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NEW METHODS OF CELESTIAL MECHANICS

VOLUME III. INTEGRAL INVARIANTS, PERIODIC SOLUTIONS OF THE SECOND TYPE, DOUBLY ASYMPTOTIC SOLUTIONS

By H. Poincaré

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TABLE OF CONTENTS

VOLUME III

CHAPTER XXII

INTEGRAL INVARIANTS

CHAPTER XXIII

FORMATION OF INVARIANTS

CHAPTER XXIV

USE OF INTEGRAL INVARIANTS

CHAPTER XXV

INTEGRAL INVARIANTS AND ASYMPTOTIC SOLUTIONS

Page

CHAPTER XXX

FORMULATION OF SOLUTIONS OF THE SECOND TYPE

CHAPTER XXXI

PROPERTIES OF SOLUTIONS OF THE SECOND TYPE

CHAPTER XXXII

PERIODIC SOLUTIONS OF THE SECOND TYPE

CHAPTER XXXIII

DOUBLY ASYMPTOTIC SOLUTIONS

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NEW METHODS OF CELESTIAL MECHANICS

VOLUME III

H. POINCARÉ

ABSTRACT

Integral invariants are introduced using the steady motion of the fluid as an example. The usefulness of invariants in celestial mechanics is demonstrated. Various forms of the three-body problem are treated. Poisson stability is defined for the steady motion of a liquid, the general and restricted three-body problem. The theory of "consequents" is introduced in the discussion. The existence, stability, and properties of periodic solutions of the second type are treated. These are related to the principle of least action and the Darwin orbits. The concepts of kinetic focuses and Maupertius focuses are introduced in the discussion. Periodic solutions of the second type are treated. Homoclinous and heteroclinous doubly asymptotic solutions are discussed for the three-body problem.

CHAPTER XXII

INTEGRAL INVARIANTS

Steady Motion of a Fluid

233. In order to clarify the origin and importance of the idea /1* of integral invariants, it is useful to start with a study of a particular example from the field of physics.

Let us consider an arbitrary fluid, and let u, v, w be the three velocity components of the molecule which has the coordinates x, y, z at time t.

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* Numbers given in the margin indicate pagination in the original foreign text.

We will consider u, v, w as functions of t, x, y, z, and we will assume that these functions are given.

If u, v, w are independent of t and only depend on x, y, z, the motion of the fluid is said to be steady. We will assume that this condition is satisfied.

The trajectory of an arbitrary molecule of the fluid is therefore a curve which is defined by the differential equation

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}. \tag{1}$$

If it were possible to integrate these equations, one would obtain /2

$$x = \varphi_1(t, x_0, y_0, z_0),$$
$$y = \varphi_2(t, x_0, y_0, z_0),$$
$$z = \varphi_3(t, x_0, y_0, z_0),$$

such that x, y and z would be expressed as a function of time t and their initial values x_0, y_0, z_0.

If the initial position of a molecule were known, one could deduce the position of this same molecule at time t.

Let us consider fluid molecules the group of which forms a certain figure F_0 at the initial instant of time; when these molecules are displaced, their group will form a new figure which will move while being continuously deformed, and at the time t the group of molecules under consideration will form a new figure F.

We will assume that the movement of the fluid is continuous, i.e., u, v, w are continuous functions of x, y, z; there are therefore certain relationships between the figures F_0 and F which are obvious from the conditions of continuity.

If the figure F_0 is a curve or a continuous surface, the figure F will be a curve or a continuous surface.

If the figure F_0 is a simply connected volume, the figure F will be a simply connected volume.

If the figure F_0 is a curve or a closed surface, the same will hold true for the figure F.

In particular, let us examine the case of liquids where the fluid is incompressible, i.e., where the volume of a mass of fluid is invariable.

Let us assume that the figure F_0 is a volume. At time t the mass of fluid which fills out this volume will occupy a different volume which will be nothing else than the figure F.

The volume of the mass of fluid did not change; thus, F_0 and F have the same volume. Therefore, one can write

$$\iiint dx\,dy\,dz = \iiint dx_0\,dy_0\,dz_0; \qquad (2)$$

The first integral is extended over the volume F and the other over the volume F_0.

We will then say that the integral /3

$$\iiint dx\,dy\,dz$$

is an <u>integral</u> <u>invariant</u>.

It is known that the condition of incompressibility can be expressed by the equation

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0. \qquad (3)$$

The two equations (2) and (3) are thus equivalent.

Let us again consider the case of a gas, i.e., the case where the volume of a mass of fluid is variable. Thus, the mass becomes invariable, such that if one calls ρ the density of the gas, one has

$$\iiint \rho\,dx\,dy\,dz = \iiint \rho_0\,dx_0\,dy_0\,dz_0. \qquad (4)$$

The first integral is extended over the volume F, the second over the volume F_0. In other words, the integral

$$\iiint \rho\,dx\,dy\,dz$$

is an integral invariant.

In this case, where the motion is steady, the equation of continuity can be written as

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} = 0. \qquad (5)$$

The conditions (4) and (5) remain equivalent.

234. The theory of vortices of Helmholtz provides us with a second example.

Let us assume that the figure F_0 is a closed curve. The same will hold true for the figure F.

Let us assume that the fluid, whether it is compressible or not, is at a constant temperature, and is only subjected to forces which have a potential. In order that the motion remains steady, it is necessary that u, v, w satisfy certain conditions. It is not useful to develop these conditions here.

Let us assume that they are satisfied. /4

Under this assumption, let us consider the integral

$$\int (u\,dx + v\,dy + w\,dz).$$

As the theorem of Helmholtz shows, it has the same value along the curve F and along the curve F_0.

In other words, this integral is an integral invariant.

Definition of Integral Invariants

235. Due to the nature of the question, the examples which I have just presented readily lead one to the consideration of integral invariants.

It is clear that these invariants can be used by generalizing their definition for cases which are much broader, in which it is not possible to give a simple physical meaning to the invariants.

Let us consider differential equations of the form

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = dt, \tag{1}$$

X, Y, Z are given functions of x, y, z.

If they could be integrated, one would obtain x, y, z as a function of t and their initial values x_0, y_0, z_0.

If we assume that the time is represented by t and x, y, z represent the coordinates of a moving point M in space, equations (1) define the laws of motion of this moving point.

If these equations are integrated once, one can find the position of the point M at time t, if its initial position M_0, given by the coordinates x_0, y_0, z_0, is known.

If one considers moving points which obey the same law and the group of which forms a figure F_0 at the initial instant of time, the group of these same points will form a different figure F at time t which will be a line, a surface, or a volume depending on whether the figure F_0 is a line, a surface or a volume.

Let us consider a simple integral /5

$$\int (A\,dx + B\,dy + C\,dz), \tag{2}$$

where A, B, C are the known functions of x, y, and z. If F_0 is a line, it may happen that this integral (2) extended over all of the elements of the line F is a constant which is independent of time, and is consequently equal to the value of this same integral extended over all of the elements of the line F_0.

Let us now assume that F and F_0 are surfaces, and let us imagine the double integral

$$\iint (A'\,dy\,dz + B'\,dx\,dz + C'\,dx\,dy), \tag{3}$$

where A', B', C' are functions of x, y, and z. It may happen that this integral has the same value which is extended over all the elements of the surface F, or over all of those of the surface F_0.

Let us now assume that F and F_0 are volumes, and let us imagine the triple integral

$$\iiint M\,dx\,dy\,dz; \tag{4}$$

M is a function of x, y, z. It is possible that it may have the same value for F and F_0.

In these different cases, we say that the integrals (2), (3) or (4) are <u>integral invariants</u>.

It occasionally happens that the simple integral (2) will only have the same value for the lines F and F_0 if these two curves are closed, or the double integral (3) will only have the same value for the surfaces F and F_0 if these two surfaces are closed.

We may thus say that (2) is an integral invariant with respect to the closed curves and that (3) is an integral invariant with respect to the closed surfaces.

236. The geometric representation which we have employed plays no important role. We can thus lay it aside, and nothing prevents us from extending the preceding definitions to the case in which the number of variables is greater than three. Let us consider the /6 following equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt, \tag{1}$$

where $X_1, X_2, \ldots, X_n$ are the given functions of $x_1, x_2, \ldots, x_n$. If one could integrate them, one would find $x_1, x_2, \ldots, x_n$ as functions of t and of their initial values $x_1^0, x_2^0, \ldots, x_n^0$. In order to retain the same terminology, we may call point M the system of values $x_1, x_2, \ldots, x_n$, and the point M_0 the system of values $x_1^0, x_2^0, \ldots, x_n^0$.

Let us consider a group of points M_0 forming a subset F_0 and the group of corresponding points M forming another subset F[(1)].

We shall assume that F_0 and F are continuous subsets having p dimensions where $p \leq n$.

Let us consider an integral of the order p

$$\int \Sigma A \, d\omega, \tag{2}$$

where A is a function of $x_1, x_2, \ldots, x_n$, and where $d\omega$ is the product of p differentials chosen among the n differentials

$$dx_1, \; dx_2, \; \ldots, \; dx_n.$$

---

(1) The word subset is now commonly employed, so that I did not feel it was necessary to recall the definition. Every continuous group of points (or system of values) is named this way: In three-dimensional space, an arbitrary surface is a subset having two dimensions, and an arbitrary line is a subset having one dimension.

It is possible to give this integral the same value for the two subsets F and F_0. We may thus say that it is an integral invariant.

It may also happen that this integral has the same value for the two subsets F and F_0, but only under the condition that these two subsets are closed. It is thus an integral invariant with respect to the closed subsets.

Other types of integral invariants may be also assumed. For example, let us assume that p = 1 and that F and F_0 may be reduced to lines. It is possible that the integral /7

$$\int (A_1 dx_1 + A_2 dx_2 + \ldots + A_n dx_n) = \int \Sigma A_i dx_i$$

has the same value for F and F_0, and is an integral invariant. This may also be the case for the following integral

$$\int \sqrt{\Sigma B_i dx_i^2 + 2\Sigma C_{ik} dx_i dx_k},$$

where B and C are like the A of the functions of $x_1, x_2, \ldots, x_n$. As I stated, it is possible that this integral may have the same value for F and F_0, and other similar examples may be readily envisaged.

The quantity p will be called the order of the integral invariant.

Relationships Between the Invariants and the Integrals

237. Let us again consider the system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt. \tag{1}$$

If one could integrate it, all of its integral invariants could be formed.

If integration were performed, the result could be presented in the following form

$$\left\{\begin{array}{l} y_1 = C_1, \\ y_2 = C_2, \\ \ldots\ldots\ldots \\ y_{n-1} = C_{n-1}, \\ z = t + C_n, \end{array}\right. \tag{2}$$

C_1, C_2, ..., C_n are arbitrary constants, and the y's and z's are the given functions of the x's.

Let us change the variables by taking y's and z for the new variables, instead of x's.

Let us now consider an arbitrary integral invariant. Under the sign $\int$ (which will be repeated p times if the invariant is of the order p), this invariant must include a certain expression, the function of the /8 x's and of their differentials dx. After a change in the variables, this expression will become a function of the y's, z, and of their differentials dy and dz.

Without changing the y's, in order to pass from one point of the figure F_0 to a corresponding point in the figure F, it is necessary to change z into z + t. Therefore, when passing from an infinitely small arc of F_0 to the corresponding arc of F, the differentials dy and dz do not change (the quantity t which is added to z is, in effect, the same for the two ends of the arc). If one considers an infinitely small figure F_0 having an arbitrary number of dimensions and the corresponding figure F, the product of a number (equalling that of the dimensions of F_0 and F) of differentials dy or dz will not change either when one passes from one figure to the other.

In short, in order that an expression may be an integral invariant, it is necessary and sufficient that z is not contained in it; the y's, the dy's, and dz may be included in an arbitrary manner.

Let us consider an expression having the same form as that which we discussed in the preceding section

$$\int \Sigma A \, d\omega, \tag{3}$$

This expression represents an integral of the order p, A is a function of x_1, x_2, ..., x_n, $d\omega$ is a product of p differentials chosen from the n differentials

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_n.$$

We would like to know whether this is an integral invariant. By carrying out a change in variables as indicated above, we find that expression (3) becomes

$$\int \Sigma B \, d\omega',$$

B is a function of the y's and of z, $d\omega'$ is a product of p differentials chosen from the n differentials

$$dy_1, \; dy_2, \; \ldots, \; dy_{n-1}, \; dz.$$

In order that expression (3) be an integral invariant, it is necessary and sufficient that all of the B's be independent of z and only depend on the y's.

Just as in the preceding section, let us again consider the expression /9

$$\int \sqrt{\Sigma B_i dx_i^2 + 2\Sigma C_{i.k} dx_i dx_k}, \tag{4}$$

The B_i's and the $C_{i.k}$'s are functions of the x's.

After the change in the variables, this expression becomes

$$\int \sqrt{\Sigma B'_i dx'^2_i + 2\Sigma C'_{i.k} dx'_i dx'_k};$$

For greater symmetry in the notation, I have set the following

$$x'_i = y_i; \quad (i = 1, 2, \ldots, n-1); \quad x'_n = z,$$

In order that expression (4) be an integral invariant, it is necessary and sufficient that all of the B'_i's and the $C'_{i.k}$'s be independent of z, and depend only on y.

Relative Invariants

238. We are now led to attempt to form the integral invariants relative to the closed subsets. Let us first assume that p = 1, and let us determine the condition by which the simple integral

$$\int (A_1 dx_1 + A_2 dx_2 + \ldots + A_n dx_n) \tag{1}$$

is an integral invariant with respect to closed lines.

Let us carry out the change in variables as indicated above, and our integral will become

$$\int (B_1 dy_1 + B_2 dy_2 + \ldots + B_{n-1} dy_{n-1} + B_n dz),$$

which I can write again, taking the most symmetrical notation from the end of the preceding section

$$\int \Sigma B_i dx'_i. \qquad (1')$$

This simple integral, extended over a closed, one-dimensional subset -- i.e., over a closed line -- may be transformed by the Stokes theorem /10
into a double integral extended over a non-closed, two-dimensional subset -- i.e., over a non-closed surface. We then have

$$\int \Sigma B_i dx'_i = \int \sum \left(\frac{dB_i}{dx'_k} - \frac{dB_k}{dx'_i} \right) dx'_i dx'_k. \qquad (2)$$

However, the integral of the second member of (2) must be an absolute integral invariant, and not only with respect to the closed subsets.

We can therefore conclude the following:

In order that (1) be an integral invariant with respect to the closed lines it is necessary and sufficient that the binomials

$$\frac{dB_i}{dx'_k} - \frac{dB_k}{dx'_i}$$

all be independent of z.

Similarly, and more generally, let

$$\int \Sigma A \, d\omega \qquad (3)$$

be an integral expression of the order p, having the same form as those which were discussed in the preceding sections. We would like to know whether this is an integral invariant with respect to the closed subsets of the order p.

Let us assume that this integral is extended over an arbitrary closed subset of the order p. A theorem similar to that of Stokes states that it may be transformed into an integral of the order p + 1, extended over an arbitrary subset, which may be closed or not closed, of the order p + 1. The transformed integral may be written

$$\int \Sigma \Sigma_k \pm \frac{dA}{dx_k} dx_k d\omega. \qquad (4)$$

One always takes the sign + if p is even, and the signs + and - alternately if p is odd. [For additional details, refer to my report on the residuals of double integrals (Acta Mathematica, Volume VIII), and to my report contained in the Special Centenary Edition of the Journal de l'École Polytechnique.]

The condition which is necessary and sufficient for (3) to be an integral invariant of the order p with respect to closed subsets is that (4) be an absolute integral invariant of the order p + 1. /11

239. Let us again consider expression (1) of the preceding section, and let us assume that it is a relative invariant, that is, an integral invariant with respect to closed lines.

Let us change it to the form (1') by our change in variables.

Let M_0 be a point of F_0 and

$$y_1, \; y_2, \; \ldots, \; y_{n-1}, \; z$$

be its coordinates (with the new variables).

Let M be the corresponding point of F and

$$y_1, \; y_2, \; \ldots, \; y_{n-1}, \; z+t$$

be its coordinates. The B_k's will be functions of the y's and of z, but I will make z appear, writing B_k in the following form

$$B_k(z).$$

If the line F_0 is closed, we will then have

$$\int \Sigma B_k(z+t)\, dx'_k = \int \Sigma B_k(z)\, dx'_k,$$

that is, the expression

$$\Sigma[B_k(z+t) - B_k(z)]\, dx'_k \tag{3}$$

is an exact differential which I set equal to dV. The function V will depend not only on the y's and z, but also on t. In order that t = o, it must be reduced to a constant.

If we assume that t is infinitely small and if we call $B'_k(z)$ the derivative of $B_k(z)$ with respect to z, expression (3) may be reduced to

$$\Sigma[t B'_k(z)]\, dx'_k.$$

The expression

$$\Sigma B'_k(z)\, dx'_k \qquad (4)$$

is then an exact differential which I set equal to dU. The function U which is thus defined will depend on the y's and z, but it will no longer depend on t. I shall again make z appear by writing U(z). It then happens that /12

$$\frac{dV}{dt} = \int \Sigma B'_k(z+t)\, dx'_k = \int dU(z+t) = U(z+t) + f(t),$$

f(t) is an arbitrary function of t.

However, U(z) may be regarded as the derivative with respect to z of another function W(z) which is also dependent on the y's, and we will then have

$$\frac{d}{dt} W(z+t) = U(z+t).$$

On the other hand, since V must be reduced to a constant for t = o, we may finally conclude that

$$V = W(z+t) - W(z) + \varphi(t).$$

The quantity $\phi(t)$ designates an arbitrary function of t only, and may be assumed to be zero without essentially restricting the conditions of generality.

One then finds

$$B_k(z) = \frac{d}{dx'_k} W(z) + C_k.$$

C_k is independent of z, so that the expression (1') may be reduced to

$$\int dW + \int \Sigma C_k\, dx'_k,$$

and the first integral is that of an exact differential, and the second integral is an absolute integral invariant.

240. In a similar way let us discuss a relative invariant which is of a higher order than the first. Let us assume that

$$\int \Sigma A \, d\omega$$

is this invariant which, after the change in variables, will become

$$\int \Sigma B \, d\omega'.$$

The integral

$$\int \Sigma [B(z+t) - B(z)] \, d\omega' = J \tag{1}$$

must be zero, whatever may be the closed subset of order p over which it is extended.

It must therefore satisfy certain "integrability conditions" which /13 are similar to those stating that a total differential of the first order is an exact differential.

Let us now consider a subset V of p dimensions, which is not closed and limited by a subset v of p - 1 dimensions which will serve as the boundary for it.

The integral (1), which is extended over the subset V, will not be zero. However, if it is calculated for other similar subsets V', V'', etc., having the same boundary v, one will obtain the same value -- i.e., the value of the integral (1) only depends on the boundary v.

It equals an integral of the order p - 1

$$J = \int \Sigma C \, d\omega'' \tag{2}$$

which is extended over the subset v and where dω" designates an arbitrary product of p - 1 differentials, while C is a function of the y's, z and t.

This integral (2) is clearly a function of t, which depends in addition on the subset v. Let us consider its derivative with respect to t. We will have

$$\frac{dJ}{dt} = \int \sum \frac{dC}{dt} \, d\omega'' = \int \Sigma B'(z+t) \, d\omega'.$$

As its last expression shows, this derivative does not change when one changes t into t - h or when, at the same time, one transforms V (or v) by changing z everywhere into z + h.

It can be concluded that J has the following form

$$J = \int \Sigma D(z+t)\,d\omega' - \int \Sigma D(z)\,d\omega',$$

D(z) is a function of x, y, z.

The integral

$$\int \Sigma D(z)\,d\omega' \tag{3}$$

is of the order p - 1, but it may be readily transformed into an integral of the order p. It is sufficient to apply the transformation which, in section No. 238, allowed us to change from the integral (3) to the integral (4), and which is the opposite of that by which, in the present section, we changed from the integral (1) to the integral (2). /14

The integral (3), extended over the subset v, is therefore equal to the integral of the order p

$$\int \Sigma E(z)\,d\omega' \tag{4}$$

extended over the subset V.

By analogy with the terminology employed for simple integrals, we may say that the integral (4) is an exact differential integral. And, in effect:

1. It is zero for every closed subset;

2. It may be reduced to an integral of lesser order.

Under this assumption, we will have

$$J = \int \Sigma E(z+t)\,d\omega' - \int \Sigma E(z)\,d\omega',$$

and the integrals are extended over the subset V.

However, this equation may also be written as follows

$$\int \Sigma[B(z+t) - E(z+t)]\,d\omega' = \int \Sigma[B(z) - E(z)]\,d\omega',$$

and it is valid for an arbitrary subset V.

This means that

$$\int \Sigma [B(z) - E(z)]\, d\omega'$$

is an absolute integral invariant.

We therefore arrive at the following result:

Every relative integral invariant is the sum of an exact differential integral and an absolute integral invariant.

241. In Section No. 238, we have seen how an absolute invariant of the order p + 1 may be deduced from a relative invariant p.

The same procedure may also be applied to absolute invariants, so that one could be tempted to continue to apply it and to construct invariants of the order p + 2, p + 3, ... successively.

However, this procedure would have to be abandoned very quickly.

There is a case in which the procedure in question is illusory; /15
this is the case in which the invariant which one wishes to transform is an exact differential integral. The integral invariant to which the transformation would lead would then be also zero.

If an invariant of the order p is transformed, one obtains an invariant of the order p + 1, but this invariant is an exact differential integral, so that if one wishes to transform it again, one obtains a result which is also zero.

Relationship Between the Invariants and the Variational Equation

242. Let us again consider the system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt. \tag{1}$$

We may form the corresponding variational equations as they were defined at the beginning of Chapter IV.

In order to form these equations, in equations (1) we change x_i into $x_i + \xi_i$, and we disregard the squares of the ξ_i's. One thus obtains the system of linear equations

$$\frac{d\xi_k}{dt} = \frac{dX_k}{dx_1}\xi_1 + \frac{dX_k}{dx_2}\xi_2 + \ldots + \frac{dX_k}{dx_n}\xi_n. \tag{2}$$

There is a close relationship which may be readily perceived between the integrals of equations (2) and the integral invariants of equations (1).

Let

$$F(\xi_1, \xi_2, \ldots, \xi_n) = \text{const.},$$

be an arbitrary integral of equations (2). This will be a homogeneous function of the ξ's, which depends on the x's in an arbitrary manner. I can always assume that this function F is homogeneous of degree 1 with respect to the ξ's. Because, if this were not the case, I would only have to increase F to a suitable power in order to obtain a homogeneous function of degree 1.

Let us now consider the following expression

$$\int F(dx_1, dx_2, \ldots, dx_n), \tag{3}$$

which is an integral invariant of system (1).

We should first note that the quantity under the sign $\int$ /16

$$F(dx_1, dx_2, \ldots, dx_n)$$

is an infinitesimal quantity of the first order, since dx_1, dx_2 ..., dx_n are infinitesimal quantities of the first order, and that F is homogeneous of the first order with respect to the quantities.

The simple integral (3) is therefore finite.

Under this assumption, let us first assume that the figure F_0 may be reduced to an infinitesimal line, whose extremities have the following coordinates

$$x_1, \quad x_2, \quad \ldots, \quad x_n,$$

$$x_1 + \xi_1, \quad x_2 + \xi_2, \quad \ldots, \quad x_n + \xi_n.$$

The integral (3) may be reduced to a single element, and consequently will equal

$$F(\xi_1, \xi_2, \ldots, \xi_n).$$

Due to the fact that this expression is an integral of equations (2), it will remain constant and will have the same value for the line F_0 and for the line F.

If the line F_0, and consequently the line F, are finite, we may divide the line F_0 into infinitesimal parts. The integral (3), extended over one of these infinitesimal parts of F_0, will equal the integral (3) extended over the corresponding infinitesimal part of F. The integral extended over the entire line F_0 will equal the integral extended over the entire line F.

Therefore, the integral (3) is an integral invariant.

q.e.d.

Conversely, let us assume that (3) is an integral invariant of the first order, and

$$F(\xi_1, \xi_2, \ldots, \xi_n)$$

will be an integral of the equations (2).

In reality, the integral (3) must be the same for the line F_0 and for the line F, whatever these lines may be, and particularly if F_0 is reduced to an infinitesimal element whose ends have the following coordinates

$$x_i \quad \text{and} \quad x_i + \xi_i.$$

As we have seen, the integral (3) may be reduced to /17

$$F(\xi_1, \xi_2, \ldots, \xi_n). \tag{4}$$

Since the integral is an invariant, this expression (4) must be constant.

It is therefore an integral of equations (2). q.e.d.

Thus, an integral of equations (2) corresponds to each integral invariant of the first order of equations (1), and vice versa.

243. Let us now determine to what the invariants of an order higher than the first correspond.

Let us consider two particular arbitrary solutions of the equations (2). Let

$$\begin{cases} \xi_1, & \xi_2, \ldots, & \xi_n, \\ \xi'_1, & \xi'_2, \ldots, & \xi'_n, \end{cases} \tag{5}$$

be these two solutions.

The following functions may exist

$$F(x_i, \xi_i, \xi'_i)$$

which depend on the x_i's, the ξ_i's, and the ξ'_i's at the same time. No matter what the two chosen solutions, these functions may be reduced to constants which are independent of time.

In other words, the function F will be an integral of the system

$$\left\{\begin{aligned} \frac{d\xi_k}{dt} &= \frac{dX_k}{dx_1}\xi_1 + \frac{dX_k}{dx_2}\xi_2 + \ldots + \frac{dX_k}{dx_n}\xi_n, \\ \frac{d\xi'_k}{dt} &= \frac{dX_k}{dx_1}\xi'_1 + \frac{dX_k}{dx_2}\xi'_2 + \ldots + \frac{dX_k}{dx_n}\xi'_n, \end{aligned}\right. \tag{6}$$

which the ξ_i's and the ξ'_i's satisfy.

Let us formulate a more definite hypothesis, and let us assume that F has the form

$$\Sigma A_{ik}(\xi_i\xi'_k - \xi_k\xi'_i),$$

and the A_{ik}'s are functions of the x's alone.

It may then be stated that the double integral

$$J = \int \Sigma A_{ik}\, dx_i\, dx_k$$

is an integral invariant of the equations (1).

Let us assume that the figure F_0 may be reduced to an infinitesimal parallelogram whose corners have the following coordinates at $t = o$ /18

$$x_i, \quad x_i + \xi_i, \quad x_i + \xi'_i, \quad x_i + \xi_i + \xi'_i.$$

The figure F will also be similar to an infinitesimal parallelogram whose corners will have the following coordinates at $t = t$

$$x_i, \quad x_i + \xi_i, \quad x_i + \xi'_i, \quad x_i + \xi_i + \xi'_i.$$

The integral J will be reduced to a single element which will have precisely the following value

$$\Sigma A_{ik}(\xi_i\xi'_k - \xi_k\xi'_i),$$

and -- since, under the hypothesis, this expression is an integral of the system (6) -- it will have the same value for the two figures F and F_0.

Let us now assume that F and F_0 are two finite surfaces. Let us divide F_0 into infinitesimal parallelograms, to each of which an elementary parallelogram of F will correspond. The value of J is therefore the same for each element of F_0 and for the corresponding element of F. It is therefore the same even for the entire surface F_0 and for the entire surface F.

The integral J is therefore an integral invariant. q.e.d.

The converse of this may be proven in the same way as in the preceding section.

244. The theorem is obviously general, and may be applied to invariants of a high order than two. Let us present it for those of the third order. Let us consider three special solutions of the equations (2), ξ_i, ξ'_i, ξ''_i. These three solutions must satisfy the system

$$\begin{cases} \frac{d\xi_k}{dt} = \sum \frac{dX_k}{dx_i} \xi_i, \\ \frac{d\xi'_k}{dt} = \sum \frac{dX_k}{dx_i} \xi'_i, \\ \frac{d\xi''_k}{dt} = \sum \frac{dX_k}{dx_i} \xi''_i. \end{cases} \tag{7}$$

If the system (7) includes an integral of the form

$$\Sigma A_{i,k,l} \begin{vmatrix} \xi_i & \xi'_i & \xi''_i \\ \xi_k & \xi'_k & \xi''_k \\ \xi_l & \xi'_l & \xi''_l \end{vmatrix}, \tag{8}$$

where the A's are functions of the x's, the triple integral /19

$$\int \Sigma A_{ikl}\, dx_i\, dx_k\, dx_l \tag{9}$$

will be an integral invariant of the equations (1), and vice versa.

Transformation of the Invariants

245. With the invariants thus reduced to the integrals of the variational equation, one may readily find several procedures which make it possible to transform these invariants.

If one knew a certain number of integral invariants of the equations

$$\frac{dx_i}{dt} = X_i, \tag{1}$$

one could deduce from each of them an integral of the variational equations

$$\frac{d\xi_i}{dt} = \sum \frac{dX_k}{dx_i} \xi_i. \tag{2}$$

By combining these different integrals, we will obtain a new integral of equations (2), from which one may deduce a new invariant of the equations (1).

Let us commence by studying the case of first-order invariants.

Let

$$\Phi_1, \quad \Phi_2, \quad \ldots, \quad \Phi_p,$$

be a certain number of integrals of equations (1). These integrals will be functions of the x_i's alone.

Now let

$$\int F_1(dx_i), \quad \int F_2(dx_i), \quad \ldots, \quad \int F_q(dx_i),$$

be q integral invariants of the first order of these same equations (1).

The functions under the sign $\int$

$$F_1(dx_i), \quad F_2(dx_i), \quad \ldots, \quad F_q(dx_i)$$

will depend on the x_i's and their differentials dx_i's. They will depend on the x_i's in an arbitrary manner. However, with respect to the differentials /20

$$dx_1, \quad dx_2, \quad \ldots, \quad dx_n,$$

they must be homogeneous and of the first order.

Then

$$F_1(\xi_i), \quad F_2(\xi_i), \quad \ldots, \quad F_q(\xi_i)$$

will be integrals of the equations (2) and will be homogeneous and of the first order with respect to the ξ_i's.

Now let

$$\theta(\Phi_1, \Phi_2, \ldots, \Phi_p; F_1, F_2, \ldots, F_q) = \theta[\Phi_k, F_l],$$

be a function of the Φ's and of the F's, which depends on the Φ's in an arbitrary manner, but which is homogeneous and of the first order with respect to the F's.

Then

$$\theta[\Phi_k, F_l(\xi_i)]$$

will be a new integral of the equations (2). In addition, this will be a homogeneous function and of the first order with respect to the ξ_i's.

It thus results that

$$\int \theta[\Phi_k, F_l(dx_i)]$$

is an integral invariant of the first order of the equations (1).

The same result could be readily achieved when transforming the invariants by changing the invariables of No. 237.

For example,

$$\int (F_1 + F_2 + \ldots + F_q)$$

and

$$\int \sqrt{F_1^2 + F_2^2 + \ldots + F_q^2}$$

will be integral invariants.

246. The same calculation may be applied to invariants of a higher order.

Let

$$\Phi_1, \quad \Phi_2, \quad \ldots, \quad \Phi_p,$$

be the p integrals of the equations (1), and let /21

$$\int F_1(dx_i\,dx_k), \quad \int F_2(dx_i\,dx_k), \quad \ldots, \quad \int F_q(dx_i\,dx_k),$$

be the q integral invariants of the second order. The F's will be functions of the x_i's and the products of the differentials

$$dx_i\, dx_k.$$

They will be homogeneous and of the first order with respect to these products.

Then

$$F_l(\xi_i \xi'_k - \xi_k \xi'_i)$$

will be integrals of the system (6).

If

$$\Theta[\Phi_\mu, F_l]$$

is an arbitrary function of the Φ's and of the F's, which is homogeneous of the first order with respect to the F's, the expression

$$\Theta[\Phi_k, F_l(\xi_i \xi'_k - \xi'_i \xi_k)]$$

will be an integral of the equations (6). It will be homogeneous in addition, and of the first order with respect to the determinants

$$\xi_k \xi'_i - \xi'_k \xi_i.$$

As a result, the double integral

$$\int \Theta[\Phi_\mu, F_l(dx_i\, dx_k)]$$

will be an integral invariant of the second order of equations (1).

247. Knowing several invariants of the same order, we thus have the means of combining them to obtain other invariants of the same order.

When several invariants of the same order are known, the same procedure makes it possible to obtain new invariants of a different order.

For example, let

$$\int F_1(dx_i), \quad \int F_2(dx_i),$$

be two integral invariants of the first order. I assume, which is the most general case, that F_1 and F_2 are linear and homogeneous functions of the differentials dx_i. /22

The expressions

$$F_1(\xi_i), \quad F_2(\xi_i)$$

will be homogeneous and of the first order with respect to the ξ_i's, and these will be integrals of equations (2).

In the same way,

$$F_1(\xi_i'), \quad F_2(\xi_i')$$

will be integrals of the equations (6).

As a result,

$$F_1(\xi_i)F_2(\xi_k') - F_1(\xi_k)F_2(\xi_i') \tag{10}$$

will be an integral of the system (6).

Since F_1 and F_2 are linear with respect to the ξ_i's, we will have

$$F_1(\xi_i + \xi_i') = F_1(\xi_i) + F_1(\xi_i'); \qquad F_2(\xi_i + \xi_i') = F_2(\xi_i) + F_2(\xi_i').$$

As a result, expression (10), which changes sign when one exchanges the ξ_i's and the ξ_i''s, does not change when one changes ξ_i into $\xi_i + \xi_i'$.

We may thus conclude that this expression (10) is a linear and homogeneous function of the determinants

$$\xi_i\xi_k' - \xi_k\xi_i',$$

and the coefficients depend on the x's alone, but not on the ξ's and the ξ''s.

An integral invariant of the second order of the equations (1) may be therefore deduced from this expression (10).

Now let

$$\int F_1(dx_i), \quad \int F_2(dx_i\, dx_k)$$

be two integral invariants of equations (1); the first is of the first order and the second is of the second order. I shall assume that F_1 and F_2 are linear and homogeneous functions, the first with respect to the n differentials dx_i, the second with respect to the $\frac{n(n-1)}{2}$ products

$$dx_i\, dx_k.$$

The functions /23

$$F_1(\xi_l), \quad F_2(\xi_i \xi'_k - \xi_k \xi'_i)$$

will be integrals of the system (6).

The expression

$$\left\{ \begin{aligned} & F_1(\xi_1) F_2(\xi'_i \xi''_k - \xi'_k \xi''_i) + F_1(\xi'_l) F_2(\xi''_i \xi_k - \xi''_k \xi_i) \\ & \quad + F_1(\xi''_l) F_2(\xi_i \xi'_k - \xi_k \xi'_i) \end{aligned} \right. \tag{11}$$

will be an integral of the system (7).

On the other hand, it may be readily verified that it will be linear and homogeneous with respect to the determinants

$$\begin{vmatrix} \xi_i & \xi'_i & \xi''_i \\ \xi_k & \xi'_k & \xi''_k \\ \xi_l & \xi'_l & \xi''_l \end{vmatrix}.$$

An integral invariant of the third order may thus be deduced from it.

Now let

$$\int F_1(dx_i\, dx_k), \quad \int F_2(dx_i\, dx_k)$$

be two invariants of the second order of equations (1).

We can deduce from it two integrals of equations (6) -- that is,

$$F_1(\xi_i \xi'_k - \xi_k \xi'_i), \quad F_2(\xi_i \xi'_k - \xi_k \xi'_i),$$

which I can write as follows, for purposes of brevity,

$$F_1(\xi\xi'), \quad F_2(\xi\xi').$$

Then the expression

$$\left\{ \begin{aligned} & F_1(\xi\xi') F_2(\xi''\xi''') + F_1(\xi''\xi''') F_2(\xi\xi') \\ & + F_1(\xi\xi'') F_2(\xi'''\xi') + F_1(\xi'''\xi') F_2(\xi\xi'') \\ & + F_1(\xi\xi''') F_2(\xi'\xi'') + F_1(\xi'\xi'') F_2(\xi\xi''') \end{aligned} \right. \tag{12}$$

will be an integral of the system obtained by adding the following equations to the equations (7)

$$\frac{d\xi_k'''}{dt} = \sum \frac{dX_k}{dx_i} \xi_i'''.$$

In addition, this will be a linear and homogeneous function with respect to the determinants formed with four of the quantities ξ_i and the corresponding quantities ξ_i', ξ_i'', ξ_i'''. /24

I shall continue to assume that F_1 and F_2 are homogeneous and linear with respect to the products $dx_i dx_k$.

An integral invariant of the fourth order could thus be deduced from expression (12).

It should be noted that this invariant does not become exactly equal to zero when we set

$$F_1 = F_2.$$

Expression (12), divided by 2, may be then reduced to

$$F_1(\xi\xi')F_1(\xi''\xi''') + F_1(\xi\xi'')F_1(\xi'''\xi') + F_1(\xi\xi''')F_1(\xi'\xi'').$$

An invariant of the fourth order may always be deduced from an invariant of the second order. An invariant of the sixth order would be obtained by the same procedure. More generally, an invariant of the order 2p would be obtained from it (2p being an arbitrary even number).

248. In general, let

$$\int F_1, \int F_2$$

be two arbitrary invariants of equations (1); the first is of the order p, and the second is of the order q.

I shall assume that F_1 and F_2 are linear and homogeneous functions, the first with respect to the products of p differentials dx, and the second with respect to the products of q differentials.

Let

$$\xi_i^{(1)}, \ \xi_i^{(2)}, \ \ldots, \ \xi_i^{(p+q)}$$

be p + q solutions of equations (2). These solutions will satisfy the system of differential equations

$$\frac{d\xi_k^{(\mu)}}{dt} = \sum \frac{dX_k}{dx_i} \xi_i^{(\mu)} \quad (i, k = 1, 2, \ldots, n; \mu = 1, 2, \ldots, p+q). \tag{13}$$

Then let F'_1 be the quantity which F_1 becomes when each product of p differentials is replaced by the corresponding determinant formed by means of the p solutions

$$\xi_i^{(1)}, \ \xi_i^{(2)}, \ \ldots, \ \xi_i^{(p)}.$$

In the same way, let F'_2 represent the quantity which F_2 becomes when each product of q differentials is replaced by the corresponding determinant formed by means of the q solutions /25

$$\xi_i^{(p+1)}, \ \xi_i^{(p+2)}, \ \ldots, \ \xi_i^{(p+q)}.$$

Then the product

$$F'_1 F'_2$$

will be an integral of system (13).

Under this assumption, let us make the p + q letters

$$\xi_i^{(1)}, \ \xi_i^{(2)}, \ \ldots, \ \xi_i^{(p+q)}$$

undergo an arbitrary permutation. The product $F'_1 F'_2$ will become

$$F''_1 F''_2$$

and this will still be an integral of system (13).

We shall give this product the sign +, if the permutation under consideration belongs to the alternate group -- i.e., if it may be reduced to an even number of permutations between two letters.

On the other hand, we shall assign the product the - sign, if the permutation does not belong to the alternate group -- i.e., if it may be reduced to an odd number of permutations between two letters.

In any case, the expression

$$\pm F''_1 F''_2 \tag{14}$$

will be an integral of system (13).

We have (p + q)! possible permutations; we will therefore obtain (p + q)! expressions similar to (14). However, we shall only have

$$\frac{(p+q)!}{p!\,q!}$$

which will be different. This is due to the fact that expression (14) does not change when the p letters which are included in F''_1 are only interchanged among them, and, on the other hand, when the q letters which are included in F''_2 are only interchanged among them.

Let us now take the sum of all the expressions (14). We shall have an integral of system (13). However, this integral will be linear and homogeneous with respect to determinants of the order p + q, which can be formed with the letters

$$\xi_i^{(1)},\ \xi_i^{(2)},\ \ldots,\ \xi_i^{(p+q)}.$$

<u>An invariant of the order p + q of equations (1) may thus be deduced</u>. /26

If p = q and if F_1 is identical to F_2, the invariant thus obtained will be equal to zero if p is odd. However, this will no longer be the case if p is even, as I explained at the end of the preceding section.

<u>Other Relationships Between the Invariants and the Integrals</u>

249. Based on the knowledge of a certain number of invariants, let us now trace the manner in which we may deduce one or several integrals.

I shall first assume that we know two invariants of the n^{th} order

$$\int M\,dx_1\,dx_2\ldots dx_n,$$

and

$$\int M'\,dx_1\,dx_2\ldots dx_n,$$

where M and M' are functions of the x's. It may be stated that the ratio $\frac{M'}{M}$ will be an integral of equations (1).

Let us consider the variational equations (2) and let

$$\xi_i^{(1)},\ \xi_i^{(2)},\ \ldots,\ \xi_i^{(n)}$$

be n arbitrary solutions which are linearly independent of these equations.

These n solutions will satisfy a system of differential equations, which is similar to systems (6) and (7), which I shall designate as system s.

Let Δ be the determinant formed by means of the n^2's letters $\xi_i^{(k)}$. Then

$$M\Delta \quad \text{and} \quad M'\Delta$$

will be integrals of system S. The same will also hold for the ratio

$$\frac{M'}{M}$$

and, since this ratio only depends on the x's, and not on the ξ's, it must be an integral of equations (1).

The same result may be obtained in another manner. /27

Let us perform the change in variables as was done in No. 237. Our two integral invariants will become

$$\int MJ\, dy_1\, dy_2 \ldots dy_{n-1}\, dz,$$

and

$$\int M'J\, dy_1\, dy_2 \ldots dy_{n-1}\, dz$$

J designates the Jacobian or the working determinant of the old variables $x_1, x_2, \ldots, x_n$ with respect to the new variables $y_1, y_2, \ldots, y_{n-1}, z$.

According to No. 237, MJ and M'J must only depend on

$$y_1,\ y_2,\ \ldots,\ y_{n-1},$$

and this also holds for the ratio $\frac{M'}{M}$. Since every function of the y_i's is an integral of equations (1), this ratio is an integral of equations (1).

q.e.d.

250. This procedure may be varied in several ways.

For example, let

$$\int F_1(dx_i), \quad \int F_2(dx_i), \quad \ldots, \quad \int F_p(dx_i)$$

be the p linear invariants of the first order. Let us assume that we also have

$$F_1 = M_2 F_2 + M_3 F_3 + \ldots + M_p F_p,$$

and the M_i's depend only on the x's, and not on the differentials dx.

It may be stated that the M_i's, if $p \leq n + 1$, will be integrals of equations (1).

Let A_{ik} be the coefficient of dx_k in F_i. We must then have

$$A_{1.k} = M_2 A_{2.k} + M_3 A_{3.k} + \ldots + M_p A_{p.k}.$$

Let us perform the change in variables as was done in No. 237. Our invariants then become

$$\int F'_1(dx'_i), \quad \int F'_2(dx'_i), \quad \ldots, \quad \int F'_p(dx'_i).$$

If we also set /28

$$F'_i = \Sigma A'_{ik} \, dx'_k,$$

we must have

$$A'_{1.k} = M_2 A'_{2k} + M_3 A'_{3k} + \ldots + M_p A'_{pk}.$$

We shall then have n linear equations, from which we may obtain the M_i's, provided that $p \leq n + 1$.

According to No. 237, the A'_{ik}'s depend only on the y's, and not on z. The same is therefore true for the M_i's, that is, the M_i's are integrals of equations (1).

251. Now let

$$F(x_1, x_2, \ldots, x_n)$$

be an integral. It is apparent that

$$\int \left(\frac{dF}{dx_1} dx_1 + \frac{dF}{dx_2} dx_2 + \ldots + \frac{dF}{dx_n} dx_n \right)$$

will be an integral invariant of the first order.

One may then formulate the following question:

Let us consider an integral invariant of the first order

$$\int (A_1\, dx_1 + A_2\, dx_2 + \ldots + A_n\, dx_n)$$

and let us assume that the term under the $\int$ sign is an exact differential. What will be the relationship between the integral of this exact differential and the integrals of equations (1)?

In order to determine this, let us make the change in variables of No. 237; our invariant will become

$$\int dU = \int (B_1\, dy_1 + B_2\, dy_2 + \ldots + B_{n-1}\, dy_{n-1} + C\, dz).$$

The B's and the C's must depend on the y's, but not on z.

If this expression dU is an exact differential, the function U must therefore have the following form

$$U = U_0 + z U_1;$$

U_0 and U_1 are integrals of equation (1). We will then have /29

$$\frac{dU}{dt} = U_1.$$

If we return to the old variables x_i, we will have

$$\frac{dU}{dt} = \frac{dU}{dx_1} X_1 + \frac{dU}{dx_2} X_2 + \ldots + \frac{dU}{dx_n} X_n.$$

It therefore results that

$$\frac{dU}{dx_1} X_1 + \frac{dU}{dx_2} X_2 + \ldots + \frac{dU}{dx_n} X_n$$

is an integral of equations (1). If this expression is zero, we have

$$U_1 = 0, \qquad U = U_0,$$

and U is an integral of equations (1).

252. We could cite numerous examples of this type. I shall only

present one example.

Let us consider an invariant of the first order having the form

$$\int\sqrt{\Sigma B_i dx_i^2 + 2\Sigma C_{ik} dx_i dx_k} = \int\sqrt{\Phi}.$$

Let Δ be the discriminant of the quadratic form Φ.

Let us make the change in variables according to No. 237, and our invariant will become

$$\int\sqrt{\Sigma B_i' dx_i'^2 + 2\Sigma C_{ik}' dx_i' dx_k'} = \int\sqrt{\Phi'}.$$

Let Δ' be the discriminant of the quadratic form Φ'.

Let J be the Jacobian or the working determinant of the x's with respect to the x''s. We will have

$$\Delta' = \Delta J^2.$$

The quantity Δ' will obviously be (like the B''s and the C''s) an integral of equations (1).

Now let an invariant of the n^{th} order be

$$\int M\, dx_1 dx_2 \ldots dx_n.$$

After the change in the variables according to No. 237, it becomes /30

$$\int MJ\, dx_1' dx_2' \ldots dx_n',$$

and MJ must be an integral of equations (1).

I may conclude from this that

$$\frac{\Delta'}{M^2J^2},$$

i.e.,

$$\frac{\Delta}{M^2}$$

must be an integral of equations (1).

Change in Variables

253. When the variables x_i are changed in an arbitrary manner, without affecting the variable t which represents time, it is only necessary to apply the customary rules for the variable change for single or multiple definite integrals to the integral invariants. This is the procedure we have already followed several times.

However, when the variable t is changed, greater difficulty is encountered. It would even appear a priori that this transformation cannot lead to any result.

Let us consider the system

$$dt = \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \dots = \frac{dx_n}{X_n}. \qquad (1)$$

Let us introduce a new variable t_1 defined by the relationship

$$\frac{dt}{dt_1} = Z;$$

Z is the given function of $x_1, x_2, \dots, x_n$.

System (1) must become

$$dt_1 = \frac{dx_1}{ZX_1} = \frac{dx_2}{ZX_2} = \dots = \frac{dx_n}{ZX_n}. \qquad (2)$$

Let us assume that the initial values $x_1^0, x_2^0, \dots, x_n^0$ represent the coordinates of a certain point M_0 in space having n dimensions.

If the motion of this point is defined by equations (1), with t representing time, at the time $t = \tau$ this point will move to M. /31

On the other hand, if the motion is defined by equations (2), with t_i representing time, at the time $t_1 = \tau$ the point M_0 will move to M'.

Let us now consider a figure F_0 occupied at the time zero by different points M_0.

If the motion and the deformation of this figure are defined by equations (1), at the time $t = \tau$ it will become a new figure F.

If the motion is defined by equations (2), at the time $t_1 = \tau$ the figure F_0 will become a new figure F' which is different from F.

Not only will F' be different from F, but in general it will no longer coincide with one of the positions occupied by F at a time which is different from the time $t = \tau$.

It thus appears that we have profoundly changed the given quantities of the problem, and we must not expect that the invariants of (2) may be deduced from the invariants of (1). However, this is what occurs for invariants of order n.

Let us make the change in variables of No. 237. System (1) will become

$$dt = \frac{dy_1}{0} = \frac{dy_2}{0} = \ldots = \frac{dy_{n-1}}{0} = \frac{dz}{1}, \quad (1')$$

and system (2)

$$dt_1 = \frac{dy_1}{0} = \frac{dy_2}{0} = \ldots = \frac{dy_{n-1}}{0} = \frac{dz}{Z}. \quad (2')$$

We must then assume that Z is expressed as functions of the y's and of z.

Let us then set

$$z_1 = \int \frac{dz}{Z},$$

with integration being performed with respect to z (the y's are assumed to be constants), and starting with an arbitrary origin which may depend on the y's.

System (2) becomes

$$dt_1 = \frac{dy_1}{0} = \frac{dy_2}{0} = \ldots = \frac{dy_{n-1}}{0} = \frac{dz_1}{1} \quad (2'')$$

and will have the same form as (1').

Then let /32

$$\int M\, dx_1 dx_2 \ldots dx_n,$$

be an invariant of the order n of equations (1). When the variables are changed according to No. 237, it becomes

$$\int MJ\, dy_1 dy_2 \ldots dy_{n-1} dz;$$

J is the Jacobian of the x's with respect to the y's and z; MJ must be a function of the y's.

