{z}}{z}+\frac{a^2\dot{z}}{3}$ is the Fluxion of the Solid generated by the Space $cQGB$. And if $\frac{p}{r}\times\frac{a^2b^3\dot{z}}{3z^3}$ be taken from the Fluxion of the Solid described by the Space ACP, the Remainder $\frac{p}{r}\times\overline{\frac{a^2b^2\dot{z}}{z^2}+\frac{a^2b\dot{z}}{z}+\frac{a^2\dot{z}}{3}}$ is the Fluxion of the other Solid $BGPC$.

from

From which substracting the Fluxion of the Solid $eQGB$, and the Remainder $\frac{p}{r} \times \frac{2a^2b\dot{z}}{z}$ is the Difference of the said Solids; so that $\frac{p}{r} \times \frac{a^2b\dot{z}}{z}$ is the Fluxion of $\frac{1}{2}$ the Difference of the Solids. Which may be compar'd with that of the first Form in the Tables of Mr. *Cotes*, (having first substituted $a-x$ for z, and $-\dot{x}$ for $\dot{z}$, the Fluxion becoming $\frac{p}{r} \times \frac{1a^2b\dot{x}}{a-x}$). For making $z=x$, $\theta=1$, $n=1$, $d=a^2b$, $e=a$, $f=-1$, the Fluxion $\frac{d\dot{z}z^{\theta n-1}}{e+fz^n}$ will be $= \frac{p}{r} \times \frac{a^2b\dot{x}}{a-x}$, and the Fluent $\frac{d}{nf}\left|\frac{e+fz^n}{e}\right.$ becomes $= -\frac{p}{r}aab\left|\frac{a-x}{a}\right. = \frac{p}{r}aab\left|\frac{a}{a-x}\right.$; since the Logarithm of the Ratio of a to $a-x$ with an affirmative Sign, is equal to the Logarithm of the Ratio of $a-x$ to a with a negative Sign.

The Fluents or Quantities of $\frac{1}{2}$ the Sum and $\frac{1}{2}$ the Difference of the Solids being thus found we proceed next to their Construction, beginning with that of the half Sum. From what has been already said, it is easy to find the Fluxion of the Sector of the Sphere described by the circular Sector AEF, and so the Fluent or Quantity of that Sector; the Fluxion being $= \frac{p}{r} \times \frac{a}{3} \times y \times \frac{a\dot{z}}{y} = \frac{p}{r} \times \frac{a^2\dot{z}}{3}$, and the Fluent $= \frac{p}{r} \times \frac{a^2z}{3} = \frac{p}{r} \times \frac{a^3-a^2x}{3}$, by substituting $a-x$ for z; and making $x=0$, the said Fluent will become $\frac{p}{r} \times \frac{a^3}{3}$, to be substracted from

the

the other. Whence the true Fluent is $\frac{p}{r} \times \frac{-a^2x}{3}$ or $\frac{p}{r} \times \frac{a^2x}{3}$ = Solidity of the Sector of the Sphere described as aforesaid.

This being granted, make $\overline{AE}^2 (aa) : 3 AB \times AG + \overline{AE}^2 (3b \times \frac{ab}{a-x} + aa) ::$ Sector Sphe. $\frac{p}{r} \times \frac{a^2x}{3}$: Solidity of $\frac{1}{2}$ the Sum of the Solids $= \frac{p}{r} \times \overline{ab^2x + \frac{a^3x}{3} - \frac{a^2x^2}{3}}$ being the same Expression as that before found.