And then

$$\int \mathrm{MJ}\, dy_1 dy_2 \dots dy_{n-1} dz_1$$

will be an invariant of equations (2");

$$\int \frac{\mathrm{MJ}}{\mathrm{Z}} dy_1 dy_2 \dots dy_{n-1} dz$$

will be an invariant of equations (2'), and finally

$$\int \frac{\mathrm{M}}{\mathrm{Z}} dx_1 dx_2 \dots dx_n$$

will be an invariant of equations (2).

General Remarks

253'. Let us consider a system of differential equations

$$dx_i = \mathrm{X}_i\, dt, \tag{1}$$

and their variational equations

$$d\xi_i = \Xi_i\, dt. \tag{2}$$

Let us assume that equations (1) include an integral invariant of the first order

$$\int \Sigma \mathrm{A}_i\, dx_i.$$

Expression $\Sigma A_i \xi_i$ will be an integral of equations (2).

On the other hand, these equations (2) will have the solution /33

$$\xi_i = \varepsilon \mathrm{X}_i,$$

with ε being an arbitrary infinitesimal constant.

Let

$$x_i = \varphi_i(t)$$

be an arbitrary solution of equations (1). If ε is a very small constant

$$x_i = \varphi_i(t+\varepsilon) = \varphi_i(t) + \varepsilon \frac{dx_i}{dt}$$

will still be a solution of equations (1), and

$$\xi_i = \varphi_i(t+\varepsilon) - \varphi_i(t) = \varepsilon \frac{dx_i}{dt} = \varepsilon X_i$$

will be a solution of equations (2).

It thus results that

$$\Sigma A_i \xi_i = \varepsilon \Sigma A_i X_i$$

must be a constant.

Therefore, $\Sigma A_i X_i$ is an integral of equations (1).

Let us now assume that equations (1) include an integral invariant of the second order

$$\int\int \Sigma A_{ik}\, dx_i\, dx_k.$$

Then

$$\Sigma A_{ik}(\xi_i \xi'_k - \xi_k \xi'_i)$$

will be an integral of equations (2) and of equations (2'), which may be deduced by changing the ξ_i's into ξ'_i.

Let us set

$$\xi'_i = \varepsilon X_i,$$

with ε being a constant. This is permissible, because $\xi'_i = \varepsilon X_i$ is a solution of (2').

Then

$$\Sigma A_{ik}(\xi_i X_k - X_i \xi_k)$$

will be an integral of (2). This shows that

$$\int \Sigma A_{ik}(X_k\, dx_i - X_i\, dx_k)$$

is an integral invariant of the first order of equations (1).

This procedure makes it possible to obtain an invariant of the order n - 1, when one knows an invariant of the order n. The procedure may sometimes be illusory, because the invariant which is thus obtained may be equal to zero. /34

Let us now envisage an invariant having the following form

$$\int \Sigma (A_i + t B_i)\, dx_i,$$

where A_i and B_i are functions of the x's. We shall encounter invariants having this form below.

Then

$$\Sigma (A_i + t B_i) \xi_i$$

will be an integral of equations (2). As a result,

$$\Sigma (A_i + t B_i) X_i$$

must be a constant.

For purposes of brevity, let us set

$$\Phi = \Sigma A_i X_i; \qquad \Phi_1 = \Sigma B_i X_i,$$

and the expression

$$\Phi + t \Phi_1$$

must be a constant, which entails the condition

$$\frac{\partial \Phi}{\partial t} + t \frac{\partial \Phi_1}{\partial t} + \Phi_1 = 0,$$

or

$$\sum \frac{d\Phi}{dx_i} X_i + \Phi_1 + t \sum \frac{d\Phi_1}{dx_i} X_i = 0. \tag{3}$$

The X_i's, the A_i's, and the B_i's are functions of the x's. The same holds true for

$$\Phi, \quad \Phi_1, \quad \sum \frac{d\Phi}{dx_i} X_i, \quad \sum \frac{d\Phi_1}{dx_i} X_i.$$

The identity (3) can only occur if we also have identically

$$\sum \frac{d\Phi_1}{dx_i} X_i = 0$$

and

$$\sum \frac{d\Phi}{dx_i} X_i + \Phi_1 = 0.$$

The first relationship shows us that Φ_1 is an integral of equations (1). /35

253". Let

$$\Phi = \text{const.}$$

be an integral of equations (2). The function Φ must be of a specific form, a whole and homogeneous polynomial with respect to the ξ_i's, where the coefficients depend on the x_i's in an arbitrary manner.

Let m be the degree of this polynomial. The expression

$$\int \sqrt[m]{\Phi'}$$

(where Φ' is nothing else than Φ, where the ξ_i's were replaced by the differentials dx_i) will be an integral invariant of equations (1).

Under this assumption, let I be an arbitrary invariant of the specific form Φ.

Let us make the change in variables according to No. 237, and the equations (1) will become

$$\frac{dy_i}{dt} = 0, \qquad \frac{dz}{dt} = 1, \tag{1'}$$

and, if one employs η_i and ζ to designate the variations of y_i and z, the variational equations of (1') will be reduced to

$$\frac{d\eta_i}{dt} = \frac{d\zeta}{dt} = 0.$$

With these new variables, Φ will have the specific form Φ_0, which is whole, homogeneous, and has the degree m with respect to the η_i's and ζ. The coefficients may be arbitrary functions of the y_i's. However, according to the theory presented in No. 237, since we are dealing with an integral invariant, these coefficients cannot depend on z.

The x_i's are functions of the y's and of z, and the following relationships between the variations may be deduced

$$\xi_i = \sum \frac{dx_i}{dy_k} \eta_k + \frac{dx_i}{dz} \zeta. \tag{4}$$

The ξ's are therefore linear functions of the η's and of ζ, and the determinant of these linear equations (4) is nothing else than the Jacobian of the x's with respect to y and to z. I have called the Jacobian J. /36

One then passes from the form Φ to the form Φ_0 by linear substitution (4), whose determinant is J.

Let I_0 be the invariant of Φ_0, which corresponds to the invariant I of Φ. We will have

$$I = I_0 J^p$$

with p being the degree of the invariant.

However, I_0 is a function of the coefficients of Φ_0 and, consequently, a function of the y's, which is independent of z. It is therefore an integral of equations (1).

Let M be the last multiplier of equations (1), in such a way that we have

$$\sum \frac{dMX_i}{dx_i} = 0$$

and that

$$\int M\, dx_1\, dx_2 \dots dx_n$$

is an integral invariant of the order n.

We have seen in No. 252 that MJ will be an integral of equations (1). Therefore,

$$I_0(MJ)^p = IM^p$$

will be an integral of equations (1). An integral of these equations therefore corresponds to each invariant of the form Φ.

Now let C be a covariant of the form Φ, having the degree p with respect to the coefficients of Φ, and the degree q with respect to the variables ξ.

If C_0 is the corresponding covariant of Φ_0, we will have

$$C = C_0 J^p.$$

The coefficients of C_0 are functions of the coefficients of Φ_0, and they are therefore independent of z. The same holds true for those of

$$C_0(MJ)^p = CM^p.$$

Therefore, CM^p is an integral of equations (2); therefore,

$$\int \sqrt[q]{C'M^p}$$

is an integral invariant of equations (1), where C' is none other than C, where the ξ_i's have been replaced by dx_i. /37

We therefore have a method of forming a great number of integral invariants. The particular case in which p is zero (i.e., the case of the so-called absolute invariants or covariants) merits particular attention. If C, for example, is an absolute covariant of Φ

$$\int \sqrt[q]{C'}$$

will be an integral invariant of equations (1). One may therefore form a new integral invariant without knowing the last multiplier M.

The same procedure may be applied to integral invariants of higher order. For example, let

$$\int \Sigma A_{ik}\, dx_i\, dx_k$$

be an integral invariant of the second order. The bilinear form

$$\Phi = \Sigma A_{ik}(\xi_i \xi'_k - \xi_k \xi'_i)$$

which is an integral of equations (2) and (2') is connected with this integral invariant.

Every invariant or covariant of this form, multiplied by one appropriate power of M, will be an integral of equations (2), (2') and will consequently produce a new integral invariant.

In the same way, if one has a system of integral invariants, a system of forms which are similar to Φ may be deduced from it, which will be integrals of equations (2), (2'). An integral of equations (1) will correspond to every invariant of this system of forms. An integral invariant of equations (1) will correspond to every covariant of this system of forms.

For example, let F and F_1 be two quadratic forms with respect to the ξ's. They become F' and F'_1 when the ξ_i's are replaced by the differentials dx_i. Let us assume that F and F_1 are integrals of (2) and that, consequently,

$$\int\sqrt{F'},\ \int\sqrt{F'_1}$$

are integral invariants of (1).

Let us consider the form /38

$$F - \lambda F_1$$

where λ is an unknown. When stating that the discriminant of this form is zero, we shall obtain an algebraic equation of degree n in λ, for which the n roots will obviously be absolute invariants of the system of forms F, F_1. These will therefore be integrals of equations (1).

However, this is not all. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be these roots, and F and F_1 can be written in the form

$$F = \lambda_1 A_1^2 + \lambda_2 A_2^2 + \ldots + \lambda_n A_n^2,$$
$$F_1 = A_1^2 + A_2^2 + \ldots + A_n^2,$$

with $A_1, A_2, \ldots, A_n$ being the linear forms which may be determined by purely algebraic operations.

The quantities $A_1, A_2, \ldots, A_n$ may be regarded as the covariants of zero degree of the F, F_1 system, so that

$$\int A'_1,\ \int A'_2,\ \ldots,\ \int A'_n$$

are the integral invariants of equations (1), if A'_i designates what A_i becomes when the ξ_i's are replaced by the differentials dx_i.

However, there would be an exception if the equation for λ had multiple roots. For example, if λ_1 were equal to λ_2, it could no longer be stated that

$$\int A'_1,\ \int A'_2$$

are integral invariants, but only that

$$\int \sqrt{A'^2_1 + A'^2_2}$$

is an integral invariant.

Now let

$$\int \Sigma A_{ik}\, dx_i dx_k,\quad \int \Sigma B_{ik}\, dx_i dx_k$$

be two integral invariants of the second order. The two bilinear forms /39

$$\Phi = \Sigma A_{ik}(\xi_i \xi'_k - \xi_k \xi'_i)$$
$$\Phi_1 = \Sigma B_{ik}(\xi_i \xi'_k - \xi_k \xi'_i)$$

will be integrals of (2) and (2').

The most interesting case is that in which n is even; therefore, let n = 2m.

Let us consider the form

$$\Phi - \lambda \Phi_1$$

and let us make its determinant equal to 0. We shall have an algebraic equation for λ of degree n = 2m. However, the first term in this equation is a perfect square, so that it may be reduced to an equation of order m. The m roots

$$\lambda_1, \lambda_2, \ldots, \lambda_m$$

will be integrals of equations (1), for the same reason as above.

Now Φ and Φ_1 can be written in the form

$$\Phi = \sum_{i=1}^{i=m} \lambda_i (P_i Q'_i - Q_i P'_i)$$

$$\Phi_1 = \Sigma (P_i Q'_i - Q_i P'_i)$$

and the P_i's and the Q_i's are 2m linear polynomials with respect to the ξ's. The P'_i's and the Q'_i's are the same polynomials, where the ξ_m's have been replaced by the ξ''s .

Then the expressions

$$P_1 Q'_1 - Q_1 P'_1, \quad P_2 Q'_2 - Q_2 P'_2, \quad \ldots, \quad P_m Q'_m - Q_m P'_m$$

will be covariants of the system Φ, Φ_1, and consequently integrals of (2), (2') to which the integral invariants will correspond.

There would be an exception to this if the equation for λ had multiple roots.

If we have, for example,

$$\lambda_1 = \lambda_2$$

it could no longer be stated that the two expressions /40

$$P_1 Q'_1 - P'_1 Q_1, \quad P_2 Q'_2 - P'_2 Q_2$$

are integrals of (2), (2'), but only that their sum

$$P_1 Q'_1 - P'_1 Q_1 + P_2 Q'_2 - P'_2 Q_2$$

is an integral of (2), (2').

CHAPTER XXIII.

FORMATION OF INVARIANTS

Use of the Last Multiplier

254. There is an integral invariant which may be formed very readily when the last multiplier of the differential equations is known. /41

Let

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n} = dt, \qquad (1)$$

be our differential equations.

Let us assume that there is a function M of x_1, x_2, ..., x_n, so that we also have, identically

$$\frac{d(MX_1)}{dx_1} + \frac{d(MX_2)}{dx_2} + \ldots + \frac{d(MX_n)}{dx_n} = 0.$$

This function M is called the last multiplier.

It may then be stated that the integral of the order n

$$J = \int M\, dx_1\, dx_2 \ldots dx_n$$

is an integral invariant. Let us assume that equations (1) have been integrated; expressing x_1, x_2, ..., x_n as functions of t and of n integration constants

$$\alpha_1, \quad \alpha_2, \quad \ldots, \quad \alpha_n,$$

the integral J will become

$$J = \int M\, \Delta d\alpha_1\, d\alpha_2 \ldots d\alpha_n.$$

The quantity Δ is the Jacobian or the functional determinant of the x's with respect to the α's. We will then have /42

$$\frac{dJ}{dt} = \int \frac{dM\Delta}{dt}\, d\alpha_1\, d\alpha_2 \ldots d\alpha_n.$$

However,

$$\frac{dM\Delta}{dt} = M\frac{d\Delta}{dt} + \Delta\frac{dM}{dt};$$

$$\frac{dM}{dt} = \Sigma X_i \frac{dM}{dx_i}.$$

On the other hand, $$\Delta = \left| \frac{dx_1}{d\alpha_1}, \frac{dx_2}{d\alpha_1}, \ldots, \frac{dx_n}{d\alpha_1} \right|.$$

I may only write the first line of this determinant; the others may be deduced from it by changing α_1 to $\alpha_2, \alpha_3, \ldots, \alpha_n$.

Therefore, $\Delta + dt\frac{d\Delta}{dt}$ must be the Jacobian of the

$$x_i + dt\frac{dx_i}{dt} = x_i + X_i\,dt$$

with respect to the α's. This will be the product of the Jacobian of the x_i's with respect to the α's -- i.e., of Δ, and the Jacobian of the $x_i + X_i dt$'s with respect to the x_i's, which I shall call D. I may write

$$\Delta + dt\frac{d\Delta}{dt} = \Delta . D.$$

However, the Jacobian D may be readily formed. The elements of the principal diagonal are finite, and that belonging to the i^{th} line and to the i^{th} column may be written

$$1 + dt\frac{dX_i}{dx_i}.$$

The other elements are infinitely small; that belonging to the i^{th} line and to the k^{th} column ($i \gtrless k$) may be written

$$dt\frac{dX_i}{dx_k}.$$

It thus results that, neglecting terms on the order of dt^2, we will have

$$D = 1 + dt\sum\frac{dX_i}{dx_i},$$

from which it follows that /43

$$\frac{d\Delta}{dt} = \Delta \sum \frac{dX_i}{dx_i}.$$

One may conclude that

$$\frac{dM\Delta}{dt} = \Delta\Sigma X_i \frac{dM}{dx_i} + \Delta\Sigma M \frac{dX_i}{dx_i} = \Delta \sum \frac{d(MX_i)}{dx_i} = 0,$$

from which we finally have

$$\frac{dJ}{dt} = 0.$$

q.e.d.

Equations of Dynamics

255. In the case of equations of dynamics, a great number of integral invariants may be readily formed. From Sections 56 on, we learned how to form a certain number of integrals of the variational equations, and in the preceding chapter we learned how to deduce integral invariants from them.

The first integral (equations 3, Vol. I, p. 167) is as follows

$$\eta'_1 \xi_1 - \xi'_1 \eta_1 + \eta'_2 \xi_2 - \xi'_2 \eta_2 + \ldots = \text{const.}$$

The integral invariant which may be deduced from it is as follows

$$J_1 = \int (dx_1\, dy_1 + dx_2\, dy_2 + \ldots + dx_n\, y_n).$$

It is of the second order and is of the greatest importance for the statements which will follow. A little farther on (still p. 167, Vol. I), I obtained a second integral which I may write

$$\Sigma \begin{vmatrix} \xi_i & \xi'_i & \xi''_i & \xi'''_i \\ \eta_i & \eta'_i & \eta''_i & \eta'''_i \\ \xi_k & \xi'_k & \xi''_k & \xi'''_k \\ \eta_k & \eta'_k & \eta''_k & \eta'''_k \end{vmatrix} = \text{const.}$$

The integral invariant which I may deduce from it is of the fourth order and may be written as follows

$$J_2 = \int \Sigma\, dx_i\, dy_i\, dx_k\, dy_k.$$

The summation indicated by the sign Σ may be extended over the $\frac{n(n-1)}{2}$ /44
combinations of the indices i and k.

In the same way, the integral

$$J_3 = \int \Sigma \, dx_i \, dy_i \, dx_k \, dy_k \, dx_l \, dy_l,$$

where the summation is extended over $\frac{n(n-1)(n-2)}{6}$ combinations of the three indices i, k and l, will still be an invariant, and so on.

We thus obtain n integral invariants if we have n pairs of conjugate variables. One of these invariants J_1 will be of the second order; another J_2 will be of the fourth order; another J_3 will be of the sixth order, ..., and the last J_n will be of the order 2n.

However, it is not necessary to assume that these invariants are all different. At the end of No. 247, I stated that from an invariant of the second order, one can always deduce an invariant of the fourth order, an invariant of the sixth order, and so on. The invariants J_1, J_2, ..., J_n which I have just defined are none other than those which may be deduced from the first of them.

These invariants may be considered in another way. At the beginning of page 169, Volume I, I demonstrated the manner in which one could deduce the Poisson theorem from the integral (3) on page 157, or -- which amounts to the same thing -- from the integral invariant J_1.

Following the same procedure with the invariant J_2, one would obtain a theorem similar to that of Poisson.

Let

$$\Phi, \quad \Phi_1, \quad \Phi_2, \quad \Phi_3,$$

be four integrals of the equations of dynamics.

Let

$$\Delta_{ik}$$

be the Jacobian of these four integrals with respect to

$$x_i, \quad y_i, \quad x_k, \quad y_k.$$

The expression

$$\Sigma \Delta_{ik},$$

where the summation is extended over all combinations of the indices i,k, will still be an integral.

A similar theorem would be obtained by commencing with any of the invariants J_3, J_4, ..., J_n. /45

However, according to the statements which I have just made, none of the theorems is different from that of Poisson in reality.

However, from among all of these invariants, great importance may be attributed to the last of them

$$J_n = \int dx_1 \, dy_1 \, dx_2 \, dy_2 \dots dx_n \, dy_n.$$

It could be obtained by the procedure given in the preceding section. It is known that the equations of dynamics have unity as the last multiplier.

256. I shall now assume that the x's designate the rectangular coordinates of n points in space, and I shall employ the notation given on page 169 of Vol. 1.

On page 170, we obtained the following integral of the variational equations

$$\sum \frac{y\eta}{m} - \sum \frac{dV}{dx} \xi = \text{const.}$$

The corresponding integral invariant may be written

$$\int \sum \left(\frac{y \, dy}{m} - \frac{dV}{dx} \, dx \right).$$

In the same way, the invariant

$$\int (dy_{11} + dy_{12} + \dots + dy_{1n})$$

corresponds to the integral

$$\Sigma \eta_{1i} = \text{const.}$$

The invariant

$$\int \Sigma (x_{1i} \, dy_{2i} - y_{1i} \, dx_{2i} - x_{2i} \, dy_{1i} + y_{2i} \, dx_{1i})$$

corresponds to the integral

$$\Sigma_i(x_{1i}\eta_{2i} - y_{1i}\xi_{2i} - x_{2i}\eta_{1i} + y_{2i}\xi_{1i}) = \text{const.}$$

However, none of these invariants is of great interest, since they may be immediately deduced from the integrals of energy, center of gravity, and area.

This does not hold for the following, which occurs when the function V is homogeneous with respect to the x's. /46

In No. 56, we learned that if V is homogeneous of degree -1, the variational equations have the integral

$$\Sigma(2x_{ki}\eta_{ki} + y_{ki}\xi_{ki}) = 3t\left[\sum\left(\frac{y_{ki}\eta_{ki}}{m_i} - \frac{dV}{dx_{ki}}\xi_{ki}\right)\right] + \text{const.},$$

or, removing the indices, we have

$$\Sigma(2x\eta + y\xi) = 3t\sum\left(\frac{y\eta}{m} - \frac{dV}{dx}\xi\right) + \text{const.}$$

It may be stated more generally that if V is homogeneous of order p, the same procedure leads to the following integral

$$\Sigma(2x\eta - py\xi) = (2-p)t\sum\left(\frac{y\eta}{m} - \frac{dV}{dx}\xi\right) + \text{const.},$$

from which we obtain the integral invariant

$$J = \int\Sigma(2x\,dy - py\,dx) + (p-2)t\int\sum\left(\frac{y\,dy}{m} - \frac{dV}{dx}dx\right),$$

an invariant which has a very special nature since it depends on time.

The second integral may be written

$$\int d\sum\left(\frac{y^2}{2m} - V\right),$$

and is therefore an integral of an exact differential. It may be readily seen that

$$\sum\left(\frac{y^2}{2m} - V\right)$$

is none other than the energy constant, which I shall call C.

The invariant J is of the first order; it is therefore an integral taken along an arc of an arbitrary curve. Let C_0 and C_1 be the values for the energy constant at the two ends of this arc.

This arc is the figure which we have called F_0 in the preceding chapter. When this figure is deformed to become F, C_0 and C_1 do not /47 change, as I explained in the preceding chapter.

As a result, we have

$$J = \int \Sigma(2x\,dy - py\,dx) + (p-2)t(C_1 - C_0).$$

The integral

$$\int \Sigma(2x\,dy - py\,dx)$$

is therefore not constant when figure F (which is reduced to an arc of a curve here) is deformed; however, these variations are proportional to time.

The integral is constant, if the two ends of the arc correspond to a single value for the energy constant.

In particular, this is also true if the arc of the curve is closed. This integral is therefore a relative invariant, as I designated it in the preceding chapter.

However, if one assumes that the arc of the curve is closed, an arbitrary exact differential may be added under the $\int$ sign without changing the value of the integral. For example, we may add

$$\Sigma(x\,dy + y\,dx),$$

with an arbitrary constant coefficient.

Thus, the integrals

$$\int \Sigma y\,dx, \quad \int \Sigma x\,dy$$

are also relative invariants.

We saw in No. 238 that an absolute invariant of the second order may always be deduced from a relative invariant of the first order. The invariant of the second order which is thus obtained is none other than

$$J_1 = \int \Sigma\, dx\, dy,$$

which we studied above.

This is the case in which the expression

$$\Sigma(2x\, dy - py\, dx),$$

which appears under the $\int$ sign, becomes an exact differential. This /48 is the case in which p = -2, which would occur if the attraction, instead of following Newton's law, followed the inverse of the cube of the distance. We then have

$$\int \Sigma(2x\, dy - py\, dx) = \Sigma 2xy.$$

The quantity $\Sigma 2xy$ is therefore a polynomial of the first degree with respect to time, and since

$$\Sigma 2xy = \Sigma 2mx \frac{dx}{dt} = \frac{d}{dt} \Sigma mx^2,$$

the expression Σmx^2 is a polynomial of the second degree with respect to time.

Jacobi reached this result at the beginning of his Vorlesungen.*

However, in general,

$$\Sigma(2x\, dy - py\, dx)$$

is not an exact differential.

In the special case of Newtonian attraction, our invariant takes the following form

$$\int \Sigma(2x\, dy + y\, dx) - 3t(C_1 - C_0).$$

Integral Invariants and Characteristic Exponents

257. It may be asked whether there are other algebraic integral invariants in addition to those which we have just formed.

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* Translator's Note: English title is Lectures.

Either the method of Bruns, or the method which I employed in Chapters IV and V, may be employed. As we have seen, the integral invariants correspond to the integrals of the variational equations, and the same procedures could be applied to these equations as are applied to the equations of motion themselves.

However, it may be more advantageous to modify these procedures, at least with respect to form.

Let us set an arbitrary system of differential equations

$$\frac{dx_i}{dt} = X_i, \tag{1}$$

and their variational equations /49

$$\frac{d\xi_i}{dt} = \sum \frac{dX_i}{dx_k} \xi_k. \tag{2}$$

Let us first try to determine the integral invariants of first order having the form

$$\int (B_1 dx_1 + B_2 dx_2 + \ldots + B_n dx_n), \tag{3}$$

in which the expression under the sign $\int$ is linear with respect to the differentials dx, and where the B's are algebraic functions of the x's.

These invariants correspond to the linear integrals of equations (2).

What are the conditions under which equations (2) have integrals which are linear with respect to the ξ's and algebraic with respect to the x's?

Let us assume that values are assigned to the x's which correspond to a periodic solution of period T. The coefficients of equations (2) will be the known functions of t, which will be periodic and have the period T. One can then derive the general solution of equations (2) in the following form

$$\xi_i = \Sigma_k A_k e^{\alpha_k t} \psi_{i,k} \ . \tag{4}$$

The $\psi_{i,k}$'s are periodic functions of t, the α_k's are characteristic exponents, and the A_k's are integration constants.

We can then solve the linear equations (4) with respect to the unknowns $A_k e^{\alpha_k t}$, and we will obtain

$$A_k e^{\alpha_k t} = \Sigma_i \xi_i \theta_{i,k} \tag{5}$$

The $\theta_{i,k}$'s are periodic functions of t.

There will therefore be n relationships of the form (5) between the ξ's, and there will be no others.

If equations (1) and (2) include q <u>different</u> integrals which are linear with respect to the ξ's and algebraic with respect to the x's, some of these q integrals may cease to be different when the x's are replaced by the values corresponding to <u>one</u> of the periodic solutions of equations (1).

What may then be done? /50

Let

$$H_i = B_{1i}\xi_1 + B_{2i}\xi_2 + \dots + B_{ni}\xi_n = \text{const.} \qquad (i = 1, 2, \dots, q)$$

be these q linear integrals, where the B's will be algebraic functions of the x's, which will correspond to q integral invariants of the form (3).

They are different -- i.e., there are no identical relationships between them having the following form

$$\beta_1 H_1 + \beta_2 H_2 + \dots + \beta_q H_q = 0, \tag{6}$$

where the coefficients β are constants. Neither does the following form occur

$$\psi_1 H_1 + \psi_2 H_2 + \dots + \psi_q H_q = 0, \tag{6'}$$

with the ψ's being integrals of equations (1).

Is it then possible that there may be a relationship between them having the following form

$$\varphi_1 H_1 + \varphi_2 H_2 + \dots + \varphi_q H_q = 0, \tag{6''}$$

with the ϕ's being arbitrary functions of the x's alone. According to No. 250, if the same relationships hold, the ratios of the functions ϕ must be the integrals of equations (1).

We will therefore have

$$\frac{\varphi_1}{\psi_1} = \frac{\varphi_2}{\psi_2} = \dots = \frac{\varphi_q}{\psi_q},$$

with the ψ's being integrals, and consequently

$$\psi_1 H_1 + \psi_2 H_2 + \dots + \psi_q H_q = 0,$$

which is contrary to our hypothesis.

An identity relationship of the form (6") cannot therefore exist between the H_i's.

However, if the values corresponding to one special solution, whether it is periodic or not, are assigned to the x's, it could happen that the first term of (6) vanishes identically. It could happen even if equation (6), which is not identically satisfied <u>whatever may be the x's</u>, would hold when the x's are replaced by the appropriately chosen functions of t, that is, by those functions which correspond to a special solution. /51

Every special solution under which this phenomenon is produced, I shall designate as a <u>singular</u> solution.

Under this assumption, two cases may be presented.

The case in which the periodic solutions of equations (1) are all singular;

Or, the case in which they are not all singular.

258. Let us consider a singular solution S. Let us set

$$\beta_1 B_{k.1} + \beta_2 B_{k.2} + \dots + \beta_q B_{k.q} = B_k,$$

from which it follows

$$B_1 \xi_1 + B_2 \xi_2 + \dots + B_n \xi_n = \beta_1 H_1 + \beta_2 H_2 + \dots + \beta_q H_q.$$

Since relationship (6) was not <u>identically</u> verified, we do not have identically

$$B_1 = B_2 = \dots = B_n = 0 . \quad (7)$$

However, since relationship (6) must be verified by the solution S, these relationships (7) (which are algebraic, according to our hypotheses) must be satisfied for the values of the x's which correspond to the solution S.

Now let us set

$$B'_i = X_1 \frac{dB_i}{dx_1} + X_2 \frac{dB_i}{dx_2} + \ldots + X_n \frac{dB_i}{dx_n},$$

and thus

$$B''_i = X_1 \frac{dB'_i}{dx_1} + X_2 \frac{dB'_i}{dx_2} + \ldots + X_n \frac{dB'_i}{dx_n}, \qquad \ldots.$$

The solution S must obviously satisfy the relationships

$$B'_i = 0 \qquad (i = 1, 2, \ldots, n), \tag{7'}$$

and the relationships

$$B''_i = 0 \qquad (i = 1, 2, \ldots, n) \tag{7''}$$

and so on.

We shall therefore successively form the relationships (7), (7'), (7"), etc., and we shall stop when we have arrived at a system of relationships which will only be the result of those which will have been /52 previously formed.

Relationships (7), (7'), (7"), etc., will be algebraic according to our hypotheses, and all of them together will form what I have called in No. 11 a system of invariant relationships.

Therefore, if a system of differential equations permits a singular, periodic solution, it will permit a system of algebraic invariant relationships.

It is probable that the three-body problem permits no other algebraic invariant relationships except those which are already known. I am still not able to prove this.

Under this assumption, let us assume that we have several singular solutions. For each of them, we must have

$$\beta_1 B_{i.1} + \beta_2 B_{i.2} + \ldots + \beta_q B_{i.q} = 0. \tag{8}$$

Only the constants β will not be the same for two different singular solutions. It is therefore not apparent that these two singular solutions must satisfy one and the same system of invariant relationships. However, this is what takes place, as we shall prove.

In order to formulate our ideas, let us assume that $q = 4$; the line of reasoning would be the same in the case of $q > 4$. Let us consider the n relationships

$$\beta_1 B_{i1} + \beta_2 B_{i2} + \beta_3 B_{i3} + \beta_4 B_{i4} = 0 \qquad (i = 1, 2, \ldots, n). \tag{17}$$

Let us form the Table T of the 4 n coefficients B. All of the determinants formed by means of the four columns in this table must be zero.

If this is not the case, we shall obtain one or more relationships which must be satisfied by all the singular solutions, which will include only the x's and which will not include the indeterminate β's.

If they are identically equal to zero, let us consider three of the relationships (17), and we may deduce the following from them

$$\frac{M_1}{\beta_1} = \frac{M_2}{\beta_2} = \frac{M_3}{\beta_3} = \frac{M_4}{\beta_4},$$

The M's are minors of the first order of Table T.

We will therefore have /53

$$M_1 H_1 + M_2 H_2 + M_3 H_3 + M_4 H_4 = 0. \tag{18}$$

This relationship (18) must be identical, because the coefficient of ξ_k is one of the determinants of Table T, which I assume to be identically zero.

We shall therefore have a relationship of the form (6"), which is opposed to our hypothesis, unless one only assumes that all of the M's are identically zero.

If all of the minors of the first order of Table T are identically zero, let us form the minors of the second one.

Let M'_1, M'_2, M'_3 be three of these minors obtained by taking three arbitrary columns in the table and by cancelling the lines 1 and 4 for M'_1, 2 and 4 for M'_2, 3 and 4 for M'_3.

It will become

$$M'_1 H_1 + M'_2 H_2 + M'_3 H_3 = 0. \tag{19}$$

This relationship must be identical, because the coefficient of ξ_k in the first member is one of the minors of the first order of T which I have assumed to be identically zero.

This would still be a relationship of the form (6"), unless one only assumes that all of the minors of the second order M' are identically zero.

If this is the case, it will become identically

$$B_{i2}H_1 - B_{i1}H_2 = 0,$$

which is still a relationship of the form (6").

It may therefore only be the case that all of the determinants of Table T vanish identically. We shall therefore have at least one relationship (and, consequently, a system of invariant relationships) which must be satisfied by all the singular solutions of equations (1).

It may be immediately concluded that all of the solutions of equations (1) cannot be singular.

But this is not all; we may expand our definition of singular solutions.

We have just defined the singular solutions with respect to q integrals H_i of equations (2) which are linear with respect to the ξ's and which correspond to q invariants (linear and of the first order) of equations (1). /54

In the same way, we may provide a definite definition of the singular solutions with respect to q arbitrary integrals

$$H_1, \ H_2, \ \ldots, \ H_q$$

of equations (2) and of equations (2') obtained by replacing the ξ's by the ξ''s.

These integrals must be homogeneous and of the same order, both with respect to the ξ's and with respect to the ξ''s. They will be whole polynomials with respect to these variables, but they will not be necessarily linear with respect to the ξ's. They may therefore correspond to integral invariants of a higher order, or to integral invariants of the first order, but which are not linear.

In addition, these integrals must be different -- i.e., they must not satisfy identically a relationship of the form (6), (6') or (6").

I may then state that a special solution S is singular if a relationship (6) is satisfied for the values of x which correspond to this solution.

We shall then have

$$H_i = \Sigma B_{k.i} A_k,$$

The quantity A_k is a monomial formed by the product of a certain number of factors $\xi_1, \xi_2, \ldots, \xi_n, \xi'_1, \xi'_2, \ldots, \xi'_n$ raised to a suitable power, and the $B_{k.i}$'s are algebraic functions of the x's.

We shall first set, as was done above,

$$B_i = \beta_1 B_{i.1} + \beta_2 B_{i.2} + \ldots + \beta_q B_{i.q},$$

and no changes need be made in the line of reasoning pursued above. We shall arrive at the same conclusion.

Every singular solution with respect to the q integrals H_i satisfies one and the same system of algebraic invariant relationships.

These results are still valid if one envisages the integrals in the following form

$$H_i = B_{1.i}\xi_1 + B_{2.i}\xi_2 + \ldots + B_{n.i}\xi_n + B_{n+1.i}t\xi_1 + B_{n+2.i}t\xi_2 + \ldots + B_{2n.i}t\xi_n.$$

The definition of the singular solutions, with respect to these integrals, will still be the same, and these singular solutions will satisfy one and the same system of algebraic invariant relationships. /55

The proof presented above need only be repeated, without any changes. The coefficients of the quantities $B_{k.i}$ -- which will play the same role in this proof as the ξ_i's -- may be either the ξ_i's, the products of ξ_i and of ξ'_i, or the products of the form $t\xi_i$.

259. I do not wish to delve into the reasons for my belief that all periodic solutions cannot be singular solutions in the case of the three-body problem.

This would take me too far afield from my subject; I shall return to this later. In the meantime, I shall provisionally assume that this proposition is correct, only observing that it is very unlikely that all of the periodic solutions of the three-body problem satisfy a system of invariant relationships, which would be necessary -- according to the preceding section -- in order that they may be singular. We shall again employ the notation and the numbering of equations in No. 257.

If equations (1) and (2) include q different integrals which are linear with respect to the ξ's and algebraic with respect to the x's, these q integrals will still be different when the x's are replaced by the values corresponding to a non-singular periodic solution.

By stating that these q integrals are constants, and by replacing the x's by the values corresponding to a periodic solution in the equations thus obtained, one will obtain q equations of form (5), but where the exponent α_k will be zero. These q equations must therefore be included among equations (5). Therefore, in order that equations (1) include q different integral invariants which are linear with respect to the x's, it is necessary that q of the characteristic exponents α_k be zero for every non-singular periodic solution.

Let us now try to determine the integral invariants of the form

$$\int\sqrt{\Sigma A_i dx_i^2 + 2B_{ik} dx_i dx_k} = \int\sqrt{F(dx_i)}. \tag{7}$$

These invariants will correspond to the integrals of equations (1) and /56 (2) which are quadratic with respect to the ξ's. The integral

$$F(\xi_i) = \text{const.}$$

will correspond to the invariant (7); this integral must be quadratic with respect to the ξ's and algebraic with respect to the x's. In this equation, let us replace the x's by the values corresponding to a non-singular periodic solution. We shall have

$$F^*(\xi_i) = \text{const.}, \tag{8}$$

where F* is a quadratic polynomial which is homogeneous with respect to the ξ's, whose coefficients are periodic functions of t.

It must be possible to deduce all equations of the form (8) from equations (5) in the following manner.

When dealing with a problem of dynamics -- in particular, in the case of the three-body problem -- we have seen that the characteristic exponents are pairwise equal and have the opposite sign. We can therefore group equations (5) by pairs. Let us set

$$A_k e^{\alpha_k t} = \Sigma_i \xi_i \theta_{ik}, \tag{5'}$$

$$B_k e^{-\alpha_k t} = \Sigma_i \xi_i \theta'_{ik}. \tag{5''}$$

When multiplying equations (5') and (5") by each other, we will obtain an equation of the form (8), and all equations of the form (8) must be linear combinations of the equations thus obtained.

If we therefore assume that equations (1) have the canonical form of the equations of dynamics, and that they contain p pairs of conjugate variables, we shall have p pairs of equations similar to (5') and (5"). Consequently, for each periodic solution, we shall have p equations of the form (8) which are linearly independent.

Let us choose one equation from these p equations and their linear combinations, for instance, $F^*(\xi_i)$. Let us follow the same procedure for all of the other periodic solutions. We shall then have a certain polynomial $F^*(\xi_i)$ which is homogeneous and of the second degree with respect to the ξ's, whose coefficients will be functions of the x's which are only defined for values of x which correspond to a periodic solution. /57

We must now determine whether the selection may be made in such a way that the coefficients of F* are algebraic functions of the x's, or even of the known functions of the x's. I shall simply pose this problem, without attempting to solve it at the present time.

Let us now try to determine the invariants of the second order -- i.e., those having the form of a double integral

$$\iint F,$$

where F is a linear function of the products $dx_i dx_k$ (the coefficients of this linear function are naturally functions of the x's). These invariants of the second order will have the following significance.

Let us select equations (1) and (2) once again (we shall always retain the numbering given in No. 257), and let us form in addition the equations

$$\frac{d\xi'_i}{dt} = \sum \frac{dX_i}{dx_k}\xi'_k. \tag{2a}$$

They will lead us to equations which are similar to (5), and which I may write as follows

$$A'_k e^{\alpha_k t} = \Sigma \xi'_i \theta_{ik}. \tag{5a}$$

They only differ from equations (5) because the letters are accented.

According to the preceding chapter, the invariants of the second order will then correspond to those of the integrals of (1), (2) and (2a), which are linear with respect to the determinants

$$\xi_i \xi'_k - \xi_k \xi'_i,$$

and algebraic with respect to the x's.

Let $$F(\xi_i \xi'_k - \xi_k \xi'_i)$$

be one of these integrals. If the x's are replaced by the values corresponding to a periodic solution, we will obtain an equation having the form /58

$$F^*(\xi_i\xi'_k - \xi_k\xi'_i) = \text{const.}, \tag{9}$$

where F* will be a linear function with respect to the determinants

$$\xi_i\xi'_k - \xi_k\xi'_i$$

and whose coefficients will be periodic functions of t.

We have now determined the manner in which all relationships of the form (9), relative to a given periodic solution, may be formed.

In the case of equations of dynamics, equations (5a) may be divided into pairs like equations (5). Let

$$A'_k e^{\alpha_k t} = \Sigma \xi'_i \theta_{ik}, \tag{5a'}$$

$$B'_k e^{-\alpha_k t} = \Sigma \xi'_i \theta'_{ik}. \tag{5a''}$$

be one of these pairs. Let us multiply (5a') by (5''), (5a'') by (5'), and let us subtract. We shall obtain an equation having the form (9). Each pair of equations will give us one, and all other equations of the form (9) will only be linear combinations of those which thus may be formed.

Let us choose one equation from among all equations of the form (9) thus obtained. Let us follow the same procedure for all other periodic solutions. We shall then have a relationship

$$F^*(\xi_i\xi'_k - \xi_k\xi'_i) = \text{const}.$$

whose first term will be a linear function of the determinants. The coefficients of this linear function will be functions of the x's which are only defined for values of the x's corresponding to a periodic solution.

We must now determine whether the selection may be made so that these coefficients are algebraic functions or even the known functions of the x's.

Let us now return to the linear invariants of the first order. According to No. 29, the form of equations (4), and consequently that of equations (5), is modified when two or more characteristic exponents become equal.

If, for example, nine of these exponents equal zero, we may write the corresponding equations (5) in the following form /59

$$P_k = \Sigma_i \xi_i \theta_{ik}. \tag{10}$$

The quantity P_k designates a whole polynomial with respect to t, having constants for coefficients.

These polynomials are of the degree q - 1 at most. In order to specify this more precisely, the number of polynomials is q. The first may be reduced to a constant, the second is of degree one at most, the third is of degree two at most, and so on, and finally the last is of degree q - 1 at most.

In the case in which the degree of this last polynomial reaches its maximum and is equal to q - 1, the polynomial before the last is a derivative of the last, the q - 2nd one the derivative of the q - 1st one, and so on.

In every case, the q polynomials may be divided into several groups. In each group, the first polynomial may be reduced to a constant, and each of them is the derivative of the following.

In order that there may be p linear integral invariants, it is not sufficient that p of the characteristic exponents are zero. It is necessary that p of the polynomials P_k be reduced to constants (or, which is the same thing, that these polynomials be at least divided into p groups).

From the point of view of our study, what is then the significance of equations (10) where P_k may not be reduced to a constant?

In No. 216 we defined an integral invariant whose role is very important. This invariant has the form

$$\int F + t \int F_1.$$

where F and F_1 are functions which are algebraic with respect to the x's, and linear with respect to the differentials dx.

A similar invariant corresponds to an integral of equations (2) having the following form

$$F + tF_1 = \text{const.},$$

where F and F_1 are functions which are algebraic with respect to the x's, and linear with respect to the ξ's.

In this integral, if I replace the x's by the values which correspond to a periodic solution, we shall have /60

$$F^* + tF_1^* = \text{const.}, \qquad (11)$$

where F^* and F_1^* are functions which are linear with respect to the ξ's, whose coefficients are periodic functions of t.

We have now determined the manner in which we may obtain all relationships of (11) starting with equations (10).

Let us consider two polynomials P_k, the first being reduced to a constant, and the second being of the first degree; the first is the derivative of the second. The corresponding equations (10) may be written

$$A_1 = \Sigma \xi_i \theta_i, \qquad (10')$$

$$A_2 + A_1 t = \Sigma \xi_i \theta'_i, \qquad (10'')$$

where the θ_i's and the θ'_i's are periodic in t. We may thus deduce

$$\Sigma \xi_i \theta'_i - t \Sigma \xi_i \theta_i = \text{const.},$$

which is a relationship of the form (11).

We should note that equation (10'), raised to the square, provides us with a relationship of form (8), and that a relationship of form (9) may be deduced from equations (10') and (10"), that is,

$$(\Sigma \xi_i \theta_i)(\Sigma \xi'_i \theta'_i) - (\Sigma \xi'_i \theta_i)(\Sigma \xi_i \theta'_i) = \text{const.}$$

260. Let us apply this procedure to the three-body problem, and let us determine what may be the maximum number of integral invariants, of the several types studied in the preceding section, for this problem. That is:

The first type: linear invariants with respect to the differentials dx;

The second type: invariants where the function under the sign $\int$ is the square root of a second-degree polynomial with respect to the differentials of the x's;

The third type: invariants of the second order, which are linear with respect to products of the differentials $dx_i dx_k$;

The fourth type: invariants having the form considered at the end of the preceding section -- i.e., having the form /61

$$\int F + t \int F_1.$$

These different types of invariants correspond to different types of integrals of equations (2) and (2a), that is:

The first type: linear integrals with respect to ξ's;

The second type: quadratic integrals with respect to the ξ's;

The third type: linear integrals with respect to the determinants $\xi_i \xi'_k - \xi_k \xi'_i$;

The fourth type: integrals having the form

$$F + tF_1,$$

where F and F_1 are linear with respect to the ξ's.

We may assume that it is extremely probable that none of the periodic solutions of the three-body problem is singular.

In the three-body problem, the number of degrees of freedom is six; the number of characteristic exponents is twelve. According to the ideas presented in No. 78, there are six, and six alone, which vanish; the six others are equal pairwise, and have the opposite sign. There are therefore six equations of form (10) and six polynomials P_k, of which four are of degree zero and two are of degree one. Or, there are three pairs of equations having the form (5'), (5"), four equations having the form (10'), and two equations having the form (10").

Let us therefore determine how many independent invariants of each type there will be.

I shall state more precisely what I mean. I do not regard n invariants of the first type as independent

$$\int F_1, \int F_2, \ldots, \int F_n,$$

or n invariants of the second type

$$\int \sqrt{F_1}, \int \sqrt{F_2}, \ldots, \int \sqrt{F_n},$$

or n invariants of the third type

$$\iint F_1,\ \iint F_2,\ \ldots,\ \iint F_n,$$

or n invariants of the fourth type /62

$$\int F_1,\ \int F_2,\ \ldots,\ \int F_n \qquad (F_i = F_i' + i F_i''),$$

when there is an identical relationship between F_1, F_2, ..., F_n having the form

$$\Phi_1 F_1 + \Phi_2 F_2 + \ldots + \Phi_n F_n = 0,$$

where Φ_1, Φ_2, ..., Φ_n are integrals of equations (1).

It is apparent that we cannot have more than four invariants of the first type, i.e., no more than the number of equations (10') already known.

We cannot have more than thirteen invariants of the second type, of which three will come from the three pairs of equations having the form (5') and (5"), and the six others will be obtained by means of the squares of the four equations (10') and of their products by pairs. These last ten exist in actuality. However, they are not independent of the four invariants of the first type, since they may be deduced by the procedure given in No. 245. We may therefore have three new invariants.

We cannot have more than eleven invariants of the third type, of which three will come from the three pairs of equations having the form (5') (5"). Six will be obtained by combining the four equations (10') by pairs; two will be obtained by combining the two equations (10") with the corresponding equation (10').

Seven of these invariants are known. One is the invariant J_1 of No. 255; the six others are those which may be deduced from the four equations (10'), but they may not be regarded as independent of the four invariants of the first type, since they may be deduced by the procedure given in No. 247.

We may therefore have four new invariants of the third type.

Finally, we may not have more than two invariants of the fourth type, i.e., no more than the number of equations (10").

One of these invariants is known, that of No. 256; we may still have a new invariant.

It is probable that these new invariants, the possibility of which was not excluded in the preceding discussion, do not exist. However, in order to prove this, we must resort to other procedures -- for example, procedures similar to those of the method advanced by Bruns. /63

Use of Kepler Variables

261. The invariant of the fourth type in No. 256 may be written in still another form.

Let us set an arbitrary system of canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}. \tag{1}$$

Let us consider the following integral taken along an arbitrary curve arc

$$J = \int (x_1\,dy_1 + x_2\,dy_2 + \ldots + x_n\,dy_n).$$

Let us assume that we are writing the equations of the curve arc along which integration is performed, expressing the x's and the y's as a function of the parameter α, and that the values of this parameter which correspond to the ends of the arc are α_0 and α_1. The integral J will equal

$$J = \int_{\alpha_0}^{\alpha_1} \left[\Sigma x \frac{dy}{d\alpha}\right] d\alpha.$$

Let us assume that we are considering our curve arc like the figure F in the preceding chapter, which varies with time and may be reduced to F_0 for t = 0.

Then the x's, the y's and the functions of the x's and the y's, such as F, $\frac{dF}{dx}$, $\frac{dF}{dy}$, ..., will be functions of α and of t.

We shall have

$$\frac{dJ}{dt} = \int \left[\sum \frac{dx}{dt}\frac{dy}{d\alpha}\right] d\alpha + \int \left[\Sigma x \frac{d^2 y}{dt\,d\alpha}\right] d\alpha$$

or

$$\frac{dJ}{dt} = \int \left[\sum \frac{dF}{dy} \frac{dy}{d\alpha} \right] d\alpha - \int \left[\Sigma x \frac{d}{d\alpha} \left(\frac{dF}{dx} \right) \right] d\alpha$$

when integrating by parts

$$\frac{dJ}{dt} = \int \left[\sum \frac{dF}{dy} \frac{dy}{d\alpha} \right] d\alpha + \int \left[\sum \frac{dF}{dx} \frac{dx}{d\alpha} \right] d\alpha - \left[\Sigma x \frac{dF}{dx} \right].$$

However, /64

$$\sum \frac{dF}{dy} \frac{dy}{d\alpha} + \sum \frac{dF}{dx} \frac{dx}{d\alpha} = \frac{dF}{d\alpha},$$

and therefore

$$\frac{dJ}{dt} = \left[F - \Sigma x \frac{dF}{dx} \right]_{\alpha=\alpha_0}^{\alpha=\alpha_1}. \tag{2}$$

If we assume that F is homogeneous and has the degree p with respect to the x's, it will become

$$\Sigma x \frac{dF}{dx} = pF.$$

Let C be the energy constant, so that the equation of energy may be written

$$F = C.$$

Let C_0 and C_1 be the values of this constant which correspond to α_0 and α_1; it will become

$$\frac{dJ}{dt} = (1 - p)(C_1 - C_0). \tag{3}$$

Therefore, strictly speaking, J is not an invariant. However, its derivative, with respect to time, is constant and -- to use the expression defined in the preceding section, it is an invariant of the fourth type.

262. Let us now assume that F presents another type of homogeneity.

Let us divide the pairs of conjugated variables into two classes, and let us use x_i, y_i to designate the pairs of conjugated variables of the first class, and let us use x'_i, y'_i to designate the pairs of conjugated variables of the second class.

I shall assume that F is homogeneous of the order p with respect to the x_i's, to the $(x'_i)^2$'s, and to the $(y'_i)^2$'s, so that we have

$$\Sigma x \frac{dF}{dx} + \frac{1}{2} \Sigma \left(x' \frac{dF}{dx'} + y' \frac{dF}{dy'} \right) = p\,F.$$

Let us then set

$$= \int \left[\Sigma (x\, dy) + \frac{1}{2} \Sigma (x'\, dy' - y'\, dx') \right]$$

or /65

$$J = \int_{\alpha_0}^{\alpha_1} \left[\Sigma x \frac{dy}{d\alpha} + \frac{1}{2} \Sigma \left(x' \frac{dy'}{d\alpha} - y' \frac{dx'}{d\alpha} \right) \right] d\alpha,$$

from which it follows that

$$\frac{dJ}{dt} = \int \left[\sum \frac{dx}{dt} \frac{dy}{d\alpha} + \frac{1}{2} \sum \left(\frac{dx'}{dt} \frac{dy'}{d\alpha} - \frac{dy'}{dt} \frac{dx'}{d\alpha} \right) \right] d\alpha + \int \left[\Sigma x \frac{d^2 y}{d\alpha\, dt} + \frac{1}{2} \Sigma \left(x' \frac{d^2 y'}{d\alpha\, dt} - y' \frac{d^2 x'}{d\alpha\, dt} \right) \right] d\alpha$$

or

$$\frac{dJ}{dt} = \int \left[\sum \frac{dF}{dy} \frac{dy}{d\alpha} + \frac{1}{2} \sum \left(\frac{dF}{dx'} \frac{dx'}{d\alpha} + \frac{dF}{dy'} \frac{dy'}{d\alpha} \right) \right] d\alpha - \int \left[\Sigma x \frac{d}{d\alpha} \frac{dF}{dx} + \frac{1}{2} \Sigma \left(x' \frac{d}{d\alpha} \frac{dF}{dx'} + y' \frac{d}{d\alpha} \frac{dF}{dy'} \right) \right] d\alpha,$$

or, integrating by parts,

$$\frac{dJ}{dt} = \int_{\alpha_0}^{\alpha_1} \sum \left(\frac{dF}{dx} \frac{dx}{d\alpha} + \frac{dF}{dy} \frac{dy}{d\alpha} + \frac{dF}{dx'} \frac{dx'}{d\alpha} + \frac{dF}{dy'} \frac{dy'}{d\alpha} \right) d\alpha - \left[\Sigma x \frac{dF}{dx} + \frac{1}{2} \Sigma \left(x' \frac{dF}{dx'} + y' \frac{dF}{dy'} \right) \right]_{\alpha=\alpha_0}^{\alpha=\alpha_1}$$

or

$$\frac{dJ}{dt} = [F - pF]_{\alpha=\alpha_0}^{\alpha=\alpha_1}$$

or finally

$$\frac{dJ}{dt} = (1-p)(C_1 - C_0),$$

which shows that J is still an invariant of the fourth type.

263. Let us apply the preceding statements to the three-body problem, and let us determine the change in the invariant of No. 256

with the different variables chosen.

In No. 11, we used the following as variables

$$\begin{matrix} \beta L, & \beta G, & \beta\theta, & \beta' L', & \beta' G', & \beta'\theta', \\ l, & g, & \theta, & l', & g', & \theta'. \end{matrix}$$

F is homogeneous of degree -2 with respect to the variables of the first series. Therefore,

$$\int [\beta(L\,dl + G\,dg + \theta\,d\theta) + \beta'(L'\,dl' + G'\,dg' + \theta'\,d\theta')] + 3t(C_1 - C_0)$$

will be an invariant.

The same homogeneity remains if the following variables are chosen, /66 as in No. 12,

$$\begin{matrix} \Lambda, & H, & Z, & \Lambda', & H', & Z', \\ \lambda, & h, & \zeta, & \lambda', & h', & \zeta'. \end{matrix}$$

Therefore,

$$\int (\Lambda\,d\lambda + H\,dh + Z\,d\zeta + \Lambda'\,d\lambda' + H'\,dh' + Z'\,d\zeta') + 3t(C_1 - C_0)$$

will be an invariant.

If the following are chosen as variables (see No. 12)

$$\begin{matrix} \Lambda, & \Lambda', & \xi, & \xi', & p, & p', \\ \lambda, & \lambda', & \eta, & \eta', & q, & q', \end{matrix}$$

the function F will be homogeneous of degree -2 with respect to the Λ's, to the ξ's, to the η's, to the p's, and to the q's.

As a result,

$$\int \Sigma(2\Lambda\,d\lambda + \xi\,d\eta - \eta\,d\xi + p\,dq - q\,dp) + 6t(C_1 - C_0)$$

is an invariant.

The sign Σ indicates that the term which is deduced when the letters are accented must be added to each term. Thus, we have

$$\Sigma\xi\,d\eta = \xi\,d\eta + \xi'\,d\eta'.$$

If finally we select the variables of Nos. 131 and 137

$$\Lambda, \quad \Lambda'; \quad \tau_i,$$
$$\lambda, \quad \lambda'; \quad \sigma_i,$$

we shall see that

$$\int [2\Lambda\, d\lambda + 2\Lambda'\, d\lambda' + \Sigma(\tau_i\, d\sigma_i - \sigma_i\, d\tau_i)] + 6t(C_1 - C_0)$$

will be an invariant of the fourth type.

Remarks on the Invariant Given in No. 256

264. In No. 256, we considered the case in which the x's designate the coordinates of n points in space, and in which the equations of dynamics take the following form /67

$$m \frac{d^2 x}{dt^2} = \frac{dV}{dx},$$

where V is homogeneous of degree p with respect to the x's.

We have seen that in this case

$$J = \int \Sigma(2x\, dy + py\, dx) + (p - 2)t(C_1 - C_0)$$

is an invariant of the fourth type.

Two special cases merit particular attention. Let us assume that

$$p = 2,$$

and we then have

$$J = 2\int \Sigma(x\, dy - y\, dx)$$

and J is an invariant of the first type.

In particular, this is what occurs when one assumes several material points which attract each other in direct ratio to distance. This may be readily verified.

In this case, we have

$$x = \Lambda \cos \lambda t + B \sin t$$

and

$$y = -m\lambda A \sin\lambda t + m\lambda B \cos\lambda t,$$

where λ is an absolute constant, while A and h are integration constants which are different for different pairs of conjugated variables. It then becomes

$$dx = \cos\lambda t\, dA + \sin\lambda t\, dB,$$
$$dy = -m\lambda \sin\lambda t\, dA + m\lambda \cos\lambda t\, dB,$$

from which it follows that

$$x\,dy - y\,dx = m\lambda(A\,dB - B\,dA),$$

which shows that

$$J = 2\lambda \int \Sigma m(A\,dB - B\,dA)$$

is an invariant, since time has disappeared, and that only the integration constants and their differentials enter.

Now let us set p = -2. This is the case which holds when several material points attract each other in inverse ratio to the cube of their distance. /68

The invariant J then becomes

$$J = 2\int \Sigma(x\,dy + y\,dx) - 4t(C_1 - C_0).$$

Here, the quantity under the sign ∫ is the exact differential of the expression

$$S = \Sigma\, xy,$$

so that if the values of S corresponding to the two ends of the integration arc are designated by S_0 and S_1, it becomes

$$J = (2S_1 - 4C_1 t) + (2S_0 - 4C_0 t).$$

In particular, if we assume that one of the ends of the integration arc corresponds to a special situation of the system, where the n material points are at rest and are located at a very great distance from each other, the mutual forces will be very small, so that the velocities of these material points will remain very small for a very long period of time, and the distances will remain very large. As a result, C_0 will be zero, as well as S_0, both for all values of t and for t = 0, and we will still have

$$J = 2S_1 - 4C_1 t.$$

We will therefore have

$$S = 2Ct + B,$$

where B is a new constant, and C is the energy constant, or else

$$\Sigma xy = 2Ct + B,$$

or

$$\Sigma mx \frac{dx}{dt} = 2Ct + B,$$

or, performing integration,

$$\sum \frac{mx^2}{2} = Ct^2 + Bt + A,$$

where A is a third constant.

This is the result which Jacobi obtained at the beginning of his Vorlesungen über Dynamik (Lectures on Dynamics).

Case of the Reduced Problem /69

265. We may reconsider the question which we discussed in No. 260, considering the problems pertaining to the three-body problem, which are, however, somewhat simplified.

I shall first consider what I have designated as the restricted problem, i.e., the problem discussed in No. 9 where two masses describe concentric circumferences, while the third, infinitesimal mass moves in the plane of these two circumferences.

There are then two degrees of freedom. There is one pair having the form (5'), (5"), one equation (10') and one equation (10") (see No. 259).

Therefore, we can have at best an invariant of the first type, which is already known, two invariants of the second type, of which one is known, two invariants of the third type, of which one is known, and one invariant of the fourth type, which is already known.

We can also consider the plane problem -- i.e., the problem of three bodies moving in one plane.

Finally, we may assume that the number of degrees of freedom has been reduced by the procedure given in No. 16. Let us make this assumption in the case of the general problem. We shall then arrive at what I have designated as the general reduced problem. Let us assume that this is true in the case of the plane problem; we will then arrive at what I have designated as the plane reduced problem.

A resumé of the discussion which would be followed in these different cases is given in the following table. /70

	Problems				
	Re-stricted	Plane	General	Reduced Plane	Reduced General
Number of degrees of freedom	2	4	6	3	4
Number of pairs (5'), (5'')	1	2	3	2	3
Number of equations (10')	1	2	4	1	1
Number of equations (10'')	1	2	2	1	1
Maximum number of possible invariants:					
First type	1	2	4	1	1
Second type	2	5	13	3	4
Third type	2	5	11	3	4
Fourth type	1	2	2	1	1
Maximum number of possible new invariants:					
First type	0	0	0	0	0
Second type	1	2	3	2	3
Third type	1	3	4	2	3
Fourth type	0	1	1	0	0

CHAPTER XXIV

USE OF INTEGRAL INVARIANTS

Test Procedures

266. In Volume II we discussed different procedures for finding series which formally satisfy the equations of the three-body problem. Since these series may be of great practical importance and since they are only attained at the price of long and difficult computations, every method which one may find to verify these computations may be very valuable. The consideration of integral invariants provides us with one method which is of interest. /71

Let us call x_i ($i = 1, 2, 3, 4, 5, 6$) the coordinates of two planets (as we stated in Section No. 11 and as we have always done since, the first must be related to the Sun, the second to the center of gravity of the first and of the Sun). On the other hand, let us call Y_i the components of their momentum. These quantities x_i and y_i may be developed in series in the following manner.

Let us recall the results of Chapters XIV and XV, in particular, those obtained in No. 155. In these chapters, instead of the twelve variables x_i and y_i which I have just defined, in order to define the positions of two planets we employed twelve other variables

$$\Lambda, \quad \Lambda', \quad \lambda_1, \quad \lambda'_1, \quad \sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \sigma_4, \quad \tau_1, \quad \tau_2, \quad \tau_3, \quad \sigma_4.$$

In addition, we introduced six arguments

$$w_1, \quad w_2, \quad w'_1, \quad w'_2, \quad w'_3, \quad w'_4,$$

while setting

$$w_i = n_i t + \varpi_i; \qquad w'_i = n'_i t + \varpi'_i,$$

and six other integration constants

$$\Lambda_0, \quad \Lambda'_0, \quad x'^0_1, \quad x'^0_2, \quad x'^0_3, \quad x'^0_4,$$

and we found that the equations of motion could be satisfied in the following way. /72

The quantities

$$\Lambda, \quad \Lambda', \quad \lambda_1 - w_1, \quad \lambda'_1 - w_2, \quad \sigma_i, \quad \tau_i$$

may be developed in powers of μ and of the x'^0_i's . Each term is periodic with respect to the w's and the w''s, and depends in addition on the two integration constants Λ_0 and Λ'_0.

The constants n_i and n'_i may be developed in powers of μ and of the x'^0_i's , and depend on Λ_0 and Λ'_0 in addition.

The $\overline{\omega}_i$'s and the $\overline{\omega}'_i$'s are six integration constants.

Finally,

$$\Lambda\, d\lambda + \Lambda' d\lambda'_1 + \Sigma \sigma_i\, d\tau_i$$

is an exact differential when the twelve variables Λ, λ, σ and τ are replaced by their expansions, and when the w's and the w''s are regarded as six independent variables and the quantities Λ_0, Λ'_0, x'^0_i are regarded as constants in these expansions.

Our quantities x_i and y_i which I have just defined may be expressed readily by means of the twelve variables Δ, λ, σ and τ.

It may be concluded that x_i and y_i may be developed in series in powers of μ and of the x'^0_i's , as well as according to the cosines and the sines of multiples of the w's and the w''s . In addition, each coefficient depends on Λ and Λ'_0.

The expression

$$\Sigma x_i\, dy_i$$

will be an exact differential, if the w's and the w''s are regarded as six independent variables and Λ_0, Λ'_0, x'^0_i are regarded as constants.

We need barely point out that the series thus obtained are not convergent. They are only of value with respect to formal calculations, which gives them, however, a certain practical utility as I explained in Chapter VIII.

Nevertheless, if we substitute these expansions for the x_i's and the y_i's in the expression of an integral invariant, the result of this substitution must, from the formal point of view, satisfy the conditions which must be satisfied by an integral invariant. This provides me with the verification procedure to which I wish to draw attention.

267. We saw above that /73

$$\int \Sigma(2x\,dy + y\,dx) - 3t(c_1 - c_0) \tag{1}$$

is an integral invariant.

In order that we may make use of this invariant, we are going to perform a change of variables which is similar to that given in No. 237.

In order to have greater symmetry in the notation, let us set

$$w'_i = w_{i+2} \quad (i = 1, 2, 3, 4), \qquad n'_i = n_{i+2}, \qquad \varpi'_i = \varpi_{i+2},$$
$$\Lambda_0 = \xi_1, \quad \Lambda'_0 = \xi_2; \qquad x'^0_i = \xi_{i+2}.$$

We have seen that we may develop the x's and the y's in series depending on the w's, the w''s, the Λ_0, Λ'_0, and the x'^0_i's -- i.e., with our new notation, the w_i's and the ξ_i's (i = 1, 2, 3, 4, 5, 6).

For new variables we may then take the ξ_i's and the w_i's, and then the differential equations of motion will take the form

$$\frac{d\xi_i}{0} = \frac{dw_i}{n_i} = dt \tag{2}$$

[just as in No. 237, equations (1) become, after the change in variables,

$$\frac{dy_i}{0} = \frac{dz}{1} = dt$$

as we have seen].

The n_i's are functions of the ξ_i's alone.

However, it is more advantageous to select other variables. Due to the fact that the six n_i's are only functions of the six ξ_i's, nothing prevents us from taking the n_i's and the w_i's as variables, instead of the ξ_i's and the w_i's, so that the differential equations become

$$\frac{dn_i}{0} = \frac{dw_i}{n_i} = dt. \tag{3}$$

An integral invariant of the first order will take the form

$$J = \int (\Sigma A_i dn_i + \Sigma B_i dw_i),$$

where A and B are functions of the n_i's and the w_i's.

I may assume that figure F is a curve arc for which the equations, /74
which are variable with time, have the following form

$$n_i = f_i(\alpha, t); \qquad w_i = f'_i(\alpha, t),$$

where the variables n_i and w_i are expressed as functions of time t and of a parameter α which varies from α_0 to α_1 when the arc F is entirely traversed. The equation of the arc F_0 will then be

$$n_i = f_i(\alpha, 0); \qquad w_i = f'_i(\alpha, 0).$$

With these stipulations, I may then write

$$J = \int_{\alpha_0}^{\alpha_1} \left(\Sigma A_i \frac{dn_i}{d\alpha} + \Sigma B_i \frac{dw_i}{d\alpha} \right) d\alpha,$$

from which it follows that

$$\frac{dJ}{dt} = \int d\alpha \sum \left(\frac{dA_i}{dt} \frac{dn_i}{d\alpha} + \frac{dB_i}{dt} \frac{dw_i}{d\alpha} + A_i \frac{d^2 n_i}{dt\, d\alpha} + B_i \frac{d^2 w_i}{dt\, d\alpha} \right).$$

However, we have

$$\frac{dA_i}{dt} = \Sigma n_k \frac{dA_i}{dw_k},$$
$$\frac{dB_i}{dt} = \Sigma n_k \frac{dB_i}{dw_k},$$
$$\frac{d^2 n_i}{dt\, d\alpha} = 0; \qquad \frac{d^2 w_i}{dt\, d\alpha} = \frac{dn_i}{d\alpha},$$

from which it finally follows that

$$\frac{dJ}{dt} = \int \Sigma_i \left[dn_i \left(\Sigma_k n_k \frac{dA_i}{dw_k} + B_i \right) + dw_i \Sigma_k n_k \frac{dB_i}{dw_k} \right].$$

If J is an absolute integral invariant, we must therefore have

$$\Sigma_k n_k \frac{dB_i}{dw_k} = 0, \tag{4}$$

$$\Sigma_k n_k \frac{dA_i}{dw_k} = -B_i. \tag{5}$$

Let us now determine what occurs in the case when the A's and the B's are periodic functions of the w's and may be, consequently, developed in trigonometric series.

Let us first consider equation (4), and let us set

$$B_i = \Sigma[b\cos(m_1 w_1 + \ldots + m_6 w_6) + b'\sin(m_1 w_1 + \ldots + m_6 w_6)],$$

where the b's and the b''s depend on the n_i's.

Equation (4) becomes /75

$$\Sigma(m_1 n_1 + \ldots + m_6 n_6)[-b\sin(m_1 w_1 + \ldots + m_6 w_6) + b'\cos(m_1 w_1 + \ldots + m_6 w_6)] = 0,$$

which may only hold if

$$m_1 n_1 + \ldots + m_6 n_6 = 0, \tag{6}$$

or if

$$b = b' = 0.$$

However, the m's are integer constants, and the n's are our independent variables between which no linear relationship may hold. Equation (6) therefore entails the following

$$m_1 = m_2 = \ldots = m_6 = 0.$$

This means that the trigonometric expansion of B_i may be reduced to its known term -- i.e., B_i is a function of the n_i's alone, and is independent of the w's.

Let us now pass to equation (5). Let us set

$$A_i = \Sigma(a\cos\omega + a'\sin\omega),$$

writing ω, for purposes of brevity, instead of

$$m_1 w_1 + \ldots + m_6 w_6.$$

Equation (5) may then be written

$$\Sigma(m_1 n_1 + \ldots + m_6 n_6)(-a\sin\omega + a'\cos\omega) = -B_i.$$

Let us first consider a term which is dependent on the w's, i.e., such that m_1, m_2, ..., m_0 are not zero at the same time. We shall then have

$$m_1 n_1 + \ldots + m_6 n_6 \gtrless 0.$$

In the second term, B_i does not depend on the w's. This second term contains neither a term for cosω, nor a term for sinω. As a result, we have

$$a = a' = 0.$$

Therefore, A_i does not depend on the w's, and may be reduced to the known term of its trigonometric expansion, a term which depends only on the n_i's.

However, equation (5) may then be reduced to /76

$$B_i = 0.$$

In general, every linear absolute, integral invariant of the first order, where the term under the sign $\int$ is algebraic with respect to the x's and the y's and, consequently, periodic with respect to the w's, must have the following form

$$\int \Sigma A_i \, dn_i,$$

where the A_i's depend only on the n_i's. In reality, this is what occurs for the absolute invariants which we know and which are obtained by differentiating the integrals of area, energy or motion of the center of gravity.

However, the relative invariant

$$J = \int \Sigma (2x \, dy + y \, dx)$$

deserves more attention. We have seen that

$$J - 3t(C_1 - C_0)$$

(where C_0 and C_1 are the values of the energy constant at the two ends of the arc F_0) is an integral invariant. We shall therefore have

$$\frac{dJ}{dt} = 3(C_1 - C_0) = 3 \int dC. \tag{7}$$

If we set

$$J = \int \Sigma (A_i \, dn_i + B_i \, dw_i),$$

equation (7) becomes

$$\int \Sigma_i \left[dn_i \left(\Sigma_k n_k \frac{dA_i}{dw_k} + B_i \right) + dw_i \, \Sigma_k n_k \frac{dB_i}{dw_k} \right] = 3 \int \sum_i \frac{dC}{dn_i} \, dn_i,$$

because the energy constant C is only a function of the n_i's.

Equations (4) and (5) must therefore be replaced by the following equations /77

$$\Sigma_k n_k \frac{dB_i}{dw_k} = 0, \tag{4'}$$

$$\Sigma_k n_k \frac{dA_i}{dw_k} = 3\frac{dC}{dn_i} - B_i. \tag{5'}$$

The A's and the B's must be periodic functions of the w's.

If we treat equations (4') and (5') just the same as we treated equations (4) and (5), we find the following:

1. The B_i's are independent of the w's;

2. The A_i's are independent of the w's;

3. And that

$$3\frac{dC}{dn_i} = B_i.$$

We finally obtain

$$\Sigma(2x\,dy + y\,dx) = \Sigma A_i dn_i + 3\sum \frac{dC}{dn_i}\,dw_i,$$

where the A_i's depend only on the n_i's.

In other words, expressions

$$\Sigma_i\left(2x_i \frac{dy_i}{dn_k} + y_i \frac{dx_i}{dn_k}\right)$$

or

$$\Sigma_i\left(2x_i \frac{dy_i}{d\xi_k} + y_i \frac{dx_i}{d\xi_k}\right) \tag{8}$$

do not depend on the w's and are only functions of either the ξ's or the n's, depending on whether everything is expressed as a function of the ξ's and the w's, or as a function of the n's and the w's.

In the same way, we shall have

$$\Sigma_i\left(2x_i \frac{dy_i}{dw_k} + y_i \frac{dx_i}{dw_k}\right) = 3\frac{dC}{dn_k}. \tag{9}$$

As I have already stated, the x_i's, the y_i's, and C are developed in powers of μ and of the $x_i'^0$'s. Expressions (8) and the two terms in equations (9) may therefore also be developed in powers of these quantities.

All of the expansion terms of expressions (8), which are expanded in powers of μ and of the $x_i'^0$'s, must therefore be independent of the w's. /78

On the other hand, each expansion term of the first member of (9) must equal the corresponding term of the second member.

We thus have numerous procedures for verifying our computations.

268. I have stated that

$$\Sigma x_i\, dy_i$$

is an exact differential, if the ξ_i's are regraded as constants, and the w's are regarded as independent variables.

We then obtain

$$\int \Sigma(2x\,dy + y\,dx) = 3\int \Sigma \frac{dC}{dn_i}\,dw_i,$$

or, since the $\frac{dC}{dn_i}$'s depend only on the ξ's, they must be consequently regarded as constants

$$\int \Sigma(2x\,dy + y\,dx) = 3\sum \frac{dC}{dn_i}\,w_i;$$

from which it follows that

$$\int \Sigma x\,dy + \int \Sigma(x\,dy + y\,dx) = 3\sum \frac{dC}{dn_i}\,w_i,$$

from which we finally have

$$\int \Sigma x\,dy = 3\sum \frac{dC}{dn_i}\,w_i - \Sigma xy. \tag{10}$$

Let us briefly return to the notation given in No. 162. In this section, just as in No. 152, we chose the following as variables

$$\begin{cases} \Lambda, & \Lambda', & \sigma_i, \\ \lambda_1, & \lambda'_1, & \tau_i, \end{cases} \tag{11}$$

and we set

$$dS = (\Lambda - \Lambda_{00})\, d\lambda_1 + (\Lambda' - \Lambda'_{00})\, d\lambda'_1 + \Sigma \sigma_i\, d\tau_i - d(\sigma_i^{0\,1} \tau_i).$$

On the other hand, the variables (11), just as the variables x_i, y_i, are conjugate variables. As a result, just as I have explained several times, the expression

$$\Sigma x_i\, dy_i - \Lambda\, d\lambda_1 - \Lambda'\, d\lambda'_1 - \Sigma \sigma_i\, d\tau_i = dU$$

is an exact differential. I should add that the function U may be /79
readily formed, which may be consequently regarded as a known function of the x_i's and the y_i's.

We then have

$$S = 3 \sum \frac{dC}{dn_i} w_i - \Sigma xy - U - \Lambda_{00} \lambda_1 - \Lambda'_{00} \lambda'_1 - \sigma_i^{0\,1} \tau_i. \tag{12}$$

Just as when the procedure outlined in Chapter XV is applied, one is led to formulate the function S, and equation (12) furnishes us with the desired verification in a new form.

Relationship to a Jacobi Theorem

269. It is known that at the beginning of his Vorlesungen über Dynamik, Jacobi demonstrated the fact that, in the case of Newtonian attraction, the mean value of the kinetic energy equals, with the exception of a constant factor, the mean value of the potential energy, assuming that the coordinates may be expressed by the trigonometric series having the same form as those which we are presently studying.

This Jacobi theorem is directly related to the preceding statements. The equations of motion may be written

$$m_i \frac{dx_i}{dt} = y_i, \qquad \frac{dy_i}{dt} = \frac{dV}{dx_i},$$

from which it follows that

$$\sum \frac{y_i^2}{2m_i} - V = C.$$

Then -V represents the potential energy, C the total energy, and

$$\sum \frac{y_i^2}{2m_i}$$

the kinetic energy.

On the other hand, due to the fact that V is homogeneous of degree -1, we shall have

$$-V = \sum \frac{dV}{dx_i} x_i = \Sigma x_i \frac{dy_i}{dt},$$

$$\sum \frac{y_i^2}{2m_i} = \frac{1}{2} \Sigma y_i \frac{dx_i}{dt}.$$

The energy equation may therefore be written /80

$$\frac{1}{2} \Sigma y_i \frac{dx_i}{dt} + \Sigma x_i \frac{dy_i}{dt} = C.$$

Let us take equations (9) from No. 217 and let us add them, after having multiplied them respectively by n_k. We shall have

$$\Sigma_i \left(2x_i \Sigma_k n_k \frac{dy_i}{dw_k} + y_i \Sigma_k n_k \frac{dx_i}{dw_k} \right) = 3 \Sigma_k n_k \frac{dC}{dn_k}.$$

If we note that

$$\Sigma n_k \frac{dx}{dw_k} = \frac{dx}{dt}$$

(since $\frac{dw_k}{dt} = n_k$), we may conclude that

$$\Sigma \left(2x_i \frac{dy_i}{dt} + y_i \frac{dx_i}{dt} \right) = 3 \Sigma n_k \frac{dC}{dn_k}.$$

When making a comparison with the energy equation, we find that

$$\Sigma n_k \frac{dC}{dn_k} = \frac{2}{3} C,$$

which shows that C must be homogeneous of degree $\frac{2}{3}$ with respect to the n_k's, which could be seen directly. The mean value of a function U, which I shall designate by the notation [U], will be zero if U is the derivative of a periodic function. We shall therefore have

$$\Sigma\left[y_i \frac{dx_i}{dt} + x_i \frac{dy_i}{dt}\right] = 0$$

and, connecting this with the energy equation, we obtain

$$\left[\frac{1}{2} \Sigma y_i \frac{dx_i}{dt}\right] = -C,$$

$$\left[\Sigma x_i \frac{dy_i}{dt}\right] = 2C,$$

from which we have

$$\frac{\left[\sum \frac{y_i^2}{4\,2\,m_i}\right]}{[-V]} = -\frac{1}{2}.$$

This is the Jacobi theorem.

If the partial derivatives $\frac{dx}{dw_k}$ are considered instead of the total /81 derivatives $\frac{dx}{dt}$, similar results would be obtained. We would obtain

$$\Sigma_i\left(x_i \frac{dy_i}{dw_k} + y_i \frac{dx_i}{dw_k}\right) = 0,$$

and consequently

$$\left[\Sigma_i x_i \frac{dy_i}{dw_k}\right] = 3 \frac{dC}{dn_k},$$

$$\left[\Sigma_i y_i \frac{dx_i}{dw_k}\right] = -3 \frac{dC}{dn_k}.$$

Application to the Two-Body Problem

270. In particular, the preceding considerations may be applied to the two-body problem. Let us consider a planet and the Sun, and let us refer the planet to axes having fixed directions and passing through the Sun. Consequently, let us consider the relative motion of the planet with respect to the Sun.

Let x_1, x_2, x_3 be the three coordinates of the planet; let y_1, y_2, y_3 be the three components of angular momentum.

Let ξ, η, ζ be the three coordinates of the planet with respect to particular axes, i.e.: The major axis of the orbit, a parallel line to the minor axis, and a perpendicular line to the orbital plane. We shall have

$$x_1 = h_1\xi + h'_1\eta + h''_1\zeta,$$
$$x_2 = h_2\xi + h'_2\eta + h''_2\zeta,$$
$$x_3 = h_3\xi + h'_3\eta + h''_3\zeta,$$

where the h's are constants which are connected by the well-known relationships which indicate that the transformation of coordinates is orthogonal.

In the same way, we shall have

$$y_i = \mu h_i \frac{d\xi}{dt} + \mu h'_i \frac{d\eta}{dt} + \mu h''_i \frac{d\zeta}{dt},$$

where μ is the mass of the planet.

It is now evident that ζ is zero, and that ξ and η are functions of one single argument w, which is the mean anomaly, and of two constants, which are the major axis a and the eccentricity e.

In addition, the h's are the functions of the three Euler angles, or more generally, of three arbitrary functions ω_1, ω_2, ω_3 of these three angles. /82

Thus, the x's and the y's are functions of w, a, e, and of the ω's.

If we designate C as the energy constant and n as the mean motion, we shall then have

$$\sum\left(2x_i \frac{dy_i}{dw} + y_i \frac{dx_i}{dw}\right) = 3\frac{dC}{dn}$$

and, in addition, the expressions

$$\sum\left(2x_i \frac{dy_i}{da} + y_i \frac{dx_i}{da}\right)$$
$$\sum\left(2x_i \frac{dy_i}{de} + y_i \frac{dx_i}{de}\right)$$
$$\sum\left(2x_i \frac{dy_i}{d\omega_k} + y_i \frac{dx_i}{d\omega_k}\right)$$

must be independent of w.

Some of the statements were apparent beforehand, and provide us with no new verification.

In actuality, the $\frac{dx_i}{d\omega_k}$'s are linear functions of the x_i's whose coefficients depend on the ω's and are such that

$$\Sigma_i x_i \frac{dx_i}{d\omega_k} = 0.$$

As a result, we may write the following identity

$$\alpha_1 \frac{dx_1}{d\omega_k} + \alpha_2 \frac{dx_2}{d\omega_k} + \alpha_3 \frac{dx_3}{d\omega_k} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ x_1 & x_2 & x_3 \\ \varphi_1^k & \varphi_2^k & \varphi_3^k \end{vmatrix}$$

where the α's are <u>arbitrary</u> constants and the ϕ_i^k's are the given functions of the ω's. In the same way, we shall have

$$\alpha_1 \frac{dy_1}{d\omega_k} + \alpha_2 \frac{dy_2}{d\omega_k} + \alpha_3 \frac{dy_3}{d\omega_k} = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ y_1 & y_2 & y_3 \\ \varphi_1^k & \varphi_2^k & \varphi_k^3 \end{vmatrix}.$$

As a result, we have

$$\sum\left(2x_i \frac{dy_i}{d\omega_k} + y_i \frac{dx_i}{d\omega_k}\right) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ \varphi_1^k & \varphi_2^k & \varphi_3^k \end{vmatrix}.$$

This expression must be reducible to a constant which is independent of /83
w, and -- since we have three similar relationships which one obtains by setting k = 1, 2, 3 -- we may write

$$y_3 x_2 - y_2 x_3 = \text{const.}$$
$$y_1 x_3 - y_3 x_1 = \text{const.}$$
$$y_2 x_1 - y_1 x_2 = \text{const.}$$

However, this is not a new result; these are the area equations.

Let us now investigate the expression

$$\sum_{i}\left(2x_i \frac{dy_i}{da} + y_i \frac{dx_i}{da}\right).$$

Let us determine the manner in which the x's and the y's depend on a. The x's include a as a factor, and the y's include an, because we have

$$y_i = \mu \frac{dx_i}{dt} = \mu n \frac{dx_i}{dw}.$$

We therefore have

$$\frac{dx_i}{da} = \frac{x_i}{a}; \qquad \frac{dy_i}{da} = \frac{y_i}{a} + \frac{y_i}{n}\frac{dn}{da}.$$

Our expression therefore becomes

$$\Sigma x_i y_i \left(\frac{3}{a} + \frac{2}{n}\frac{dn}{da}\right).$$

It may be readily verified that it is zero. According to the third law of Kepler, we have

$$n^2 a^3 = \text{const.},$$

from which it follows that

$$\frac{2\,dn}{n} + \frac{3\,da}{a} = 0.$$

We have still not obtained a new procedure for the proof.

We must now examine the two expressions

$$\sum\left(2x_i\frac{dy_i}{dw} + y_i\frac{dx_i}{dw}\right) = \text{W},$$

$$\sum\left(2x_i\frac{dy_i}{de} + y_i\frac{dx_i}{de}\right) = \text{E}.$$

e and w remain to be varied. We can therefore only vary the ω's, i.e., the direction of the major axis of the orbit. We may /84
therefore choose particular axes and may set

$$x_1 = \xi = a\left[-\frac{3}{2}e + \Sigma J_{p-1}(pe)\frac{\cos pw}{p}\right],$$

$$x_2 = \eta = a\sqrt{1-e^2}\left[\Sigma J_{p-1}(pe)\frac{\sin pw}{p}\right],$$

$$x_3 = \zeta = 0.$$

The functions J are Bessel functions. The index p under the sign Σ includes all the integer values from $-\infty$ to $+\infty$, with the exception of 0.

We may therefore deduce the following

$$y_1 = -\mu a n \Sigma J_{p-1}(pe)\sin pw$$

$$y_2 = \mu a n\sqrt{1-e^2}\,\Sigma J_{p-1}(pe)\cos pw.$$

Expression W becomes the following, if the common factor $\mu a^2 n$ is removed,

$$3e\Sigma J_{p-1}p\cos pw - 2\Sigma J_{p-1}\frac{\cos pw}{p}\Sigma J_{p-1}p\cos pw + [\Sigma J_{p-1}\sin pw]^2$$
$$-2(1-e^2)\Sigma J_{p-1}\frac{\sin pw}{p}\Sigma J_{p-1}p\sin pw + (1-e^2)[\Sigma J_{p-1}\cos pw]^2 = W'.$$

For purposes of brevity, I have written J_{p-1} everywhere, instead of $J_{p-1}(pe)$.

We must then have

$$W = 3\frac{dC}{dn}.$$

However,

$$C = -\frac{m\mu}{2a}, \qquad n^2a^3 = m,$$

where m designates the mass of the Sun plus that of the planet. We therefore have

$$C = -\frac{\mu}{2} m^{\frac{2}{3}} n^{\frac{2}{3}}$$

and

$$3\frac{dC}{dn} = -\mu m^{\frac{2}{3}} n^{-\frac{1}{3}} = -\mu a^2 n.$$

However, since

$$W = \mu a^2 n W',$$

we find

$$W' = -1.$$

When identifying the similar terms, we have a series of relationships between the Bessel functions J. /85

A study of expression E leads us to a series of relationships which are similar, in which the Bessel functions J and their first derivatives will be included this time.

271. Numerous examples of these particular applications could be provided. For example, after having treated the case of Keplerian motion as we have just done in the preceding section -- i.e., after having taken into account terms of the degree zero with respect to the disturbed masses -- one could apply the same principles to the entire

group of terms of degree 1. There is no doubt that this would lead to interesting results.

Using the same procedure, we could also study the secular variational equations which we discussed in Chapter X. In place of the integral invariant

$$\int \Sigma(2x_i dy_i + y_i dx_i),$$

we would have the advantage of employing similar invariants which we defined in Nos. 261, 262, 263.

We shall put these questions aside.

Application to Asymptotic Solutions

272. Let us apply these principles to asymptotic solutions. Let us take the coordinates x_i and the

$$y_i = m_i \frac{dx_i}{dt}$$

as the variables. Let us consider the invariant

$$J = \int \Sigma(2x\, dy + y\, dx).$$

We know that if C is the energy constant, and C_1 and C_0 are the values of this constant at the two ends of the integration line, we shall have

$$J - 3t(C_1 - C_0) = \text{const.} \tag{1}$$

If we consider a system of asymptotic solutions, it will have the following form: The x_i's and the y_i's will be developed in powers of /86

$$A_1 e^{\alpha_1 t}, \quad A_2 e^{\alpha_2 t}, \quad \ldots, \quad A_k e^{\alpha_k t},$$

where the coefficients are periodic in $t + h$, where

$$A_1, \quad A_2, \quad \ldots, \quad A_k, \quad h$$

are k + 1 arbitrary constants.

If these values of the x_i's and the y_i's are substituted in the energy equation, the first member is always developed in powers of

$$A_1 e^{\alpha_1 t}, \quad A_2 {}^{\alpha_2 t}, \quad \ldots, \quad A_k e^{\alpha_k t},$$

where the coefficients are periodic in $t + h$. Since it must be independent of t, it will also be independent of $A_1, A_2, \ldots, A_k$ and h.

If the values of the x_i's and y_i's are substituted in equation (1), we shall have

$$C_1 = C_0,$$

and, consequently,

$$J = \text{const.}$$

In J, the expression under the sign $\int$, is developed in powers of

$$A_1 e^{\alpha_1 t}, \quad A_2 e^{\alpha_2 t}, \quad \ldots, \quad A_k e^{\alpha_k t};$$

The coefficients are periodic in $t + h$; it depends linearly on the $k + 1$ differentials

$$dA_1, \quad dA_2, \quad \ldots, \quad dA_k, \quad dh.$$

We must therefore have

$$\begin{cases} \Sigma\left(2x \dfrac{dy}{dA_i} + y \dfrac{dx}{dA_i}\right) = \text{const.}, \\ \Sigma\left(2x \dfrac{dy}{dh} + y \dfrac{dx}{dh}\right) = \text{const.} \end{cases} \tag{2}$$

The first terms of equations (2) are developed in powers of the $A_i e^{\alpha_i t}$'s. All terms of this development must be zero, except for the known term. One thus obtains a multitude of relationships between the coefficients of the development of the x_i's in powers of the $A_i e^{\alpha_i t}$'s. /87

By way of an example, I shall confine myself to considering the first term, and I shall write

$$x_i = X_i + Z_i A e^{\alpha t}$$

where X_i and Z_i are periodic in $t + h$.

We may deduce

$$y_i = m_i [X'_i + A e^{\alpha t}(Z'_i + \alpha Z_i)]$$

where X'_i and Z'_i designate the derivatives of X_i and Z_i.

Neglecting all terms in $e^{2\alpha t}$, etc., we then have

$$\Sigma\left(2x\frac{dy}{d\lambda}+y\frac{dx}{d\lambda}\right)=\Sigma me^{\alpha t}[2X(Z'+\alpha Z)+X'Z].$$

We therefore have

$$\Sigma m(2XZ'+2\alpha XZ+X'Z)=0,$$

which provides us with the first relationship between the coefficients X and Z_i.

The relationship

$$\Sigma\left(2x\frac{dy}{dh}+y\frac{dx}{dh}\right)=\text{const.}$$

furnishes us with another one which, in reality, would not differ from the first, since -- when it is combined with the first relationship -- an equation is obtained which is an immediate consequence of the energy principle.

CHAPTER XXV

INTEGRAL INVARIANTS AND ASYMPTOTIC SOLUTIONS

Return to the Method of Bohlin

273. Before proceeding any further, I must supplement some of the results given in Chapters VII, XIX and XX. I would first like to sum up the results which I wish to compare and which will serve as my point of departure. /88

We saw in Chapter VII that if a system

$$\frac{dx_i}{dt} = X_i \qquad (i = 1, 2, \ldots, n) \tag{1}$$

has a periodic solution

$$x_i = x_i^0, \tag{2}$$

and if we set

$$x_i = x_i^0 + \xi_i,$$

the ξ_i's may be developed in increasing powers of

$$A_1 e^{\alpha_1 t}, \quad A_2 e^{\alpha_2 t}, \quad \ldots, \quad A_n e^{\alpha_n t}, \tag{3}$$

where the coefficients are periodic functions of t. The A_i's are integration constants; the α_i's are the characteristic exponents of the periodic solution (2).

The series always satisfy equations (1) formally. They are convergent under certain conditions, which we have discussed in No. 105.

There is an exception in the case where we have a relationship having the following form between the exponents α

$$\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i = 0 \tag{4}$$

where the coefficients β are whole, positive, or zero, and the coefficient γ is whole, positive, or negative. (See Volume I, page 338, line 5. When writing this relationship, I assumed that the unit of /89 time was chosen so that the period of the solution (2) equalled 2π).

If there is a relationship having the form (4), the ξ's cannot be developed in powers of the quantities (3), but in powers of these quantities (3) and of t.

This is precisely what occurs if the equations (1) have the canonical form of the equations of dynamics. In actuality, in this case two of the exponents are zero, and the others are equal in pairs and have the opposite sign.

In the case of equations of dynamics [or, more generally, when there is a relationship having the form (4)], we were still able to obtain a result. It is sufficient to give special values to the integration constants A, so as to cancel the values of these constants corresponding to a zero exponent, and one of the two corresponding to each pair of equal exponents having opposite signs. [More generally, the constant A corresponding to one of the exponents included in the relationship having the form (4) would be cancelled, so that there would no longer be a relationship having this form between the exponents corresponding to the constants A which are not zero.]

For example, if

$$\alpha_1 = \alpha_2 = 0, \quad \alpha_3 = -\alpha_4, \quad \alpha_5 = -\alpha_6, \quad \dots, \quad \alpha_{n-1} = -\alpha_n \quad (n \text{ even}),$$

we would make

$$A_1 = A_2 = 0, \quad A_3 = 0, \quad A_5 = 0, \quad \dots, \quad A_{n-1} = 0.$$

The ξ's may then be developed in powers of those quantities (3) which are not zero. However, we shall no longer have the general solution of equations (1), but a special solution depending on the number of arbitrary constants which is less than n (i.e., $\frac{n-1}{2}$ in the general case of the equations of dynamics).

We have thus arrived at the asymptotic solutions: We have done this by cancelling a certain number of constants A, not only those which we have set equal to zero for the reason which I have just given, but also those which we had to cancel in order to satisfy the convergence conditions given in No. 105. /90

For the time being, I shall not deal with the development of the ξ's in powers of μ or of $\sqrt{\mu}$.

In Chapter XIX I studied the method derived by M. Bohlin, which is basically only an application of the Jacobi method, since the problem is reduced to obtaining a function S which satisfies an equation with partial derivatives. Only this function S has a form which is particularly suitable for the case in which there is approximately a linear relationship having whole coefficients between the mean motions. The cases which are of greatest interest to us are those which are similar to that which I have designated as the limiting case (No. 207). In this section, we saw that the function S may be developed in powers of $\sqrt{\mu}$, in the following form

$$S = S_0 + \sqrt{\mu} S_1 + \mu S_2 + \dots$$

and that

$$\frac{dS_p}{dy_k}$$

is periodic with the period 2π with respect to

$$y_2, \; y_3, \; \dots, \; y_n$$

(employing the notation in the section indicated above).

However, the results may be simplified by performing the change in variables which was discussed in No. 209 and 210.

In section No. 206, I defined n + 1 functions

$$\eta, \; \zeta, \; \xi_i$$

which are periodic with respect to the variables

$$y_2, \; y_3, \; \dots, \; y_n,$$

and which I regarded as generalizations of periodic solutions.

In No. 210, we set the following

$$x_1 = x'_1 + \eta, \qquad y'_1 = y_1 - \zeta, \qquad y'_i = y_i \qquad (i > 1)$$
$$x_i = x'_i + \xi_i + y'_1 \frac{d\eta}{dy_i} - x'_1 \frac{d\zeta}{dy_i}.$$

The equations retain the canonical form with the new variables x'_i, y'_i. Only the new equations have the following invariant relationships /91

$$x'_1 = x'_i = y'_1 = 0,$$

which, with respect to the new canonical equations, may be regarded as generalizations of periodic solutions, just as is the case for

$$x_1 = \eta, \quad y_1 = \zeta, \quad x_i = \xi_i,$$

with respect to the old ones.

Without limiting the conditions of generality, we may assume that our canonical equations imply the following invariant relationships

$$x_1 = x_i = y_1 = 0.$$

If this is the case, we saw in No. 210 that $y_1 = 0$ is a simple zero for the derivatives $\frac{dS_p}{dy_1}$, and a double zero for the derivatives $\frac{dS_p}{dy_i}$ $(i > 1)$.

Thus S, or rather $S - S_0$, may be developed in powers of y, and the expansion will begin with a term of the second degree. We shall have

$$S = S_0 + \Sigma_2 y_1^2 + \Sigma_3 y_1^3 + \Sigma_4 y_1^4 + \cdots \tag{5}$$

where the Σ's are series depending on $y_2, y_3, \ldots, y_n$ and are developed in powers of $\sqrt{\mu}$. In addition, it may be seen that the Σ's are periodic functions of $y_2, y_3, \ldots, y_n$.

Unfortunately, this is not sufficient for our purposes.

The function S, which is defined by equation (5), depends only on n - 1 arbitrary constants

$$x_2^0, \; x_3^0, \; \ldots, \; x_n^0,$$

whereas n would be required for the complete solution of the problem.

In order to pursue the study in greater detail, we shall resort to the change in variables, given in No. 206. If we employ the notation given in this section -- i.e., if we set

$$z_1 = \frac{1}{2}\int \frac{dy_1}{\sqrt{x_1' - \psi}}, \qquad z_i = y_i - \frac{y_1}{\Lambda}\frac{dB}{dx_i'}, \qquad \ldots,$$

and if we define, just as in the indicated section, the variables x_i',

u_1, v_1, and the functions T and V, the derivatives of V with respect to v_1 and to the z_i's will be periodic functions of the z_i's (see Volume II, p. 361). /92

Let us examine in greater detail the equations which appear at the beginning of page 363 (Vol. II) and which are written

$$y_1 = \theta(v_1, y_2, y_3, \ldots, y_n),$$
$$x_k = \zeta_k(v_1, y_2, y_3, \ldots, y_n)$$

Regarding y_2, y_3, ..., y_n as constants, let us consider the following equations (always just as in the indicated section)

$$y_1 = \theta(v_1), \qquad x_1 = \zeta_1(v_1).$$

When we vary v_1, the point (x_1, y_1) will describe a curve which I wish to study. Let us assume that we vary x'_1, instead of varying the constants x'_2, x'_3, ..., x'_n, and we shall obtain an infinity of curves corresponding to different values of x'_1.

We assumed above that the following invariant relationships hold

$$x_1 = x_i = y_1 = 0$$

which are like a generalization of periodic solutions.

The following point will correspond to these relationships

$$x_1 = y_1 = 0$$

i.e., the origin of the coordinates. I would like to study our curves in the vicinity of this point.

Let us assign to x'_1 the value corresponding to the special function S defined by equation (5), and we shall have

$$x_1 = 2\Sigma_2 y_1 + 3\Sigma_3 y_1^2 + \ldots.$$

The corresponding curve passes through the origin. By changing $\sqrt{\mu}$ into $-\sqrt{\mu}$, we would obtain a second curve passing through the origin.

We therefore have two curves crossing at the origin. The center curves may pass near the origin, without reaching it and without intersecting each other, so that all of our curves together will look like (in terms of their general form in the immediate vicinity of the origin) the figure formed by a series of hyperbolas having the same asymptotes /93 and formed by their asymptotes.

274. In order to study these curves and their corresponding functions S in greater detail, let us limit ourselves to the case in which there are only two degrees of freedom.

Let us assume that the change in variables of No. 208 was performed in such a way that

$$x_1 = x_2 = y_1 = 0$$

is a periodic solution, which amounts to stating that for

$$x_1 = x_2 = y_1 = 0$$

we have

$$\frac{dF}{dy_1} = \frac{dF}{dy_2} = \frac{dF}{dx_1} = 0.$$

Let us develop F in increasing powers of x_1, x_2, and y_1. The term of degree 0 would only depend on y_2, and since we must have

$$\frac{dF}{dy_2} = 0$$

it will be reduced to a constant. Since F is only defined up to a constant, we may assume that this term of degree zero is zero.