Lastly, To construct the Expression $\frac{p}{r} aab \,|\, \frac{a}{a-x}$, we know that $\frac{p}{2r} aa$ is the Area of a Circle, having $BC (a)$ for a Radius; which drawn into $b \,|\, \frac{a}{a-x}$ will be the Solidity of $\frac{1}{2}$ the Difference. But because of the similar Triangles AHF, ABG, the Ratio of $a = AF$ to $a-x = AH$, is equal to the Ratio of AG to AB; therefore the Value of $\frac{1}{2}$ the Difference of the Solids to be cubed is equal to a Cylinder, the Diameter of whose Base is cC $(2a)$; and Altitude the Measure of the duplicate Ratio of AG to AB, the Module being $AB (b)$. for $\frac{p}{2r} aab \,|\, \frac{a^2}{\overline{a-x}^2}$ is $= \frac{p}{r} aab \,|\, \frac{a}{a-x}$; because the Logarithm of any Ratio doubled is that of the duplicate Ratio.

PROB. XIV.

FIG. 78. 123. *TO find the Nature of a Curve* AMC *being such, that if a Vessel be described by the Revolution of the same about the Axis* AB *perpendicular to the Horizon; and then filled with Water, which afterwards runs out of a small round Hole in the Bottom* A, *the Surface of the Water shall descend equal Spaces in equal Times; admitting the Velocity of the Water running out, to be as the square Root of the Altitude of its Surface above the Hole.*

Let $AB=a$, $AP=x$ be any indeterminate Altitude of the Water, and let $PM=y$. Also let the Surface of the Hole be $=b$, the Velocity $=v$, and the Time of Descent of the Surface $=t$. Which from the Condition of the Problem is as $a-x$, but $\frac{\dot{x}}{\dot{t}}$ is as v; that is, as $\sqrt{x}$ or as $\sqrt{ax}$, from the Condition of the Problem; therefore $\frac{\dot{x}}{\dot{t}} : \sqrt{ax} ::$ Surf. Hole b: Surf. Water y^2. But since t may be taken $= a-x$, therefore $\dot{t}=\dot{x}$; and so $1 : \sqrt{ax} :: b^2 : y^2$. Whence $b\sqrt{ax}=y^2$, and $ab^2x=y^4$. Consequently the Curve ACM is a biquadratick Parabola.

PROB. XV.

FIG. 79. 124. *IF* AC *be a horizontal Line upon the Point* C *of which stands an upright Parallelepipedon* CD; *one of the plain Surfaces of which is perpendicular to* AC; *it is required to find the Angle* CAB, *in the Point* A *of which one End* A *of a long Solid*

Solid AB *being set, so as with its other End* B *it may bear against the Parallelepipedon* CD; *the same shall press perpendicularly against* CD *with a greater Force than if it was set at any other Point besides* A.

Let F be the Centre of Gravity of AB. From F draw FE perpendicular to AC, and from E, EI perpendicular to AB. Also from B, BH perpendicular to AB, and $= AE$; and from H, HG perpendicular to BD. Make $AB = a$, $AF = b$, and $CB = x$.

Now the Pressure of the Solid AB in the Point B, and Direction HB against CD will be as AE, and so may be represented by it; therefore since $BH = AE$, the direct Pressure against CD in the Point B will be $= HG$. This is shewn in the Principles of Mechanicks. Whence $CB \times HG$ is the Effect of the perpendicular Pressure of the Solid AB against CD in the Point B, which must be a *Maximum*. Again, $AB\ (a) : AC\ (\sqrt{a^2 - x^2}) :: AF\ (b) : AE = \frac{b}{a}\sqrt{a^2 - x^2}$. And since the Triangles AEI, ACB are similar, therefore $AB\ (a) : BC\ (x) :: AE \left(\frac{b}{a}\sqrt{a^2 - x^2}\right) : EI = HG = \frac{bx}{a^2}\sqrt{a^2 - x^2}$. Whence $CB \times HG = \frac{bx^2}{a^2}\sqrt{a^2 - x^2}$. The Fluxion $\frac{4a^2b^2x^3\dot{x} - 6b^2x^5\dot{x}}{2a^4 \times \frac{bx^2}{a^2}\sqrt{a^2 - x^2}}$ of which must be $= 0$; therefore $2a^2 = 3x^2$, and so $x = \sqrt{\frac{2aa}{3}}$. Whence as AB to $\sqrt{\frac{2}{3}}\overline{AB}$, so is the Radius to the Sine of the Angle $CAB = 54^\circ.44'$.