Let us try to find the terms of the first degree. Since

$$\frac{dF}{dx_1} = \frac{dF}{dy_1} = 0$$

there will be no other terms of the first degree except for a term for x_2.

Let us now set

$$x_1 = \varepsilon x'_1, \qquad y_1 = \varepsilon y'_1, \qquad x_2 = \varepsilon^2 x'_2, \qquad y_2 = y'_2.$$

It can be seen that F may be divided by ε^2 and that, if one sets

$$F = \varepsilon^2 F',$$

the equations retain the canonical form and become

$$\frac{dx'_i}{dt} = \frac{dF'}{dy'_i}, \qquad \frac{dy'_i}{dt} = -\frac{dF'}{dx'_i}. \tag{1}$$

In addition, F' will be developed in powers of ε in the form /94

$$F' = F'_0 + \varepsilon F'_1 + \varepsilon^2 F'_2 + \dots;$$

F' can be developed, on the other hand, in powers of x'_1, x'_2, y'_1. The coefficients are periodic functions of y'_2. We shall have finally

$$F'_0 = H x'_2 + A x'^2_1 + 2B x'_1 y'_1 + C y'^2_1$$

where H, A, B and C are periodic functions of y'_2.

We shall apply a method which is similar to that of Bolin to our equations. In this method, the parameter ε will play the same role that the parameter μ played in Chapter XIX.

Let us remove our accents which have become useless, and let us write x_i, y_i, F, F_i instead of x'_i, y'_i, F', F'_i.

I would first like to state that I may always assume

$$H = 1.$$

If this were not the case, I would choose the following for new variables

$$x^*_2 = H x_2, \quad y^*_2 = \int \frac{dy_2}{H}.$$

The canonical form of the equations would not be changed, since

$$x^*_2 \, dy^*_2 - x_2 \, dy_2 = 0$$

is an exact differential.

In addition, y^*_2 increases by a constant when y_2 increases by 2π. I may always select the unit of time in such a way that this constant equals 2π. Then every periodic function of y_2 having the period 2π will be a periodic function of y^*_2 with period 2π. The form of the function F will not be changed; only the first term Hx_2 will be reduced to x^*_2.

Let us therefore assume that H = 1.

I may now state that we may assume

$$A = C = 0, \quad B = \text{const.}$$

Let us form our canonical equations (1) assuming that ε = 0, and we have /95

$$\frac{dy_2}{dt} = -1; \quad \frac{dx_1}{dt} = -\frac{dx_1}{dy_2} = 2(Bx_1 + Cy_1)$$

$$\frac{dy_1}{dt} = -\frac{dy_1}{dy_2} = -2(Ax_1 + By_1)$$

and an equation for $\frac{dx_2}{dt}$ which I may replace by the equation of energy

$$x_2 + Ax_1^2 + 2Bx_1y_1 + Cy_1^2 = \text{const.}$$

The equations

$$\frac{dx_1}{dy_2} = -2(Bx_1 + Cy_1), \quad \frac{dy_1}{dy_2} = 2(Ax_1 + By_1)$$

are linear equations having periodic coefficients. In virtue of No. 29, they will have the following for a general solution

$$x_1 = w\varphi + w_1\psi \qquad y_1 = w\varphi_1 + w_1\psi_1,$$
$$w = \alpha e^{ay_2}, \qquad w_1 = \beta e^{by_2},$$

where ϕ, ψ, ϕ_1, ψ_1 are periodic functions of y_2; α and β are integration constants, and a and b are constants.

It may be readily seen that $b = -a$ and that $\phi\psi_1 - \psi\phi_1$ is a constant, which I may set equal to 1.

Under this assumption, let us make a new change in variables, setting

$$x_1 = x_1'\varphi + y_1'\psi; \qquad y_1 = x_1'\varphi_1 + y_1'\psi_1,$$
$$x_2 = x_2' + Hx_1'^2 + 2Kx_1'y_1' + Ly_1'^2; \qquad y_2 = y_2',$$

where H, K, L are functions of y_2, chosen in such a way that the canonical form of the equations is not changed. For this purpose, it is sufficient that

$$x_1\,dy_1 - x_1'\,dy_1' + x_2\,dy_2 - x_2'\,dy_2'$$

be an exact differential.

It may be seen that $x_1dy_1 - x_1'dy_1'$ equals an exact differential increased by the amount

$$-\frac{dy_2}{2}\left[x_1'^2(\varphi_1\varphi' - \varphi\varphi_1') + y_1'^2(\psi_1\psi' - \psi\psi_1') + 2x_1'y_1'(\varphi_1\psi' - \varphi\psi_1')\right];$$

The quantities ϕ', ϕ_1', ... designate the derivatives of ϕ, ϕ_1, ... with respect to y_2.

In order that the canonical form of the equations is not changed, it is sufficient to set /96

$$2H = \varphi_1\varphi' - \varphi\varphi'_1; \quad 2K = \varphi_1\psi' - \varphi\psi'_1; \quad 2L = \psi_1\psi' - \psi\psi'_1.$$

It may be seen that H, K, L are periodic functions of y_2, from which it follows that the form of the function F will not be changed either.

However, if we set $\varepsilon = 0$, our equations must have the solution

$$x'_1 = \alpha e^{+ay_1}; \quad y'_1 = \beta e^{-ay_1},$$

from which it follows that $B = -\frac{a}{2}, \quad A = C = 0.$

Without limiting the conditions of generality, we may assume that

$$H = 1, \quad A = C = 0, \quad B = \text{const.}$$

from which it follows (since we have removed the accents)

$$F_0 = x_2 + 2Bx_1y_1.$$

We shall follow this procedure from this point on.

Let us perform a change in variables, setting

$$x_1y_1 = u, \quad \log\frac{y_1}{x_1} = 2v.$$

Since

$$x_1\,dy_1 - u\,dv = \frac{d(x_1y_1)}{2}$$

is an exact differential, the canonical form will not be changed.

We then thave

$$x_1 = e^v\sqrt{u}; \quad y_1 = e^{-v}\sqrt{u}.$$

The function F may then be developed in powers of

$$\varepsilon, \ x_2, \ \sqrt{u}, \ e^v, \ e^{-v}, \ e^{iy_2}, \ e^{-iy_2}.$$

We have

$$F_0 = x_2 + 2Bu.$$

Let us write F in the following form

$$F(x_2, u; y_2, v)$$

and let us define a function S by the Jacobi equation /97

$$F\left(\frac{dS}{dy_2}, \frac{dS}{dv}; y_2, v\right) = C,$$

where C is a constant. Let us develop S and C in powers of ε

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \dots,$$
$$C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \dots.$$

In order to determine S_0, S_1, S_2, ..., by a recurrence method, we shall have the following equations

$$\begin{cases} \dfrac{dS_0}{dy_2} + 2B\dfrac{dS_0}{dv} = C_0, \\ \dfrac{dS_1}{dy_2} + 2B\dfrac{dS_1}{dv} = \Phi + C_1, \\ \dfrac{dS_2}{dy_2} + 2B\dfrac{dS_2}{dv} = \Phi + C_2. \\ \dots\dots\dots\dots\dots\dots \end{cases} \qquad (2)$$

As I have already done previously, I shall designate every known function by Φ. In the second equation (2), I assume that S_0 is known. In the third equation, I assume that S_0 and S_1 are known, and so on.

Let us set

$$S_0 = \alpha_0 y_2 + \beta_0 v$$

with the condition

$$\alpha_0 + 2B\beta_0 = C_0.$$

Since C_0 is arbitrary, the two constants α_0 and β_0 may be chosen arbitrarily. Nevertheless, it is important that we do not set $\beta_0 = 0$. Following is the reason for this.

Let us assume that it has been shown that

$$\frac{dS_0}{dv} + \varepsilon\frac{dS_1}{dv} + \dots + \varepsilon^p\frac{dS_p}{dv}$$

may be developed in powers of

$$\varepsilon, \quad e^{\pm v}, \quad e^{\pm i y_2}.$$

We may conclude (if β_0 is not zero) that the same holds true for

$$\sqrt{\frac{dS_0}{dv} + \varepsilon \frac{dS_1}{dv} + \dots + \varepsilon^p \frac{dS_p}{dv}}$$

since the quantity under the radical may be reduced to β_0 for $\varepsilon = 0$. /98
This conclusion could not be reached if β_0 were zero. It is important that this conclusion may be reached, due to the presence of the radical $\sqrt{u}$ in F.

Let us now consider the second equations (2). The function Φ which it includes depends on v and on y_2, and has the following form

$$\Phi = \Sigma A_{m,n} e^{mv+iny_2} + A_{00}.$$

The coefficients A are constants which may depend on α_0 and on β_0. The indices m and n may take all whole, positive, negative, or zero values. When removing it from the sign Σ, I have shown the term in which these two indices are zero.

The second equation (2) then gives us the following

$$S_1 = \alpha_1 y_2 + \beta_1 v + \sum \frac{A_{mn} e^{mv+iny_2}}{in + 2Bm}$$

with the condition

$$\alpha_1 + 2B\beta_1 = A_{00} + C_1.$$

Except for this condition, the constants α_1, β_1 and C_1 are arbitrary. I shall therefore assume that

$$\alpha_1 = \beta_1 = 0.$$

I shall determine S_2 by the third equation (2). Due to the fact that this equation has exactly the same form as the second, it will be treated in the same manner, and so on.

To sum up, the derivatives $\frac{dS}{dy_2}$ and $\frac{dS}{dv}$ may be developed in powers of

$$\varepsilon, \quad e^{\pm v}, \quad e^{\pm i y_2}.$$

If one compares this analysis with that given in No. 125, it may be seen that there is an exact analogy between them. However, instead of having only imaginary exponentials

$$e^{\pm i y_1}, \quad e^{\pm i y_2}, \quad \ldots, \quad e^{\pm i y_n},$$

we here have real exponentials

$$e^{\pm v}.$$

275. Once the function S has been determined, by applying the Jacobi method, we may arrive at series which are similar to those given in No. 127.

The function S depends on v, on y_2, and the two constants α_0 and β_0. The energy constant /99

$$C = C_0 + \varepsilon C_1 + \ldots$$

is a function of α_0 and β_0.

As a solution of our canonical differential equations, we then have the following equations

$$x_2 = \frac{dS}{dy_2}; \quad u = \frac{dS}{dv}; \quad n_1 t + \varpi_1 = \frac{dS}{d\alpha_0}; \quad n_2 t + \varpi_2 = \frac{dS}{d\beta_0};$$
$$n_1 = -\frac{dC}{d\alpha_0}; \quad n_2 = -\frac{dC}{d\beta_0},$$

where $\bar{\omega}_1$ and $\bar{\omega}_2$ are two new integration constants.

It may be first seen that n_1 and n_2, which depend on α_0 and β_0, may be developed in powers of ε.

In addition, S may be developed in powers of ε and, if I set $\varepsilon = 0$, I have the following as the first approximation

$$x_2 = \frac{dS_0}{dy_2} = \alpha_0; \quad u = \frac{dS_0}{dv} = \beta_0;$$
$$n_1 t + \varpi_1 = \frac{dS_0}{d\alpha_0} = y_2; \quad n_2 t + \varpi_2 = \frac{dS_0}{d\beta_0} = v.$$

We have four equations from which we may obtain x_2, u, y_2 and v developed in powers of ε, depending on α_0, β_0, $n_1 t + \bar{\omega}_1$, $n_2 t + \bar{\omega}_2$.

By pursuing a line of reasoning exactly like that given in No. 127, we may see that

$$x_2, \quad u, \quad y_2 - (n_1 t + \varpi_1), \quad v - (n_2 t + \varpi_2)$$

may be developed in powers of

$$\varepsilon, \quad e^{\pm i(n_1 t+\varpi_1)}, \quad e^{\pm(n_2 t+\varpi_2)}.$$

The same will hold true for $\sqrt{u}$, x_1, and y_1.

I would like to add that all of these quantities may be developed in powers of

$$\varepsilon, \quad \alpha_0, \quad e^{\pm i(n_1 t+\varpi_1)}, \quad \sqrt{\beta_0}\, e^{(n_2 t+\varpi_2)}, \quad \sqrt{\beta_0}\, e^{-(n_2 t+\varpi_2)},$$

and $S - S_0$ may be developed in powers of

$$\varepsilon, \quad \alpha_0, \quad e^{\pm i y_2}, \quad \sqrt{\beta_0}\, e^{v}, \quad \sqrt{\beta_0}\, e^{-v}.$$

If we set for the time being /100

$$y_2 - (n_1 t + \varpi_1) = z_2, \qquad v - (n_2 t + \varpi_2) = z_3,$$

the two equations

$$n_1 t + \varpi_1 = \frac{dS}{d\alpha_0}, \qquad n_2 t + \varpi_2 = \frac{dS}{d\beta_0}$$

will take the form

$$z_2 = \varepsilon\psi_2, \qquad z_3 = \varepsilon\psi_3, \tag{3}$$

where ψ_2 and ψ_3 may be developed in powers of

$$\varepsilon, \quad \alpha_0, \quad e^{\pm i(n_1 t+\varpi_1)}, \quad \sqrt{\beta_0}\, e^{(n_2 t+\varpi_2)}, \quad \sqrt{\beta_0}\, e^{-(n_2 t-\varpi_2)}, \quad z_2, \quad z_3$$

[and, for example, we have

$$\sqrt{\beta_0}\, e^{v} = \sqrt{\beta_0}\, e^{n_2 t+\varpi_2}\left(1 + \frac{z_3}{1} + \frac{z_3^2}{1.2} + \frac{z_3^3}{1.2.3} + \dots\right),$$

and similar formulas for $e^{\pm i y_2}$, and $\sqrt{\beta_0} e^{-v}$].

In order to prove the postulate presented above, it is sufficient to apply the theorem given in No. 30 to equations (3).

Let us now compare the results obtained with that given in Chapter VII, which I reiterated at the beginning of this chapter.

We saw in Chapter VII that, in the vicinity of the periodic solution

$$x_1 = y_1 = x_2 = 0,$$

the variables x_1, y_1, x_2, y_2 may be developed in powers of

$$e^{\pm i(n'_1 t+\varpi_1)}, \qquad A e^{n'_2 t}, \qquad A' e^{-n'_2 t}, \qquad \text{et} \qquad t;$$

where A, A' are integration constants. n'_1 and n'_2 are absolute constants, depending only on the period of the periodic solution and the characteristic exponents.

We have just seen that these same variables must be developed in powers of

$$e^{\pm i(n_1 t+\varpi_1)}, \qquad \sqrt{\beta_0}\, e^{(n_2 t+\varpi_2)}, \qquad \sqrt{\beta_0}\, e^{-(n_2 t+\varpi_2)}.$$

The two results clearly are in agreement. We may first set

$$A = \sqrt{\beta_0} e^{\overline{\omega}_2}, \qquad A' = \sqrt{\beta_0} e^{-\overline{\omega}_2}.$$

In addition, n_1 and n_2 are constants, but constants which may be /101
developed in powers of ε, α_0 and β_0, and which may be reduced to n'_1 and n'_2 for $\varepsilon = \alpha_0 = \beta_0 = 0$.

We may then write, for example,

$$e^{n_2 t} = e^{n'_2 t} \cdot e^{(n_2 - n'_2)t},$$

and may then develop the second factor in powers of ε, α_0, β_0. In addition, the second factor will then be developed in powers of t.

It is for this reason that we saw in Chapter VII the time t and its powers emerge from the exponential and trigonometric signs, which could have led to a certain amount of difficulty in certain cases. The preceding analysis shows that this difficulty was entirely artificial.

If I now wish to compare out result with those given in Chapter XIX, I shall consider the curves

$$y_1 = \theta(v_1), \qquad x_1 = \zeta_1(v_1)$$

whose definition I presented at the end of No. 273. In order to obtain the equations for these curves, I need only take the expressions of x_1 and y_1 and assign a constant value to α_0, β_0, $n_1 t + \overline{\omega}_1$. Then y_1 and x_1 may be developed in powers of

$$e^{\pm(n_2 t+\varpi_2)}.$$

When $n_2 t + \overline{\omega}_2$ is varied, it may be seen that the curves have the form which I described at the end of No. 273.

In conclusion, I should point out that all of these results are only valid from the formal point of view. The series only converge in the case of asymptotic solutions, for which one obtains the equations by setting

$$\beta_0 = 0, \quad \varpi_2 = +\infty;$$

I mean by this, setting

$$\sqrt{\beta_0}\, e^{\varpi_2} = A, \quad \sqrt{\beta_0}\, e^{-\varpi_2} = 0,$$

or even setting

$$\beta_0 = 0, \quad \varpi_2 = -\infty;$$

I mean by this, setting /102

$$\sqrt{\beta_0}\, e^{\varpi_2} = 0, \quad \sqrt{\beta_0}\, e^{-\varpi_2} = A',$$

where A and A' designate the finite constants.

276. Let us proceed to the case in which there are more than two degrees of freedom. The preceding results may be generalized in two different manners.

In order to explain this, it is sufficient to assume three degrees of freedom. It may happen that we may wish to study our equations in the vicinity of a system of invariant relationships

$$x_1 = x_2 = x_3 = y_1 = 0,$$

which play the role of a generalization of the periodic solutions, in the sense of that given in No. 209.

It may also happen that we wish to study them in the vicinity of a true periodic solution

$$x_1 = x_2 = x_3 = y_1 = y_2 = 0.$$

In the first case, there are four invariant relationships and one linear relationship between the mean motions, a relationship which we have represented in the following form, employing the change in variables of No. 202 if necessary

$$n_1 = 0.$$

In the second case, there are five invariant relationships and two linear relationships between the mean motions, which we have represented in the following form

$$n_1 = 0, \quad n_2 = 0.$$

We shall begin with the first case, and we shall set

$$F = \varepsilon^2 F'; \quad x_1 = \varepsilon x'_1, \quad y_1 = \varepsilon y'_1, \quad x_2 = \varepsilon^2 x'_2, \quad x_3 = \varepsilon^2 x'_3 \cdot$$

The equations remain canonical equations, and F' may be developed in powers of ε, in the following form

$$F' = F'_0 + \varepsilon F'_1 + \ldots .$$

We then have

$$F'_0 = h_2 x'_2 + h_3 x'_3 + A x'^2_1 + 2 B x'_1 y'_1 + C y'^2_1,$$

or, removing the accents which have become useless, we have /103

$$F_0 = h_2 x_2 + h_3 x_3 + A x_1^2 + 2 B x_1 y_1 + C y_1^2.$$

The functions h_2, h_3, A, B, C depend only on y_2 and y_3, and are periodic with the period 2π with respect to these two variables.

I am going to perform the change in variables of No. 274 again. Everything which I have stated remains valid, but only from the formal point of view.

In order that I may apply the principles of formal calculation, it is necessary that there be a parameter with respect to the powers of which the expansions may be performed. This will be the parameter μ.

F and, consequently, h_2, h_3, A, B, C may be developed in whole powers of μ. I should add that, for $\mu = 0$, B and C may be reduced to 0 and that h_2, h_3, A may be reduced to constants which I designate as h_2^0, h_3^0 and A_0.

Let us try to integrate the following equations

$$\frac{dy_2}{dt} = -h_2, \quad \frac{dy_3}{dt} = -h_3. \tag{1}$$

I shall try to perform integration in such a manner that

$$y_2 - y'_2, \quad y_3 - y'_3$$

are periodic functions having the period 2π of the two new variables y'_2 and y'_3 which must themselves have the following form

$$y'_2 = n_2 t + \varpi_2, \quad y'_3 = n_3 t + \varpi_3 .$$

The quantities n_2 and n_3 are constants which may be developed in powers of μ; ϖ_2 and ϖ_3 are integration constants.

Equations (1) then take the form

$$n_2 \frac{dy_2}{dy_2'} + n_3 \frac{dy_2}{dy_3'} = -h_2, \qquad n_2 \frac{dy_3}{dy_2'} + n_3 \frac{dy_3}{dy_3'} = -h_3. \tag{2}$$

We shall set

$$\begin{aligned} h_i &= h_i^0 + \mu h_i^{(1)} + \mu^2 h_i^{(2)} + \dots, \\ y_i &= y_i^0 + \mu y_i^{(1)} + \mu^2 y_i^{(2)} + \dots, \\ n_i &= n_i^0 + \mu n_i^{(1)} + \mu^2 n_i^{(2)} + \dots, \end{aligned}$$

and we shall assume that the $n_i^{(k)}$'s are constants, that the $h_i^{(k)}$'s are periodic functions of y_2 and of y_3 (the h_i^0's may be reduced to constants, as we have seen), and finally that the $y_i^{(k)}$'s are periodic functions of y_2' and y_3', except for the y_i^0's, which may be reduced to y_i'. /104

In equations (2), we shall equate the equations having similar powers of μ, and we shall have a series of equations which will enable us to determine the $y_i^{(k)}$'s and $n_i^{(k)}$'s by a recurrence method.

These equations may be written

$$\left\{ \begin{aligned} & n_2^0 \frac{dy_2^0}{dy_2'} + n_3^0 \frac{dy_2^0}{dy_3'} = -h_2^0, \\ & n_2^0 \frac{dy_2^{(1)}}{dy_2'} + n_3^0 \frac{dy_2^{(1)}}{dy_3'} + n_2^1 \frac{dy_2^{(0)}}{dy_2'} + n_3^1 \frac{dy_2^0}{dy_3'} = \Phi, \\ & n_2^0 \frac{dy_2^{(2)}}{dy_2'} + n_3^0 \frac{dy_2^{(2)}}{dy_3'} + n_2^{(2)} \frac{dy_2^0}{dy_2'} + n_3^{(2)} \frac{dy_2^0}{dy_3'} = \Phi, \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \right. \tag{3}$$

I shall designate every known function by Φ. In the second equation, I assumed that the $y_i^{(0)}$'s and the $n_i^{(0)}$'s are known; in the third equation the y_i^0's, the $y_i^{(1)}$'s, the n_i^0's, and the $n_i^{(1)}$'s, and so on.

We then have

$$y_2^0 = y_2', \qquad y_3^0 = y_3'; \qquad n_2^0 = -h_2^0, \qquad n_3^0 = -h_3^0,$$

so that equations (3) may be reduced to

$$\left\{ \begin{aligned} & n_2^0 \frac{dy_2^{(1)}}{dy_2'} + n_3^0 \frac{dy_2^{(1)}}{dy_3'} + n_2^{(1)} = \Phi, \\ & n_2^0 \frac{dy_2^{(2)}}{dy_2'} + n_3^0 \frac{dy_2^{(2)}}{dy_3'} + n_2^{(2)} = \Phi, \\ & \dots\dots\dots\dots\dots\dots\dots\dots \end{aligned} \right. \tag{3'}$$

to which we must add the following equations

$$\left\{\begin{array}{l} n_2^0 \dfrac{dy_3^{(1)}}{dy'_2} + n_3^0 \dfrac{dy_3^{(1)}}{dy'_3} + n_3^{(1)} = \Phi, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{3''}$$

which may be deduced from the second equation (2), just as equations (3') are from the first equation (2).

All of these equations may be integrated in the same manner. Let us take, for example, the first equation (3'). The function Φ which it contains (like all the other functions Φ) is periodic in y'_2 and y'_3. We shall set $n_2^{(1)}$ equal to the mean value of this function, and by em- /105 ploying the procedure which we have already applied several times we shall be able to satisfy our equation by a function $y_2^{(1)}$ which is periodic in y'_2 and y'_3.

Having thus determined y_2 and y_3 as functions of y'_2 and y'_3, I may set

$$x'_2 = x_2 \frac{dy_2}{dy'_2} + x_3 \frac{dy_3}{dy'_2},$$
$$x'_3 = x_2 \frac{dy_2}{dy'_3} + x_3 \frac{dy_3}{dy'_3}.$$

It is apparent that

$$x'_2\,dy'_2 + x'_3\,dy'_3 - x_2\,dy_2 - x_3\,dy_3,$$

which is zero, is an exact differential and, consequently, that the canonical form of the equations is not changed when one takes x'_2, x'_3, y'_2, y'_3 for new variables, instead of x_2, x_3, y_2, y_3.

The form of the function F is not changed either, but it may be seen that we have the identity

$$-n_2 x'_2 - n_3 x'_3 = h_2 x_2 + h_3 x_3,$$

which shows that the coefficients of x'_2 and of x'_3 may be reduced to constants.

I may therefore assume that h_2 and h_3 are constants.

I shall make this assumption from this point on.

Let us now integrate the equations

$$\frac{dx_1}{dt} = 2(Bx_1 + Cy_1), \qquad \frac{dy_1}{dt} = -2(Ax_1 + By_1),$$

or, which is the same thing

$$\begin{cases} h_2 \dfrac{dx_1}{dy_2} + h_3 \dfrac{dx_1}{dy_3} = -2(Bx_1 + Cy_1), \\ h_2 \dfrac{dy_1}{dy_2} + h_3 \dfrac{dy_1}{dy_3} = \quad 2(Ax_1 + By_1). \end{cases} \tag{4}$$

Let us try to satisfy these equations by setting

$$x_1 = e^{at} z, \qquad y_1 = e^{at} s,$$

where a is a constant, z and s are periodic functions of y_2 and y_3. The equations become /106

$$\begin{cases} h_2 \dfrac{dz}{dy_2} + h_3 \dfrac{dz}{dy_3} - az = -2(Bz + Cs), \\ h_2 \dfrac{ds}{dy_2} + h_3 \dfrac{ds}{dy_3} - as = \quad 2(Az + Bs). \end{cases} \tag{4'}$$

Let us develop A, B, C in powers of μ in the following form

$$A = A_0 + \mu A_2 + \ldots,$$
$$B = B_0 + \mu B_2 + \ldots,$$
$$C = C_0 + \mu C_2 + \ldots.$$

We should point out that A_0 is a constant and that $B_0 = C_0 = 0$. In the same way, let us develop h_2 and h_3

$$h_i = h_i^0 + \mu h_i^1 + \ldots.$$

The coefficients of these expansions are known quantities. On the other hand, let us develop the unknowns z, s and a in increasing powers of $\sqrt{\mu}$ in the following form

$$a = a_1\sqrt{\mu} + a_2\mu + a_3\mu\sqrt{\mu} + \ldots,$$
$$z = z_1\sqrt{\mu} + z_2\mu + z_3\mu\sqrt{\mu} + \ldots,$$
$$s = s_0 + s_1\sqrt{\mu} + s_2\mu + \ldots.$$

In order to present the equations in a more symmetrical form, I shall write the expansion of A in the form

$$A = A_0 + A_1\sqrt{\mu} + A_2\mu + A_3\mu\sqrt{\mu} + A_4\mu^2 + \ldots.$$

We need only recall that A_1, A_3, A_5, ... are zero. The same holds true for the expansions of B and C.

Under this assumption, in equations (4'), I shall equate the coefficients having similar powers of μ. I shall employ (4' p) to designate the two equations obtained by equating, on the one hand, the coefficients of $\mu^{\frac{p+1}{2}}$ in the first equation (4') and, on the other hand, the coefficients of $\mu^{\frac{p}{2}}$ in the second equation (4').

The equations (4' 0) and (4' 1) will determine a_1, s_0 and z_1;

The equations (4' 1) and (4' 2) will determine a_2, s_1 and z_2;

The equations (4' 2) and (4' 3) will determine a_3, s_2 and z_3;

and so on.

I mean by this that equations (4' p) will determine s_p and z_{p+1} /107
up to a constant, that they will determine a_p, and will complete the determination of s_{p-1} and z_p, which are determined by equations (4' p-1), up to a constant.

If we recall that

$$B_0 = B_1 = C_0 = C_1 = 0,$$

it may be seen that equations (4' 0) may be written

$$\begin{cases} h_2^0 \dfrac{dz_1}{dy_2} + h_3^0 \dfrac{dz_1}{dy_3} = 0, \\ h_2^0 \dfrac{ds_0}{dy_2} + h_3^0 \dfrac{ds_0}{dy_3} = 0; \end{cases} \tag{4' 0}$$

and equations (4' 1) may be written

$$\begin{cases} h_2^0 \dfrac{dz_2}{dy_2} + h_3^0 \dfrac{dz_2}{dy_3} - a_1 z_1 = -2C_2 s_0, \\ h_2^0 \dfrac{ds_1}{dy_2} + h_3^0 \dfrac{ds_1}{dy_3} - a_1 s_0 = 2A_0 z_1; \end{cases} \tag{4' 1}$$

and equations (4' 2) may be written

$$\begin{cases} h_2^0 \dfrac{dz_3}{dy_2} + h_3^0 \dfrac{dz_3}{dy_3} - a_2 z_1 - a_1 z_2 = -2B_2 z_1 - 2C_2 s_1 - 2C_3 s_0 + \Phi, \\ h_2^0 \dfrac{ds_2}{dy_2} + h_3^0 \dfrac{ds_2}{dy_3} - a_2 s_0 - a_1 s_1 = \quad 2A_0 z_2 + 2A_1 z_1 + 2B_2 s_0 + \Phi \end{cases} \tag{4' 2}$$

[the letters Φ designate the known periodic functions in y_2 and y_3, which are zero in equations (4' 2), but which I have written nevertheless because they will appear in the following equations].

Equations (4' 0) indicate that z_1 and s_0 are constants. Let us now proceed to equations (4' 1), and let us equate the mean values of the two terms. We have

$$-a_1 z_1 = -2 s_0 [C_2],$$
$$-a_1 s_0 = 2 A_0 z_1,$$

which determines a_1, s_0 and z_1. For a_1, we obtain two equal values having the opposite sign. Equations (4' 1) then determine, up to constant terms, z_2 and s_1, which are periodic functions of y_2 and y_3. /108
We can therefore assume that the following are known

$$z_2 - [z_2] \text{ and } s_1 - [s_1].$$

Let us turn to equations (4' 2) and let us equate the mean values of the two terms. We shall obtain two equations, from which we may obtain a_2, $[z_2]$ and $[s_1]$.

If the mean values of the two terms are equal, equations (4' 2) will provide us with z_3 and s_2, up to constant terms, in the form of periodic functions of y_2 and y_3.

This procedure may then be continued.

Since we have found two values for a_1, the equations (4') will have two solutions. Let

$$a = a, \qquad z = \varphi, \qquad s = \varphi_1,$$
$$a = -a, \qquad z = \psi, \qquad s = \psi_1$$

be these two solutions. The general solution of equations (4) will be

$$x_1 = A e^{at} \varphi + B e^{-at} \psi,$$
$$y_1 = A e^{at} \varphi_1 + B e^{-at} \psi_1.$$

We may always assume

$$\varphi \psi_1 - \varphi_1 \psi = 1.$$

We will then see, as was the case in No. 274, that if we set

$$x_1 = x'_1 \varphi + y'_1 \psi, \qquad y_1 = x'_1 \varphi_1 + y'_1 \psi_1,$$
$$x_2 = x'_2 + H_2 x'^2_1 + 2 K_2 x'_1 y'_1 + L_2 y'^2_1, \qquad y_2 = y'_2,$$
$$x_3 = x'_3 + H_3 x'^2_1 + 2 K_3 x'_1 y'_1 + L_3 y'^2_1, \qquad y_3 = y'_3,$$

and if H_2, K_2, L_2, H_3, K_3, L_3 are the suitably chosen periodic functions of y_2 and y_3, the canonical form of the equations will not be changed.

The form of F will not be changed either, but B will be reduced to a constant, and A and C will be reduced to 0.

We may always set

$$B = \text{const.}, \quad A = C = 0.$$

The rest of the computation may be performed as was done in Nos. 274 and 275, and the following conclusion will finally be reached.

The variables x_i and y_i may be developed in powers of $\varepsilon, \sqrt{\mu}$, of three constants α_0, α'_0 and β_0, of $e^{\pm i(n_1 t+\varpi_1)}$, of $e^{\pm i(n'_1 t-\varpi'_1)}$, and of $\sqrt{\beta_0}\, e^{(n_2 t+\varpi_2)}$, $\sqrt{\beta_0}\, e^{-(n_2 t+\varpi_2)}$. The constants n_1, n'_1 and n_2 may themselves be developed in powers of ε, $\sqrt{\mu}$, α_0, α'_0, and β_0. /109

277. Let us proceed to the second generalization method, and let us assume that we wish to study the equations in the vicinity of a true periodic solution having the form

$$x_1 = x_2 = x_3 = y_1 = y_2 = 0.$$

We shall set

$$F = \varepsilon^2 F', \quad x_1 = \varepsilon x'_1, \quad y_1 = \varepsilon y'_1, \quad x_2 = \varepsilon x'_2, \quad y_2 = \varepsilon y'_2,$$
$$x_3 = \varepsilon^2 x'_3, \quad y_3 = y'_3,$$

from which it follows that

$$F = F'_0 + \varepsilon F'_1 + \ldots.$$

The equations remain canonical equations, and we have

$$F'_0 = h x'_3 + \Phi(x'_1, y'_1, x'_2, y'_2),$$

where Φ is a homogeneous quadratic form in x'_1, y'_1, x'_2, y'_2. The coefficients of Φ and h are periodic functions of $y_3 = y'_3$.

However, we shall remove the accents which have become useless, and we shall simply write

$$F_0 = h x_3 + \Phi(x_1, y_1, x_2, y_2).$$

Just as in No. 274 and 276, it may be shown that we may always assume that h may be reduced to a constant.

Let us now consider the equations

$$\frac{dy_3}{dt} = -h, \quad \frac{dx_1}{dt} = \frac{d\Phi}{dy_1}, \quad \frac{dy_1}{dt} = -\frac{d\Phi}{dx_1},$$
$$\frac{dx_2}{dt} = \frac{d\Phi}{dy_2}, \quad \frac{dy_2}{dt} = -\frac{d\Phi}{dx_2}.$$

They are linear and have periodic coefficients. Their general solution will have the form

$$\begin{aligned}
x_1 &= A_1 e^{at}\varphi_{1.1} + A_2 e^{-at}\varphi_{2.1} + A_3 e^{bt}\varphi_{3.1} + A_4 e^{-bt}\varphi_{4.1},\\
y_1 &= A_1 e^{at}\varphi_{1.2} + A_2 e^{-at}\varphi_{2.2} + A_3 e^{bt}\varphi_{3.2} + A_4 e^{-bt}\varphi_{4.2},\\
x_2 &= A_1 e^{at}\varphi_{1.3} + A_2 e^{-at}\varphi_{2.3} + A_3 e^{bt}\varphi_{3.3} + A_4 e^{-bt}\varphi_{4.3},\\
y_2 &= A_1 e^{at}\varphi_{1.4} + A_2 e^{-at}\varphi_{2.4} + A_3 e^{bt}\varphi_{3.4} + A_4 e^{-bt}\varphi_{4.4}.
\end{aligned}$$

The A's are integration constants, and the ϕ's are periodic functions /110
of y_3.

It may be readily shown that expression

$$\varphi_{i.1}\varphi_{k.2} - \varphi_{i.2}\varphi_{k.1} + \varphi_{i.3}\varphi_{k.4} - \varphi_{i.4}\varphi_{k.3}$$

is zero, except in the two following cases

$$i = 1, \quad k = 2; \quad i = 3, \quad k = 4.$$

In these two cases, this expression may be reduced to a constant, which I may set equal to 1.

Let us now set

$$\begin{aligned}
x_1 &= x'_1\varphi_{1.1} + y'_1\varphi_{2.1} + x'_2\varphi_{3.1} + y'_2\varphi_{4.1},\\
y_1 &= x'_1\varphi_{1.2} + y'_1\varphi_{2.2} + x'_2\varphi_{3.2} + y'_2\varphi_{4.2},\\
x_2 &= x'_1\varphi_{1.3} + y'_1\varphi_{2.3} + x'_2\varphi_{3.3} + y'_2\varphi_{4.3},\\
y_2 &= x'_1\varphi_{1.4} + y'_1\varphi_{2.4} + x'_2\varphi_{3.4} + y'_2\varphi_{4.4}.
\end{aligned}$$

It may then be seen that

$$\begin{aligned}
&x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2\\
&\quad = x'_1\,dy'_1 - y'_1\,dx'_1 + x'_2\,dy'_2 - y'_2\,dx'_2 + \psi\,dy_3,
\end{aligned}$$

where ψ is a homogeneous quadratic form with respect to x'_1, y_1, x'_2, y'_2 whose coefficients are periodic functions of y_3.

If we then set

$$x_3 = x'_3 - \frac{\psi}{2}, \quad y_3 = y'_3,$$

the expression

$$x_1\,dy_1 + x_2\,dy_2 + x_3\,dy_3 - x'_1\,dy'_1 - x'_2\,dy'_2 - x'_3\,dy'_3$$

will be an exact differential, and the canonical form of the equations is not changed.

The form of the function F is not changed, only F_0 may be reduced to

$$h\,x'_3 + A\,x'_1 y'_1 + B\,x'_2 y'_2,$$

where h, A and B are constants.

We shall then set

$$x_1 y'_1 = u_1, \qquad \log\frac{y'_1}{x'_1} = 2v_1,$$
$$x'_2 y'_2 = u_2, \qquad \log\frac{y'_2}{x'_2} = 2v_2,$$

and the calculation may be performed as was done in Nos. 275 and 276. The following conclusion will be reached. /111

The x_i's and the y_i's may be developed in powers of ε of three constants α_0, β_0 and β'_0, of $e^{\pm\sqrt{-1}(n_1 t+\varpi_1)}$, and of

$$\sqrt{\beta_0}\,e^{n_2 t+\varpi_2}, \quad \sqrt{\beta_0}\,e^{-(n_2 t+\varpi_2)},$$
$$\sqrt{\beta'_0}\,e^{n'_2 t+\varpi'_2}, \quad \sqrt{\beta'_0}\,e^{-(n'_2 t+\varpi'_2)}.$$

The exponents n_1, n_2 and n'_2 may themselves be developed in powers of ε, α_0, β_0 and β'_0.

This generalization may be directly applied when there are n degrees of freedom. The first case, which is the case given in the preceding section, corresponds to that in which there are n + 1 invariant relationships and one single linear relationship between the mean motions. This is what we discussed in Chapter XIX.

The second case, which is what we are discussing in the present section, corresponds to that in which there are 2n - 1 invariant relationships describing a true periodic solution, and where there are n - 1 linear relationships between the mean motions. This is the case of asymptotic solutions which we discussed in Chapter VII.

However, there are intermediate cases in which we have n - q invariant relationships and q linear relationships between the mean motions. Then the x_i's and the y_i's may be developed in positive or negative powers of q real exponentials and of n - q imaginary exponentials.

Relationship with Integral Invariants

278. Let us assume that the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}; \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \ldots, n) \tag{1}$$

have a periodic solution with the following form

$$x_i = \varphi_i(t + h), \quad y_i = \psi_i(t + h),$$

where h is an integration constant. Let T be the period, in such a way that ϕ_i and ψ_i may be developed in series of sines and cosines of the multiples of $\frac{2\pi}{T}$ (t + h).

Let us consider the solutions which are near this periodic /112
solution. According to the preceding statements, they may be written in the following form: x_i and y_i will be developed in powers of 2n - 2 quantities which are conjugate by pairs, and which I shall call

$$\begin{array}{ll} A_1 e^{\alpha_1 t}, & A'_1 e^{-\alpha_1 t} \\ A_2 e^{\alpha_2 t}, & A'_2 e^{-\alpha_2 t} \\ \ldots\ldots, & \ldots\ \ldots \\ A_{n-1} e^{\alpha_{n-1} t}, & A'_{n-1} e^{-\alpha_{n-1} t}. \end{array}$$

The A's and the A''s are arbitrary integration constants. The exponents α may themselves be developed in powers of $A_1A'_1$, $A_2A'_2$, ..., $A_{n-1}A'_{n-1}$.

In addition, the expansion coefficients of x_i and of y_i are periodic functions of t + h, having period T. These coefficients (just like the exponents α) depend, in addition, on the energy constant C.

We know that there is an integral invariant

$$\int \Sigma\, dx_i\, dy_i, \tag{2}$$

from which it follows that, if β and γ are two integration constants, we must have

$$\sum_i \left(\frac{dx_i}{d\beta} \frac{dy_i}{d\gamma} - \frac{dx_i}{d\gamma} \frac{dx_i}{d\beta} \right) = \text{const.}$$

We could write this equation in another form. Let us assume that β is increased by δβ, and that as a result for x_i, y_i, $A_i e^{\alpha_i t}$, ..., we have the following increases:

$$\delta x_i, \quad \delta y_i, \quad \delta A_i e^{\alpha_i t}, \quad \ldots$$

On the other hand, let us assume that γ is increased by $\delta'\gamma$, and that as a result we have the following increases for x_i, y_i, ...

$$\delta' x_i, \quad \delta' y_i, \quad \ldots$$

Our equation may be written

$$\Sigma(\delta x_i \delta' y_i - \delta y_i \delta' x_i) = \text{const.} \tag{3}$$

The second number is a constant. By this I mean that it is a function of the integration constants multiplied by $\delta\beta\delta'\gamma$.

We obviously have /113

$$\delta A e^{\alpha t} = e^{\alpha t}(\delta A + t\delta\alpha).$$

On the other hand, we have

$$\delta x_i = \frac{dx_i}{dC}\delta C + \frac{dx_i}{dh}\delta h + \sum \frac{dx_i}{d(A_k e^{\alpha_k t})}\delta A_k e^{\alpha_k t} + \sum \frac{dx_i}{d(A'_k e^{-\alpha_k t})}\delta A'_k e^{-\alpha_k t},$$

$$\delta\alpha = \frac{d\alpha}{dC}\delta C + \sum \frac{d\alpha}{d(A_K A'_K)}\delta(A_K A'_K).$$

It can thus be seen that δx_i and δy_i have the following form

$$\delta y_i = \eta_i + t\eta_{1.i}; \qquad \delta' y_i = \eta'_i + t\eta'_{1.i}$$
$$\delta x_i = \xi_i + t\xi_{1.i}; \qquad \delta' x_i = \xi'_i + t\xi'_{1.i},$$

where ξ_i, $\xi_{1.i}$, η_i, $\eta_{1.i}$ are linear with respect to δC, δh, and to the $\delta A e^{\alpha t}$'s and $\delta A' e^{-\alpha t}$'s. In addition, they may be developed in powers of the $Ae^{\alpha t}$'s and the $A'e^{-\alpha t}$'s and the sines and the cosines of the multiples of $\frac{2\pi}{T}(t + h)$. The expressions of $\delta' x_i$, $\delta' y_i$ may be readily obtained. It is sufficient to change δ into δ' in those of δx_i and δy_i. It may be then seen that equation (3) may be written in the following form

$$D + Et + Ft^2 = \text{const.},$$

from which it follows that

$$D = \Sigma(\xi_i \eta'_i - \xi'_i \eta_i),$$
$$E = \Sigma(\xi_i \eta'_{1.i} - \xi'_i \eta_{1.i} + \xi_{1.i}\eta'_i - \xi'_{1.i}\eta_i),$$
$$F = \Sigma(\xi_{1.i}\eta'_{1.i} - \xi'_{1.i}\eta_{1.i})$$

are developed in powers of the $Ae^{\alpha t}$, $A'e^{-\alpha t}$'s and the sines and cosines of the multiples of $\frac{2\pi}{T}(t + h)$, and they are bilinear with respect to the

$$\delta A e^{\alpha t}, \quad \delta A' e^{-\alpha t}, \quad \delta C, \quad \delta h,$$
$$\delta' A e^{\alpha t}, \quad \delta' A' e^{-\alpha t}, \quad \delta' C, \quad \delta' h.$$

The first term must be independent of t, and we shall have

$$E = F = 0,$$

which has already provided us with certain verification relationships which must be satisfied by the expansions of the x_i's and the y_i's.

Thus, D must be independent of t. It will therefore be linear with respect to the following determinants /114

$$\left\{\begin{array}{c} \delta A_k \delta' A'_k - \delta' A_k \delta A'_k, \\ A'_k A'_j (\delta A_k \delta' A_j - \delta A_j \delta' A_k), \\ A'_k (\delta A_k \delta' C - \delta' A_k \delta C), \\ A'_k (\delta A_k \delta' h - \delta' A_k \delta h) \\ (\delta C \delta' h - \delta' \delta C h) \end{array}\right. \tag{4}$$

(or with respect to similar determinants determined from the former by interchanging A_k with A'_k, or A_j with A'_j).

The coefficients will be developed in powers of the $A_k A'_k$'s, and will depend in addition on C.

The time must disappear. The exponentials must therefore disappear, which can only happen if each factor $Ae^{\alpha t}$ is multiplied by a factor $A'e^{-\alpha t}$ or $\delta A' e^{-\alpha t}$, or $\delta' A' e^{-\alpha t}$.

A new series of verification relationships may thus be deduced from this.

279. Among the α_k exponents, some are imaginary, and others are real. Among the real exponents, some are positive, and others are negative. However, since I may arbitrarily choose an exponent which I may call α_k from between two exponents which are equal and have opposite sign, I shall not limit the conditions of generality by assuming that α_k is positive if it is real.

Let us now cancel the coefficients A_k which correspond to an imaginary exponent, or to a positive exponent.

We will then have the following, if α_k is real

$$A_k = 0, \qquad A'_k \gtrless 0$$

and if α_k is imaginary

$$A_k = A'_k = 0.$$

In addition, I shall set

$$C = C_0,$$

where C_0 is the value of the energy constant which corresponds to the periodic solution under consideration.

Our series will then be convergent, and will represent the asymptotic solutions which we studied in Chapter VII. They include h and the A'_k's, which correspond to negative exponents, as arbitrary constants.

We shall therefore have 2n equalities which will express the x_i's /115 and the y_i's as functions of t and of these constants h and A'_k. If we eliminate t, h and the A'_k's between these 2n equalities, we shall have a certain number of invariant relationships between the y_i's.

If a group of values of the x_i's and the y_i's is regarded as representing a point in space having 2n dimensions, these invariant relationships will represent a certain subset V of this space. This is what I shall designate as the asymptotic subset.

Let us reconsider the integral invariant

$$\int \Sigma \, dx_i \, dy_i$$

and let us extend the integration over a portion of this asymptotic subset V. In other words, let us assume that every system of values of the x_i's and the y_i's, which form a part of the integration region, satisfies our invariant relationships.

I may state that the integral invariant will be zero.

It is sufficient for me to demonstrate the fact that

$$\Sigma(\delta x_i \delta' y_i - \delta y_i \delta' x_i) = 0,$$

and this is apparent, because we have

$$A_k = 0, \qquad C = C_0,$$

from which it follows that

$$\delta A_k = 0, \quad \delta C = 0,$$
$$\delta' A_k = 0, \quad \delta' C = 0,$$

which shows that all of the expressions (4) are cancelled. We can also set

$$C = C_0,$$
$$A_k \gtrless 0, \quad A'_k = 0 \quad \text{(for real } \alpha_k\text{)},$$
$$A_k = A'_k = 0 \quad \text{(for imaginary } \alpha_k\text{)}.$$

We shall have obtained a new series of asymptotic solutions and, consequently, a new asymptotic subset to which the same conclusions will apply.

The procedure which we followed for the invariant (2) could be followed for an arbitrary bilinear invariant (invariant of the third type, No. 260), i.e., having the form /116

$$\int\int \Sigma B \, dx_i \, dx_k, \tag{5}$$

where B is a function of the x_i's and of the y_i's and where one or two of the differentials dx_i, dx_k may be replaced by dy_i or dy_k under the sign Σ.

The expression

$$\Sigma B (\delta x_i \delta' x_k - \delta x_k \delta' x_i)$$

will still be linear with respect to the quantities (4). This would still apply to a quadratic invariant (invariant of the second type, No. 260) having the form

$$\int \sqrt{\Sigma B \, dx_i \, dx_k}, \tag{6}$$

where B is a function of the x_i's and the y_i's, and where one or two of the differentials dx_i, dx_k may be replaced by dy_i, dy_k under the sign Σ.

It may be seen that the expression

$$\Sigma B \, \delta x_i \, \delta x_k$$

must be linear with respect to the expressions

$$\left\{ \begin{array}{c} \delta A_k \, \delta A'_k, \\ A'_k A'_j \, \delta A_k \, \delta A_j, \\ A'_k \, \delta A_k \, \delta C, \\ \delta C \, \delta h \end{array} \right. \tag{4'}$$

and to those which may be deduced from them when interchanging A_k and A'_k, A_j and A'_j.

For every asymptotic subset, the invariant (5), like the invariant (6), must be cancelled.

Another Discussion Method

280. This same study may be pursued farther, while presenting it in a different form.

For example, we shall assume that we are dealing with a problem of dynamics, that the x_i's are the coordinates of different points of matter of the system, and that the conjugate variables y_i are the com- /117
ponents of their momentum. We plan to study the integral invariants which are algebraic with respect to the x_i's and to the y_i's, and to determine whether one may exist in addition to the one which is known, and which is written

$$\int\int \Sigma \, dx_i \, dy_i.$$

We have seen that, in the vicinity of a periodic solution, the x_i's and the y_i's may be developed in powers of the $Ae^{\alpha t}$'s,.... We are going to consider these expansions again, but we shall assume that the value of the energy constant corresponding to the periodic solution is zero, so that the expansions will not only proceed in powers of the $Ae^{\alpha t}$'s, but even in powers of C. In addition, they will depend on t + h.

By equating the x_i's and the y_i's to these expansions, we obtain 2n equations, which we shall solve with respect to the $Ae^{\alpha t}$'s, C and t + h.

We have

$$\left\{\begin{aligned} A_k e^{\alpha_k t} &= f_k, \\ A'_k e^{-\alpha_k t} &= f'_k, \\ C &= \Phi, \\ \alpha_0 t + \beta_0 = \frac{2\pi}{T}(t+h) &= \theta. \end{aligned}\right. \tag{7}$$

We should point out that α_0, like α_k, may be developed in powers of C and of the $A_k A'_k$'s. It may be seen that f_k, f'_k, Φ, $\cos\Theta$, $\sin\Theta$ are uniform functions of the x_i's and the y_i's in the vicinity of the periodic solution. In addition, the x_i's and the y_i's may be developed in powers of the f_k's, the f'_k's, and Φ, and according to the sines and cosines of the multiples of Θ.

On the other hand, the expression

$$\Sigma(\delta x_i \delta' y_i - \delta y_i \delta' x_i) \tag{3}$$

which corresponds to the invariant (2), or the similar expressions which would correspond to another bilinear invariant of the form (5), must be developed in powers of the f_k, f'_k, Φ's and be bilinear with respect to /118

$$\begin{matrix} \delta f_k, & \delta f'_k, & \delta\Phi, & \delta\Theta, \\ \delta' f_k, & \delta' f'_k, & \delta'\Phi, & \delta'\Theta. \end{matrix}$$

In addition, when we replace f_k, f'_k, Φ, Θ by their values (7), this expression must be independent of t. The time t may be introduced in three different ways:

1. In the exponential form;

2. In the form of the cosine or sine of the multiples of (t + h);

3. Outside of the exponential and trigonometric expressions (and, as we shall see, of the second degree and more).

It must not enter in any of these three ways.

1. In order that it does not enter in the exponential form, it is necessary and sufficient that the expression be linear with respect to the following quantities which are similar to (4)

$$\left\{ \begin{array}{c} \delta f_k \delta' f'_k - \delta' f_k \delta f'_k, \\ f'_k f'_j (\delta f_k \delta' f_j - \delta f_j \delta' f_k), \\ f'_k (\delta f_k \delta'\Phi - \delta' f_k \delta\Phi), \\ f'_k (\delta f_k \delta'\Theta - \delta' f_k \delta\Theta), \\ (\delta\Phi \delta'\Theta - \delta'\Phi \delta\Theta), \end{array} \right. \tag{8}$$

where the coefficients may be developed in powers of the f_k's, f'_k's, and of Φ.

2. In order that t does not enter in the trigonometric form, it is necessary and sufficient that our expression does not depend on Θ, but only on its variations $\delta\Theta$, $\delta'\Theta$.

3. We must now determine the condition under which t does not enter outside the exponential and trigonometric expressions. We should point out that we have

$$\begin{cases} \delta f_k = e^{\alpha_k t}(\delta A_k + A_k t\,\delta\alpha_k), \\ \delta f'_k = e^{-\alpha_k t}(\delta A'_k - A'_k t\,\delta\alpha_k), \\ \delta\rho = \delta C, \qquad \delta\Theta = \delta\beta_0 + t\,\delta\alpha_0. \end{cases} \tag{9}$$

We may distinguish five types of terms in our expression, depending on whether they contain as a factor a quantity (8) included in the first, second, third, fourth, or fifth line of the table (8).

Under this assumption, if we replace δf_k, ... by their values (9), we shall see that the five types of terms include as a factor, respec- /119 tively,

$$\begin{cases} (\delta A_k \delta' A'_k - \delta A'_k \delta' A_k) + t[\delta\alpha_k \delta'(A_k A'_k) - \delta'\alpha_k \delta(A_k A'_k)], \\ A'_k A'_j(\delta A_k \delta' A_j - \delta A_j \delta' A_k) \\ \quad + A'_k A'_j t[A_k(\delta\alpha_k \delta' A_j - \delta'\alpha_k \delta A_j) - A_j(\delta\alpha_j \delta' A_k - \delta'\alpha_j \delta A_k)] \\ \quad + A_k A'_k A_j A'_j t^2(\delta\alpha_k \delta'\alpha_j - \delta\alpha_j \delta'\alpha_k), \\ A'_k(\delta A_k \delta' C - \delta' A_k \delta C) + A_k A'_k t(\delta\alpha_k \delta' C - \delta'\alpha_k \delta C), \\ A'_k(\delta A_k \delta'\beta_0 - \delta' A_k \delta\beta_0) + A_k A'_k t(\delta\alpha_k \delta'\beta_0 - \delta'\alpha_k \delta\beta_0) \\ \quad + A'_k t(\delta A_k \delta'\alpha_0 - \delta' A_k \delta\alpha_0) + A_k A'_k t^2(\delta\alpha_k \delta'\alpha_0 - \delta'\alpha_k \delta\alpha_0), \\ (\delta C \delta'\beta_0 - \delta' C \delta\beta_0) + t(\delta C \delta'\alpha_0 - \delta' C \delta\alpha_0). \end{cases} \tag{10}$$

It may be seen that the time can enter in the second power.

Let us first make the terms for t^2 disappear. They may only begin with terms of the second type or of the fourth type.

It may be stated that the coefficient of

$$t^2(\delta\alpha_k \delta'\alpha_j - \delta\alpha_j \delta'\alpha_k)$$

must be zero.

In actuality, due to the fact that the virtual displacements in the constants are arbitrary, we may assume that all the $\delta\alpha_i$'s vanish, with the exception of $\delta\alpha_k$, and in the same way it may be assumed that all the $\delta'\alpha_j$'s vanish with the exception of $\delta'\alpha_j$.

All the terms in t^2 cancel, with the exception of the term in

$$t^2(\delta\alpha_k \delta'\alpha_j - \delta\alpha_j \delta'\alpha_k).$$

There would be an exception if there were a relationship between the n - 1 exponents α_i. We could no longer assume that all the $\delta\alpha_i$'s cancel except one, unless the last one itself cancels too.

There are now four terms of the second type which result in terms in

$$t^2(\delta\alpha_k \delta'\alpha_j - \delta\alpha_j \delta'\alpha_k).$$

For purposes of brevity, I may write them in the following form

$$\psi_1\omega_1 + \psi_2\omega_2 + \psi_3\omega_3 + \psi_4\omega_4;$$

The ψ's are developed in powers of the f_i, f'_i and of Φ. I have employed ω_1 to designate the expression which appears in the second line of the table (8): /120

ω_2 may be deduced from ω_1 by interchanging f_k and f'_k,

ω_3 may be deduced from ω_1 by interchanging f_j and f'_j,

ω_4 may be deduced from ω_1 by making these two permutations at the same time.

In order that the terms in t^2 disappear, it is necessary and sufficient that

$$\psi_1 - \psi_2 - \psi_3 + \psi_4 = 0. \tag{11}$$

If this condition is fulfilled, our four terms

$$\psi_1\omega_1 + \psi_2\omega_2 + \psi_3\omega_3 + \psi_4\omega_4$$

will provide us with the following terms in t

$$\begin{aligned}(\psi_1 - \psi_2)\, t A_k A'_k[\delta\alpha_k \delta'(A_j A'_j) - \delta'\alpha_k \delta(A_j A'_j)] \\ + (\psi_3 - \psi_1)\, t A_j A'_j[\delta\alpha_j \delta'(A_k A'_k) - \delta'\alpha_j \delta(A_k A'_k)].\end{aligned}$$

Let us now consider terms of the fourth type, which we shall group together by pairs. Let the following be one group of two terms

$$\psi_1\omega_1 + \psi_2\omega_2,$$

where ψ_1 and ψ_2 may be developed in powers of C and of the $A_k A'_k$'s, where ω_1 is the expression included on the fourth line of the table (10), and where ω_2 is that which is deduced by interchanging A_k and A'_k and changing α_k to $-\alpha_k$.

In order that the terms in t^2 disappear, it is necessary that

$$\psi_1 = \psi_2$$

and then the terms in t may be reduced to

$$\psi_1 t[\delta(A_k A'_k)\delta'\alpha_0 - \delta'(A_k A'_k)\delta\alpha_0].$$

281. Our terms in t now proceed in powers of C, and of the $A_k A'_k$'s, and according to the δ's and the δ''s of α_0, α_k, C, $A_k A'_k$. We must now make these terms vanish. I shall state that they are zero when we set

$$C = 0, \qquad A_k A'_k = 0,$$

without assuming that δC, $\delta' C$, $\delta' A_k A'_k$, $\delta' A_k A'_k$ are zero.

In our invariant, let B_k be that which the coefficient of the term in $(\delta f_k \delta' f'_k - \delta f'_k \delta' f_k)$ becomes when we set $C = A_k A'_k = 0$.

Let D_k be that which the coefficient of the term containing /121

$$f'_k(\delta f_k \delta'\theta - \delta\theta\,\delta' f_k)$$

becomes, and D_0 that which the coefficient of the term containing

$$(\delta\Phi\,\delta'\theta - \delta\theta\,\delta'\Phi).$$

becomes. We must also have identically

$$\begin{aligned} &\Sigma B_k[\delta\alpha_k \delta'(A_k A'_k) - \delta'\alpha_k \delta(A_k A'_k)] \\ &\quad + \Sigma D_k[\delta(A_k A'_k)\delta'\alpha_0 - \delta'(A_k A'_k)\delta\alpha_0] + D_0(\delta C\,\delta'\alpha_0 - \delta' C\,\delta\alpha_0) = 0. \end{aligned}$$

For purposes of brevity, let us write γ_k instead of $A_k A'_k$, γ_0 instead of C and

$$\partial(u, v)$$

instead of

$$\delta u\,\delta' v - \delta v\,\delta' u;$$

We have

$$\Sigma B_k \partial(\alpha_k, \gamma_k) + \Sigma D_k \partial(\gamma_k, \alpha_0) + D_0 \partial(\gamma_0, \alpha_0) = 0$$

or

$$\Sigma\Sigma B_k \frac{d\alpha_k}{d\gamma_j}\partial(\gamma_j, \gamma_k) + \Sigma\Sigma D_k \frac{d\alpha_0}{d\gamma_j}\partial(\gamma_k, \gamma_j) + \Sigma D_0 \frac{d\alpha_0}{d\gamma_j}\partial(\gamma_0, \gamma_j) = 0.$$

Under the sign Σ or $\Sigma\Sigma$, k may take on the values 1, 2, ..., n-1 and j may take on the values 0, 1, 2, ..., n-1.

When setting the coefficient of $\partial(\gamma_j, \gamma_k)$ equal to zero, we obtain

$$B_k \frac{d\alpha_k}{d\gamma_j} - B_j \frac{d\alpha_j}{d\gamma_k} - D_k \frac{d\alpha_0}{d\gamma_j} + D_j \frac{d\alpha_0}{d\gamma_k} = 0. \tag{12}$$

By setting the coefficient of $\partial(\gamma_0, \gamma_j)$ equal to zero, we have

$$B_j \frac{d\alpha_j}{d\gamma_0} - D_j \frac{d\alpha_0}{d\gamma_0} + D_0 \frac{d\alpha_0}{d\gamma_j} = 0. \tag{12'}$$

These equations indicate that

$$-D_0 \alpha_0 d\gamma_0 + \Sigma(B_k \alpha_k - D_k \alpha_0) d\gamma_k \tag{13}$$

is an exact differential.

We must set $\gamma_j = 0$ in equations (12) and (12'). The $\frac{d\alpha}{d\gamma}$'s are therefore constants. The α_j's are therefore linear functions of the γ's. In actuality, as we have seen, the α's may be developed in powers of the γ's. However, the result which we have just obtained is only valid if we neglect the squares of the γ's, and if we stop the expansions of the α's at the terms of the first degree. In addition, the B's and D's are constants. The expression (13) is therefore the exact differential of a polynomial of the second degree. /122

In order to carry this investigation further, let us express the α_k's not only as functions of

$$\gamma_0, \gamma_1, \ldots, \gamma_{n-1},$$

but also as functions of

$$\alpha_0, \gamma_1, \ldots, \gamma_{n-1}.$$

In order to avoid any confusion, let us employ ∂ to designate the derivatives chosen with respect to the new variables, and the d's to designate the derivatives chosen with respect to the old variables.

It may then be seen that

$$\Sigma B_k \alpha_k d\gamma_k + d\alpha_0 \Sigma D_j \gamma_j$$

is an exact differential, which entails the following conditions

$$B_k \frac{\partial \alpha_k}{\partial \gamma_i} = B_i \frac{\partial \alpha_i}{\partial \gamma_k}. \tag{14}$$

If one knew the relationships between the α's and the γ's, these equations would allow us to determine the coefficients B_i.

We can express $\Sigma D_j \gamma_j$ as a function of the variables

$$\alpha_0, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}$$

while writing

$$\Sigma D_j \gamma_j = E_0 \alpha_0 + \Sigma E_k \gamma_k.$$

The E_k's will be given by the equations

$$E_k = B_k \frac{\partial \alpha_k}{\partial \alpha_0}, \tag{14'}$$

and E_0 may be chosen arbitrarily.

It is necessary that equations (14) be compatible, which requires certain conditions in the case of $n > 3$

$$\frac{\partial \alpha_k}{\partial \gamma_i} \frac{\partial \alpha_i}{\partial \gamma_j} \frac{\partial \alpha_j}{\partial \gamma_k} = \frac{\partial \alpha_i}{\partial \gamma_k} \frac{\partial \alpha_j}{\partial \gamma_i} \frac{\partial \alpha_k}{\partial \gamma_j}. \tag{15}$$

These conditions (15) will always be fulfilled, since there is always an integral invariant /123

$$\int \Sigma \, dx_i dy_i.$$

If there are several integral invariants which do not vanish identically for the periodic solution under consideration, a system of values of the coefficients B_i and E_i must correspond to each of these invariants.

If equations (14) have q solutions which are linearly independent, we may calculate the corresponding values of the E_k's by means of equations (14'). Since E_0 remains arbitrary, we shall have q + 1 systems of values, which are linearly independent, of the coefficients B_i and E_i.

We may therefore have q + 1 different integral invariants (if the periodic solution under consideration is not singular, with the meaning attributed to this word in No. 257), but we cannot have any more.

282. I stated above that conditions (15) were definitely fulfilled; there may still be some doubt on this point. If equations (14) have q different solutions, we may have q + 1 invariants. If there is only one invariant, we could assume that q = 0. The presence of a single invariant

$$\int \Sigma \, dx_i dy_i$$

would not enable us to state that equations (14) definitely have a solution.

This is the doubt which I wish to dispel.

I would first like to note that in the case of the three-body problem, there are not one, but two integral invariants.

In Volume I, Chapter IV, we studied the variational equations of this problem.

On pages 170 and 172 we obtained the following integrals

$$\sum \frac{y\eta}{m} - \sum \frac{dV}{dx}\xi = \text{const.} \tag{1}$$

$$\Sigma(2x\eta + y\xi) - 3t\left(\sum \frac{y\eta}{m} - \sum \frac{dV}{dx}\xi\right) = \text{const.} \tag{2}$$

In the same way, we could obtain /124

$$\sum \frac{y\eta'}{m} - \sum \frac{dV}{dx}\xi' = \text{const.} \tag{1'}$$

$$\Sigma(2x\eta' + y\xi') - 3t\left(\sum \frac{y\eta'}{m} - \sum \frac{dV}{dx}\xi'\right) = \text{const.} \tag{2'}$$

Let us multiply (2') by (1), (1') by (2), and let us subtract. We then have

$$\left\{\begin{aligned} &\sum\left(\frac{y\eta}{m} - \frac{dV}{dx}\xi\right)\Sigma(2x\eta' + y\xi') \\ &\quad -\sum\left(\frac{y\eta'}{m} - \frac{dV}{dx}\xi'\right)\Sigma(2x\eta + y\xi) = \text{const.}\end{aligned}\right. \tag{16}$$

The first term is linear with respect to the determinants having the form

$$\eta_i\eta'_k - \eta_k\eta'_i, \quad \eta_i\xi'_k - \eta'_k\xi_i, \quad \xi_i\xi_k - \xi_k\xi'_i.$$

We therefore have an integral of the variational equations, and we may deduce from it a new bilinear integral invariant.

In the case of the three-body problem, we therefore have at least q=1, and it may be stated that conditions (15) are fulfilled.

283. Is this still true in the general case? Let us assume that it is not. Then all the coefficients which we have called B_i must be zero, as well as all of the E_k's, with the exception of E_0.

Therefore, when we attribute the values corresponding to the periodic solution under consideration to the x_i's and the y_i's, i.e., when we set

$$C = A_k A'_k = 0,$$

the coefficients of the terms in $\delta f_k \delta' f'_k - \delta f'_k \delta f_k$ must vanish, and only the terms in

$$f'_k(\delta f_k \delta'\theta - \delta\theta\delta' f_k)$$
$$(\delta\Phi\delta'\theta - \delta\theta\delta'\Phi).$$

remain.

Our invariant must therefore vanish when we have

$$\delta\theta = \delta'\theta = 0.$$

This is not the case for the invariant /125

$$\int \Sigma dx_i dy_i$$

to which the following expression corresponds

$$\Sigma(\delta x_i \delta' y_i - \delta y_i \delta' x_i):$$

Let us set

$$\delta\theta = \Sigma a_i \delta x_i + \Sigma b_i \delta y_i,$$
$$\delta'\theta = \Sigma a_i \delta' x_i + \Sigma b_i \delta' y_i.$$

We must have an equation of the form

$$\Sigma(\delta x_i \delta' y_i - \delta y_i \delta' x_i) = \Sigma(a_i \delta x_i + b_i \delta y_i)\Sigma(c_i \delta' x_i + e_i \delta' y_i) - \Sigma(a_i \delta' x_i + b_i \delta' y_i)\Sigma(c_i \delta x_i + e_i \delta y_i).$$

However, this is impossible, since the first term is a bilinear form with determinant 1, and the second is a bilinear form with determinant 0.

We must therefore conclude that conditions (15) are always fulfilled.

284. Let us now try to determine whether equations (14) may have several solutions.

Let

$$B_1, \quad B_2, \quad \ldots, \quad B_n.$$
$$B'_1, \quad B'_2, \quad \ldots, \quad B'_n$$

be these two solutions and let us assume that we do not have

$$\frac{B_k}{B'_k} = \frac{B_i}{B'_i};$$

and then the two equations

$$B_k \frac{\partial x_k}{\partial y_i} = B_i \frac{\partial x_i}{\partial y_k},$$
$$B'_k \frac{\partial x_k}{\partial y_i} = B'_i \frac{\partial x_i}{\partial y_k}$$

will imply

$$\frac{\partial \alpha_k}{\partial \gamma_i} = \frac{\partial \alpha_i}{\partial \gamma_k} = 0.$$

Then the indices

$$1, \ldots, 2, n$$

will be divided into a certain number of groups, as many groups as there different values for the ratio $\frac{B_i}{B'_i}$. Two indices will belong to the same group, if they correspond to the same value of the ratio $\frac{B_i}{B'_i}$. /126

In order that α_k depends on γ_i (or α_i on γ_k), it is necessary that the indices i and k belong to the same group.

In order to formulate these ideas clearly, let us assume that there are only two groups containing the indices, respectively,

$$1, 2, \ldots, p,$$

$$p+1, p+2, \ldots, n-1.$$

Then

$$\alpha_1, \alpha_2, \ldots, \alpha_p$$

will depend only on

$$\alpha_0, \gamma_1, \gamma_2, \ldots, \gamma_p,$$

and

$$\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{n-1}$$

will depend only on

$$\alpha_0, \gamma_{p+1}, \gamma_{p+2}, \ldots, \gamma_{n-1}.$$

It then appears that the characteristic exponents α_k form several independent groups, in such a way that the α_k's of one group do not depend on the products $A_j A'_j$ corresponding to another group.

The periodic solutions for which this condition will be produced (or for which there would be one relationship between the α_k's) may be called <u>particular</u> solutions.

We therefore arrive at the following conclusion:

<u>In order that there be another algebraic invariant, in addition to</u>

those which we know, it would be necessary that all the periodic solutions be particular solutions, or that they all be singular solutions, with the meaning given in No. 257.

I shall not try to demonstrate the fact that this condition could not occur in the three-body problem, but this would seem to be very unlikely.

Quadratic Invariants /127

285. Let us now study the quadratic invariants from the same point of view, i.e., the integral invariants having the form

$$\int \sqrt{F},$$

where F is a quadratic form with respect to the differentials dx_i, dy_i.

Let us set

$$F = \Sigma H \, dx_i \, dx_k,$$

where the H's are functions of the x's and the y's, and where the product $dx_i dx_k$ may be replaced in certain terms by the product $dx_i dy_k$ or $dy_i dy_k$.

We may then write the following equation which is similar to equation (3) of No. 278

$$\Sigma H \, \delta x_i \, \delta x_k = \text{const.} \tag{1}$$

On the other hand, we find in No. 278 that

$$\delta x_i = \xi_i + t\xi_{1,i}, \qquad \delta y_i = \eta_i + t\eta_{1,i}.$$

We may then write equation (1) in the form

$$D + Et + Ft^2 = \text{const.},$$

where D, E, F may be developed in powers of the $Ae^{\alpha t}$'s, $A'e^{-\alpha t}$'s and of the sines and cosines of the multiples of $\frac{2\pi}{T}(t + h)$, and where D, E, F are quadratic with respect to the

$$\delta A e^{\alpha t}, \quad \delta A' e^{-\alpha t}, \quad \delta C, \quad \delta h.$$

We must therefore have

$$E = F = 0,$$

and, in addition, D must be independent of t, which shows that D must be /128 linear with respect to the following expressions

$$\begin{gathered}\delta A_k\,\delta A'_k,\\ A'_k A'_j\,\delta A_k\,\delta A_j,\\ A'_k\,\delta A_k\,\delta C,\\ A'_k\,\delta A_k\,\delta h,\\ \delta C\ \ \delta h,\end{gathered}$$

or with respect to the expressions which may be deduced by interchanging A_k and A'_k, or A_j and A'_j.

The coefficients will be developed in powers of the products $A_kA'_k$ and of C (if one assumes that the periodic solution corresponds to the zero value of the energy constant).

286. Let us return to equations (7) given in No. 280, and let us pursue the same line of reasoning as given in No. 280. We shall find that the expression

$$\Pi = \Sigma H\,\delta x_i\,\delta x_k,$$

must satisfy the following conditions when the x_i's and y_i's are replaced by their expansions as functions of the f_k, f'_k, Φ and Θ's:

1. It must be linear with respect to the following quantities:

$$\left\{\begin{gathered}\delta f_k\,\delta f'_k,\\ f'_k f'_j\,\delta f_k\,\delta f_j,\\ f'_k\,\delta f_k\,\delta\Phi,\\ f'_k\,\delta f_k\,\delta\Theta,\\ \delta\Phi\ \ \delta\Theta,\\ \delta\Phi^2\,\delta\Theta^2,\end{gathered}\right. \qquad (8')$$

where the coefficients are developed in powers of the $f_kf'_k$'s and of Φ.

2. It will not depend on Θ, but only on $\delta\Theta$.

3. If these conditions are fulfilled, expression Π will not include the time, neither in the exponential form nor in the trigonometric form.

We must now determine the condition under which the time is not included outside either the exponential or trigonometric terms.

Let us consider equations (9) again from Section No. 280. We shall find that the following terms will correspond to the different terms given in the Table (8'): /129

$$
\left\{
\begin{array}{l}
\delta A_k \delta A'_k + t(A_k \delta A'_k \delta\alpha_k - A'_k \delta A_k \delta\alpha_k) - A_k A'_k t^2 (\delta\alpha_k)^2, \\
A'_k A'_j \delta A_k \delta A_j + A'_k A'_j t (A_k \delta\alpha_k \delta A_j + A_j \delta\alpha_j \delta A_k) \\
\quad + A_k A'_k A_j A'_j t^2 \delta\alpha_k \delta\alpha_j, \\
A'_k \delta A_k \delta C + A_k A'_k t \delta\alpha_k \delta C, \\
A'_k \delta A_k \delta\beta_0 + A'_k t (\delta A_k \delta\alpha_0 + A_k \delta\alpha_k \delta\beta_0) + A_k A'_k t^2 \delta\alpha_k \delta\alpha_0, \\
\delta C \delta\beta_0 + t \delta C \delta\alpha_0; \qquad \delta C^2, \qquad \delta\beta_0^2 + 2t \delta\beta_0 \delta\alpha_0 + t^2 \delta\alpha_0^2.
\end{array}
\right. \tag{10'}
$$

Let us first make the terms in t^2 vanish.

The entire group of these terms is a quadratic form with respect to

$$\delta\alpha_0, \quad \delta\alpha_1, \quad \ldots, \quad \delta\alpha_{n-1}.$$

This quadratic form must be zero.

The coefficient of $\delta\alpha_k \delta\alpha_j t^2$ must therefore be zero. However, there are four terms which could introduce the product $t^2 \delta\alpha_k \delta\alpha_j$; these are the terms in .

$$f'_k f'_j \delta f_k \delta f_j, \quad f'_k f_j \delta f_k \delta f'_j, \quad f_k f'_j \delta f'_k \delta f_j, \quad f_k f_j \delta f'_k \delta f'_j.$$

For purposes of brevity, let us designate these four expressions by ω_1, ω_2, ω_3, ω_4. The entire group of our four terms may then be written

$$\psi_1 \omega_1 + \psi_2 \omega_2 + \psi_3 \omega_3 + \psi_4 \omega_4,$$

where ψ_1, ψ_2, ψ_3 and ψ_4 may be developed in powers of the $f_k f'_k$'s and of Φ. In order that the coefficient of $t^2 \delta\alpha_k \delta\alpha_j$ vanish, we must have identically

$$\psi_1 + \psi_2 + \psi_3 + \psi_4 = 0.$$

In the same way, the coefficient of $t^2 \delta^2 \alpha_k$ must vanish. It arises from terms in

$$\delta f_k \delta f'_k, \quad f_k^2 \delta f'^2_k, \quad f'^2_k \delta f_k^2.$$

For purposes of brevity, let us designate these three expressions by ω'_1, ω'_2, ω'_3, and the entire group of the three terms by

$$\psi'_1 \omega'_1 + \psi'_2 \omega'_2 + \psi'_3 \omega'_3,$$

where ψ'_1, ψ'_2, ψ'_3 may be developed in powers of the $f_k f'_k$'s and of Φ.

In order that the coefficient of $t^2 \delta^2 \alpha_k$ may vanish, we must have /130

$$f_k f'_k (\psi'_2 + \psi'_3) - \psi'_1 = 0. \tag{11}$$

For the periodic solution, we must have

$$f_1 = f'_1 = f_2 = f'_2 = \ldots = f_{n-1} = f'_{n-1} = 0.$$

All the terms including as a factor one of the expressions appearing on the 2nd, 3rd, or 4th lines of the Table (8') must then vanish, because each of these expressions includes f_k or f'_k as a factor.

The only terms of expression Π which do not vanish for the periodic solution are therefore the terms in

$$\delta f_k \, \delta f_k, \quad \delta\Phi \, \delta\Theta, \quad \delta\Phi^2, \quad \delta\Theta^2.$$

Equation (11) shows that ψ'_1 contains $f_k f'_k$ as a factor. Therefore, the term $\psi'_1 \delta f_k \delta f'_k$ must also vanish. We then have only the terms in

$$\delta\Phi^2, \quad \delta\Phi \, \delta\Theta, \quad \delta\Theta^2.$$

The first does not include t, the second includes it in the first power , and the third includes it in the second power.

Due to the fact that this third term is the only one which includes t^2, it must be zero. If it is zero, the second term will also be zero, due to the fact that it is the only one which includes t.

Finally, all the terms of Π vanish for the periodic solution, except the term in $\delta\Phi^2$.

In the general problem of dynamics, just as in the case of the three-body problem which we have designated as the restricted problem, the general reduced problem, and the planar reduced problem, we have a quadratic invariant, but no more than one.

I may write the energy equation in the following form

$$F = \text{const}.$$

This invariant is nothing else than

$$\int \sqrt{(dF)^2},$$

and the term in $\delta\Phi^2$ which does not vanish corresponds to this invariant.

If there is a quadratic invariant, other than that which is known, /131 this invariant must vanish for all points of the periodic solution.

In other words, this periodic solution must be singular in the sense of the meaning given in No. 257.

There would be an exception, if the n exponents

$$\alpha_0, \quad \alpha_1, \quad \alpha_2, \quad \ldots, \quad \alpha_{n-1}$$

were not independent of each other, but if there were one relationship between them. In this case, the coefficient of t^2, which is a quadratic form with respect to the n variables

$$\delta\alpha_0, \quad \delta\alpha_1, \quad \ldots, \quad \delta\alpha_{n-1},$$

could vanish without all of its coefficients being zero, since these n variables will no longer be independent.

To sum up, <u>in order that there may be other quadratic invariants, in addition to those which we are acquainted with, it is necessary that all periodic solutions be singular or particular</u>.

It is very unlikely that this will be the case for the three-body problem.

<u>Case of the Restricted Problem</u>

287. We may conceive of another discussion method which we shall only apply to the case of the restricted problem. The discussion presented in No. 257 has presented the possibility of two quadratic invariants, of which one is known. Let us assume that these two quadratic invariants exist, and let Π be the quadratic form corresponding to one of these invariants. According to the preceding statements, Π may include terms in

$$\left\{ \begin{array}{l} \delta f_1 \delta f'_1, \quad f'_1 \delta f_1 \delta\Phi, \quad f_1 \delta f'_1 \delta\Phi, \quad f'_1 \delta f_1 \delta\Theta, \quad f_1 \delta f'_1 \delta\Theta, \\ \qquad f'^2_1 \delta f^2_1, \quad f^2_1 \delta f'^2_1, \quad \delta\Theta^2, \quad \delta\Phi \delta\Theta, \quad \delta\Phi^2. \end{array} \right. \tag{1}$$

On the other hand, Π is a quadratic form with respect to the quantities

$$\delta x_1, \quad \delta x_2, \quad \delta y_1, \quad \delta y_2,$$

whose coefficients are the algebraic functions of x_1, x_2, y_1, y_2. /132

Following are the variables x_i and y_i which we shall select. In this problem, which I have called the restricted problem, two of the bodies describe concentric circumferences, and the third (whose mass is zero) moves in the plane of these circumferences. I shall refer this third body to moving axes turning uniformly around the center of gravity of the first two. One of these axes will constantly coincide with the line joining these two first bodies. I shall use x_1 and x_2 to designate the coordinates of the third body with respect to these moving axes, and y_1 and y_2 to designate the projections of the absolute velocity on the moving axes.

Let us then set

$$\Phi = F + \omega G,$$

where F and G designate the energy function and the area function in the absolute motion, and where ω designates the angular rotational velocity of the two first bodies around their common center of gravity. The equations take the canonical form

$$\frac{dx_i}{dt} = \frac{d\Phi}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{d\Phi}{dx_i}.$$

The integral Φ = const. is nothing else than "the Jacobi integral" (see Volume I, No. 9, page 23).

Under this assumption, our expression Π will be a quadratic form in

$$\delta x_1, \quad \delta x_2, \quad \delta y_1, \quad \delta y_2,$$

for which the coefficients will be algebraic in x_i and y_i. If we assume that the four variables x and y are related by the relationship

$$\Phi = \text{const.},$$

which entails the following condition

$$\delta\Phi = 0,$$

our four variables δx_i, δy_i will no longer be independent. One of them could be eliminated, and Π will become a ternary quadratic form.

Let us consider one point of the periodic solution. For this point, we shall have

$$f_1 = f'_1 = 0.$$

All the expressions (1) will therefore vanish with the exception of /133

$$\delta f_1 \delta f'_1, \quad \delta\Theta^2, \quad \delta\Phi\delta\Theta \quad \text{and} \quad \delta\Phi^2.$$

If we set $\delta\Phi = 0$, they will all vanish with the exception of

$$\delta f_1 \delta f'_1 \quad \text{and} \quad \delta\Theta^2.$$

Therefore, for a point of the periodic solution, let us set

$$\Pi = B\,\delta f_1 \delta f'_1 + C\,\delta\Theta^2.$$

The entire group of terms for t^2 will therefore be reduced, for this same point, to

$$-Bf_1f'_1t^2\delta\alpha_1^2 + Ct^2\delta\alpha_0^2$$

(see, supra, Table 10') and, since $f_1 = f'_1 = 0$, may be reduced to

$$Ct^2\delta\alpha_0^2.$$

The terms in t^2 must vanish. The latter is the only one which does not vanish for the point under consideration; all the others are zero, even when the condition $\delta\Phi = 0$ is not imposed, because $\delta\Phi\delta\Theta$ and $\delta\Phi^2$ do not provide terms in t^2.

However, $\delta\alpha_0$ is not also zero. For one point of the periodic solution, we have

$$\frac{d\alpha_0}{df_1} = \frac{d\alpha_0}{df'_1} = \frac{d\alpha_0}{d\theta} = 0,$$

but we cannot be sure of having $\frac{d\alpha_0}{d\Phi} = 0$. This would assume that there is a continuous infinity of periodic solutions having the same period, which does not occur.

Nevertheless, it may be noted that $\frac{d\alpha_0}{d\Phi}$ includes the small quantity which I may designate by μ as a factor, i.e., the mass of the second body. Consequently, it may be noted that $\delta\alpha_0$ vanishes for $\mu = 0$, i.e., in Keplerian motion.

The terms in t^2 can only vanish if we have

$$C = 0,$$

from which it follows

$$\Pi = B\delta f_1\delta f'_1.$$

However, this latter equation would indicate that Π may be reduced to a binary quadratic form and, consequently, that its discriminant is zero. Thus, the discriminant Δ of Π must vanish for every point of every periodic solution. /134

288. However, an algebraic relationship such as

$$\Delta = 0$$

cannot be valid, unless it is reduced to an identity, for every point of every periodic solution.

If the relationship $\Delta = 0$

is supplemented by two other relationships

$$F = \beta, \qquad G = \gamma \tag{3}$$

(where β and γ are two arbitrary constants, and F and G are the two functions which were designated in the preceding section) and a fourth arbitrary algebraic relationship

$$H = o, \tag{4}$$

the number of solutions of these four algebraic equations will be limited whatever the constants β and γ may be.

Let us now consider a periodic solution, and the variables x_i and y_i will be developed in powers of μ in the following form

$$\begin{cases} x_i = x_i^0 + \mu x_i^1 + \ldots \\ y_i = y_i^0 + \mu y_i^1 + \ldots \end{cases} \tag{5}$$

In the same way, F will be developed in powers of μ, and we shall have

$$F = F_0 + \mu F_1 + \ldots$$

and G and H will be independent of μ.

The quantity Δ remains. It may be stated that this function, which is algebraic in x_i and y_i under the terms of the hypothesis, also depends algebraically on μ.

If we state that

$$\int \sqrt{\Pi}$$

is an integral invariant, we will be led to certain relationships which include the coefficinets of Π, their derivatives, and the coefficients of the differential equations of motion.

We assumed that Π is an algebraic function of the x_i's and the y_i's. We may assume that this algebraic function is included as a special case /135
in a definite type, not containing μ explicitly, but depending algebraically on a certain number of arbitrary parameters. The quantity $\int \sqrt{\Pi}$ will not be an integral invariant no matter what these parameters may be, but only when these parameters take on certain special values depending on μ.

When stating that $\int \sqrt{\Pi}$ is an integral invariant, one is led to certain algebraic equations between μ and these parameters. These equations must be compatible, and it is apparent that the parameters will be obtained as algebraic functions of μ.

The coefficients of the form Π and Δ will also be algebraic in μ.

The equation $\Delta = 0$ is therefore algebraic in μ, and we may assume that it has undergone a transformation in such a way that the first term is a whole polynomial in μ.

We may therefore write

$$\Delta = \Delta_0 + \mu\Delta_1 + \mu^2\Delta_2 + \ldots.$$

In addition, Δ_0 will not be identically zero, unless Δ is. If Δ_0 would vanish, Δ would contain a factor μ which could be made to vanish.

The function Δ must vanish when the x_i's and y_i's are replaced by the expansions (5). It may then be developed in powers of μ and, due to the fact that the term which is independent of μ must vanish we shall have

$$\Delta_0(x_i^0, y_i^0) = 0. \tag{2'}$$

We should now point out that we must have

$$\begin{cases} F_0(x_i^0, y_i^0) = \beta_0, \\ G(x_i^0, y_i^0) = \gamma_0, \end{cases} \tag{3'}$$

where β_0 and γ_0 are constants. In order that this may be the case, it is sufficient to recall that, for $\mu = 0$, the motion may be reduced to Keplerian motion.

Now, for example, let us take /136

$$H = x_1^2 + x_2^2 - 1$$

and let us write the equation

$$(x_1^0)^2 + (x_2^0)^2 = 1. \tag{4'}$$

If we set $\mu = 0$, we may then observe that the third body will describe a Keplerian ellipse. Let ξ and η be the coordinates of this body, not with respect to the moving axes, but with respect to the axes of symmetry of this ellipse.

The equations of the Keplerian ellipse will then be written

$$\begin{cases} \xi = \xi_0 + \xi_1 \cos\varphi + \xi_2 \cos 2\varphi + \ldots, \\ \eta = \quad\quad \eta_1 \sin\varphi + \eta_2 \sin 2\varphi + \ldots. \end{cases} \tag{6}$$

The coefficients ξ_k, η_k will depend on two constants which are the major axis and the eccentricity of the ellipse and, consequently, on β_0

and γ_0. We shall have

$$\varphi = n_1 t + \varpi_1,$$

where the mean motion n_1 depends on β_0 and where $\bar{\omega}_1$ is a new integration constant.

The intersection of the ellipse (6) with the circle

$$\xi^2 + \eta^2 = 1$$

will occur at two points which will be given by the equations

$$\xi = \cos\theta, \qquad \eta = \pm \sin\theta, \qquad \varphi = \pm \varphi_0. \tag{7}$$

We will then have

$$\begin{cases} x_1^0 = \xi \cos(\omega t + \varpi_2) + \eta \sin(\omega t + \varpi_2), \\ x_2^0 = \xi \sin(\omega t + \varpi_2) - \eta \cos(\omega t + \varpi_2), \end{cases} \tag{8}$$

where $\overline{\omega}_2$ is a new integration constant.

We shall obtain solutions of the equation (4') by combining equations (7) and (8), which yields

$$x_1^0 = \cos\left[\theta + \frac{\omega}{n_1}(\varphi_0 + 2k\pi - \varpi_1) + \varpi_2\right],$$

$$x_1^0 = \cos\left[-\theta + \frac{\omega}{n_1}(-\varphi_0 + 2k\pi - \varpi_1) + \varpi_2\right]$$

(k is an arbitrary whole number).

In order that the solution be periodic, it is necessary and suffi- /137 cient that the ratio $\frac{\omega}{n_1}$ be commensurable. Let us write this ratio in the form of a fraction reduced to its most simple expression, and let D be its denominator. It may be seen that equation (4') has 2D different solutions.

Equations (2'), (3'), and (4') must have only a limited number of solutions, no matter what the constants β_0 and γ_0 may be. I may choose β_0 in such a way that $\frac{\omega}{n_1}$ has the value which I desire, and consequently that D may also be as large as I desire.

This can only occur if Δ_0, and consequently if Δ, are identically zero.

Consequently, the discriminant having the form Π is identically zero, and this form must be reduced to a binary form.

It could be shown in the same way that, in the sense of No. 257, it is impossible that every periodic solution be a singular solution.

This has only been proven in a very special case, but it is possible that this proof may be extended to the general case.

289. The form Π regarded as a binary form, must be reducible to

$$B\,\delta f_1\,\delta f_1'$$

for one point of a periodic solution. The binary form will therefore be definite (i.e., equal to the sum of two squares) if the periodic solution is stable -- i.e., if the characteristic exponents are imaginary. It will be indefinite (i.e., equal to the difference of two squares) if the periodic solution is unstable -- i.e., if the characteristic exponents are real.

Let us assume that μ is very small, and let us reconsider equation (4').

According to the principles outlined in Chapter III (No. 42), for a /138
given value of β_0, we shall have at least two periodic solutions, of which one is stable and one is unstable. Let

$$\varpi_1', \quad \varpi_2'; \qquad \varpi_1'', \quad \varpi_2''$$

be the corresponding values of the constants $\overline{\omega}_1$ and $\overline{\omega}_2$.

Let us set

$$0 + \frac{\omega}{n_1}(\varphi_0 - \varpi_1') + \varpi_2' = \psi',$$

$$0 + \frac{\omega}{n_1}(\varphi_0 - \varpi_1'') + \varpi_2'' = \psi'',$$

and equation (4') will give us, for the first periodic solution,

$$x_1^0 = \cos\left(\psi' + \frac{2k\omega\pi}{n_1}\right)$$

and for the second

$$x_1^0 = \cos\left(\psi'' + \frac{2k\omega\pi}{n_1}\right).$$

Without restricting the conditions of generality, we may assume that $\psi'' > \psi'$ and that ψ' and ψ'' are contained between zero and $\frac{2\pi}{D}$. Then the form Π will be definite for $x_1^0 = \cos\left(\psi' + \frac{2\pi}{D}\right)$,

indefinite for $x_1^0 = \cos\left(\psi'' + \frac{2\pi}{D}\right)$,

definite for $x_1^0 = \cos\left(\psi' + \frac{4\pi}{D}\right)$,

indefinite for $x_1^0 = \cos\left(\psi'' + \frac{4\pi}{D}\right)$,

..............................,

definite for $x_1^0 = \cos(\psi' + 2\pi)$,
indefinite for $x_1^0 = \cos(\psi'' + 2\pi)$;

which shows that the discriminant of Π, considered as a binary form, must vanish at least 2D times. Just as above, it may be concluded from this that it is identically zero.

The form Π may therefore be reduced to a square term. Therefore, since it must equal

$$B\,\delta f_1\,\delta f'_1$$

for every point of a periodic solution, it must vanish for all of these points. /139

The same line of reasoning would show that it is identically zero.

To sum up, there is no other quadratic invariant except the one which is known, at least for the special case of the restricted problem.

CHAPTER XXVI

POISSON STABILITY

Different Definitions of Stability

290. The word stability has been understood to have several dif- /140
ferent meanings, and the difference between these meanings is clearly apparent if we recall the history of science.

Lagrange has shown that, if the squares of the masses are neglected, the major axes of the orbits are invariant. This means that, with this degree of approximation, the major axes may be developed in series whose terms have the following form

$$A \sin(\alpha t + \beta),$$

where A, α and β are constants.

If these series are uniformly convergent, this results in the fact that the major axes are contained between certain limits. The system of stars cannot therefore pass through every situation which is compatible with the integrals of energy and area, and furthermore it will repass an infinite number of times as close as desired to the initial situation.

This is complete stability.

Carrying the approximation further, Poisson demonstrated that the stability continues to exist when one takes into accout the squares of the masses and when the cubes are neglected.

However, this does not have the same meaning.

He meant that the major axes may be developed in series, containing not only terms having the form

$$A \sin(\alpha t + \beta),$$

but also terms having the form /141

$$A t \sin(\alpha t + \beta).$$

The value of the major axis then undergoes continuous oscillations, but nothing indicates that the amplitude of these oscillations does not increase indefinitely with time.

We may state that the system will always repass an infinite number of times as close as desired to the initial situation. However, we may not state that it does not recede from it very much.

The word stability does not therefore have the same meaning for Lagrange as for Poisson.

It is advantageous to point out that the theorems of Lagrange and Poisson include one important exception: They are no longer valid if the ratio of the mean motion is commensurable.

The two scientists concluded from it that stability exists, because the probability that they are commensurable is infinitely small.

It is therefore advantageous to provide an exact definition of stability.

In order that there be complete stability in the three-body problem, the three folowing conditions are necessary:

1. None of the three bodies can recede indefinitely;

2. Two of the bodies cannot collide with each other, and the distance of these two bodies cannot desend below a certain limit;

3. The system repasses an infinite number of times as desired to the initial situation.

If the third condition alone is fulfilled, without knowing whether the first two conditions are fulfilled, I would say that there is only Poisson stability.

A case has been known to exist for a long time for which the first condition is fulfilled. We shall see that the third condition is fulfilled also. I can say nothing with respect to the second condition.

This is the case given in the problem of Section No. 9, where I assumed that the three-bodies move in the same plane, that the mass of the third is zero, and that the first two describe concentric circumferences around the common center of gravity. For purposes of brevity, I shall call this the restricted problem.

Motion of a Liquid

/142

291. In order to provide a better explanation of the principle underlying the proof, I am now going to present a simple example.

Let us consider a liquid which is enclosed in a vessel having an invariable form and which is completely filled. Let x, y, z be the coordinates of a liquid molecule, u, v, w the velocity components, in such a way that the equations of motion may be written

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt. \tag{1}$$

The components u, v, w are functions, which I assume to be given functions, of x, y, z and t.

I shall assume that the motion is steady, in such a way that u, v, w depend only on x, y and z.

Since the liquid is incompressible, we shall have

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

In other words, the volume

$$\int dx\, dy\, dz$$

is an integral invariant.

Let us study the trajectory of an arbitrary molecule. I may state that this molecule will repass an infinite number of times as close as desired to its initial position. More precisely, let U be an arbitrary volume inside of the vessel, which is as small as desired. It may be stated that there will be molecules crossing this volume an infinite number of times.

Let U_0 be an arbitrary volume inside of the vessel. The liquid molecules which fill this volume at the time 0 will fill a certain volume U_1 at the time τ, a certain volume U_2, ..., at the time 2τ, and a certain volume U_n at the time nτ.

The incompressibility of the liquid or, which is the same thing, the existence of the integral invariant, indicates to us that all the volumes /143

$$U_0, \quad U_1, \quad U_2, \quad \ldots, \quad U_n$$

are equal.

Let V be the total volume of the vessel, and if

$$V < (n+1)U_0,$$

we shall have

$$V < U_0 + U_1 + U_2 + \ldots + U_n.$$

It is therefore impossible that all the volumes U_0, U_1, ..., U_n are all exterior to each other. It is necessary that at least two of them, U_i and U_k, for example, have a part in common.

It may be stated that if U_i and U_k have a part in common, the same will hold true for U_0 and U_{k-i} (assuming, for example, $k > i$). Let M be a point in common to U_i and U_k. The molecule which is at the point M at the time $i\tau$ is, at the time 0, at a point M_0 belonging to U_0, since the point M belongs to U_i.

In the same way, the molecule which is at the point M at the time $k\tau$ is, at the time $(k-i)\tau$, at the point M_0, since the motion is steady. On the other hand, it is at the time 0 at a point M_1 belonging to U_0, since M belongs to U_k, and we must conclude from this that M_0 belongs to U_{k-i}.

Therefore, U_{k-i} and U_0 have points in common.

q.e.d.

Therefore, it is possible to choose the number α in such a way that U_0 and U_α have a part in common.

Let U'_0 be the part in common, and let us form U'_1, U'_2, ..., with U'_0, as we formed U_1, U_2, ..., with U_0. We may obtain a number β in such a way that U'_0 and U'_β have a part in common.

Let U''_0 be this part in common.

We may obtain a number γ in such a way that U''_0 and U''_γ have a part in common.

This procedure may then be continued.

As a result, U'_0 is part of U_0, U'' of U'_0, and U'''_0 of U''_0, ... In general, $U_0^{(p+1)}$ will be part of $U_0^{(p)}$. When the number p increases indefinitely, the volume $U_0^{(p)}$ must therefore become smaller and smaller.

According to a well-known theorem, there will be at least one point, perhaps several, or perhaps an infinity, which belong at the same time /144 to U_0, to U'_0, to U''_0, ..., and to $U_0^{(p)}$, however large p may be.

This group of points, which I shall call E, will be in a measure the limit toward which the volume $U_0^{(p)}$ tends, when p increases indefinitely.

It may be composed of isolated points; however, it may be somewhat different. For example, it may happen that E is a region in space having a finite volume.

A molecule which will be inside of U_0', and, consequently, inside of U_α, at the time zero, will be inside of U_0 at the time $-\alpha\tau$.

A molecule which will be inside of U_0'' and, consequently, inside of U_β' at the time zero, will be inside of U_0' at the time $-\beta\tau$, and, consequently inside of U_0 at the time $-(\alpha + \beta)\tau$.

A molecule which will be inside of U_0''' at the time zero will be inside of U_0'' at the time $-\gamma\tau$, inside of U_0' at the time $-(\beta + \gamma)\tau$, and inside of U_0 at the time $-(\alpha + \beta + \gamma)\tau$.

Since U_0''', U_0'', U_0' are part of U_0, this molecule will be inside of U_0 at four different times (multiples of τ).

In the same way, and more generally, a molecule which is inside of $U_0^{(p)}$ at the time zero will be inside of U_0 at p different previous times (which will equal the negative multiples of τ).