 PROB.

PROB. XVI.

125. *TO find the Nature of a Curve* ACEDB, *along which if a Body freely falls by its own Weight from the given Point* A *to the given Point* B; *the Time of the Descent shall be less than the Time of the Descent of a Body from* A *to* B *along any other Curve passing thro' the given Points* A *and* B.

FIG. 80.

Let C and D be two given Points in the Curve infinitely near each other. Also let the intermediate Point E of the Curve be taken such, that drawing DI and EH perpendicular, and AK, CL, EM parallel to the Horizon, the small Line EL may be $= DM$.

Now since $ACEDB$ is supposed to be the Curve along which the Body falls from A to B in the least Time possible; it will fall from C to D, any Part of the said Curve, in the least Time possible. For if it does not, suppose it to fall along CGD in a less Time than along CED; then the Curve $ACGDB$ will be described in a less Time than the Curve $ACEDB$. Which is absurd.

Because the Points C and D are given in Position, therefore the Lines HE, ID, $EL = DM$, are all given or invariable; and CL, CE; EM, ED are variable. Make $EL = DM = m$, $HE = b$, $ID = p$; also $CL = u$, $EM = z$; then the Time of the Description of the small Line CE (being as CE directly, and the Velocity inversly) will be as $\frac{CE}{\sqrt{HE}}$ $\left(\sqrt{\frac{m^2 + n^2}{b}}\right)$, the Velocity during the Descri-

ption

ption of infinitely ſmall right Lines being looked upon as equable; ſo likewiſe the Time of the Body's deſcribing the ſhort Line ED is as $\frac{ED}{\sqrt{ID}}\left(\sqrt{\frac{m^2+z^2}{p}}\right)$. Conſequently the Sum of theſe Times muſt be a *Minimum*, viz. $\sqrt{\frac{m^2+u^2}{b}}+\sqrt{\frac{m^2+z^2}{p}}$; and ſo the Fluxion of the ſame, which is $\frac{u\dot{u}}{b^{\frac{1}{2}}\sqrt{m^2+u^2}}+\frac{z\dot{z}}{p^{\frac{1}{2}}\sqrt{m^2+z^2}}$ muſt be $=0$, but $CN=u+z$ is invariable; whence $\dot{u}+\dot{z}=0$, and ſo $\dot{u}=-\dot{z}$; therefore $\frac{u}{b^{\frac{1}{2}}\sqrt{m^2+u^2}}=\frac{z}{p^{\frac{1}{2}}\sqrt{m^2+z^2}}$.

Now if AH be $=x$, and $HE=y$; then will $EM=\dot{x}$, and $MD=\dot{y}$; and ſo $ED=\sqrt{\dot{x}^2+\dot{y}^2}$. Conſequently the Fluxion of the Curve is always as $\frac{\dot{x}}{y^{\frac{1}{2}}}$; that is, in the direct Ratio of the Fluxion of the Abſciſs $AH\,(x)$, and the reciprocal ſubduplicate Ratio of the correſpondent Ordinate $HE\,(y)$.

Now a Curve that has this Property will be found to be a Cycloid, paſſing through the given Points A and B, with the Vertex downwards, as may be eaſily ſhewn thus: Suppoſe A and B to be ſo poſited, that ACD be a Semi-cycloid, &c.

Deſcribe the generating Semicircle KPB. Continue out EM to P and Q. Draw the Tangent CET to the Curve in E, and from B draw the Chord BP. Let $BK=a$; then will KQ be $=y$, and $BQ=a-y$. Now it is a noted Property of the Cycloid for the Tangent ET to be parallel to the Chord BP of of the correspondent Arch of the generating Circle. Whence the Triangles BPQ, and ECL are similar; therefore PQ $(\sqrt{ay-y^2})$: BP $(\sqrt{a^2-y^2})$ :: CL $(\dot{x})$: $CE = \frac{\dot{x}\sqrt{a^2-ay}}{\sqrt{ay-y^2}} = \frac{\dot{x}\times\sqrt{a}\times\sqrt{a-y}}{\sqrt{y}\times\sqrt{a-y}} = \frac{\dot{x}\times\sqrt{a}}{\sqrt{y}}$. Which Expression is as $\frac{\dot{x}}{y^{\frac{1}{2}}}$, because $\sqrt{a}$ is given or invariable; therefore the Cycloid $AGEDB$ is the Curve of the swiftest Descent.

PROB. XVII.

FIG. 82. 126. *To find the Nature of a Curve* DM *being such, that the Solid described by the Revolution of it about the Axis* AP, *moving in a Fluid (such as that in* Art. 119.) *in the Direction of the Axis, shall be less resisted than any other Superficies described by what Curve soever terminating in the given Points* D *and* M *about the same Axis* AP, *and moving in the same manner.*

FIG. 81. Let the right Lines MN, NO be supposed to be two infinitely small Parts of the Curve sought. Now the Surfaces described by these will meet with a less Resistance than the two Superficies described by any two Parts drawn from the Points O and M to any Point besides N. This is very plain: for if the same be denied, the Consequence will be that N is not in the Curve sought.

This being granted, from the Points M, N, O, draw MP, NQ, OH perpendicular to the Axis; and from M draw MF parallel to the same; also thro' N draw another indefinite Parallel

Parallel GNT cutting OH in G. Now the Resistance of the little Superficies described by MN, is to the Resistance of the little *Annulus* described by FN, as the Square of the Sine of the Angle of Incidence to the Square of the Radius; that is, (because the Angle FMN is $=$ Angle of Incidence TNM) making MN the Radius, as $\overline{FN}^2$ to $\overline{MN}^2$. And if the Resistance of the *Annulus*, described by FN, be represented by itself; or rather by $\overline{QF}^2 + 2QF \times FN + \overline{FN}^2$ *minus* $\overline{QF}^2$ (being the Difference of $\overline{QN}^2$ and $\overline{QF}^2$) which Remainder is $2QF \times FN + \overline{FN}^2$; this Difference being always as that *Annulus*. But because FN is infinitely less than QF, therefore $\overline{FN}^2$ is infinitely less than $2QF \times FN$; and so it may be rejected: And the Resistance of the *Annulus* aforesaid will be $2QF \times FN$; or $QF \times FN$; or $PM \times FN$. Whence at length making $\overline{MN}^2 : \overline{NF}^2 :: PM \times FN : \frac{NF^3 \times PM}{\overline{MN}^2}$. And this will express the Resistance of the Superficies described by MN; so will $\frac{GO^3 \times QN}{\overline{ON}^2}$ be that described by ON.

Again, let the Points O, M, and the right Line GT be given in Position; then we are to enquire into the Situation of MN, NO, being such that the Resistance of the Superficies described by them, be less than the Superficies described by any other Lines On, nM. In order to this, let $PM = a$, $FN = b$, $GO = c$, $NQ = e$. These are all given, and invariable; and

and let the variable Quantities MN be $= x$, and $NO = z$.

Then will the Refiftance of the Superficies defcribed by MN be $= \frac{ab^3}{x^2}$, as has been fhewn above. In like manner $\frac{ec^3}{z^2}$ will be the Refiftance of the Superficies defcribed by ON. Whence $\frac{ab^3}{x^2} + \frac{ec^3}{z^2}$ muft be a *Minimum*; that is, $\frac{ab^3x\dot{x}}{x^4} + \frac{ec^3z\dot{z}}{z^4}$, or $\frac{ab^3\dot{x}}{x^3} + \frac{ec^3\dot{z}}{z^3} = 0$. Whence $-\frac{ab^3\dot{x}}{x^3} = \frac{ec^3\dot{z}}{z^3}$.