Since E is part of $U_0^{(p)}$, however large p may be, as a result a molecule which, at the time zero, is part of E will cross U_0 an infinite number of different times, which all equal a negative multiple of τ.

There are therefore molecules which cross the volume U_0 an infinite number of times, however small this volume may be.

q.e.d.

The equations

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = dt$$

become

$$\frac{dx}{-u} = \frac{dy}{-v} = \frac{dz}{-w} = dt,$$

when t is changed into -t. They therefore retain the same form.

As a consequence, according to the same reasoning which we have just employed to show that there are molecules which cross U_0 an infinite number of times before the time zero, we should be able to show that there are molecules which cross U_0 an infinite number of times after the time zero. /145

The preceding line of reasoning provided us with the times at which U_0 is crossed by a molecule which, at the time zero, is part of E. Due to the fact that it is inside of E and, consequently, inside of U'_0 and of U_α, at the time zero, it will be inside of U_0 at the time

$$-\alpha\tau.$$

Due to the fact that it is inside of E and, consequently, inside of U''_0 and of U'_β at the time zero, it will be inside of U'_0 and U_α at the time

$$-\beta\tau,$$

and inside of U_0 at the time

$$-(\alpha+\beta)\tau.$$

It will therefore be inside of U_0 at two times $-\beta\tau$ and $-(\alpha+\beta)\tau$.

Since it is part of E and of U'''_0 at the time zero, it will be part of U''_0 at the time $-\gamma\tau$, of U'_0 at the time $-(\beta+\gamma)\tau$, and part of U_0 at the time $-(\alpha+\beta+\gamma)\tau$, so that it will cross U_0 at three times

$$-\gamma\tau, \quad -(\beta+\gamma)\tau, \quad -(\alpha+\beta+\gamma)\tau.$$

At the time $-\gamma\tau$ it is part of U''_0 and, consequently, part of U'_0 and of U_α. At the time

$$-(\alpha+\gamma)\tau$$

it will therefore be part of U_0.

To sum up, this molecule must cross U_0 at different times

$$\begin{array}{llll} -\alpha\tau, & -\beta\tau, & -\gamma\tau, & \dots, \\ -(\alpha+\beta)\tau, & -(\beta+\gamma)\tau, & -(\alpha+\gamma)\tau, & \dots, \\ -(\alpha+\beta+\gamma)\tau, & \dots\dots, & \dots\dots, & \dots, \\ \dots\dots, & \dots\dots, & \dots\dots, & \dots, \end{array}$$

where the coefficient of $-\tau$ is an arbitrary combination of the numbers $\alpha, \beta, \gamma, \dots$.

Among all of these times, there are now times when the molecule will not only be inside of U_0, but also inside of U'_0. /146

It may be readily seen that it is sufficient to select combinations which do not include the number α.

The times at which the molecule will be inside of U_0'' will correspond in the same way to the combinations which do not include either the number α or the number β.

292. Let us again consider the volumes

$$U_0, \quad U_1, \quad U_2, \quad \ldots, \quad U_n. \tag{1}$$

For purposes of brevity, I would like to state that each of them is the consequent of that preceding it in the series (1) and the antecedent of that following it.

Thus, U_2, U_3 will be the second and the third consequent of U_0.

I may continue the series (1) beyond U_n, compiling the successive consequents of U_n

$$U_{n+1}, \quad U_{n+2}, \quad \ldots.$$

I may also extend it to the left, and may compile the successive antecedents of U_0

$$U_{-1}, \quad U_{-2}, \quad \ldots,$$

in such a way that the molecules which are in U_0 at the time zero will be in U_{-1} at the time $-\tau$, and in U_{-2} at the time -2τ.

Under this assumption, I shall always use V to designate the total volume of the vessel and k to designate an arbitrary whole number. If we have

$$kV < (n+1)U_0,$$

there will be points which are part of k + 1 volumes of the series (1) at the same time.

The sum of the volumes of the series (1) is equal to $(n + 1)U_0$. If no point could be part of more than k of these volumes at the same time, this sum must be smaller than kV.

We may therefore obtain k + 1 volumes in the series (1)

$$U_{\alpha_0}, \quad U_{\alpha_1}, \quad U_{\alpha_2}, \quad \ldots, \quad U_{\alpha_k}$$

which will have a part in common.

I may conclude from this that the k + 1 volumes /147

$$U_0,\ U_{\alpha_1-\alpha_0},\ U_{\alpha_2-\alpha_0},\ \ldots,\ U_{\alpha_k-\alpha_0}$$

have a part in common.

For example, let us set k = 2

$$2V < (n+1)U_0,$$

and we may obtain three volumes

$$U_\alpha,\ U_\beta,\ U_\gamma$$

which will have a part in common. The indices α, β, γ will satisfy the conditions

$$0 \leq \alpha \leq n; \quad 0 \leq \beta \leq n; \quad 0 \leq \gamma \leq n; \quad \alpha < \beta < \gamma.$$

It may be deduced from this that the three volumes

$$U_0,\ U_{\beta-\alpha},\ U_{\gamma-\alpha}$$

have a part in common, and that the same holds true for the three volumes

$$U_{\alpha-\beta},\ U_0,\ U_{\gamma-\beta}$$

or the three volumes

$$U_{\alpha-\gamma},\ U_{\beta-\gamma},\ U_0.$$

293. We saw above that there are molecules which cross U_0 an infinite number of times <u>before the time zero</u>, and others which cross an infinite number of times after the time zero. I propose to establish the fact that there are as many which cross U_0 <u>before the time zero as after the time zero</u> an infinite number of times.

Let U_0 be an arbitrary volume. According to the preceding section, we may always obtain two numbers, a and α, the first negative and the second positive, and such that the three volumes

$$U_a,\ U_0,\ U_\alpha$$

have a part in common. Let U_0' be this part in common.

Every molecule which will be in U_0' at the time zero will be in U_0 at the three times

$$-\alpha\tau,\quad 0,\quad -a\tau.$$

/148

Of these three times, the first is negative and the last is positive.

Our molecule will therefore cross U_0 at least once before the time zero, and at least once after this time.

Following the same procedure with U'_0 as with U_0, we shall obtain two numbers b and β, the first negative and the second positive, so that the three volumes

$$U'_b, \quad U'_0, \quad U'_\beta$$

have a part in common. Let U''_0 be this part in common.

Every molecule which will be in U''_0 at the time zero will be in U'_0 at the three times

$$-\beta\tau, \quad 0, \quad -b\tau,$$

and, consequently, in U_0 at the five times

$$-(\alpha+\beta)\tau, \quad -\beta\tau, \quad 0, \quad -b\tau, \quad -(a+b)\tau.$$

Of these times, the first two are negative, and the last two are positive.

Every molecule which will be in U''_0 at the time 0 will cross U_0 at least two times before the time zero, and at least two times after this time.

This procedure will then be continued.

One could form U'''_0 with U''_0, U^{IV}_0 with U'''_0, and it could be seen that every molecule which will be in $U^{(p)}_0$ at the time 0 will cross U_0 at least p times before the time zero, and at least p times after this time.

However, U'_0 is part of U_0, U''_0 of U'_0, and so on. We shall therefore have a group of points E(containing at least one point) which will be part of all the volumes $U^{(p)}_0$ at the same time, wherever p may be.

Every molecule which, at the time zero, will be inside of E will also be inside of

$$U_0, \quad U'_0, \quad U''_0, \quad \ldots, \quad U^{(p)}_0, \quad \textit{ad inf.}$$

since E is part of all these volumes.

Therefore, it will cross U_0 an infinite number of times before the time 0, and an infinite number of times after this time.

There are therefore molecules which cross U_0 an infinite number of times both before and after the time zero.

q.e.d.

294. The group E, which was defined in No. 291 (just like the group E considered in the preceding section) may be composed of a single point (be that as it may, there is always an infinity of molecules crossing U_0 an infinity of times). /149

It may be composed of a finite number of points, or of an infinite number of discrete points.

It could be assumed that this group E has a finite volume. Let us see what the consequences of this hypothesis would be. Let us discuss the group E defined in No. 291.

I shall consider the series of whole numbers

$$\alpha, \ \beta, \ \ldots,$$

which were defined in this section, and it may be stated that we have

$$\beta \geqq \alpha.$$

The quantity U_α is the first of the consequents of U_0 which has a common part with U_0.

U' is the first of the consequents of U'_0 which β has a part in common with U'_0.

However, U'_0 is part of U_0, and U'_β is part of U_β. Therefore, if U'_β has a common part with U'_0, U_β is one of the consequents of U_0 which has a part in common with U_0. This entails the inequality

$$\alpha \leqq \beta.$$

In the same way we would obtain

$$\beta \leqq \gamma \leqq \delta \leqq \ldots.$$

The numbers α, β, γ, δ, ... are therefore always increasing or, at least, never decrease.

On the other hand, according to No. 291, we have

$$1 + \alpha < \frac{V}{U_0}; \quad 1 + \beta < \frac{V}{U'_0}; \quad 1 + \gamma < \frac{V}{U''_0}.$$

We clearly have

$$U_0 > U'_0 > U''_0 > \ldots,$$

and, if E has a finite volume which I may also call E, no matter what p may be, we have

$$E < U_0^{(p)}$$

since E is part of $U_0^{(p)}$.

The numbers α, β, γ, ... are therefore all smaller than /150

$$\frac{V}{E} - 1.$$

Therefore, they cannot increase beyond any limit, and we may conclude that, starting with a certain order, all the terms are equal in the series of numbers α, β, ...

Let us assume that all the terms are equal to λ, starting with the p^e order.

Then $U_0^{(p)}$ and $U_\lambda^{(p)}$ will have a part in common which will be $U_0^{(p+1)}$, and $U_0^{(p+1)}$ and $U_\lambda^{(p+1)}$ will have a part in common which will be $U_0^{(p+2)}$, and so on.

Let E_λ be the λ^e consequent of E.

E is the group of points which are part of U_0, U_0', U_0'', ..., ad inf. at the same time. E_λ will be the group of points which are part of U_λ, U_λ', U_λ'', ..., at the same time. It may also be stated that E is the group of points which are part of

$$U_0^{(p+1)}, \quad U_0^{(p+2)}, \quad \ldots \tag{1}$$

at the same time, since each of the regions U_0, U_0' is only a portion of the preceding region. In the same way, E_λ is the group of points which are part of

$$U_\lambda^{(p)}, \quad U_\lambda^{(p+1)}, \quad \ldots . \tag{2}$$

at the same time.

However, $U_0^{(p+1)}$ is a part of $U_\lambda^{(p)}$, and $U_0^{(p+2)}$ is a part of $U_\lambda^{(p+1)}$. Each term in series (2) is a part of the corresponding term in series (1). Therefore E is a part of E_λ, or coincides with E_λ.

However, we assumed that E is a certain region in space having a finite volume. Due to the fact that the fluid is incompressible, its λ^{th} consequent E_λ must also be a certain region in space having the <u>same</u> volume. E cannot therefore be a part of E_λ. Therefore, E and E_λ coincide.

If we assume that E is a certain region in space having a finite volume, it must be stated that E coincides with one of its consequents.

295. Following are some theorems which are all but obvious, and I shall confine myself to discussing these theorems. Let /151

$$U_{\alpha_1}, \ U_{\alpha_2}, \ \ldots, \ U_{\alpha_\mu}, \ \ldots$$

be those consequents of U_0 which have a part in common with U_0. The numbers α_μ are arranged in order of increasing magnitude. We shall have

$$1 + \alpha_\mu \leq \mu \frac{V}{U}.$$

Then let

$$U_{\gamma_1}, \ U_{\gamma_2}, \ \ldots, \ U_{\gamma_\mu}$$

be the μ consequents of U_0, which have a part in common with each other and with U_0. I have chosen these numbers γ in such a way that γ_μ is as small as possible. We shall have

$$1 + \alpha_\mu \leq 1 + \gamma_\mu \leq \mu \frac{V}{U}.$$

Let us employ the notation given in No. 291 once again, and let us employ U_α to designate the first consequent which has a part in common with U_0, U_0' to designate this common part, U_β' to designate the first consequent of U_0' which has a part in common with U_0'. If β is not equal to α, we shall have

$$\beta \geq 2\alpha,$$

and $U_{\beta-\alpha}$ will have a part in common with U_0.

Probabilities

296. We saw in No. 291 that there are molecules which cross U_0 an infinite number of times. On the other hand, there are others which cross U_0 only a finite number of times. I plan to demonstrate the fact that these latter molecules must be regarded as unusual or, to state this more precisely, the <u>probability</u> that one molecule crosses U_0 only a finite number of times is infinitely small, <u>if it is assumed</u> that this molecule is inside of U_0 at the initial instant of time. However, I must clarify the meaning which I am here attributing to the word <u>probability</u>. Let $\phi(x, y, z)$ be a positive, arbitrary function of the three coordinates x, y, z. I may state that the probability that a molecule may be located within a certain volume at the time t = 0 is proportional to the integral /152

$$J = \int \varphi(x, y, z)\, dx\, dy\, dz$$

extended over this volume. Consequently, it equals the integral J divided by the same integral extended over the entire vessel V.

We may arbitrarily choose the function ϕ, and the probability is thus absolutely definite. Since the trajectory of a molecule depends only on its initial position, the probability that a molecule behaves in a certain way is a completely definite quantity, as soon as the function ϕ has been chosen.

Under this assumption, I shall simply set $\phi = 1$, and I shall try to find the probability p that a molecule does not cross the region U_0 more than k times between the time $-n\tau$ and the time zero.

Therefore, let σ_0 be a region which is part of U_0 and which is defined by the following property. Every molecule which will be within σ_0 at the initial instant of time will not cross U_0 more than k times between the times $-n'\tau$ and 0.

If we assume that our molecule is within U_0 at the time zero, the desired probability will be

$$p = \frac{\sigma_0}{U_0}. \tag{1}$$

Let

$$\sigma_1, \quad \sigma_2, \quad \ldots, \quad \sigma_n$$

be the n first consequents of σ_0. It is not possible to have a region common to more than k of the n + 1 regions

$$\sigma_0, \quad \sigma_1, \quad \sigma_2, \quad \ldots, \quad \sigma_n,$$

since, without this stipulation, every molecule which was located in this common region at the time zero would cross σ_0, and consequently U_0, more than k times between the times $-n\tau$ and 0.

We therefore have

$$(n+1)\sigma_0 < kV,$$

and, consequently,

$$p < \frac{kV}{(n+1)U_0}.$$

No matter how small U_0 may be, or how large k may be, we may always take n large enough, so that the second term of this inequality is as small as desired. Therefore, when n tends toward infinity, p tends toward zero. /153

Therefore the probability is infinitely small that a molecule which is located in the region U_0 at the initial instant of time does not cross this region more than k times between the times $-\infty$ and 0.

In the same way, the probability is infinitely small that this molecule does not cross this region more than k times between the times 0 and $+\infty$.

Let us now set $n = k^3 + x$. The probability that our molecules does not cross U_0 more than k times between the times $-(k^3 + x)\tau$ and 0, will be smaller than

$$\frac{kV}{(k^3 + x + 1)U_0}.$$

It tends toward zero when k increases indefinitely.

The probability P that our molecule does not cross U_0 an infinite number of times between the times $-\infty$ and 0 is therefore infinitely small.

In reality, this probability P is the sum of the probabilities that the molecule crosses U_0 only once, that it crosses U_0 twice and only twice, that it crosses U_0 three times and only three times, etc.

However, the probability that the molecule crosses U_0 k times and k times only, between the times $-\infty$ and 0, is obviously smaller than the probability that it will cross U_0 k times or less than k times between the times $-(k^3 + x)\tau$ and 0 -- it is consequently smaller than

$$\frac{kV}{(k^3 + x + 1)U_0}.$$

The total probability P is therefore smaller than

$$P < \frac{V}{(x+2)U_0} + \frac{2V}{(x+9)U_0} + \ldots + \frac{kV}{(k^3 + x + 1)U_0} + \ldots$$

The series of the second term is uniformly convergent. Each of the terms tends to zero when x tends to infinity. Therefore the sum of the /154 series tends to zero, and P is infinitely small.

In the same way, the probability is infinitely small that our molecule does not cross U_0 an infinite number of times between the times 0 and $+\infty$.

The same results are obtained when any other choice is made for the function ϕ, instead of setting $\phi = 1$.

Equation (1) must then be replaced by the following

$$p = \frac{J(\sigma_0)}{J(U_0)},$$

where $J(\sigma_0)$ and $J(U_0)$ designate the integral J extended over the regions σ_0 and U_0, respectively.

I shall assume that the function ϕ is continuous; consequently, it does not become infinite, and I may assign an upper limit μ to it. We then have

$$J(\sigma_0) < \mu\sigma_0,$$

and since

$$(n+1)(\sigma_0) < kV,$$

we may deduce the following

$$p < \frac{\mu k V}{(n+1) J(U_0)}.$$

No matter how small $J(U_0)$ is, or how large k is, we may always take n large enough that the second term of this inequality is also as small as desired. We again obtain the same results which are therefore independent of the choice of the function ϕ.

To sum up, the molecules which cross U_0 only a finite number of times are unusual, in the same way as the commensurable numbers which are only an exception in the series of numbers, while the incommensurable numbers are the rule.

Therefore, if Poisson could provide an affirmative answer to the stability question which was posed, although he had excluded the cases in which the ratio of the mean motion is commensurable, we have the right to state that the stability which we have defined has been proven, although we are forced to exclude the unusual molecules which we have just discussed.

I would like to add that the existence of asymptotic solutions provides sufficient proof for the fact that these unusual molecules exist in reality. /155

Extension of the Preceding Results

297. Up to the present time, we have limited ourselves to a very special case -- that in which an incompressible liquid is enclosed in a vessel, i.e., -- to employ analytical language -- the case of the

following equations

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z},$$

where X, Y, Z are three functions which are interrelated by the following relationship

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dX}{dz} = 0$$

and such that on every point of a closed surface (that of the vessel) we have

$$lX + mY + nZ = 0,$$

where l, m, n are the direction cosines of the normal to this closed surface.

However, all of the preceding results are still valid even in the more extended cases without changing a thing, including the line of reasoning leading to these results.

Let the n variables $x_1, x_2, \ldots, x_n$, satisfy the differential equations

$$pt = \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}, \tag{1}$$

where $X_1, X_2, \ldots, X_n$ are n arbitrary, uniform functions satisfying the condition

$$\frac{dMX_1}{dx_1} + \frac{dMX_2}{dx_2} + \ldots + \frac{dMX_n}{dx_n} = 0,$$

in such a way that equations (1) include the integral invariant

$$\int M\, dx_1\, dx_2 \ldots dx_n. \tag{2}$$

In addition, I shall assume that M is positive. We may then state that equations (1) have a positive integral invariant. /156

I shall assume that equations (1) are such that, if the point $(x_1, x_2, \ldots, x_n)$ is located within a certain region V at the initial instant of time (which plays the same role that the vessel played just recently where the liquid is enclosed), it will remain indefinitely within this region.

Finally, I shall assume that the integral

$$\int M \, dx_1 \, dx_2 \ldots dx_n$$

extended over this region is finite.

Under these conditions, if we consider a region U_0 contained in V, we may select the initial position of the point $(x_1, x_2, \ldots, x_n)$ in an infinite number of ways, so that this point crosses this region U_0 an infinite number of times. If the choice of the initial position is made at random within U_0, the probability that the point $(x_1, x_2, \ldots, x_n)$ will not cross the region U_0 an infinite number of times will be infinitely small.

In other words, if the initial conditions are not unusual -- with respect to the meaning I attributed to this word above -- the point $(x_1, x_2, \ldots, x_n)$ will return as close as desired to its initial position an infinite number of times.

Nothing needs be changed in the preceding proof. For example, we may obtain the following inequality again.

$$V < (n+1) U_0,$$

where V and U_0 designate the integral (2) extended over the regions V and U_0, respectively.

The same results may be deduced from this. Due to the fact that the integral (2) is basically positive by hypothesis, it will have the same properties as the volume, namely, when extended over the entire region it will be larger than when extended over only a part of this region.

298. How may we now determine whether there is a region V such that the point $(x_1, x_2, \ldots, x_n)$ always remains within this region if it occurs at the initial instant of time?

Let us assume that equations (1) have an integral /157

$$F(x_1, x_2, \ldots, x_n) = \text{const.}$$

Let us consider the region V defined by the inequalities

$$h < F < h',$$

where h and h' are two arbitrary constants which may be as close as desired.

It is apparent that if these inequalities are satisfied at the initial instant of time, they will be always satisfied. The region V therefore closely satisfies the proposed conditions.

Application to the Restricted Problem

299. We shall apply these principles to the restricted problem given in No. 9. -- a zero mass, the circular motion of two other masses, and zero inclination. If we refer the zero mass, whose motion we are studying, to two moving axes turning around the common center of gravity of the other two masses, with a constant angular velocity n equalling that of the two other masses, if we employ ξ, η to designate the coordinates of the zero mass with respect to the two moving axes, and if we employ V to designate the force potential, we may write the equations of motion as follows

$$\frac{d\xi}{dt} = \xi', \quad \frac{d\eta}{dt} = \eta',$$
$$\frac{d\xi'}{dt} = 2n\eta' + n^2\xi + \frac{dV}{d\xi}, \tag{1}$$
$$\frac{d\eta'}{dt} = -2n\xi' + n^2\eta + \frac{dV}{d\eta} .$$

It may be immediately seen that they have a positive integral invariant

$$\int d\xi\, d\eta\, d\xi'\, d\eta'. \tag{2}$$

On the other hand, they have the Jacobi integral

$$\frac{\xi'^2 + \eta'^2}{2} = V + \frac{n^2}{2}(\xi^2 + \eta^2) + h, \tag{3}$$

where h is a constant.

Since $\xi'^2 + \eta'^2$ is necessarily positive, we must have /158

$$V + \frac{n^2}{2}(\xi^2 + \eta^2) > -h. \tag{4}$$

We are therefore led to compile the following curves

$$V + \frac{n^2}{2}(\xi^2 + \eta^2) = \text{const.}$$

The first term in relationship (4) is necessarily positive, because we have

$$V = \frac{m_1}{r_1} + \frac{m_2}{r_2},$$

where m_1 and m_2 are the masses of the two principal bodies, and r_1 and r_2 are their distances to the zero mass. The first term of (4) becomes infinite for $r_1 = 0$, for $r_2 = 0$, as well as at infinity. It must therefore have at least a minimum, and two points where its two first derivatives vanish without there being a maximum or a minimum.

More generally, if there are n relative minima or maxima, there will be n + 1 points where the two derivatives vanish without there being a maximum or a minimum.

However, it is apparent that these points, where the two derivatives vanish, correspond to the special solutions of the three-body problem which Laplace studied in Chapter VI of Book X of his Mecanique Celeste (Celestial Mechanics).

Two of these points may be obtained by constructing an equilateral triangle on m_1m_2, either above or below the line m_1m_2 which we shall use for the axis of the ξ's. The third apex of this triangle represents one of the solutions in question.

All the other points satisfying the condition are located on the axis of the ξ's. It may be readily seen that the first term of (4) has three minima, and only three minima, when ξ varies between $-\infty$ and $+\infty$. The first minimum is located between infinity and the mass m_1, the second is located between the two masses m_1 and m_2, and the third is located between infinity and the mass m_2.

The derivative $\frac{dV}{d\xi} + n^2\xi$ only vanishes (for $\eta = 0$) once in each of these intervals, since it is the sum of three terms which all increase.

The equations /159

$$\frac{dV}{d\xi} + n^2\xi = \frac{dV}{d\eta} + n^2\eta = 0$$

indicating that the first derivatives of the first term of (4) are zero, have only five solutions, namely, the points B_1 and B_2 which are the apexes of the equilateral triangles, and the points A_1, A_2 and A_3 located on the axis of the ξ's. We shall assume that these points occur in the following order

$$-\infty,\quad A_1,\quad m_1,\quad A_2,\quad m_2,\quad A_3,\quad +\infty.$$

We must now determine which of these points correspond to a minimum, and we know in advance that there are two.

We should point out that if we vary the two masses m_1 and m_2 continuously, any of the five points A and B will always correspond to a minimum, or will never correspond to one. One may only proceed from one case to another if the Hessian of the first term of (4) vanishes, i.e., if two of the points A and B coincide, which will never occur.

It is sufficient to examine a special case -- for example, that in which $m_1 = m_2$. In this case, the symmetry is sufficient for indicating to us that the two solutions A_1 and A_3 must have the same nature, just like the two solutions B_1 and B_2. It is therefore A_1 and A_3 alone, or B_1 and B_2 alone, which correspond to a minimum. Therefore, A_2 does not correspond to a minimum.

It can be seen that A_1 does not correspond to a minimum.

The two minima correspond therefore to B_1 and B_2.

Let us now assume that m_1 is a great deal smaller than m_2, which is the case in nature.

For sufficiently large values of -h, the curve

$$V + \frac{n^2}{2}(\xi^2 + \eta^2) = -h$$

will be composed of three closed branches C_1 encircling m_1, C_2 encircling m_2, and C_3 encircling C_1 and C_2. For smaller values, it will be composed of two closed branches, C_1 encircling m_1 and m_2, and C_2 encircling C_1.

For values which are still smaller, we shall have only one closed branch leaving m_1 and m_2 on the outside, and encircling B_1 and B_2.

Finally, for even still smaller values, we shall have two closed symmetrical curves, each of which encircles B_1 and B_2, respectively. /160

The statements below will only apply to the two first cases; we shall therefore put the last two cases aside.

In the first case, the group of points satisfying the inequality (4) may be divided into three partial groups: The group of points which are inside of C_1, the group of points which are inside of C_2, and the group of points which are outside of C_3.

In the second case, the group of points satisfying (4) may be divided into two partial groups: The group of points which are inside of C_1, and the group of points which are outside of C_2.

The statements below do not apply either in the first case to the group of points which are outside of C_3, nor in the second case to the group of points which are outside of C_2.

On the contrary, in the first case this applies to the group of points which are inside of C_1 or to the group of points which are inside of C_2 and, in the second case, to the group of points which are inside of C_1.

In order to formulate these ideas more clearly, let us consider the first case and the group of points which are inside of C_2.

As the region V we shall take the region defined by the inequalities

$$+h+\varepsilon > \frac{\xi'^2+\eta'^2}{2} - V - \frac{n^2}{2}(\xi^2+\eta^2) > +h-\varepsilon. \quad (5)$$

We shall assume that ε is small and that h has a value which we have employed in the first case. Finally, in order to conclude the definition of the region V, we shall impose the condition that the point (ξ, η) is located within the curve C_2.

It is then clear that, if the point (ξ, η, ξ', η') is located in the region V at the initial instant of time, it will always remain there.

In order to illustrate the fact that the results presented in the preceding paragraphs may be applied to the case which we are discussing, we must now show that the integral

$$\int d\xi\, d\eta\, d\xi'\, d\eta' \quad (2)$$

extended over the region V is finite.

How may this integral become infinite? Due to the fact that the curve C_2 is closed, ξ and η are limited. The integral can therefore only become infinite if ξ' and η' are infinite. However, because of the inequalities (5), ξ' and η' may only become infinite if /161

$$V + \frac{n^2}{2}(\xi^2+\eta^2)$$

becomes infinite, or -- since ξ and η are limited -- if V becomes infinite.

However, V becomes infinite for $r_1 = 0$ and for $r_2 = 0$. Since the point m_1 is outside of C_2, we need only examine the case of

$$r_2 = 0.$$

Let us therefore evaluate the portion of the integral which is in the vicinity of the point m_2. If r_2 is very small, $\xi^2 + \eta^2$ is equal to $(O\, m_2)^2$, and the term $\frac{m_1}{r_1}$ is also constant, so that if we set

$$h + \frac{n^2}{2}(\xi^2 + \eta^2) + \frac{m_1}{r_1} = H,$$

H will be regarded as a constant.

If we then set

$$(\xi - O m_2) = r_2 \cos\omega, \quad \eta = r_2 \sin\omega; \qquad \xi' = \rho\cos\varphi, \quad \eta' = \rho\sin\varphi,$$

inequalities (5) will become

$$H + \varepsilon > \frac{\rho^2}{2} - \frac{m_2}{r_2} > H - \varepsilon \tag{5'}$$

and the integral (2) will become

$$\int \rho r_2 \, d\rho \, dr_2 \, d\omega \, d\varphi. \tag{2'}$$

We shall add the inequality

$$r_2 < \alpha$$

to the inequalities (5'), where α is very small, since it is the part of the integral which is close to m_2 which must be evaluated, and since the other part is definitely finite.

If we integrate with respect to ω and ϕ, the integral (2') will become

$$4\pi^2 \int \rho r_2 \, d\rho \, dr_2. \tag{2''}$$

Let us integrate first with respect to ρ. We must calculate the integral /162

$$\int \rho \, d\rho = \frac{\rho^2}{2},$$

which is chosen between the limits

$$\rho = \sqrt{2\left(H - \varepsilon + \frac{m_2}{r_2}\right)} \quad \text{and} \quad \rho = \sqrt{2\left(H + \varepsilon + \frac{m_2}{r_2}\right)},$$

which provides us with ε.

The integral (2") may therefore be reduced to

$$4\pi^2\varepsilon\int_0^\alpha r_2\,dr_2 = 2\pi^2\varepsilon\alpha^2.$$

It is therefore finite.

The theorems which were proven above may be applied to the case which we are discussing. The zero mass will repass its initial position as close as may be desired an infinite number of times, if one does not impose certain unusual, initial conditions for which the probability is infinitely small.

In the restricted problem, if we assume that the initial conditions are such that the point ξ, η must remain within a closed curve C_1 or C_2, the first of the stability conditions, which were defined in No. 290, is fulfilled.

However, the third condition is also fulfilled; therefore, Poisson stability exists.

300. The result will clearly be the same whatever the law of attraction may be.

If the motion of a material point ξ, η is governed by the equations

$$\frac{d^2\xi}{dt^2} = \frac{dV}{d\xi}, \qquad \frac{d^2\eta}{dt^2} = \frac{dV}{d\eta}$$

or, in the case of relative motion, by the equations

$$\frac{d^2\xi}{dt^2} - 2n\frac{d\eta}{dt} = \frac{dV}{d\xi},$$
$$\frac{d^2\eta}{dt^2} + 2n\frac{d\xi}{dt} = \frac{dV}{d\eta},$$

in such a way that the energy integral may be written /163

$$\frac{1}{2}\left[\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2\right] - V = h$$

and if the function V and the constant h are such that the values of ξ and of η remain limited, we shall have Poisson stability.

However, this is not all. The same holds true in the more extended case.

Let x_1, x_2, ..., x_n be the coordinates of $\frac{n}{3}$ material points.

Let V be the force potential depending on these n variables.

Let m_1, m_2, ..., m_n be the corresponding masses, in such a way that we may employ m_1, m_2 or m_3 at random to designate the mass of the material point whose coordinates are x_1, x_2 and x_3.

The equations may be written

$$m_i \frac{d^2 x_i}{dt^2} = \frac{dV}{dx_i}$$

and the energy integral may be written

$$\sum \frac{m_i}{2} \left(\frac{dx_i}{dt}\right)^2 = V + h.$$

In virtue of this equation, if the function V and the constant h are such that the coordinates x_i are limited, there will be Poisson stability.

What must be demonstrated is the fact that the integral invariant

$$\int dx'_1\, dx'_2 \ldots dx'_n\, dx_1\, dx_2 \ldots dx_n \qquad \left(x'_i = \frac{dx_i}{dt}\right)$$

is finite when the integration is extended over the region I have called V, which is defined by the inequalities

$$V + h - \varepsilon < \sum \frac{m_i}{2} \left(\frac{dx_i}{dt}\right)^2 < V + h + \varepsilon. \tag{1}$$

Let us call A the integral

$$\int dx'_1\, dx'_2 \ldots dx'_n,$$

extended over the region defined by the inequality

$$\sum \frac{m_i}{2} x_i'^2 < 1.$$

The same integral extended over the region /164

$$\sum \frac{m_i}{2} x_i'^2 < R^2$$

will obviously be

$$AR^n.$$

When extended over the region defined by the inequalities (1), it will be

$$\Lambda\left[(V+h+\varepsilon)^{\frac{n}{2}}-(V+h-\varepsilon)^{\frac{n}{2}}\right],$$

or, since ε is very small,

$$n\Lambda\varepsilon(V+h)^{\frac{n}{2}-1}.$$

Our integral invariant therefore equals

$$n\Lambda\varepsilon\int(V+h)^{\frac{n}{2}-1}dx_1\,dx_2\ldots dx_n, \qquad (2)$$

and the integration must be extended over every point, such that V + h is positive.

According to my hypothesis, the region V + h > 0 is limited.

It may then be readily verified whether the integral (2) is finite or infinite.

It will always be finite if n = 2, because the exponent of V + h is then zero.

Let us now assume that n is > 2, and that V + h becomes infinitely large of the order p when the distance between the two points x_1, x_2, x_3 and x_4, x_5, x_6 becomes infinitely small of the first order.

Then the quantity under the sign $\int$ in the integral (2) is of the order

$$p\left(\frac{n}{2}-1\right).$$

The subset

$$x_1=x_4, \qquad x_2=x_5, \qquad x_3=x_6$$

has n - 3 dimensions. The integral is of the order n; the condition under which the integral is finite may therefore be written

$$n-(n-3)>p\left(\frac{n}{2}-1\right),$$

from which it follows that /165

$$p<\frac{6}{n-2}.$$

This is the condition under which there is Poisson stability.

Application to the Three-Body Problem

301. The preceding considerations apply to the case in which the following equation

$$\sum \frac{m_i}{2}\left(\frac{dx_i}{dt}\right)^2 = V + h \tag{1}$$

results in the fact that the x_i's can only vary between finite limits.

Unfortunately, this is not the case in the three-body problem. I shall employ the notation presented in No. 11. I shall use x_1, x_2, x_3 to designate the coordinates of the second body with respect to the first, x_4, x_5, x_6 to designate the coordinates of the third body with respect to the center of gravity of the first two, a, b, c to designate the distances of the three bodies, and M_1, M_2, M_3 to designate their masses. Finally, I shall employ

$$m_1 = m_2 = m_3 = \beta,$$
$$m_4 = m_5 = m_6 = \beta'$$

to designate the quantities which I have called β and β' in No. 11.

We shall then have

$$V = \frac{M_2 M_3}{a} + \frac{M_3 M_1}{b} + \frac{M_1 M_2}{c}.$$

Equation (1) entails the inequality

$$V + h > 0. \tag{2}$$

The function V is essentially positive. Therefore, if the constant h is positive, the inequality will always be satisfied. However, the question is whether we may assign small enough negative values to h so that the inequality can only be satisfied for limited values of the coordinates x_i. This amounts to inquiring whether the inequality /166

$$\frac{M_2 M_3}{a} + \frac{M_3 M_1}{b} + \frac{M_1 M_2}{c} + h > 0 \tag{3}$$

with those which are imposed at the three sides of a triangle

$$a + b > c, \quad b + c > a, \quad a + c > b \tag{4}$$

can only be satisfied for finite values of a, b, c. Let us set a = c, and assume that it is very large; we shall assume that b is very small.

The inequalities (4) will be satisfied by them.

With respect to inequality (3) which becomes

$$\frac{M_2 M_3 + M_1 M_2}{a} + \frac{M_3 M_1}{b} + h > 0,$$

no matter what h may be, it may be satisfied by arbitrarily large values of a.

No matter how small h may be, or how large a may be, we may always assume that b is small enough that the first term may be positive.

The existence of area integrals does not modify this conclusion. These integrals may be written

$$\begin{cases} \beta(x_2 x'_3 - x_3 x'_2) + \beta'(x_5 x'_6 - x_6 x'_5) = a_1, \\ \beta(x_3 x'_1 - x_1 x'_3) + \beta'(x_6 x'_4 - x_4 x'_6) = a_2, \\ \beta(x_1 x'_2 - x_2 x'_1) + \beta'(x_4 x'_5 - x_5 x'_4) = a_3. \end{cases} \tag{5}$$

In virtue of these equations, we have

$$\frac{\beta}{2}(x'^2_1 + x'^2_2 + x'^2_3) + \frac{\beta'}{2}(x'^2_4 + x'^2_5 + x'^2_6) > \frac{a_1^2 + a_2^2 + a_3^2}{2I}, \tag{6}$$

where I is the moment of inertia of a system which is formed of two material points whose masses are β and β' and the coordinates with respect to three fixed axes are x_1, x_2, x_3; x_4, x_5, x_6. I repeat, that I is the moment of inertia which this system would have with respect to the line serving as the instantaneous axis of rotation for a solid, which would coincide momentarily with this system and would rotate in such a way that the area constants are the same as for the system.

Inequality (2) must then be replaced by the following

/167

$$V + h > \frac{a_1^2 + a_2^2 + a_3^2}{2I}. \tag{2'}$$

However, this equality, just like inequality (2) itself, may be satisfied by arbitrarily large values of the x_i's, because -- for very large values of the x_i's -- the moment of inertia I is very large, and, due to the fact that the second term is very close to zero, we return to inequality (2).

We must therefore conclude that the considerations given in the preceding section are not applicable.

In order to provide a better determination of this, let us calculate the integral invariant

$$\int dx'_1\, dx'_2 \ldots dx'_6\, dx_1\, dx_2 \ldots dx_6,$$

extending it over a region defined by the following inequalities

$$\begin{cases} h - \varepsilon < T - V < h + \varepsilon, \\ a_1 - \varepsilon_1 < \quad K_1 < a_1 + \varepsilon_1, \\ a_2 - \varepsilon_2 < \quad K_2 < a_2 + \varepsilon_2, \\ a_3 - \varepsilon_3 < \quad K_3 < a_3 + \varepsilon_3. \end{cases} \tag{7}$$

The ε's are very small quantities. The K_i's are the first terms of equalities (5), and T is the reduced energy -- i.e., the first term of (6).

Let us first integrate over the x'_i's, and we obtain

$$\frac{b 4\pi}{(\beta\beta')^{\frac{3}{2}}} \varepsilon\varepsilon_1\varepsilon_2\varepsilon_3 \int \left(V + h - \frac{a_1^2 + a_2^2 + a_3^2}{2I} \right)^{\frac{3}{2}} \frac{dx_1\, dx_2 \ldots dx_6}{\sqrt{I_1 I_2 I_3}},$$

where I_1, I_2 and I_3 represent the three principal moments of inertia for this system.

In passing, I would like to note that, if the axes of the coordinates are chosen parallel to the principal axes of inertia, according to the definition of I, we shall have

$$\frac{a_1^2}{I_1} + \frac{a_2^2}{I_2} + \frac{a_3^2}{I_3} = \frac{a_1^2 + a_2^2 + a_3^2}{I}.$$

It may be seen that the integral, which is extended over every system of values such that

$$V + h - \frac{a_1^2 + a_2^2 + a_3^2}{2I} > 0$$

is infinite, although the denominator $\sqrt{I_1 I_2 I_3}$ becomes infinite when one of the points x_1, x_2, x_3 or x_4, x_5, x_6 recedes indefinitely. The inte- /168 gration field is then triply infinite, and the denominator only becomes doubly infinite.

302. Even if the considerations presented in the preceding sections are no longer applicable, we may nevertheless draw certain interesting conclusions from the existence of the integral invariant.

Let us assume that the distance b of two of the bodies becomes small, and that the third body recedes indefinitely. Due to its great distance, the third body will no longer disturb the motion of the first two, which will become essentially elliptic.

This third body will essentially describe a hyperbola around the center of gravity of the first two.

In order to elucidate this point, I shall present a simple example. I shall assume that we have a body describing a hyperbola around a fixed point. The hyperbola is composed of two branches. One of these branches is the analytical extension of the other, although the trajectory is only composed of one single branch for the engineer.

We may then inquire whether the trajectory has an analytical extension in the case of the three-body problem, and how it may be defined.

The coordinates of the second body with respect to the first are x_1, x_2, x_3; the coordinates of the third body with respect to the center of gravity of the first two are x_4, x_5, x_6, so that we must envisage the motion of two imaginary points whose coordinates, with respect to three fixed axes, are x_1, x_2, x_3 for the first and x_4, x_5, x_6 for the second.

The first of these points will essentially describe an ellipse, the second essentially a hyperbola, and it will continue receding indefinitely on one of the brances of this hyperbola. In order to obtain the desired analytical extension, let us construct the second branch of this hyperbola, and let us relate it to the ellipse described by the first point.

Let us then consider two special trajectories of our system. For the first, the initial conditions of motion will be such that, if t is positive and very large, the point x_4, x_5, x_6 will be very close to the first branch of the hyperbola and the point x_1, x_2, x_3 will be very close to the /169 ellipse, in such a way that the distances of these two points -- either to the hyperbola or to the ellipse -- tend to zero when p increases indefinitely.

Let us take the asymptote of the hyperbola as the axis of the x_4's, and let V be the velocity of the point which describes this hyperbola, for a value of t which is positive and very large. Then

$$x_4 - Vt$$

will tend toward a finite and determinate limit X when t increases indefinitely.

In the same way, let n be the mean motion on the ellipse and ℓ be the mean anomaly, and the difference

$$l - nt$$

will tend toward a finite and determinate limit ℓ_0.

If we specify the ellipse and the hyperbola and, consequently, V and n, and in addition if we specify X and ℓ_0, the initial conditions of motion corresponding to the first trajectory will be completely determined.

Let us now consider the second trajectory, and let us assume that the initial conditions of motion are such that, for t which is negative and very large, the point x_4, x_5, x_6 is very close to the second branch of the hyperbola, and the point x_1, x_2, x_3 is very close to the ellipse, and that these two points come together indefinitely from these two curves when t tends toward $-\infty$.

The differences

$$x_1 - Vt, \quad l - nt$$

tend toward the finite and determinate limits X' and ℓ'_0 when t tends toward infinity.

The initial conditions corresponding to the second trajectory are completely defined when we specify the ellipse, the hyperbola, and X' and ℓ'_0.

<u>If we have</u>

$$X = X', \qquad l_0 = l'_0,$$

<u>the two trajectories may be regarded as the analytical extension of each other</u>.

Let us now consider a system of differential equations /170

$$\frac{dx_i}{dt} = X \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where the functions X_i, which depend solely on x_1, x_2, ..., x_n, satisfy the relationship

$$\sum \frac{dX_i}{dx_i} = 0 .$$

These equations will have the integral invariant

$$\int dx_1\, dx_2 \ldots dx_n. \tag{2}$$

Let us assume that we know arbitrarily that the point x_1, x_2, ..., x_n must remain within a certain region V, which is similar to the region V which was considered in the preceding sections, but extending indefinitely so that the integral (2) extended over this region is infinite. The conclusions of Nos. 297 and 298 will no longer be applicable.

However, let us replace equations (1) by the following

$$\frac{dx_i}{dt'} = \frac{X_i}{M} = X'_i, \tag{1'}$$

where M is a given arbitrary function of $x_1, x_2, \ldots, x_n$. The point $x_1, x_2, \ldots, x_n$, whose motion is defined by equations (1'), will describe the same trajectories as that whose motion is defined by equations (1). The differential equations of these trajectories are in both cases

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}.$$

However, if I employ P to designate the point whose motion is defined by equations (1) and P' to designate that whose motion is defined by equations (1'), we may see that these two points describe the same trajectory, but obey different laws.

If I employ t to designate the time when P passes by a point of its trajectory, and t' to designate the time when P' passes by this same point, these two times will be related in the following way

$$\frac{dt}{dt'} = \frac{1}{M}.$$

We have /171

$$\sum \frac{d(MX'_i)}{dx_i} = 0,$$

which indicates that the equations

$$\frac{dx_i}{dt'} = X'_i \tag{1'}$$

have the integral invariant

$$\int M\, dx_1\, dx_2 \ldots dx_n. \tag{2'}$$

Let us assume that the function M is always positive, and that it tends toward zero when the point $x_1, x_2, \ldots, x_n$ recedes indefinitely, and recedes rapidly enough that the integral (2') extended over the region V is finite.

The conclusions presented in Nos. 297 on may be applied to equations (1'). These equations (1') therefore have Poisson stability. Since they define the same trajectories as equations (1), it may be stated in a certain sense that the trajectories of the point P also have Poisson stability.

I shall clarify this point.

We have

$$t = \int_0^{t'} \frac{dt'}{M}. \tag{3}$$

Since M is essentially positive, t increases with t'. However, since M may vanish, it may happen that the integral of the second term of (3) is infinite.

For example, let us assume that M vanishes for t' = T; then t will be infinite for

$$t' = T \text{ or for } t' > T.$$

Let us consider the trajectory of the point P'. We may divide it into two parts, the first which P' traverses from the time t' = 0 to the time t' = T; the second C' which P' traverses from the time t' = T to t' = ∞.

The point P will describe the same trajectory as P', but it will only describe the part C, because it can only reach the part C' after an infinite time t.

For the engineer, the trajectory of P would only be composed of C. For the analyst, it would be composed not only of C, but also of C', /172
which is the analytical continuation.

Let us imagine a point P_1 whose position is defined as follows: The point P_1 will occupy at the time t_1 the same position that the point P' occupies at the time t'. With respect to t_1, it will be defined by the equality

$$t_1 = \int_{t'_0}^{t'} \frac{dt'}{M} \quad (\text{where } t'_0 > T).$$

The motion of the point P_1 will conform to equations (1), and this point P_1 will describe C', in such a way that the trajectories of the points P and P_1 may be regarded as the analytical continuation of each other.

Let us now assume that the point P is within a certain region U_0 at the initial instant of time. If the initial conditions of motion are not unusual, in the sense attributed to this word in No. 296, the trajectory of the point P and its successive analytical continuations will cut across the region U_0 an infinite number of times, no matter how small it may be. However, it may happen that the point P never re-enters this region, because this region is not traversed by the trajectory, strictly speaking, of the point P, but by its analytical continuations.

303. This may be applied to the three-body problem.

We saw above that we must consider the integral

$$\int dx_1 \ldots dx_6 \, dx'_1 \ldots dx'_6,$$

which we have reduced to the sixfold integral

$$\int \left(V + h - \frac{a_1^2 + a_2^2 + a_3^2}{2I} \right)^{\frac{3}{2}} \frac{dx_1 \, dx_2 \ldots dx_6}{\sqrt{I_1 I_2 I_3}}.$$

However, we have seen that this integral, extended over the region V, is infinite, and this has prevented us from arriving at Poisson stability.

Let us write the equations of motion in the form /173

$$\frac{dx_i}{dt} = X_i, \qquad \frac{dx'_i}{dt} = Y_i,$$

where the X_i's and the Y_i's are functions of the x_i's and the x'_i's.

Then let us set

$$M = \frac{1}{(x_1^2 + x_2^2 + x_3^2 + \ldots + x_6^2 + 1)^2}$$

and let us write the new equations

$$\frac{dx_i}{dt'} = \frac{X_i}{M}, \qquad \frac{dx'_i}{dt'} = \frac{Y_i}{M}.$$

The new equations will all have the following as the integral invariant

$$\int M \, dx_1 \ldots dx_6 \, dx'_1 \ldots dx'_6$$

or

$$\int \left(V + h - \frac{a_1^2 + a_2^2 + a_3^2}{2I} \right)^{\frac{3}{2}} \frac{dx_1 \, dx_2 \ldots dx_6}{\sqrt{I_1 I_2 I_3}} M.$$

However, this integral is finite.

Therefore, if the initial situation of the system is such that the point P in space has 12 dimensions whose coordinates are

$$x_1, \; x_2, \; \ldots, \; x_6, \; x'_1, \; x'_2, \; \ldots, \; x'_6,$$

and if this point P is within a certain region U_0 at the initial instant of time, the trajectory of this point and its analytical continuations -- such as we have defined at the end of No. 302 -- will cut across this region U_0 an infinite number of times unless the initial situation of the system is not unusual, in the meaning attributed to this word in No. 296.

304. It may first appear that this result is only of interest for the analyst, and has no physical significance. However, this point of view is not entirely justified.

It may be concluded that, if the system does not repass arbitrarily close to its initial position an infinite number of times, the integral/174

$$\int_{t=0}^{t=\infty} \frac{dt}{(x_1^2 + x_2^2 + \ldots + x_6^2 + 1)^2}$$

will be finite.

This proposition is valid, if we overlook certain unusual trajectories whose probability is zero, in the meaning attributed to this word in No. 296.

If this integral is finite, it may be concluded that the time during which the perimeter of the triangle formed by the three bodies remains less than a given quantity is always finite.

CHAPTER XXVII

THEORY OF CONSEQUENTS

305. We may obtain other conclusions from the theory of integral invariants which will be of use to us below, although they will be presented in a somewhat different form. /175

Let us commence by investigating a simple example. Let us assume a point whose coordinates in space are x, y and z and whose motion is defined by the equations

$$\frac{dx}{dt} = X, \quad \frac{dy}{dt} = Y, \quad \frac{dz}{dt} = Z. \tag{1}$$

where X, Y and Z are the given, uniform functions of x, y, z. Let us assume that X and Y vanish all along the z axis, in such a way that

$$x = y = 0$$

is a solution of equations (1).

Let us then set

$$x = \rho \cos\omega, \quad y = \rho \sin\omega,$$

and equations (1) will become

$$\frac{d\rho}{dt} = R, \quad \frac{d\omega}{dt} = \Omega, \quad \frac{dz}{dt} = Z, \tag{2}$$

where R, Ω and Z are the functions of ρ, ω and z which are periodic having the period 2π with respect to ω.

It is advantageous for us to assign only positive values to ρ, and we may do this with no difficulty since x = y = 0 is a solution.

I shall now assume in addition that Ω can never vanish and, for example, always remains positive. Then ω will always increase with t.

Let us assume that equations (2) have been integrated, and that we have the solution in the following form /176

$$\rho = f_1(\omega, a, b), \quad z = f_2(\omega, a, b).$$

The letters a and b represent integration constants.

Let us set

$$\rho_0 = f_1(0, \quad a, b), \qquad z_0 = f_2(0, \quad a, b),$$
$$\rho_1 = f_1(2\pi, a, b), \qquad z_1 = f_2(2\pi, a, b).$$

Let M_0 be the point whose coordinates are

$$x = \rho_0, \qquad y = 0, \qquad z = z_0,$$

and M_1 be the point whose coordinates are

$$x = \rho_1, \qquad y = 0, \qquad z = z_1.$$

These two points both belong to the half-plane of the xz's located on the side of the positive x's.

The point M_1 will be the <u>consequent</u> of M_0.

If we consider the bundle of curves which satisfy the differential equations (1), if we pass a curve through the point M_0, and if we extend it until it encounters the half-plane $(y = 0, x > 0)$ again, the preceding definition is justified by the fact that this new encounter will occur at M_1.

If an arbitrary figure F_0 is drawn in this half-plane, the consequents of the different points of F_0 will form a figure F_1 which will be called the <u>consequent</u> of F_0.

It is evident that ρ_1 and z_1 are continuous functions of ρ_0 and z_0.

Therefore, the consequent of a continuous curve will be a continuous curve, the consequent of a closed curve will be a closed curve, and the consequent of an area which is connected n times will be an area which is connected n times.

Let us now assume that the three functions X, Y and Z are related as follows

$$\frac{dMX}{dx} + \frac{dMY}{dy} + \frac{dMZ}{dz} = 0,$$

where M is a positive, uniform function of x, y, z.

Equations (1) then have the integral invariant

$$\int M\, dx\, dy\, dz$$

and equations (2) have the following invariant /177

$$\int M\rho\, d\rho\, d\omega\, dz.$$

Let us now consider the equations

$$\frac{d\rho}{d\omega} = \frac{R}{\Omega}, \quad \frac{dz}{d\omega} = \frac{Z}{\Omega}, \quad \frac{d\omega}{d\omega} = 1, \tag{3}$$

where ω is regarded as the independent variable.

They obviously have the integral invariant

$$\int M\Omega\rho\, d\rho\, d\omega\, dz \tag{4}$$

(see No. 253).

Since it was assumed above that M, Ω and ρ are essentially positive, it is a positive integral invariant.

Let F_0 be an arbitrary area located in the half-plane

$$(y = 0, \quad x > 0),$$

and let F_1 be its consequent.

Let J_0 be the integral

$$\int M\Omega\rho\, d\rho\, dz, \tag{5}$$

extended over the planar area F_0, and let J_1 be the same integral extended over the planar area F_1.

Then let Φ_0 be the volume produced by the area F_0 when it is rotated around the z axis by an infinitely small angle ε, and the integral (4) extended over Φ_0 will be $J_0\varepsilon$.

In the same way, let Φ_1 be the volume produced by the area F_1 when it is turned around the z axis by an angle ε, and the integral (4) extended over Φ_1 will be $J_1\varepsilon$.

The integral invariant (4) must have the same value for Φ_0 as for Φ_1, and we must have

$$J_0 = J_1.$$

Thus, the integral (5) has the same value for an arbitrary area and its consequent.

This is a new form of the basic property of integral invariants.

306. Let us then assume a closed curve C_0 located in the half-plane ($y = 0$, $x > 0$) and encompassing an area F_0. Let C_1 be the consequent of C_0. This will also be a closed curve which will encompass an area F_1, and this area F_1 will be the consequent of F_0. /178

If the integral (5), extended over F_0 and over F_1, has the value J_0 and J_1, we shall have

$$J_0 = J_1,$$

from which it follows that F_0 cannot be a part of F_1, and F_1 cannot be a part of F_0.

Four hypotheses may be formulated regarding the relative position of the two closed curves C_0 and C_1.

1. C_1 is within C_0;

2. C_0 is within C_1;

3. The two curves are outside of each other;

4. The two curves intersect.

The equation $J_0 = J_1$ excludes the two first hypotheses.

If the third is also excluded, for whatever reason, the two curves will definitely intersect.

For example, let us assume that X, Y, Z depend on an arbitrary parameter μ and that for $\mu = 0$, C_0 is its own consequent. For very small values of μ, C_0 will differ very little from C_1. Therefore, it could not happen that the two curves C_0 and C_1 are outside of each other, and they must intersect.

Invariant Curves

307. Any curve which will be its own consequent will be called an invariant curve.

Invariant curves may be readily formed. Let M_0 be an arbitrary point of the half-plane, and let M_1 be its consequent. Let us connect M_0 to M_1 by an arc of an arbitrary curve C_0. Let C_1 be the consequent of C_0, C_2 be the consequent of C_1, and so on. The entire group of arcs of the curve C_0, C_1, C_2, ... will obviously constitute an invariant curve.

But we may also consider invariant curves whose formation will be more natural.

Let us assume that equations (1) have a periodic solution. Let /179

$$x = \varphi_1(t), \quad y = \varphi_2(t), \quad z = \varphi_3(t) \tag{6}$$

be the equations of this periodic solution, in such a way that the functions ϕ_i are periodic in t, having the period T.

I shall assume that when t increases by T, ω increases by 2π.

Equations (6) represent a curve. Let M_0 be the point where this curve intersects the half-plane; this point M_0 will obviously be its own consequent.

Let us now assume that there are asymptotic solutions which are very close to the periodic solution (6). Let

$$x = \Phi_1(t), \quad y = \Phi_2(t), \quad z = \Phi_3(t) \tag{7}$$

be the equations of these solutions.

The functions Φ_i may be developed in powers of $Ae^{\alpha t}$, and the coefficients are themselves periodic functions of t. In this expression, α is a characteristic exponent, and A is an integration constant.

In equations (7), the three coordinates x, y, z are therefore expressed as a function of two parameters, A and t. These equations therefore represent a surface which may be called the asymptotic surface. This asymptotic surface will pass through the curve (6), since equations (7) may be reduced to equations (6) when we set A = 0.

The asymptotic surface will intersect the half-plane along a certain curve which passes through the point M_0 and which is obviously an invariant curve.

308. Let us consider an invariant curve K. I shall assume that X, Y, Z depend on the parameter μ, as well as the curve K.

I shall assume that for $\mu = 0$, the curve K is closed, but that it ceases to be closed for small values of μ.

Let A_0 be a point of K. The position of this point will depend on μ. For $\mu = 0$, the curve K is closed, so that, after having traversed this curve starting with A_0, one returns to the point A_0. If μ is very small, this will no longer be the case, but one will pass very close to A_0. Therefore, on the curve K there will be a curve arc which is different from that where A_0 is located, but which will pass very close to A_0. Let B_0 be the point of this curve arc which is closest to A_0. /180

I shall join A_0B_0.

Let A_1 and B_1 be the consequents of A_0 and B_0. These two points will be located on K. Let A_1B_1 be the consequent curve of the small line A_0B_0.

We must consider the closed curve C_0 which is composed of the arc A_0MB_0 of curve K, included between A_0 and B_0, and of the small line A_0B_0. What will its consequent be?

In order to define our ideas more precisely, let us assume that the four points A_1, A_0, B_1, B_0 follow each other on K in the order $A_1A_0B_1B_0$.

The consequent C_1 of C_0 will be composed of the arc A_1MB_1 of the curve K and of the small arc A_1B_1, the consequent of the small line A_0B_0.

Several hypotheses may then be formulated:

1. The small curvilinear quadrilateral $A_0B_0A_1B_1$ is convex, that is, none of these curvilinear sides have a double point, and the only points which the two sides have in common are the apexes. In this hypothesis, the form of the curve would be that indicated in one of the following figures

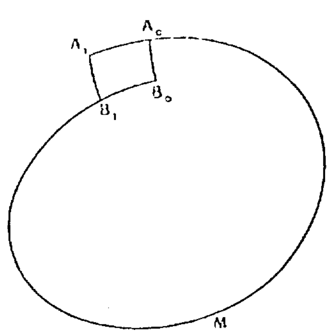

Figure 1

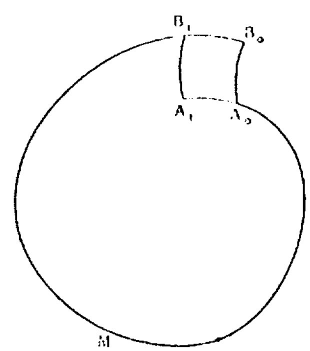

Figure 2

This hypothesis must be rejected, because it is apparent that the integral J is larger in the case of Figure 1 for C_1 than for C_0, and smaller in the case of Figure 2.

2. The arc A_0A_1 or B_0B_1 has a double point. If this were the case for the invariant curve K, there would have to be a double point on the arc joining an arbitrary point on the curve to its first consequent; /181 we shall assume that this is not the case. Actually, this condition would not occur in any of the applications which I have in mind. It does not apply, in particular, in the case of the invariant curve produced by an asymptotic surface, as I explained at the end of the preceding section. It may be readily stated that the asymptotic surface does not have a double line if we limit ourselves to the portion of this surface corresponding to small values of the quantities which I have designated as $Ae^{\alpha t}$ above.

On the other hand, the line A_0B_0 does not have a double point, and the same must be true for its consequent A_1B_1. To sum up, we shall assume that the four sides of our quadrilateral do not have a double point.

3. The arc A_0A_1 intersects the arc B_0B_1. (As a special case, this case includes that in which the curve K would be closed.) Our curves will then have the form shown in Figure 3.

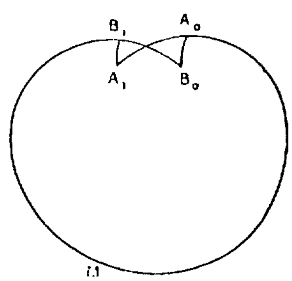

Figure 3

4. The arc A_0B_0 intersects its consequent A_1B_1. Our curves will then have the form shown in Figure 4.

There are cases in which this hypothesis must be rejected. For example, let us assume that X, Y, Z depend on one parameter μ, and that for $\mu = 0$ the curve K is closed and that each of its points is its own consequent, so that for $\mu = 0$ the four apexes of the quadrilateral coincide.

Then the four distances A_0B_0, A_1B_1, A_1A_0, B_1B_0 will be infinitely small quantities if μ is the main infinitely small quantity. Let us /182 assume that A_1A_0 is an infinitely small quantity of the order p, A_0B_0 an infinitely small quantity of the order q, and that q is

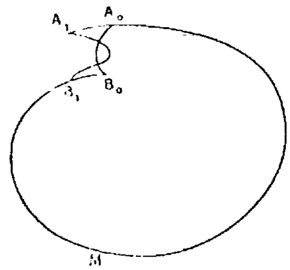

Figure 4

larger than p.

Since A_1B_1 is the consequent of A_0B_0, the length of the arc A_1B_1 must be of the order q. Then let C be one of the intersection points of A_0B_0. In the mixtilinear triangle whose two sides are the lines A_1A_0 and A_0C, and whose third side is the arc of the curve A_1C which is part of A_1B_1, the side A_1C is larger than the difference between the two others. It should therefore be of the order p, and we have seen that it must be of the order q.

The hypothesis must therefore be rejected.

5. Two adjacent sides of the quadrilateral intersect, for example, A_1A_0 and A_1B_1. It is then necessary that A_0B_0, which is the antecedent of A_1B_1, intersect K itself. If A'_0 is the intersection of A_0B_0 with K, and A'_1 is the intersection of A_1B_1 with the arc A_0A_1, A'_1 will be the consequent of A'_0, and we shall obtain the following figure.

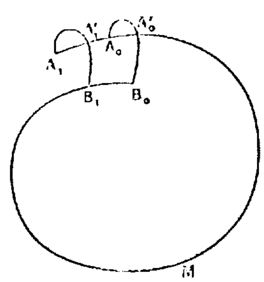

Figure 5

It is apparent that A'_0 and A'_1 may play the same role as A_0 and A_1, and that we therefore return to the first case.

This new hypothesis must therefore be rejected. /183

To sum up, the two arcs A_0A_1 and B_0B_1 will intersect every time that hypotheses 2 and 4 must be rejected, for one reason or another.

We must now examine the case in which the points A_1, A_0, B_1, B_0 follow one another in a different order on K. The orders $B_1B_0A_1A_0$, $B_0B_1A_0A_1$, $A_0A_1B_0B_1$ do not differ essentially from that which we have just studied.

Orders such as $A_1B_1B_0A_0$, $A_1B_0B_1A_0$, $A_1B_0A_0B_1$, ... will not appear in the applications which follow. We shall always assume that, if μ is very small, the distances A_0A_1 and B_0B_1 are very small with respect to the length of the arcs A_0MB_0 or A_1MB_1.

The order $A_1A_0B_0B_1$, or the equivalent orders, remain, and we shall no longer discuss them. It is apparent that if they appear, on the arc A_0MB_0 there will be a point which will be its own consequent.

309. For example, let us assume that equations (1) have a periodic solution

$$x = \varphi_1(t), \quad y = \varphi_2(t), \quad z = \varphi_3(t) \tag{6}$$

and asymptotic solutions

$$x = \Phi_1(t), \quad y = \Phi_2(t), \quad z = \Phi_3(t). \tag{7}$$

Let us assume that equations (1) depend on a very small parameter μ, and that X, Y, Z may be developed in powers of this parameter.

For $\mu = 0$, let us assume that the asymptotic solutions (7) may be reduced to periodic solutions. This may be done as follows. We have stated that the Φ_i's may be developed in powers of $Ae^{\alpha t}$, with the coefficients themselves being periodic functions of p. However, the exponent α depends on μ; let us assume that it vanishes for $\mu = 0$. Then for $\mu = 0$ the functions Φ_i will become periodic functions of t, and the solutions (7) may be reduced to periodic solutions.

The asymptotic surface intersects the half-plane along a certain curve C_0 which passes through the point M_0, which is the intersection of the half-plane with the left curve (6). /184

The curve C_0 is obviously invariant, as I stated at the end of No. 307. For $\mu = 0$, each of the points of C_0 is its own consequent.

In addition, I shall assume that the curve C_0 is closed for $\mu = 0$.

Let us refer back to Chapter VII, Volume I. We saw from Nos. 107 on that, in the case of dynamics, the characteristic exponents may be developed in powers of $\sqrt{\mu}$, and are equal pairwise and have the opposite sign. We shall assume that this is the case.

In reality, we then have two asymptotic surfaces corresponding to the two equal exponents having opposite sign α and $-\alpha$. We therefore have two curves C_0 which will intersect at the point M_0.

We may distinguish between four branches of the curve

$$C_0', \quad C_0'', \quad C_1', \quad C_1''$$

all four of which end at the point M_0; C_0' and C_0'' will correspond to the exponent α, C_1' and C_1'' to the exponent $-\alpha$.

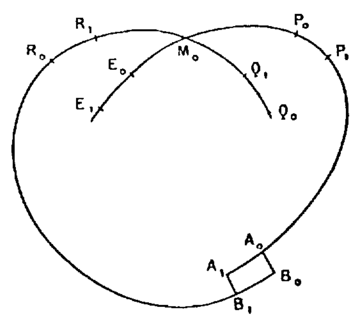

Figure 6

These different branches of the curve are shown in Figure 6. The branch C_0' is the branch $M_0P_0P_1A_0A_1$, the branch C_0'' is the branch $M_0E_0E_1$, the branch C_1' is the branch $M_0Q_1Q_0$ and the branch C_1'' is the branch $M_0R_1R_0B_1B_0$.

These four branches of the curve are obviously invariant.

Now, for $\mu = 0$, C_0' is identical to C_1', C_0'' is identical to C_1'', and (if we assume that the curve C_0 is closed for $\mu = 0$, which we shall call C_0^0,) these four branches of the curve will coincide on the closed curve C_0^0. /185

It may be deduced from this that, for very small μ, these branches of the curve will differ very little from each other, that C'_0 will deviate very little from C'_1, C''_0 will deviate very little from C''_1, and that, if C'_0 is sufficiently extended, it will pass very close to C''_1, if it is sufficiently extended.

I have indicated on the figure different points of these branches of the curve and their consequents. Thus, A_1, B_1, E_1, P_1, Q_1, R_1 are, respectively, the consequents of A_0, B_0, E_0, P_0, Q_0, R_0.

We would first like to note that the points A_1, A_0, B_1, B_0 do follow each other (as we assumed at the beginning of No. 308) in the order $A_1A_0B_1B_0$ when the invariant curve formed of the two branches C'_0 and C''_1 is traversed from A_1 to B_0.

This invariant curve is not closed, but it differs very little from the closed curve C^0_0.

In this connection, let us examine the five hypotheses of No. 308. As we have seen, the first must be rejected. The second will no longer occur.

It could only occur if the asymptotic surface (7) had a double line.

We have stated that the Φ_i's may be developed in powers of $Ae^{\alpha t}$. Therefore let us set

$$\Phi_i = \Phi_i^0 + A e^{\alpha t}\Phi_i^1 + A^2 e^{2\alpha t}\Phi_i^2 + \dots$$

If our surface had a double line, this double line would have to satisfy equations (1). Actually, the asymptotic surface is produced by an infinite number of lines satisfying these equations in such a way that, if two layers of this surface happen to intersect, the intersection could only be one of these lines.

Since Φ_i depends on the time t and the parameter A at the same time, we may show this by writing

$$\Phi_i = \Phi_i(t, A).$$

If there were a double line, we would have to have the three identities

$$\Phi_i(t, A) = \Phi_i(t', B) \qquad (i = 1, 2, 3),$$

where A and B are two constants and where t' is a function of t. These three identities would have to exist no matter what t may be. /186

Performing differentiation, we shall have

$$\frac{d\Phi_i}{dt} = \frac{d\Phi_i}{dt'}\frac{dt'}{dt}.$$

However, in view of equations (1), we shall have

$$\frac{d\Phi_1}{dt} = X[\Phi_1(t, A), \Phi_2(t, A), \Phi_3(t, A)]$$

and in the same way

$$\frac{d\Phi_1}{dt'} = X[\Phi_1(t', B), \Phi_2(t', B), \Phi_3(t', B)],$$

from which it follows that

$$\frac{d\Phi_i}{dt} = \frac{d\Phi_i}{dt'}, \qquad \frac{dt'}{dt} = 1,$$

from which we have

$$t' = t + h,$$

where h is a constant.

We would thus obtain the following

$$\Phi_i^0(t) + Ae^{\alpha t}\Phi_i^1(t) + A^2e^{2\alpha t}\Phi_i^2(t) = \Phi_i^0(t+h) + Ce^{\alpha t}\Phi_i^1(t+h) + \ldots.$$

where

$$C = Be^{\alpha h}.$$

The identity must be valid for t = —∞, from which it follows that

$$Ae^{\alpha t} = Ce^{\alpha t} = 0,$$

and we have

$$\Phi_i^0(t) = \Phi_i^0(t+h),$$

from which we have h = 0 and

$$\Phi_i^0(t) + Ae^{\alpha t}\Phi_i^1(t) + \ldots = \Phi_i^0(t) + Ce^{\alpha t}\Phi_i^1(t) + \ldots$$

or

$$A\Phi_i^1(t) + A^2e^{\alpha t}\Phi_i^2(t) + \ldots = C\Phi_i^1(t) + C^2e^{\alpha t}\Phi_i^2(t) + \ldots$$

or, setting t = —∞, we have

$$A = C = B.$$

Due to the fact that the two values A and B are equal, there is no double line. /187

The third hypothesis may be adopted.

Let us pass on to the fourth hypothesis. In order to determine whether it must be rejected, we must try to determine the order of magnitude of the distances A_1A_0 and A_0B_0. This is what we shall do in the different applications which follow.

Finally, the fifth hypothesis is always reduced to the first one, as we have seen.

Extension of the Preceding Results

310. We formulated very special hypotheses above concerning equations (1), but all of them are not equally necessary.

Let us consider a region D which is simply connected and which is part of the half-plane ($y = 0$, $x > 0$). Let us assume that we know arbitrarily that, if the point (x, y, z) is located at a point M_0 in this region at the initial instant of time, ω will constantly increase from 0 to 2π when t increases from 0 to t_0, in such a way that the curve satisfying equations (1) and passing through the point M_0 -- assuming that it is extended from this point M_0 up to its new intersection with the half-plane -- is never tangent to a plane passing through the z axis.

Just as in No. 305, we may then define the consequent of the point M_0, and it is apparent that all the preceding statements will still be applicable to the figures which are located within the region D.

It will not be necessary that the curves satisfying equations (1) and intersecting the half-plane outside of D be subjected to the condition of never being tangent to a plane passing through the z axis. It will no longer be necessary that $x = y = 0$ be a solution of equations (1).

Then, if C_0 is a closed curve inside of D and if C_1 is its consequent, the two curves will be outside of each other or will intersect.

The results given in No. 308 will be equally applicable to the invariant curves which do not leave the region D. If even one invariant curve leaves the region D when it is sufficiently extended, the results will still be applicable to the portion of this curve which is within /188 this region.

311. Let us now consider a curved surface S which is simply

connected, instead of a plane region D. Let us pass a curve γ satisfying equations (1) through a point M_0 of this curved surface, and let us extend this curve until it again intersects S. The new point of intersection M_1 may still be called the consequent of M_0.

If we consider two points M_0 and M'_0 which are very close to each other their consequents will be, in general, very close to each other. There would be an exception if the point M_1 were located at the boundary of S, or if the curve γ touched the surface at the point M_1 or at the point M_0. Except for these exceptions, the coordinates of M_1 are analytic functions of the coordinates of M_0.

In order to avoid these exceptions, I shall consider a region D which is part of S and such that the curve γ, proceeding from a point M_0 inside of D, intersects S at a point M_1 which is never located at the boundary of S -- so that the curve γ does not touch S either at M_0 or at M_1. Finally, I shall assume that this region D is simply connected.

Let us adopt a special system of coordinates which I shall call ξ, η and ζ, for example, and for which I shall only assume the following:

1. When $|\xi|$ and $|\eta|$ are smaller than 1, the rectangular coordinates x, y and z will be analytic and uniform functions of ξ, η and ζ, which are periodic with the period 2π with respect to ζ.

2. No more than one system of values of ξ, η, ζ can correspond to a point (x, y, z) in space, such that

$$|\xi| < 1, \quad |\eta| < 1, \quad 0 < \zeta < 2\pi. \tag{λ}$$

3. When we set $\zeta = 0$, or $\zeta = 2\pi$ and when we vary ξ and η between -1 and +1, the point x, y, z describes the surface S, or a portion of this surface containing the region D.

4. It results from conditions (1) and (2) that the functional determinant Δ of ξ, η, ζ with respect to x, y, z is never infinite nor zero when the inequalities (λ) are satisfied.

5. Equations (1) may be transformed by writing them in the following form /189

$$\frac{d\xi}{dt} = \Xi, \quad \frac{d\eta}{dt} = H, \quad \frac{d\zeta}{dt} = Z^*. \tag{1'}$$

I shall assume that Z^* remains positive for

$$|\xi| < 1, \quad |\eta| < 1, \quad \zeta = 0.$$

The equations (1') will have the integral invariant

$$\int \frac{M}{\Delta}\, d\xi\, d\eta\, d\zeta,$$

and the equations

$$\frac{d\xi}{d\zeta} = \frac{\Xi}{Z^*}, \qquad \frac{d\eta}{d\zeta} = \frac{H}{Z^*}, \qquad \frac{d\zeta}{d\zeta} = 1 \tag{3'}$$

will have the integral invariant

$$\int \frac{MZ^*}{\Delta}\, d\xi\, d\eta\, d\zeta.$$

Let F_0 be an arbitrary figure which is part of D and let F_1 be its consequent. Let us assume that the different points of F_0 and of F_1 move in such a way that ξ and η remain constant and that ζ increases from 0 to ε, with ε being very small. The figure F_0 will produce a volume Φ_0, and the figure F_1 will produce a volume Φ_1. The integral

$$\int \frac{MZ^*}{\Delta}\, d\xi\, d\eta\, d\zeta = \varepsilon \int \frac{MZ^*}{\Delta}\, d\xi\, d\eta$$

will have the same value for Φ_0 and for Φ_1. Therefore, the double integral

$$\int \frac{MZ^*}{\Delta}\, d\xi\, d\eta,$$

which is similar to the integral (5) of No. 305, will have the same value for F_0 and F_1. It is therefore essentially positive.

It follows from this that the results given in No. 306 may be applied to closed curves C_0 located within D, and that the results given in No. 308 may be applied to invariant curves K, or at least to the portion of these curves which is inside of D.

Even if an invariant curve leaves the region D when it is sufficiently extended, the results will still be applicable to the portion of this curve which is within this region.

Application to Equations of Dynamics

/190

312. Let F be a function of the four variables x_1, x_2, y_1, y_2. Let us formulate the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}.$$

I shall assume, as I usually do:

1. That F is a periodic function of y_1 and y_2;

2. That F depends on a parameter μ and may develop in powers of this parameter in the following form

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots;$$

3. That F_0 is a function of only x_1 and of x_2.

Under this assumption, we shall have the integral

$$F = C, \tag{2}$$

where C is a constant.

Under this assumption, let us attribute a value which is determined once and for all to C, and let M be a moving point whose rectangular coordinates are

$$[1 + \varphi(x_1)\cos y_1]\cos y_2, \quad [1 + \varphi(x_1)\cos y_1]\sin y_2, \quad \varphi(x_1)\sin y_1.$$

The function $\phi(x_1)$ is a function of x_1, of which I shall make a more comprehensive determination below.

Let us first assume that F, which will depend arbitrarily on x_2, may be developed in increasing powers of $x_1 \cos y_1$ and $x_1 \sin y_1$. For $x_1 = 0$, it will result that the function F will no longer depend on y_1 and, in addition, that the function F will not change when x_1 is changed into $-x_1$ and y_1 is changed into $y_1 + \pi$. We shall then assume that $\phi(x_1)$ is an odd function of x_1 which increases from 0 to 1 when x_1 increases from 0 to $+\infty$. We may set, for example

$$\varphi(x_1) = \frac{x_1}{\sqrt{1 + x_1^2}}.$$

If this hypothesis is adopted, the point M will always be within a torus of radius 1, which is tangent to the z axis.

An infinite number of systems of values of x_1, y_1 and y_2 will correspond to each point M within this torus. However, these systems will not differ essentially from each other, since one passes from one to the other by increasing y_1 or y_2 by a multiple of 2π, or by changing x_1 into $-x_1$ and y_1 into $y_1 + \pi$. /191

If x_1, y_1 and y_2 are given, x_2 may be deduced by means of equation (2). Let us assume that the variables x and y vary in accordance with equations (1), and the corresponding point M will describe a certain curve which I shall call the <u>trajectory</u>.

One and only one trajectory passes through each point inside the torus.

The form of these trajectories for $\mu = 0$ may be readily determined.

For $\mu = 0$, the differential equations may be reduced to

$$\frac{dx_i}{dt} = 0, \qquad \frac{dy_i}{dt} = -\frac{dF_0}{dx_i}.$$

The x_i's are therefore constants, which indicates that our trajectories are located on the tori, and the y_i's are linear functions of time, because

$$-\frac{dF_0}{dx_i} = n_i$$

depends only on the x_i's and is a constant.

If the ratio $n_1 : n_2$ is commensurable, the trajectories are closed curves. Conversely, they are not closed if this ratio is incommensurable.

Let m_1, m_2, p_1, p_2 be four whole numbers, such that

$$m_1 p_2 - m_2 p_1 = 1;$$

Let us set

$$\begin{aligned} y'_1 &= m_1 y_1 + m_2 y_2, \\ y'_2 &= p_1 y_1 + p_2 y_2, \\ x'_1 &= p_2 x_1 - p_1 x_2, \\ x'_2 &= -m_2 x_1 + m_1 x_2. \end{aligned}$$

The identity

$$x'_1 y'_1 + x'_2 y'_2 = x_1 y_1 + x_2 y_2$$

indicates that when one passes from the variables x_i, y_i to the variables x'_i, y'_i, the canonical form of the equations is not changed.

We shall assume that n_2 does not vanish when x_1 remains less than a certain limit a. Then $\frac{dy_2}{dt}$ will always retain the same sign, and we /192 shall have, for example

$$\frac{dy_2}{dt} > 0.$$

This inequality, which is valid for $\mu = 0$, will still be valid for small values of μ.

The relationships

$$y_2 = 0, \qquad |x_1| < a - \varepsilon$$

will then define a certain plane region D which will have the form of a circle.

The trajectories starting from a point in this region will never be tangent to a plane passing through the z axis, at least before having cut across the half-plane $y_2 = 0$ again. Our region may therefore play the role of region D in No. 310.

The equations (1) have the integral invariant

$$\int dx_1\, dx_2\, dy_1\, dy_2,$$

from which we may deduce the following by means of the integral F = const.

$$J = -\int \frac{dx_1\, dy_1\, dy_2}{\frac{dF}{dx_2}}.$$

However, $\frac{dF}{dx_2}$ equals $-\frac{dy_2}{dt}$, and is consequently negative. The invariant J is then a positive invariant.

The results given in Nos. 306 and 308 may therefore be applied to the curves drawn in the region D.

Under this assumption, let b be a value of x_1 which is smaller than $a - \varepsilon$, and such that the corresponding values of n_1 and of n_2 satisfy the following relationship

$$m_1 n_1 + m_2 n_2 = 0,$$

where m_1 and m_2 are two prime numbers with respect to each other.

The curve

$$x_1 = b.$$

which is a circumference will be an invariant curve for $\mu = 0$.

If we always assume that $\mu = 0$, the trajectories emanating from different points on this circumference will have the general equation /193

$$y_1 = n_1 t + \text{const.}, \qquad y_2 = n_2 t + \text{const.},$$

from which we have

$$y_1 = \frac{n_1}{n_2} y_2 + \text{const.}$$

In order to have successive consequents of a given point, it will be sufficient to set the following successively

$$y_2 = 0, \quad y_2 = 2\pi, \quad y_2 = 4\pi, \quad \ldots, \quad y_2 = 2k\pi.$$

In order to pass from a point to its consequent, it is sufficient to increase y_1 by

$$\frac{2\pi n_1}{n_2} = -\frac{2\pi m_2}{m_1},$$

from which it follows that all points on the invariant circumference $x_1 = b$ will coincide with their m_1th consequent.

This point and its $m_1 - 1$ first consequents are distributed on this circumference in a circular order, which may be readily determined when the two whole numbers m_1 and m_2 are known. I shall call the order Ω.

Let us no longer assume that $\mu = 0$. The equations (1), according to Chapter III, will still have periodic solutions which differ very little from the solutions

$$x_1 = b, \qquad y_1 = n_1 t + \text{const.}, \qquad y_2 = n_2 t + \text{const.}$$

They will have at least two, of which one is unstable and the other is stable. A closed trajectory will correspond to each of these periodic solutions. I shall consider one of these trajectories which I shall call T and which will correspond to an unstable solution, so that two asymptotic surfaces pass through T.

Let M_0 be the point where this trajectory penetrates the half-plane $y_2 = 0$, and let M_1, M_2, ... be its successive consequents (Figure 7). The point M_0 will coincide with its m_1th consequent M_m.

I shall join the point M_k to the center of the circumference $x_1 = b$. The radius which is thus drawn will intersect the circumference at a point M'_k which is very close to M_k. The different points M'_k will follow each other on the circumference in the circular order Ω.

In order to formulate these ideas more precisely, I have drawn the figure on the assumption that $m_1 = 5$, $m_2 = 2$. The closed trajectory T intersects the half-plane at five points M_0, M_1, M_2, M_3, M_4. Two asymptotic surfaces which intersect pass through this trajectory. /194

The intersection of these asymptotic surfaces with the half-plane will be composed of different curves. We shall have two curves intersecting at M_0, two at M_1, two at M_2, two at M_3, and two at M_4. All these curves are shown in the figure.

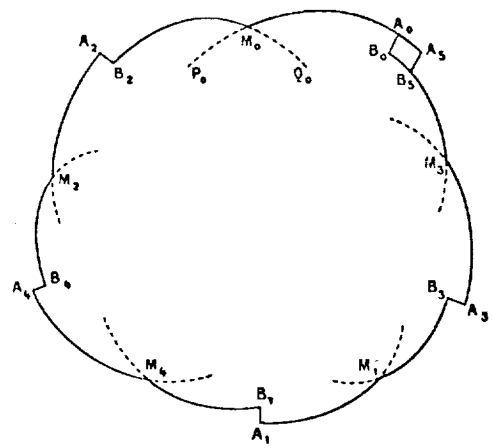

Figure 7

In particular, let us consider the two curves which pass at M_0. We may distinguish between four branches of the curve, i.e., M_0A_0, M_0B_2, M_0P_0, M_0Q_0. The first two are shown by a solid line, and the last two are shown by a dashed line. The first and the third, just like the second and the fourth, are each located in the extension of the other.

In the same way, four branches of the curve will end at each of the points M. Two of these branches are shown by solid lines and two are shown by a dashed line, and each pair is located in the extension of the other.

Let A_0 be a point of the branch M_0B_0. Let us draw a radius through A_0 going to the center of the circumference $x_1 = b$, and let us extend this radius up to B_0 where it intersects the curve shown by the solid line M_3B_0. Since μ is very small and since all of our curves differ very little from the circumference $x_1 = b$, the segment A_0B_0 will be very small. /195

We may then see that M_1A_1, M_2A_2, M_3A_3, M_4A_4, M_0A_5 are the successive consequents of M_0A_0, that M_4B_1, M_0B_2, M_1B_3, M_2B_4, M_4B_5 are the successive consequents of M_3B_0, and finally that A_1B_1, A_2B_2, ..., A_5B_5 are the successive consequents of A_0B_0.

The arcs A_1B_1, A_2B_2, ..., A_5B_5 are no longer rectilinear in general, but are very small arcs of a curve.

Figures 1 or 2 shown in No. 308 reproduce the part of the figure shown by the solid line. The entire group of our curves shown by the solid lines represents an invariant curve K.

I have drawn the figure based on the first hypothesis, which -- as we have seen -- must be rejected along with the fifth hypothesis. According to the statements I made in No. 309, this also holds true for the second hypothesis.

We must examine the fourth hypothesis in greater detail. In order to do this, let us try to determine the equation of our asymptotic surfaces. Based on the statements presented in No. 207, this equation may be obtained in the following way.

A function S is formulated which may be developed in powers of $\sqrt{\mu}$, in such a way that

$$S = S_0 + \sqrt{\mu} S_1 + \ldots + \mu^{\frac{p}{2}} S_p + \ldots.$$

Regarding S_p, it is a periodic funtion of the period 2π with respect to y'_2, and 4π with respect to y'_1.

We shall have

$$x'_1 = \frac{dS}{dy'_1}, \qquad x'_2 = \frac{dS}{dy'_2}$$

$$x_1 = m_1 \frac{dS}{dy'_1} + m_2 \frac{dS}{dy'_2}. \tag{4}$$

Equation (4) is the equation of the asympototic surface.

If the series S were convergent, the periodicity of the S_p's would entail the condition that our curves must be closed and that the two points A_0 and B_0 must coincide. However, this is not the case (see No. 225, and the following).

What significance does equation (4) have? It may only be valid from the formal point of view, i.e., if Σ_p is the sum of the p + 1 first terms of the series S, so that /196

$$\Sigma_p = S_0 + \sqrt{\mu} S_1 + \ldots + \mu^{\frac{p}{2}} S_p,$$

the equation

$$x_1 = m_1 \frac{d\Sigma_p}{dy'_1} + m_2 \frac{d\Sigma_p}{dy'_2} \tag{4'}$$

will be valid up to quantities of the order $\mu^{\frac{p+1}{2}}$.

However, equation (4') represents a closed surface, and p is arbitrarily large.

We must therefore conclude that the distance A_0B_0 is an infinitely small quantity on the order of infinity (see Nos. 225 on). In addition, the distance A_0A_5 (or B_0B_5) is on the order of $\sqrt{\mu}$, and is consequently infinitely small of the order of ½.

The distance A_0B_0 is therefore infinitely small with respect to A_0A_1, which indicates that the fourth hypothesis must be rejected.

The only possible hypothesis is therefore the third.

Therefore the two arcs A_0A_5 and B_0B_5 intersect.

Application to the Restricted Problem

313. I am going to apply the preceding principles to the problem presented in No. 9, and I shall employ the notation given in that section. Consequently, we shall have the canonical equations

$$\frac{dx'_i}{dt} = \frac{dF'}{dy'_i}, \qquad \frac{dy'_i}{dt} = -\frac{dF'}{dx'_i},$$

based on which we may set

$$\begin{cases} x'_1 = L, & x'_2 = G, \\ y'_1 = l, & y'_2 = g - t \end{cases} \tag{5}$$

and, in addition,

$$F' = R + G = F_0 + \mu F_1 + \ldots,$$

$$F_0 = \frac{1}{2x'^2_1} + x'_2.$$

Let us now set

$$x_1 = L - G, \qquad x_2 = L + G,$$
$$2y_1 = l - g + t, \qquad 2y_2 = l + g - t$$

and the equations will retain the canonical form and will become /197

$$\frac{dx_i}{dt} = \frac{dF'}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF'}{dx_i}.$$

We will have

$$F_0 = \frac{2}{(x_1 + x_2)^2} + \frac{x_2 - x_1}{2},$$

from which it follows that

$$n_1 = \frac{+4}{(x_1 + x_2)^3} + \frac{1}{2}, \qquad n_2 = \frac{+4}{(x_1 + x_2)^3} - \frac{1}{2}.$$

If we assume that the eccentricity is very small, L and G will differ very little in absolute value. Therefore, one of the two quantities x_1 and x_2 is very small.

I would like to note in addition that the equations

$$L = \sqrt{a}, \qquad G = \sqrt{a(1 - e^2)}$$

indicate that G is always smaller than L in absolute value. Therefore, x_1 and x_2 are essentially positive.

Let us assume that x_1 is very small. The function F' will be a function of a and of $\ell + g - t$ which may be developed in powers of $e \cos g$ and of $e \sin g$. Therefore, this will also be a function of x_2 and of y_2 which may be developed in powers of

$$\sqrt{x_1}\cos y_1 \quad \text{and} \quad \sqrt{x_1}\sin y_1.$$

It will be periodic with the period 2π both in y_1 and in y_2.

If, on the other hand, it is x_2 which is very small, the function F' will be a function of x_1 and of y_1, which may be developed in powers of

$$\sqrt{x_2}\cos y_2 \quad \text{and} \quad \sqrt{x_2}\sin y_2.$$

Let us now assume that our four variables x and y are related by the equation of energy

$$F = C .$$

This equation may be approximately reduced to

$$F_0 = C.$$

Let us construct the curve $F_0 = C$, taking x_1 and x_2 as the coordinates of a point in a plane.

The equation may be written

$$(x_1 + x_2)^2(2C + x_1 - x_2) = 4.$$

This curve has two asymptotes /198

$$x_1 + x_2 = 0, \qquad x_2 - x_1 = 2C$$

and it is symmetrical with respect to the first of these two asymptotes.

However, it should be noted that the only portion of the curve which is of use to us is that which is located in the first quadrant

$$x_1 > 0, \quad x_2 > 0.$$

Based on the values of C, the curve may have one of the forms shown in the two following figures

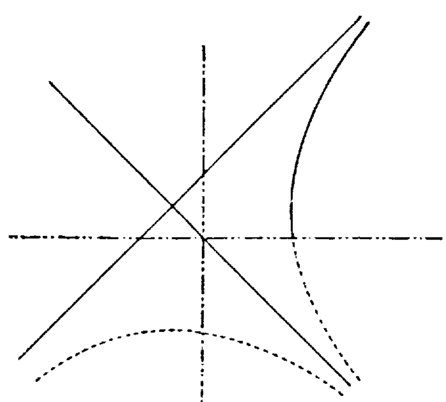

Figure 8

The axes of the coordinates are represented by the dot-dash line, the asymptotes and the utilizable portions of the curve are shown by the solid line, and the portions of the curve which are of no use are shown by the dotted line.

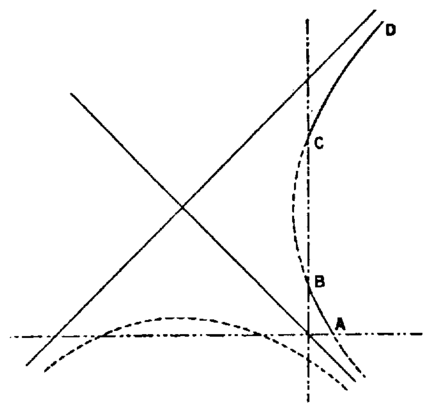

Figure 9

We shall assume that a value is assigned to C, so that the curve has the form shown in Figure 9 and so that it contains two utilizable arcs AB and CD. We shall no longer consider the arc AB.

We should point out that when one traverses this arc AB, x_1 decreases constantly from OA to zero, x_2 increases constantly from zero to OB and $\frac{x_2}{x_1}$ increases constantly from zero to $+\infty$.

If we now construct the curve F = C, assuming that y_1 and y_2 are constants and x_1 and x_2 are the coordinates of a point in a plane, the curve will differ very little from F_0 = C and can still be represented by Figure 9. It will have a utilizable arc AB, and when one traverses this arc the ratio $\frac{x_2}{x_1}$ will increase constantly from zero to $+\infty$. /199

We thus arrive at the following method of geometric representation. The location of the system will be represented by the point whose rectangular coordinates are

$$\frac{\sqrt{x_2}\cos y_2}{\sqrt{x_2+4x_1-2\sqrt{x_1}\cos y_1}}, \quad \frac{\sqrt{x_2}\sin y_2}{\sqrt{x_2+4x_1-2\sqrt{x_1}\cos y_1}},$$

$$\frac{2\sqrt{x_1}\sin y_1}{\sqrt{x_2+4x_1-2\sqrt{x_1}\cos y_1}}.$$

These three functions may be developed in powers of $\sqrt{x_1}$ cos y_1 and $\sqrt{x_1}$ sin y_1, if x_1 is very small, and may be developed in powers of

$\sqrt{x_2}$ cos y_2 and $\sqrt{x_2}$ sin y_2, if x_2 is very small. They only depend on the ratio $\frac{x_1}{x_2}$.

Thus, one and only one point in space corresponds to each system of values of y_1 and of y_2 and to each point on the utilizable arc AB.

The functional determinant of the three coordinates with respect to y_1, y_2, and with respect to $\sqrt{\frac{x_1}{x_2}}$, always retains the same sign.

We may therefore apply the results obtained in the preceding section within all of the region D where n_2 does not vanish. /200

However, n_2 vanishes for $x_1 + x_2 = 2$.

But, if we have $x_1 + x_2 = 2$, $x_1 > 0$, $x_2 > 0$, we shall obviously have

$$\frac{2}{(x_1+x_2)^2} + \frac{x_2-x_1}{2} < \frac{2}{(x_1+x_2)^2} + \frac{x_2+x_1}{2} = \frac{3}{4}.$$

However, the first term of this equation is F_0 and, when compiling the curve $F_0 = C$, we assumed that we were dealing with the case presented in Figure 9. However, the case shown in Figure 9 assumes that

$$C > \frac{3}{4}.$$

Since F_0 differs very little from F, and consequently from C, we cannot have at the same time

$$C > \frac{3}{4}, \quad F_0 < \frac{3}{4}$$

(unless C is very close to its limit $\frac{3}{4}$, which we have not assumed).

Under the conditions with which we are now dealing, we shall not have $n_2 = 0$.

Thus, the results presented in the preceding section are applicable, and if we construct the asymptotic surfaces and if we consider the intersection of these surfaces with the half-plane $y_2 = 0$, the two arcs which are similar to those which we designated as A_0A_5 and B_0B_5 above will intersect.

I would like to add one word to this.

The coordinates of the third body, with respect to the major axis and the minor axis of the ellipse which it describes, are -- according to the well-known formula

$$L^2(\cos l + \ldots),$$
$$LG(\sin l + \ldots).$$

It may thus be seen that, when G changes sign, the second of these coordinates changes sign.

As a result, the perturbed planet turns in the same direction as the perturbing planet if G is positive, and it turns in the opposite direction if G is negative.

CHAPTER XXVIII

PERIODIC SOLUTIONS OF THE SECOND TYPE

314. Let us consider a system of equations /201

$$\frac{dx_i}{dt} = X_i \quad (i = 1, 2, \ldots, n), \tag{1}$$

where the X_i's are functions of x_1, x_2, ..., x_n, and of t, which are periodic having the period T with respect to t.

Let

$$x_i = \varphi_i(t) \tag{2}$$

be a periodic solution of period T of equations (1).

We shall try to determine whether equations (1) have other periodic solutions which are very close to (2) and whose period is a multiple of T.

These solutions, if they exist, will be called periodic solutions of the second type.

Let us consider a solution of equations (1) which is very close to (2). Let

$$\varphi_i(0) + \beta_i$$

be the value of x_i for t = 0, and let

$$\varphi_i(0) + \beta_i + \psi_i = \varphi_i(kT) + \beta_i + \psi_i$$

be the value of x_i for t = kT (k is a whole number).

The β_i's and the ψ_i's, whose definition is the same as that given in Chapter III, will be very small. Just as in Chapter III, it will be found that the ψ's are functions of the β's which may be developed in increasing powers of the β's.

In order that the solution may be periodic having the period kT, it /202 is necessary and sufficient that

$$\psi_1 = \psi_2 = \ldots = \psi_n = 0. \tag{3}$$

Due to the fact that the $\phi_i(t)$'s are periodic functions, the ψ's vanish with the β's.

We shall assume that the functions X_i which appear in equations (1) depend on a certain parameter μ. Then the functions $\phi_i(t)$ will depend not only on t, but also on μ. As regards t, they will be periodic of period T, with T being a constant which is independent of μ.

Under these conditions, the functions ψ, whose definition remains the same, will depend not only on the β's, but also on μ. If we assume that

$$\beta_1, \ \beta_2, \ \ldots, \ \beta_n, \ \mu$$

are coordinates of a point in space having n + 1 dimensions, equations (3) will represent a curve in this space. A periodic solution, of period kT, will correspond to each point on this curve.

Since the ψ's all vanish when the β's all vanish at the same time, this curve will consist of the straight line

$$\beta_1 = \beta_2 = \ldots = \beta_n = 0. \tag{4}$$

The solution (2) will correspond to different points on this straight line. Due to the fact that this solution is a periodic solution of period T, it is for that reason a periodic solution of period kT.

But we must try to determine whether there are other periodic solutions which are very similar to the first or -- in other words -- if curve (3) includes, in addition to the straight line (4), other branches of the curve which are very close to the straight line(4).

In other words, are there points on the straight line (4) through which branches of the curve (3) pass, other than this line?

Let

$$\beta_1 = \beta_2 = \ldots = \beta_n = 0, \qquad \mu = \mu_0$$

be a point P of the line (4).

In order that several branches of the curve may pass through the point P, it is necessary that at this point P the functional determinant, or the Jacobian, of the ψ's, with respect to the β's, vanishes.

This condition is not sufficient, as we shall see at a later point, for several real branches of the curve to pass through the point P. /203

Let us formulate the determinant of the ψ's with respect to the β's, let us add -S to all the diagonal terms, and let us set the determinant thus obtained equal to zero. We shall thus obtain the equation which is known as the equation for S.

The roots of this equation (see No. 80) are

$$e^{k\alpha T}-1,$$

where α is one of the characteristic exponents of equation (1).

In order that the functional determinant may be zero, it is necessary and sufficient that one of the roots be zero. We must therefore have

$$e^{k\alpha T}=1.$$

which means that kαT is a multiple of 2iπ.

Therefore, in order that several branches of the curve pass through the point P, it is necessary that one of the characteristic exponents be a multiple of $\frac{2i\pi}{kT}$.

315. This condition is not sufficient, and a more extensive discussion is necessary.

Let us set

$$\mu=\mu_0+\lambda,$$

and let us try to develop the β's in whole or fractional powers of λ.

We shall assume that the Jacobian of the ψ's, with respect to the β's, is zero. This Jacobian vanishes for λ = 0, but will not be identically zero, in general. In order that this may be the case, it is necessary that one of the characteristic exponents be constant, independent of μ, and equal to a multiple of $\frac{2i\pi}{kT}$.

We shall therefore assume that the Jacobian vanishes for λ = 0, but that its derivative, with respect to λ, does not vanish.

In the same way, we shall assume that the minors of the first order of this Jacobian do not all vanish at the same time.