Now in order to get an affirmative Value of $-\dot{x}$, affected with $\dot{z}$; affume the Point n infinitely near N, and draw the right Lines On, Mn, to which draw the Perpendiculars NS, NR. Then fince Nn, and confequently NS is infinitely lefs than NM; therefore $MN = MR$, and fo the Angle MNR = Angle MRN = right Angle. Whence the Angle RNn is = Angle FNM; for either of them added to the Angle MNn makes a right Angle. Likewife fince the Triangles NSn, nGO are equiangular, they having each a right Angle OGn, NSn, and the Angle ONG common; therefore the Angle SNn is = Angle SOG = Angle NOG; fince the Angle NOS is infinitely fmall. Therefore making Nn the Radius, we have $Rn\ (-\dot{x}) : Sn\ (\dot{z}) :: $ Sine Ang. FNM: Sine Ang. GON. Suppofing MN the Radius, (that is, making $MF = m$, and $NG = n$, and affuming $NL = MN$, and drawing LK parallel to OG) as $MF(m) : NG(n)$; therefore $-\dot{x} = \frac{mz\dot{z}}{nx}$; which being put in the Equati-

on

on above, and we get $\frac{ab^3mz\dot{z}}{nx^4}=\frac{ec^3\dot{z}}{z^2}$. Whence $\frac{ab^3m}{x^4}=\frac{ec^3n}{z^4}$.

Now draw *AB* (*a*) perpendicular to the Axis *AP*, and the right Lines *BC*, *BE* parallel to the two infinitely small Subtenses *MN*, *NO*; then will $4\overline{AB}^2\times AC$ $(4a^2\times\frac{am}{b})$: $\overline{BC}^3$ $(\frac{a^3x^3}{b^3})$: : *BC* $(\frac{ax}{b})$: *MP* (*a*); and multiplying the Means and Extremes, we have $\frac{b^3m}{x^4}=\frac{1}{4}$, or $\frac{ab^3m}{x^4}=\frac{a}{4}$. In like manner, $4\overline{AB}^2\times AE:\overline{BE}^3:BE:NQ=\frac{ec^3n}{z^4}$.

Therefore the Nature of the Curve *DM* is such, that if *AB* be taken in the Line *AK* perpendicular to the Axis $=a$; and drawing *BC* parallel to a Tangent to the Curve in any Point *M*, we have always $4\overline{AB}^2\times AC:\overline{BC}^3$ $:BC:MP$ the Ordinate.

Now this Curve may be described by the Logarithmick Curve thus: Assume $AB=a$ in *AK*; and in *AP* continued out towards *A*, take $AE=\sqrt{\frac{1}{3}aa}$; and thro' the Point *E* describe the Logarithmick Curve *FEN* to the Asymptote *AK*, whose Subtangent is $=\frac{1}{4}a$. Then assuming *AC* (suppose $=s$) at pleasure, and drawing *CN* parallel to *AK*, take $AK=\frac{aa}{4s}+\frac{1}{2}s+\frac{s^3}{4aa}$, and $AP=\frac{ss}{4a}+\frac{3s^4}{16a^3}-\frac{s}{48}a$ $\pm CN$, *viz.* $-$ when *AC* is greater than *AE*, and $+$ when it is less; and draw the right Lines KM, PM parallel to AP, AK; then will their Point of Interſection M be in the Curve DM ſought.

For making $AP = x$, $PM = y$, and $AC = s$; the Property the Curve muſt have, gives AK or $PM(y) = \frac{a^4 + 2aass + s^4}{4aas}$, and conſequently $\dot{y} = \frac{1}{4}\dot{s} + \frac{3ss\dot{s}}{4aa} - \frac{aa\dot{s}}{4ss}$. Then ſince BC is parallel to the Tangent in M. Therefore $\dot{x} = \frac{s\dot{y}}{a} = \frac{s\dot{s}}{2a} + \frac{3s^3\dot{s}}{4a^3} - \frac{a\dot{s}}{4s}$, the Fluent of which is $AP(x) = \frac{ss}{4a} + \frac{3s^4}{16a^3}$, *minus* the Fluent of $\frac{a\dot{s}}{4s}$ *plus* or *minus* ſome invariable Quantity. This Quantity we will take to be $\frac{1}{48}a$, which we ſubſtracted, that ſo CN, which from the Nature of the Logarithmick Curve FEN is the Fluent of $\pm\frac{a\dot{s}}{4s}$ becoming $= 0$, $AP(x)$ may be $= 0$ alſo. Whence, &c.

When $AC = AE$, the Ordinate PM, which is then a *Minimum*, becomes $AD = \frac{1}{3}AE$, and the Tangent in D will be parallel to BE. But if AC be taken leſs than AE, the Part DO of the Curve will be deſcrib'd convex towards DM, diverging more and more from AP, AK. Therefore the Solid of the leaſt Reſiſtance may be convex or concave, or partly convex and partly concave, and the Point D is a Point of *Retrogreſſion*.

Note, The Logarithmick Curve may be deſcrib'd eaſily after this manner. For you need only take $CN =$ to the Meaſure of the Ratio between AE and AC, the Module being $\frac{1}{4}AB(a)$.

PROB.

PROB. XVIII.

127. *To find the Angle* ABC, *which the Plane of the Sail of a Windmill in Figure of a right-angled Parallelogram, whose given Breadth is* CB, *makes with the Axis* AB *being such, that a given Wind blowing in the Direction of the Axis, shall drive it round with a greater Force than if it had any other Inclination to the Axis.*

Draw CD perpendicular to the Axis. Let $CB = a$, and $DB = x$. FIG. 84.

If BC represents the Number of Particles of the Air striking the Sail when it is perpendicular to the Axis AB; then will DC be the Number of Particles striking, when the Sail is inclined to the Axis in the Angle DBC.

Now it easily follows from the Principles of Mechanicks, that the Force of the Sail in the Direction DC will be as $BD \times DC \times DC = BD \times \overline{DC}^2$; that is, the Sail will be drove round by the given Wind with a Force that is always as $BD \times \overline{DC}^2$, which consequently must be a *Maximum*. But $BD \times \overline{DC}^2$ is $= x \times \overline{aa - xx} = a^2x - x^3$. Whence $a^2\dot{x} - 3x^2\dot{x} = 0$, and $a^2 = 3x^2$; therefore $\frac{1}{3}a^2 = x^2$, and so $\sqrt{\frac{1}{3}a^2} = x$. Whence as CB to $\sqrt{\frac{1}{3}a^2}$, so is the Radius to the Sine of the Angle C; the Complement of which is the Angle ABC sought. Consequently the Angle C is 35°. 16'. and the Angle ABC, 54°. 44'.

SCHOLIUM.

THIS Problem, on account of its eaſy Solution, ſhould have been antecedent to ſome of the others, eſpecially the two laſt Problems; but before I thought of it, all the others were printed off: yet rather than leave it out, I have added the ſame here. *Monſieur Mariot*, in his *Hydroſtaticks*, has endeavour'd at a Solution of it; but his Concluſion is wrong. *Monſieur Parent* has done it, as we find in the *Hiſtory of the Royal Academy of Sciences at* Paris; but we have not his Proceſs, neither in the Hiſtory or Memoirs, as I can find.

FINIS.

ERRATA *in the* APPENDIX.

PAGE 3. line 14. for *denominates* r. *Denominators*. P. 5. l. 1. for b r. x. P. 7. l. 11 and 14. for *Surd* r. *Series*. P. 10. l. 3. for $P^{\frac{m}{n}}$ r. $P^{\frac{m}{n}}$. Ibid. l. 7. for $A\ (=P^{\frac{m}{n}})\ a^5$. r. $A\ (= P^{\frac{m}{n}}) = a^5$. P. 23. l. 21. for *will be* $x\dot{y}$, r. *will be* $\dot{x}y$. P. 24. l. 27. for *a* r. *an*. P. 48. l. 11. for $=\frac{ax+xb}{a}$ r. $=\frac{2ax+xb}{a}$. P. 57. l. 11, 12. for $\frac{b}{a}\sqrt{aa-aa}$, and $\frac{b}{a}\dot{x}\sqrt{aa-aa}$, r. $\frac{b}{a}\sqrt{xx-aa}$, and $\frac{b}{a}\dot{x}\sqrt{xx-aa}$. P. 58. l. 1. for *Art.* 11. r. *Art.* 14. Ibid. l. 2. del. $=$. Ibid. l. 10. for *PH* r. *CH*. P. 59. l. 15. for *Art.* 11. r. *Art.* 14. P. 60. l. 18. for $\sqrt{x-xx}$ r. $\dot{x}\sqrt{x-xx}$. P. 84. line 7. for $\dot{z}$ read z. Page 93. line 17. for *Quadrature* read *Rectification*. P. 104. l. 21. for $\sqrt{x^2\dot{x}^2+\dot{y}^2}$ r. $\sqrt{\dot{x}^2+\dot{y}^2}$. P. 108. l. 7. for $\sqrt{x^2+y^2}$ r. $\sqrt{\dot{x}^2+\dot{y}^2}$. P. 114. l. 11. dele *because* x *begins at* B, *and not at* A. P. 115. l. 7. for *AP* r. *AM*. P. 118. l. 11. for $\frac{px\dot{x}-px^2\dot{x}}{2r}$ r. $\frac{2prx\dot{x}-px^2\dot{x}}{2r}$. Ibidem, l. 14. r. $\frac{3prx^2-px^3}{6}$. Ibid. l. 17. for $\frac{2pr^3-8pr^3}{6r}$ r. $\frac{12pr^3-pr^3}{6r}$. P. 119. l. 2. for *Semidiameter* r. $\frac{1}{3}$ *of the Semidiameter*. P. 126. l. 7. for $\frac{pr}{y}$ r. $\frac{py}{r}$. Ibid. l. 13. for $\frac{pyyx}{2r}$ r. $\frac{pyy\dot{x}}{2r}$. Ibid. l. 15. for

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$\frac{aapx}{r}$ r. $\frac{aapx}{2r}$. P. 126. l. 19. r. $\frac{bpr}{3}$. P. 127. l. 14. for *BD* r. *PD*. P. 135. l. 22. for *and so* $\frac{n^3}{12a}$ r. *and so* $\frac{8n^3}{12a}$. Ibid. l. 23. for $\sqrt{\frac{2}{3}n^3}$ r. $\sqrt{\frac{2n^3}{3a}}$. P. 136. l. 8. add, *when lessen'd by* $\frac{1}{6}$ *of a Circle, whose Radius is* $=a$. Ibid. l. 14. r. *transverse*. Ibid. l. 15. for *PM* r. *PA*. P. 139. l. 25. for *DCE* r. *DEC*. P. 140. Art. 71. *Fig.* 54. wanting in the Margin. P. 141. l. 20, 21. dele, *but because* y *begins at the Centre* C, *and not at* A *the Vertex*. P. 145. Art. 73. for *Fig.* 5. in the Margin, r. *Fig.* 56. P. 146. l. 11, 12. dele, *since the Abscis* AP (x) *begins at* A *and not at* B. P. 190. l. 5. for *C* r. *B*. P. 191. l. 21. dele *is*.