In this case, based on the theorem in No. 30, from n - 1 of equations (3) we may derive n - 1 of the quantities β in the form of series developed in whole powers of λ and of the n^{th} quantity β, for example of β_n. /204

Let us substitute the values of

$$\beta_1, \ \beta_2, \ \ldots, \ \beta_{n-1},$$

thus derived in the n^{th} equation (3). The first term of this n^{th} equation will be developed in powers of λ and of β_n. Let us write it in the following form.

$$\theta(\lambda, \beta_n) = 0.$$

I may first point out that Θ must be divisible by β_n, because the line (4) must be part of the curve (3).

On the other hand, the derivative of Θ with respect to β_n must vanish for $\lambda = 0$, since the Jacobian vanishes. For $\lambda = 0$, Θ does not contain a term of the first degree. Let us assume that it no longer contains terms of the second degree, ..., $p - 1^{th}$ degree, but that it does contain a term of degree p.

Finally, since the derivative of the Jacobian with respect to λ does not vanish, we shall have a term containing $\lambda\beta_n$.

I may therefore write

$$\theta = A\lambda\beta_n + B\beta_n^p + C,$$

where C is the total group of terms containing β_n^{p+1}, $\lambda\beta_n^2$, or $\lambda^2\beta_n$ as a factor. A and B are constant coefficients which are not zero.

It may be seen that we may derive β_n from this in terms of a series which progresses according to the powers of $\lambda^{\frac{1}{p-1}}$, and the problem is to determine whether this series is real.

If p is even or if, p being odd, A and B have opposite signs, the series is real, and periodic solutions of the second type exist.

If p is odd, and if A and B have opposite signs, the series is imaginary, and there is no periodic solution of the second type.

I shall now assume that not only the Jacobian vanishes for $\lambda = 0$, but that the same holds true for all of its minors of the first, the second, etc., and $p - 1^{th}$ order. I shall nevertheless assume that the /205 minors of the p^{th} order are not all zero at the time.

According to the statements presented in No. 57, under these conditions, there will be not one, but p, characteristic exponents which will be multiples of $\frac{2i\pi}{kT}$.

From n - p of equations (3), we may then derive n - p of the quantities β in the form of series developed in powers of λ and of the p last quantities β.

For purposes of brevity, I shall employ the β''s to designate the n - p first quantities β, and the β'''s to designate the p last quantities β. We shall therefore have the β''s developed in powers of λ and of the β'''s.

Let us substitute these expansions in the place of the β''s in the p last equations (3), and we shall obtain p equations

$$\Theta_1 = \Theta_2 = \ldots = \Theta_p = 0, \tag{5}$$

whose first terms will be developed in powers of λ and of the β'''s.

Due to the fact that the Jacobian and its minors of the first p - 1 orders are zero, these first terms will not include terms of the first degree in β'' which are independent of λ. We must now determine whether the first terms of equations (5) will contain terms of the first degree with respect to the β'''s, and at the same time of the first degree with respect to λ.

Let θ_i be the total group of terms of Θ_i which are of the first degree with respect to the β'''s. It is apparent that θ_i may be developed in powers of λ. Let

$$\theta_i = \theta_i^0 + \lambda\theta_i^1 + \lambda^2\theta_i^{(2)} + \ldots$$

be this expansion. The $\theta_i^{(k)}$'s will be homogeneous polynomials of the first degree with respect the the β'''s.

According to the preceding statements, θ_i^0 will be identically zero, but we must now determine whether the same holds true for θ_i'.

The Jacobian of the ψ's with respect to the β's equals

$$\Pi(1 - e^{k\alpha T}),$$

The product indicated by the sign Π extends over n factors corresponding to the n characteristic exponents α.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be these n exponents, and let /206

$$\varphi(\alpha) = (1 - e^{k\alpha T}).$$

The Jacobian will equal the product

$$\varphi(\alpha_1)\varphi(\alpha_2)\ldots\varphi(\alpha_n).$$

In order that $\lambda = 0$, the Jacobian vanishes as well as its minors of $p - 1$ first orders. As a result, p of the exponents are multiples of $\frac{2i\pi}{kT}$. Therefore, p of the factors $\phi(\alpha)$ vanish for $\lambda = 0$ and are, consequently, divisible by λ. The product, i.e., the Jacobian, will therefore be divisible by λ^p.

We shall assume that for $\lambda = 0$ none of the $\frac{d\alpha}{d\lambda}$ vanishes, which is what we already assumed previously. Under these conditions, none of the $\phi(\alpha)$'s are divisible by λ^2. Therefore, the product is not divisible by λ^{p+1}.

Thus, the Jacobian is divisible by λ^p, but not by λ^{p+1}.

As a result, the determinant of the θ_i^1's is different from zero, and consequently none of the θ_i^1's vanishes identically.

The simplest case is that in which, for $\lambda = 0$, the terms of the second degree do not vanish in the expressions for Θ_i, and in which these terms of the second degree cannot vanish at the same time, unless all the β'''s vanish at the same time.

Let us assume that η_i is the total group of terms of the second degree of Θ_i for $\lambda = 0$.

It will be sufficient to consider the algebraic equations

$$\eta_i + \lambda\theta_i^1 = 0,$$

whose first terms are homogeneous polynomials of the second degree with respect to λ and the β'''s.

If these equations have real solutions, we shall have periodic solutions of the second type.

I shall not extend the discussion to the other cases, but shall complete this discussion when treating the equations of dynamics.

316. Let us assume that the functions X_i which appear in equations (1) do not depend on time t.

As we have seen in No. 61, in this case one of the characteristic exponents is always zero.

In addition, if

$$x_i = \varphi_i(t)$$

is a periodic solution of period T, the same also holds for

$$x_i = \varphi_i(t + h)$$

whatever the constant h may be.

In the preceding section, we assumed that -- no matter what μ might be -- there was a periodic solution

$$x_i = \varphi_i(t)$$

and the period could only be T, since the X_i's were periodic functions of t, of period T.

The period was therefore independent of μ.

The same is not true in this case. We shall always assume that, no matter what μ might be, equations (1) have a periodic solution

$$x_i = \varphi_i(t).$$

However, the period will depend on μ, in general. I shall call T the period, and T_0 the value of T for $\mu = \mu_0$, i.e., for $\lambda = 0$. We shall then modify the definition of the quantities β and ψ to a certain extent.

We shall always designate the value of x_i by $\phi_i(0) + \beta_i$ for t = 0. However, we shall designate the value of x_i by $\phi_i(0) + \beta_i + \psi_i$ for $t = k\ (T + \tau)$ (and not for t = kT).

Then, the ψ_i's will be functions of the n + 2 variables

$$\beta_1, \beta_2, \ldots, \beta_n, \tau, \lambda.$$

If we continue to assume that the β's and λ's are the coordinates of a point in space having n + 1 dimensions, the equations /208

$$\psi_i = 0 \qquad (3)$$

will no longer represent a curve, but will represent a surface, since we may vary the two parameters τ and λ independently and continuously.

However, we should point out that curves are drawn on this surface whose different points correspond to periodic solutions which may not be regarded as being essentially different.

If

$$x_i = f_i(t)$$

is a periodic solution, the same will hold true for

$$x_i = f_i(t + h)$$

no matter what the constant h may be, and this new solution will not differ from the first in reality.

The following point corresponds to the first

$$\beta_i = f_i(0) - \varphi_i(0),$$

and the following point corresponds to the second

$$\beta_i = f_i(h) - \varphi_i(0).$$

When h is varied continuously, the second point describes a curve whose different points do not correspond to solutions which are actually different.

In particular, let us consider the solution

$$x_i = \varphi_i(t)$$

The following point will correspond to this solution

$$\beta_i = 0$$

which belongs to the line (4).

The following point

$$\beta_i = \varphi_i(h) - \varphi_i(0), \qquad (4')$$

which belongs to a certain surface (4) making up the surface (3), will correspond to the solution

$$x_i = \varphi_i(t + h),$$

which is not actually different from the first.

We must now determine whether the surface (3) includes layers other than (4') approaching very close to (4'), i.e., whether there are points /209 on the surface (4') through which other layers of the surface (3) pass in addition to the surface (4') itself.

Without limiting the conditions of generality, we may assume that $\beta_1 = 0$ (or we may impose another arbitrary relationship between the β's).

In actuality, the solutions

$$x_i = f_i(t), \qquad x_i = f_i(t+h)$$

are not different, and it is sufficient to take one of them into consideration.

We may choose the constant h arbitrarily, and we may take it in such a way that, for example,

$$f_1(h) = \varphi_1(0),$$

from which we have

$$\beta_1 = 0.$$

q.e.d.

If we impose this condition $\beta_1 = 0$, the two surfaces (3) and (4') may be reduced to curves, and the surface (4') may be reduced to the line (4), in particular.

We would like to again determine whether another branch of the curve (3) passes through a point of the line (4).

For this purpose, let us combine equation $\beta_1 = 0$ with equations (3). These equations will represent the curve (3), or a curve of which (3) is only a part. In the region under consideration, in order that this curve may not be reduced to the line (4), it is necessary that the Jacobian $\psi_1, \psi_2, \ldots, \psi_n, \beta_1$ with respect to $\beta_1, \beta_2, \ldots, \beta_n, \tau$, and that of $\psi_1, \psi_2, \ldots, \psi_n$ with respect to $\beta_2, \beta_3, \ldots, \beta_n, \tau$, be zero for $\lambda = 0$.

Since nothing distinguishes β_1 from other β's, the Jacobians of the ψ's with respect to τ and with respect to n - 1 arbitrary β's must all vanish. That is, all the determinants included in the matrix of Nos. 38 and 63 must vanish at the same time. By pursuing a line of reasoning similar to that presented in No. 63, we may see that the equation for S must have two zero roots.

As a result, two of the characteristic exponents must be multiples of $\frac{2i\pi}{kT}$. This is already true for the one of them which is zero. A second exponent must be a multiple of $\frac{2i\pi}{kT}$.

If this condition is fulfilled, we shall formulate a system of n + 1 equations including equations (3) and $\beta_1 = 0$. We shall derive τ and the β's in the form of a series developed in whole and fractional powers of λ. /210

If the series are real, we shall have periodic solutions of the second type; if the series are imaginary, this will not be the case.

I shall not continue this discussion any further.

317. Let us now assume that the equations

$$\frac{dx_i}{dt} = X_i, \tag{1}$$

where time enters explicitly have a uniform integral

$$F = C,$$

in such a way that we have

$$\sum \frac{dF}{dx_i} X_i = 0.$$

We saw in No. 64 that in this case the Jacobian of the ψ's with respect to the β's vanishes, and that one of the characteristic exponents is zero.

The equations

$$\psi_1 = \psi_2 = \ldots = \psi_n = 0 \tag{3}$$

are not then different since we have identically

$$F[\varphi_i(0) + \beta_i + \psi_i] - F[\varphi_i(0) + \beta_i] = 0.$$

They do not represent a curve, but rather a surface.

However, according to the principles presented in Chapter III, in this case we have a double infinity of periodic solutions of period T

$$x_i = \varphi_i(t),$$

since there is one which corresponds to each value of the parameter $\mu = \mu_0 + \lambda$ and to each value of the constant C.

We shall assign a fixed value C_0 to the constant C, and we shall no longer have a simple infinity of periodic solutions of period T

$$x_i = \varphi_i(t),$$

with each of them corresponding to a value of λ.

Due to the fact that equations (3) are not different, they may be replaced by n - 1 of them -- for example, by /211

$$\psi_1 = \psi_2 = \ldots = \psi_{n-1} = 0.$$

Let us then consider the system

$$\psi_1 = \psi_2 = \ldots = \psi_{n-1} = 0, \qquad F[\varphi_i(0) + \beta_i] = C_0. \tag{3'}$$

Equations (3') no longer represent a surface, but rather a curve, part of which is formed by the line

$$\beta_i = 0. \tag{4}$$

In order that another branch of the curve may pass through a point on the line (4), it is necessary that the Jacobian of

$$\psi_1, \quad \psi_2, \quad \ldots, \quad \psi_{n-1}, \quad F,$$

with respect to the β's vanish.

This condition may be written in still another form.

Let us assume that we have solved equation

$$F(x_i) = C_0$$

with respect to x_n, and that this solution yields

$$x_n = \theta(x_1, x_2, \ldots, x_{n-1}).$$

Let us substitute θ in place of x_n in X_i, and let X'_i be the result of this substitution.

Equations (1) are thus replaced by the following

$$\frac{dx_i}{dt} = X'_i \qquad (i = 1, 2, \ldots, n-1). \tag{1'}$$

These equations (1') will have the following periodic solution

$$x_i = \varphi_i(t).$$

The number of characteristic exponents of this periodic solution, which is assumed to belong to equations (1'), will be n - 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be these n - 1 exponents. These will be the same as those for this periodic solution $x_i = \phi_i(t)$, which are assumed to belong to equations (1), suppressing the n exponents which equal zero.

In order that equations (1) have periodic solutions of the second type in the vicinity of a point on the line (4), it is necessary and sufficient that equations (1') have them, i.e., that one of the n - 1 characteristic exponents $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ is a multiple of $\frac{2i\pi}{kT}$ at a /212 point on the line (4).

Thus, the condition, which was presented above, that the Jacobian of $\psi_1, \psi_2, \ldots, \psi_{n-1}$, F is zero may be expressed in a completely different manner. In order that it may be fulfilled, it is necessary that two of the exponents be multiples of $\frac{2i\pi}{kT}$. This is always true for the one of them which is zero; this must be true for a second exponent.

Let us assume that this condition is fulfilled. From equations (3') we shall derive the β's in series which are ordered in whole and fractional powers of λ. I shall not extend this discussion, to determine whether these series are real.

318. Let us now assume that the X_i's do not depend explicitly on time and that equations (1) have an integral

$$F = C.$$

In this case, according to No. 66, two of the characteristic exponents are zero. If the equations have a periodic solution for a system of values of μ and of C, they will still have it for the adjacent values, so that we shall have a double infinity of periodic solutions

$$x_i = \varphi_i(t)$$

which depend on the two parameters μ and C. The period T will not be constant; it will be a function of μ and of C.

Let us then assign a fixed value C_0 to C, and let

$$\varphi_i(o) + \beta_i, \quad \varphi_i(o) + \beta_i + \psi_i$$

be the values of x_i for t = 0 and for t = k (T + τ).

We shall add equation $F = C_0$, and then an arbitrary relationship between the β's -- for example, $\beta_1 = 0$ -- to the equations

$$\psi_1 = \psi_2 = \ldots = \psi_n = o \qquad (3)$$

Without limiting the conditions of generality, and for the same reason as was given in No. 316, we may assume that $\beta_1 = 0$. /213

We shall thus obtain the system

$$\psi_i = o, \quad F = C_0, \quad \beta_1 = o. \qquad (3'')$$

These equations represent a curve. The number of equations equals n + 2, but the n equations (3) are not different, and may be replaced by n - 1 of them. This is justified by the same line of reasoning that was presented in the preceding section. System (3") may thus be reduced to n + 1 equations. The number of variables is n + 2 -- i.e.,

$$\beta_1, \ \beta_2, \ \ldots, \ \beta_n, \ \tau, \ \mu.$$

This curve (3") includes the line

$$\beta_i = 0. \tag{4}$$

Let $\beta_i = 0$, $\mu = \mu_0$ be a point on this line. In order that another branch of the curve may pass through this point, it is necessary that the Jacobian of the first terms of equations (3") be zero or -- which amounts to the same thing -- that the Jacobian of n - 1 of the ψ's and of F with respect to β_2, β_3, ..., β_n and τ be zero. Finally, since nothing distinguishes β_1 from the other β's, it is necessary that the Jacobians of F and of n - 1 arbitrary ψ's with respect to τ and to n - 1 arbitrary β's all be zero.

This condition may be expressed in another way.

Just as in the preceding section, we shall derive the following from the equation $F = C_0$

$$x_n = \theta(x_1, x_2, \ldots, x_{n-1}),$$

and we shall obtain the equations

$$\frac{dx_i}{dt} = X'_i \quad (i = 1, 2, \ldots, n-1). \tag{1'}$$

According to No. 316, of the n - 1 characteristic exponents, it is necessary that one of them be zero and that the other be a multiple of $\frac{2i\pi}{kT}$ [if it is assumed that the periodic solution belongs to equations (1')]. In other words -- which amounts to the same thing -- it is necessary that of the n characteristic exponents [if it is assumed that the periodic solution belongs to equations (1)], two be zero, and a third /214 be a multiple of $\frac{2i\pi}{kT}$.

Let us assume that this condition is fulfilled. We shall derive the β's and the τ from (3") in series which are ordered according to whole or fractional powers of λ. I shall still forego a discussion of this point.

Application to the Equations of Dynamics

319. I would like to discuss the equations of dynamics in greater detail. However, in order to do this I must first present an important property of these equations.

Let ξ_i and η_i be the values of x_i and y_i for $t = 0$. Let X_i and Y_i be the values of x_i and y_i for $t = T$. We know that

$$\int\int \Sigma\, dx_i\, dy_i$$

is an integral invariant. We shall therefore have

$$\int\int \Sigma(dX_i\, dY_i - d\xi_i\, d\eta_i) = 0,$$

with the double integral extending over an arbitrary area A.

This may be written as follows

$$\int \Sigma(X_i\, dY_i - Y_i\, dX_i - \xi_i\, d\eta_i + \eta_i\, d\xi_i) = 0,$$

where the simple integral is extended along the contour of the area A, i.e., an arbitrary closed contour.

In other words, the expression

$$\Sigma(X_i\, dY_i - Y_i\, dX_i - \xi_i\, d\eta_i + \eta_i\, d\xi_i)$$

is an exact differential.

As a result, we find that

$$dS = \Sigma[(X_i - \xi_i)\, d(Y_i + \eta_i) - (Y_i - \eta_i)\, d(X_i + \xi_i)]$$

is also an exact differential.

320. If we vary T, it is apparent that S will be a function of T. Let us calculate the derivative of S with respect to T by means of the /215
equations

$$\frac{dX_i}{dT} = \frac{dF}{dY_i}, \qquad \frac{dY_i}{dT} = -\frac{dF}{dX_i}$$

We have

$$\frac{dS}{dT} = \int \sum \left[\frac{dX}{dT}\, d(Y + \eta) - \frac{dY}{dT}\, d(X + \xi) + (X - \xi)\, d\frac{dY}{dT} - (Y - \eta)\, d\frac{dX}{dT} \right],$$

or

$$\frac{dS}{dT} = \int \sum \left[\frac{dF}{dY}\, d(Y + \eta) + \frac{dF}{dX}\, d(X + \xi) - (X - \xi)\, d\frac{dF}{dX} - (Y - \eta)\, d\frac{dF}{dY} \right],$$

or, integrating by parts, we have

$$\frac{dS}{dT} = -\Sigma\left[(X-\xi)\frac{dF}{dX}+(Y-\eta)\frac{dF}{dY}\right]+2\int\sum\left(\frac{dF}{dX}dX+\frac{dF}{dY}dY\right),$$

or finally

$$\frac{dS}{dT} = 2F-\Sigma\left[(X-\xi)\frac{dF}{dX}+(Y-\eta)\frac{dF}{dY}\right] + \text{arbitrary function of T.}$$

We shall set the arbitrary function of T equal to a constant -2C, and we shall have

$$\frac{dS}{dT} = 2(F-C)-\Sigma\left[(X-\xi)\frac{dF}{dX}+(Y-\eta)\frac{dF}{dY}\right]$$

For T = 0, we have dS = 0 and consequently

$$S = \text{const.}$$

We shall take this constant to be zero so that S will vanish identically for T = 0. The function S is thus completely determined.

321. Let us determine the maxima and the minima of the function S. Let us first consider T as a constant. In order that the function S has a maximum or a minimum, assuming that this function S may be regarded as a uniform function of the variables $X_i + \xi_i$ and $Y_i + \eta_i$ in the region under consideration, it is necessary that its derivatives with respect to these variables are zero -- i.e., that we have /216

$$X_i = \xi_i, \qquad Y_i = \eta_i.$$

The corresponding solution is therefore a periodic solution of period T, and this period T is one of the known quantities of the problem at hand.

We shall no longer regard T as a known quantity. In order that S has a maximum or a minimum, it would be necessary that we first have

$$X_i = \xi_i, \qquad Y_i = \eta_i,$$

and in addition

$$\frac{dS}{dT} = 0.$$

However, if X = ξ, Y = η, we still have

$$\frac{dS}{dT} = 2(F-C),$$

from which it follows that

$$F = C.$$

The corresponding solution will still be a periodic solution of period T.

However, the period T will no longer be a given quantity. The energy constant C, which did not enter the preceding case, will be a given quantity.

The two methods for determining the maxima of S are related to the two methods of interpreting the principle of least action, that of Hamilton and that of Maupertuis. This will be clear to the reader after the following chapter has been read.

322. The definition of the function S may also be modified in the following way.

In a large number of applications, F is a periodic function of period 2π with respect to the y_i's. In this case, a solution may be assumed to be periodic when $X_i = \xi_i$, and when $Y_i - \eta_i$ is a multiple of 2π.

It is then apparent that if we set

$$dS = \Sigma[(X_i - \xi_i)d(Y_i + \eta_i) - (Y_i - \eta_i - 2m_i\pi)d(X_i + \xi_i)],$$

where $m_1, m_2, \ldots, m_n$ are arbitrary whole numbers, the expression dS will still be an exact differential.

We shall thus obtain /217

$$\frac{dS}{dT} = 2F - \Sigma\left[(X - \xi)\frac{dF}{dX} + (Y - \eta - 2m\pi)\frac{dF}{dY}\right] + \text{arbitrary function of T.}$$

We shall set

$$\frac{dS}{dT} = 2(F - C) - \Sigma\left[(X - \xi)\frac{dF}{dX} + (Y - \eta - 2m\pi)\frac{dF}{dY}\right].$$

For T = 0, we have

$$dS = \Sigma 4 m_i \pi \, d\xi_i.$$

We shall set

$$S = 4\pi\Sigma m_i \xi_i,$$

which concludes the determination of the function S.

Assuming that T is a given quantity, the maxima and minima of S will be obtained by setting its derivatives equal to zero, which yields

$$X_i = \xi_i, \qquad Y_i = \eta_i + 2m_i\pi.$$

The corresponding solution is still a periodic solution, since $Y_i - \eta_i$ is a multiple of 2π. The period T is given.

If T is not given, it is first necessary that

$$X_i = \xi_i, \qquad Y_i = \eta_i + 2m_i\pi$$

and, in addition,

$$\frac{dS}{dT} = 0,$$

from which we have

$$F = C.$$

323. It is now necessary that we learn to distinguish between the real maxima and the real minima of S. Up to this point, we have only determined the condition for which the first derivatives of S are zero, but it is known that this condition is not sufficient for providing a maximum. It is still necessary that the second derivatives satisfy certain inequalities.

Let us first assume that the conditions presented in No. 319 hold, and let us regard T as given.

Let

$$x_i = \varphi_i(t), \qquad y_i = \varphi'_i(t)$$

be a periodic solution of period T, so that /218

$$\varphi_i(0) = \varphi_i(T), \qquad \varphi'_i(0) = \varphi'_i(T).$$

A maximum or a minimum of the function S may correspond to this solution.

Let

$$x_i = \varphi_i(t) + x'_i, \qquad y_i = \varphi'_i(t) + y'_i$$
$$x_i = \varphi_i(t) + x''_i, \qquad y_i = \varphi'_i(t) + y''_i$$

be two solutions which differ very little from this periodic solution.

I shall assume that x'_i, y'_i, x''_i, y''_i are small enough that we may neglect the squares and may assume that these quantities satisfy the variational equations (see Chapter IV).

Let ξ'_i and η'_i be the values of x'_i and y'_i for $t = 0$; X'_i and Y'_i -- the values of x'_i and y'_i for $t = T$.

In order to determine whether S has a maximum or a minimum, it is sufficient to study the total group of second degree terms in the expansion of S in powers of the ξ'_i's and the η'_i's.

It may be readily seen that this group of terms may be reduced to

$$\Sigma(X'_i \eta'_i - Y'_i \xi'_i).$$

Let us study the expression

$$\Sigma(x''_i y'_i - y''_i x'_i). \tag{1}$$

According to No. 56, this expression must be reduced to a constant.

What is the form of the general solution of the variational equations?

If there are n degrees of freedom, we shall have n - 1 particular solutions having the form

$$x'_i = e^{\alpha_k t}\theta_{k.i}(t), \qquad y'_i = e^{\alpha_k t}\theta'_{k.i}(t).$$

The α_k's are the characteristic exponents, and the θ's are periodic functions of period T.

We shall have n - 1 other solutions having the form

$$x'_i = e^{-\alpha_k t}\theta''_{k.i}(t), \qquad y'_i = e^{-\alpha_k t}\theta'''_{k.i}(t)$$

corresponding to the exponents $-\alpha_k$ which are equal and have the opposite sign of the n - 1 exponents α_k.

We shall have the obvious solution /219

$$x'_i = \frac{d\varphi_i}{dt}, \qquad y'_i = \frac{d\varphi'_i}{dt}$$

and finally the 2nth particular solution will be

$$x'_i = t\frac{d\varphi_i}{dt} + \psi_i, \qquad y'_i = t\frac{d\varphi'_i}{dt} + \psi'_i.$$

Therefore, the general solution may be written

$$x'_i = \Sigma A_k e^{\alpha_k t}\theta_{k.i}(t) + \Sigma B_k e^{-\alpha_k t}\theta''_{k.i}(t) + C\frac{d\varphi_i}{dt} + D\left(t\frac{d\varphi_i}{dt} + \psi_i\right),$$

$$y'_i = \Sigma A_k e^{\alpha_k t}\theta'_{k.i}(t) + \Sigma B_k e^{-\alpha_k t}\theta'''_{k.i}(t) + C\frac{d\varphi'_i}{dt} + D'\left(t\frac{d\varphi'_i}{dt} + \psi'_i\right),$$

where the A, B, C, D's are integration constants.

In the same way, we shall have

$$x''_i = \Sigma A'_k e^{\alpha_k t}\theta_{k,i}(t) + \Sigma B'_k e^{-\alpha_k t}\theta''_{k,i}(t) + C' \frac{d\varphi_i}{dt} + D'\left(t \frac{d\varphi_i}{dt} + \psi_i\right)$$

with a formula which is similar for y''_i.

The A' , B' , C' , D's are new constants.

Let us substitute these values in expression (1). This expression will become a bilinear form with respect to the two series of constants

$$\begin{matrix} A, & B, & C, & D, \\ A', & B', & C', & D'. \end{matrix}$$

Since this form must vanish identically for

$$A_k = A'_k, \quad B_k = B'_k, \quad C = C', \quad D = D'$$

this form will be a linear form with respect to the determinants contained in the matrix

$$\left\| \begin{matrix} A_1 & B_1 & A_2 & B_2 & \dots & A_{n-1} & B_{n-1} & C & D \\ A'_1 & B'_1 & A'_2 & B'_2 & \dots & A'_{n-1} & B'_{n-1} & C' & D' \end{matrix} \right\|.$$

The coefficients of this linear form must be constants, since expression (1) must be reduced to a constant.

In general, none of the characteristic exponents will be zero, and two of these exponents will not be equal to each other.

It follows from this that we cannot have a term containing one of the determinants /220

$$A_k A'_j - A_j A'_k, \quad A_k B'_j - B_j A'_k, \quad B_k B'_j - B_j B'_k,$$
$$A_k C' - C A'_k, \quad A_k D' - D A'_k, \quad B_k C' - C B'_k, \quad B_k D' - D B'_k,$$

because the coefficient of this term must contain one of the exponentials

$$e^{(\alpha_k+\alpha_j)t}, \quad e^{(\alpha_k-\alpha_j)t}, \quad e^{-(\alpha_k+\alpha_j)t}, \quad e^{\pm\alpha_k t}$$

as a factor, and cannot be reduced to a constant.

The only determinants which may enter in our form are therefore

$$A_k B'_k - B_k A'_k, \quad CD' - DC',$$

so that I may write

$$\Sigma(x''_i y'_i - y'_i x''_i) = \Sigma M_k(A_k B'_k - B_k A'_k) + N(CD' - DC'), \qquad (2)$$

where the M_k and N's are constants.

I may state that M_k cannot be zero, otherwise expression (1) would not depend on the constants A_k, A'_k, B_k, B'_k. If we then assume that all of the constants A' and B', C' and D' are zero, with the exception of the two constants A'_k and B'_k to which we may assign given values which are different from zero, we would have a relationship

$$\Sigma(x''_i y'_i - x'_i y''_i) = 0$$

which would be linear with respect to the unknowns x'_i and y'_i, and where the coefficients x''_i and y''_i would be given functions of time which are different from zero. Such a relationship cannot exist, since the 2n variables x'_i and y'_i are independent. Therefore, M_k cannot be zero.

If we change t into t + T, we shall obtain new solutions of the variational equations, and these new solutions will be obtained by changing the constants

$$A_k, \quad B_k, \quad C, \quad D$$

into

$$A_k e^{\alpha_k T}, \quad B_k e^{-\alpha_k T}, \quad C + DT, \quad D.$$

In order to have

$$\Sigma(X'_i \eta'_i - Y'_i \xi'_i),$$

it will be sufficient to set the following in expression (1)

$$A'_k = A_k e^{\alpha_k T}, \qquad B'_k = B_k e^{-\alpha_k T}, \qquad C' = C + DT, \qquad D' = D,$$

from which we have /221

$$\Sigma(X'_i \eta'_i - Y'_i \xi'_i) = \Sigma M_k (e^{-\alpha_k T} - e^{\alpha_k T}) A_k B_k - NTD^2. \tag{3}$$

324. In order to discuss equation (3), we must distinguish between several cases:

1. The exponents $\pm\alpha_k$ are real. The functions

$$\theta_{k.i}, \quad \theta'_{k.i}, \quad \theta''_{k.i}, \quad \theta'''_{k.i}$$

are also real.

2. The exponents $\pm\alpha_k$ are purely imaginary, and the square α_k^2 is real and negative.

Then the functions $\theta_{k.i}$ and $\theta''_{k.i}$, $\theta'_{k.i}$ and $\theta'''_{k.i}$ are imaginary and conjugate.

3. The exponents $\pm\alpha_k$ are complex. Among the characteristic exponents, we shall then have the exponents $\pm\alpha_j$ which will be imaginary and conjugate of the exponents $\pm\alpha_k$, and

$$\theta_{j.i}, \quad \theta'_{j.i}, \quad \theta''_{j.i}, \quad \theta'''_{j.i}$$

will be imaginary and conjugate of

$$\theta_{k.i}, \quad \theta'_{k.i}, \quad \theta''_{k.i}, \quad \theta'''_{k.i}.$$

Let us now assume that the x'_i's and the y'_i's are real. In order to calculate the constants A, B, C, D, we shall have 2n equations which we shall obtain, for example, by setting the following in the equation for x'_i

$$t = 0, \quad t = T, \quad t = 2T, \quad \ldots, \quad t = (2n-1)T.$$

These 2n equations are linear with respect to the 2n unknowns A, B, C, D. The second terms are real, and the coefficients are real or imaginary and conjugate pairwise.

When we change $\sqrt{-1}$ into $-\sqrt{-1}$:

1. A_k and B_k do not change when α_k is real;

2. A_k and B_k interchange when α_k is purely imaginary;

3. A_k and B_k change into A_j and B_j when α_k is complex and imaginary and conjugate of α_j.

Therefore:

1. A_k and B_k are real when α_k is real;

2. A_k and B_k are imaginary and conjugate when α_k is purely imaginary;

3. A_k and A_j, B_k and B_j are imaginary and conjugate when α_k is complex, and imaginary and conjugate of α_j. /222

Finally, C and D are real.

These conditions are sufficient for x'_i and y'_i to be real.

Let us assign values satisfying these conditions to the constants A_k, B_k, C, D, as well as to the constants A'_k, B'_k, C', D'. Then the second term of (2) must be real, and in order that it may be real the following is necessary:

1. That M_k is real if α_k is real;

2. That M_k is purely imaginary if α_k is purely imaginary;

3. That M_k and M_j are imaginary and conjugate if α_k and α_j are complex, and imaginary and conjugate.

Form (3) contains a term

$$M_k(e^{-\alpha_k T} - e^{\alpha_k T})A_k B_k$$

and does not contain another term depending on A_k or B_k.

If the exponent α_k is real, the presence of a term containing $A_k B_k$ is sufficient for providing that the quadratic form (3) can be defined.

Therefore, if only one of the exponents α_k is real, the function S cannot have either a maximum or a minimum.

Let us now assume that two exponents α_k and α_j are complex, and imaginary and conjugate.

Let us cancel all the constants except for

$$A_k, \quad B_k, \quad A_j, \quad B_j,$$

and the form (3) may be reduced to

$$M_k(e^{-\alpha_k T} - e^{\alpha_k T})A_k B_k + M_j(e^{-\alpha_j T} - e^{\alpha_j T})A_j B_j.$$

These two terms are imaginary and conjugate, so that form (3) is real.

Let us assume that A_k does not change, and that B_k changes sign. A_j, which is imaginary and conjugate of A_k, will change no longer, and B_j, which is imaginary and conjugate of B_k, will change into $-B_j$.

Therefore, form (3) will change sign; therefore, it cannot be defined.

Therefore, if only one of the exponents α_k is complex, the function S cannot have either a maximum or a minimum.

Let us now assume that α_k is purely imaginary. Then A_k and B_k are /223
imaginary and conjugate, and the product $A_k B_k$ is the sum of two squares.

In order that S have a maximum, it is necessary and sufficient that all of the quantities

$$\frac{M_k}{\sqrt{-1}} \sin \frac{\alpha_k T}{\sqrt{-1}}, \quad -NT$$

be negative. In order that S have a minimum, it is necessary and sufficient that all these quantities be positive.

It should be pointed out that all these quantities are real, because $\frac{M_k}{\sqrt{-1}}$ and $\frac{\alpha_k}{\sqrt{-1}}$ are real.

325. How may these results be modified if it is assumed that the energy constant is one of the given quantities of this problem? We then have identically

$$\sum\left(\frac{dF}{dx}x' + \frac{dF}{dy}y'\right) = 0,$$

where we assume that in $\frac{dF}{dx}$ and $\frac{dF}{dy}$, x_i and y_i have been replaced by the periodic functions $\phi_i(t)$ and $\phi'_i(t)$.

The constant value of the function F must be the same for the periodic solution

$$x_i = \varphi_i(t), \qquad y_i = \varphi'_i(t)$$

and for the infinitely close solution

$$x_i = \varphi_i(t) + x'_i, \qquad y_i = \varphi'_i(t) + y'_i.$$

This relationship is a linear equation between the constants

$$A_k, \quad B_k, \quad C, \quad D$$

and the coefficients must be independent of t.

It follows from this that A_k and B_k must not be included in the relationship, since these constants are always multiplied by $e^{\pm\alpha_k t}$ and since this exponential cannot vanish.

In addition, C is no longer included, since the solution

$$x_i = \varphi_i(t) + C\frac{d\varphi_i}{dt}, \qquad y_i = \varphi'_i(t) + C\frac{d\varphi'_i}{dt},$$

where C is a very small constant, may be deduced from the periodic solution by increasing the time by a small amount C. Consequently, this solution corresponds to the same value of the energy constant as does the periodic solution. /224

Our relationship, which cannot be reduced to an identity, may therefore be reduced to

$$D = 0.$$

However, if D is zero, the term $-NTD^2$ vanishes in the form (3).

In order that S may have a maximum or a minimum, it is sufficient that the quantities

$$\frac{M_k}{\sqrt{-1}} \sin \frac{\alpha_k T}{\sqrt{-1}}$$

all have the same sign.

If there are only two degrees of freedom, there is only one of these quantities.

Therefore, if there are only two degrees of freedom and if α_k is purely imaginary, the function S always has a maximum or a minimum.

326. Let us now assume that the conditions given in No. 322 hold, so that

$$dS = \Sigma[(X_i - \xi_i) d(Y_i + \eta_i) - (Y_i - \eta_i - 2 m_i \pi) d(X_i + \xi_i)]$$

and let us assume that T is a constant. In order that S may have a maximum or a minimum, it is necessary that we have a periodic solution

$$x_i = \varphi_i(t), \qquad y_i = \varphi'_i(t)$$

where

$$\varphi_i(t + T) = \varphi_i(t); \qquad \varphi'_i(t + T) = \varphi'_i(t) + 2 m_i \pi.$$

Let us then consider a close solution

$$x_i = \varphi_i(t) + x'_i; \qquad y_i = \varphi'_i(t) + y'_i,$$

and this discussion will proceed in the same way as above. The results are the same.

In order that there be a maximum or a minimum, it is necessary that all the exponents α_k are purely imaginary. It is then necessary /225
that all the quantities

$$\frac{M_k}{\sqrt{-1}} \sin \frac{\alpha_k T}{\sqrt{-1}}, \quad -NT$$

have the same sign.

If it is assumed that the energy constant is a given quantity of the problem at hand, D is zero, the term $-NTD^2$ vanishes, and it is sufficient that the quantities

$$\frac{M_k}{\sqrt{-1}} \sin \frac{\alpha_k T}{\sqrt{-1}}$$

all have the same sign.

327. What will now take place if the equations have other uniform integrals in addition to the energy integral and if, consequently, some of the characteristic exponents are zero?

A discussion similar to that presented above could still be employed.

For example, let us assume that our equations have p other uniform integrals, in addition to the energy integral:

$$F_1, \quad F_2, \quad \ldots, \quad F_p,$$

in such a way that the brackets $[F_i, F_k]$ of these integrals taken two at a time are zero. Based on the statements presented in No. 69, we then know that 2p + 2 characteristic exponents are zero. We shall assume that all the other exponents are different from zero.

We shall then have n - p - 1 pairs of constants which are similar to the constants A_k and B_k, and p + 1 pairs of constants C_k and D_k which are similar to the constants C and D.

Form (3) will then become

$$\Sigma M_k(e^{-\alpha_k T} - e^{\alpha_k T}) A_k B_k - \Sigma N_k T D_k^2,$$

where $\Sigma N_k T D_k^2$ is a sum of terms similar to the term NTD^2.

If we now assume that the values of our p + 1 integrals are given quantities of the question under consideration, the constants D_k will all be zero, the terms $N_k T D_k^2$ will vanish, and the condition under which S may have a maximum or a minimum will still stipulate that all the /226 quantities

$$M_k(e^{-\alpha_k T} - e^{\alpha_k T})$$

have the same sign.

I shall not insist upon this point, because -- in the case of the three-body problem -- either we shall be dealing with the restricted problem presented in No. 9, or we shall be able to decrease the number of degrees of freedom by applying the procedures given in Nos. 15 and 16.

In the case of the reduced problems of Nos. 9, 15 and 16, there is no more than one single uniform integral, that of energy, and there are only two zero exponents, as we saw in No. 78.

328. Let us change T successively into 2T, 3T, ..., mT, ..., . The function S defined above depends on T, and let

$$S_m = S(mT).$$

Let us try to determine the maxima and minima of S_m, assuming that T is a constant.

If we consider a periodic solution of period T, this will also be a periodic solution of period mT. Therefore, the first derivatives of S_m are zero.

In order that there may be a maximum or a minimum, it is necessary that all the exponents α_k are purely imaginary.

If all the quantities

$$\frac{M_k}{\sqrt{-1}} \sin \frac{m\alpha_k T}{\sqrt{-1}} \tag{1}$$

are negative, there will be a maximum; if they are all positive, there will be a minimum.

This is the first point to which I wish to draw attention.

If we assign all the possible whole values to the whole number m, the n - 1 quantities (1) will have in general all the possible combinations of signs.

Let us set, for purposes of brevity,

$$\frac{\alpha_k T}{\sqrt{-1}} = \omega_k,$$

and let /227

$$z_k = m\omega_k + 2m_k\pi.$$

Let us assign all the possible whole values to m and to m_k. If we assume that $z_1, z_2, \ldots, z_{n-1}$ are the coordinates of a point in space having n - 1 dimensions, we shall obtain an infinity of points. It may stated that there will be an infinity of these points in every section of space having n - 1 dimensions, no matter how small it may be.

In order to demonstrate this, I need only refer to the line of reasoning employed to establish the fact that a uniform function of n real variables cannot have n + 1 different periods.

The quantities given in the following table:

$$\begin{array}{cccc}\omega_1, & \omega_2, & \ldots, & \omega_{n-1}, \\ 2\pi, & 0, & \ldots, & 0, \\ 0, & 2\pi, & \ldots, & 0, \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots,} \\ 0, & 0, & \ldots, & 2\pi,\end{array}$$

will play the role of periods in this line of reasoning.

There would be an exception, if these periods were not different -- i.e., if one of the quantities ω were commensurable with 2π, or, more generally, if there were a linear combination of the z's which had only one single period -- i.e., if there were a relationship having the form

$$b_1\omega_1 + b_2\omega_2 + \ldots + b_{n-1}\omega_{n-1} + 2\pi b_n = 0, \tag{2}$$

where the b's are whole numbers.

Let us disregard the case of this exception. The quantities (1) will equal

$$\frac{M_k}{\sqrt{-1}} \sin z_k.$$

We may choose the whole number m in such a way that these quantities represent a combination having a given sign -- i.e., that there are numbers z_k which satisfy inequalities having the form

$$a_1 < z_1 < a_1 + \pi, \quad a_2 < z_2 < a_2 + \pi, \quad \ldots, \quad a_{n-1} < z_{n-1} < a_{n-1} + \pi, \tag{3}$$

where the a_k's equal 0 or π.

This results directly from the statements which we have just made above. /228

Let us move on to the case in which we have a relationship of the form (2). We may always assume that the whole numbers b are primes among themselves. In this case, the expression

$$b_1 z_1 + b_2 z_2 + \ldots + b_{n-1} z_{n-1} \tag{4}$$

has only the period 2π.

In order that there may be no numbers z_k satisfying the inequalities (3), it is necessary and sufficient that the difference between the largest value and the smallest value which expression (4) takes -- when all values which are compatible with the inequalities (3) are assigned to be z_k's -- is smaller than 2π, i.e., smaller than a period of this expression (4).

This difference is obviously as follows

$$\pi(|b_1| + |b_2| + \ldots + |b_{n-1}|),$$

and we must therefore have

$$|b_1| + |b_2| + \ldots + |b_{n-1}| \leq 2. \tag{5}$$

The inequality can only hold if all of the b's are zero, except for one of them which must equal ± 1.

In this case ω_k must equal a multiple of 2π. This means that α_k must be zero, since α_k is only determined up to a multiple of $\frac{2\pi\sqrt{-1}}{T}$.

We have excluded the case in which one of the α_k's is zero.

The equation can only be valid if all the b's are zero, except for two of them which must equal ± 1.

Then the sum of the difference between two of the ω_k's will be a multiple of 2π. If we note that the α_k's are only determined up to a multiple of $\frac{2\pi\sqrt{-1}}{T}$, we may express this result in another way.

Two of the characteristic exponents will be equal.

This is the only exception which still exists, and it may be readily excluded.

329. Let us now assume that the equations of dynamics under consideration depend on an arbitrary parameter μ, just as is the case for the three-body problem, as we know. /229

When we vary μ continuously, the periodic solution

$$x_i = \varphi_i(t), \quad y_i = \varphi'_i(t)$$

will also vary continuously, as we may determine from the discussion in Chapter III.

The quantities M_k will also vary continuously, but -- as was explained in No. 323 -- they can never vanish. Therefore, they will always retain the same sign, and it is their sign alone in which we are interested.

The energy constant will be regarded as one of the given quantities of the problem at hand, but this given quantity may depend on μ, and we shall choose it in such a way that the period T of the periodic solution remains constant.

The exponents α_k will also vary continuously when we vary μ continuously. Let us clarify to a certain extent the manner in which this variation should be handled in the case of the three-body problem. For $\mu = 0$, all the exponents are zero. However, as soon as μ ceases to be zero, the exponents cease to be zero also. One of these exponents can only vanish, or become equal to a multiple of $\frac{2\pi\sqrt{-1}}{T}$, or become equal to another characteristic exponent for certain special values of μ.

330. Let us consider a periodic solution of period T, such that all the exponents α_k are purely imaginary. This is what we designated above by a stable solution. In Chapters III and IV, we proved the existence of these solutions.

Let us consider one of the exponents, α_1, for example. When μ varies continuously, $\frac{\alpha_1}{\sqrt{-1}}$, -- which is real -- will become commensurable with $\frac{2\pi}{T}$ an infinity of times. Let us assign a value μ_0 to μ, such that

$$\frac{\alpha_1}{\sqrt{-1}} = \frac{2k\pi}{pT},$$

where k and p are the prime whole numbers among themselves. In addition, this value does not correspond to a maximum or a minimum of $\frac{\alpha_1}{\sqrt{-1}}$. /230

At a later point, in No. 334, we shall see why I have placed $2k\pi$ in the numerator, and not $k\pi$.

In any interval, no matter how small it may be, there is an infinite number of similar values.

If m is an arbitrary whole number, for this value μ_0 the expression

$$\frac{M_1}{\sqrt{-1}} \sin \frac{pm\alpha_1 T}{\sqrt{-1}}$$

is zero. In addition, since μ_0 does not correspond to a maximum or a minimum of $\frac{\alpha_1}{\sqrt{-1}}$, this expression will change sign when μ passes from $\mu_0 - \varepsilon$ to $\mu_0 + \varepsilon$.

For example, let us assume that it changes from being negative to being positive.

Pursuing the line of reasoning presented in No. 328, we will find that we may choose the whole number m in such a way that the expressions

$$\frac{M_k}{\sqrt{-1}} \sin \frac{p m \alpha_k T}{\sqrt{-1}} \qquad (k = 2, 3, \ldots, n-1)$$

have all possible combinations of signs, and that they are all negative.

Under this assumption, for $\mu = \mu_0 - \varepsilon$, our function $S_{m.p}$ will have a maximum, since all our expressions will be negative. However, for $\mu = \mu_0 + \varepsilon$, our periodic solution will no longer correspond to a maximum of $S_{m.p}$, since one of these expressions will have become positive.

Theorems Considering the Maxima

331. In order to pursue this subject further, it is necessary to illustrate one property of the maxima. Let V be a function of the three variables x_1, x_2 and z, which may be developed in increasing powers of these three variables. I shall assume the following:

1. For $x_1 = x_2 = 0$, V vanishes as well as its derivatives $\frac{dV}{dx_1}$, $\frac{dV}{dx_2}$, no matter what z may be;

2. For $x_1 = x_2 = 0$, V has a maximum for $z > 0$ and a minimum for /231
$z < 0$.

It may be stated that the equations

$$\frac{dV}{dx_1} = \frac{dV}{dx_2} = 0$$

have other real solutions in addition to the solution

$$x_1 = x_2 = 0.$$

Let us develop V in powers of z, and let

$$V = V_0 + z V_1 + z^2 V_2 + \ldots.$$

The functions V_0, V_1, V_2, ... may themselves be developed in powers of x_1 and of x_2. However, these expansions will contain neither terms of degree 0 nor terms of degree 1, because -- no matter what z may be -- we must have

$$V = \frac{dV}{dx_1} = \frac{dV}{dx_2} = 0$$

for $x_1 = x_2 = 0$.

In addition, V_0 does not contain terms of the second degree either.

Without the second degree terms, it is impossible to pass from the case of the maximum to the case of the minimum, when going from $z > 0$ to $z < 0$.

Conversely, V_1 will contain first degree terms, at least we shall assume this is the case. Let us then consider the equations

$$\begin{cases} 0 = \frac{dV_0}{dx_1} + z\frac{dV_1}{dx_1} + z^2\frac{dV_2}{dx_1} + \dots \\ 0 = \frac{dV_0}{dx_2} + z\frac{dV_1}{dx_2} + z^2\frac{dV_2}{dx_2} + \dots \end{cases} \quad (1)$$

which must be solved.

Let U_0 and U_1 be the lowest degree terms of V_0 and of V_1. According to the statements which we have discussed, U_1 is of the second degree, and U_0 is of the degree p -- with p being larger than 2. Let us set

$$(p-2)\mu = 1; \qquad x_1 = y_1 t, \qquad x_2 = y_2 t, \qquad V = W t^p; \qquad z = \pm t^{p-2}$$

W may be developed in powers of t. Let us set

$$W = W_0 + tW_1 + t^2 W_2 + \dots.$$

We obviously have /232

$$W_0 = \pm U_1 t^{-p} + U_0 t^{-p} = \pm U'_1 + U'_0,$$

$U'_1 = U_1 t^{-p}$ and $U'_0 = U_0 t^{-p}$ are two homogeneous polynomials in y_1 and y_2 -- one of degree 2 and the other of degree p. I shall employ the sign + or -, depending on how I have set $z = \pm t^{p-2}$. The expression

$$\frac{dV}{dx_1}\frac{dU_1}{dx_2} - \frac{dV}{dx_2}\frac{dU_1}{dx_1}$$

will also be developed in powers of t when x_1 and x_2 are replaced by $y_1 t$ and $y_2 t$. It will include a certain power of t as a factor. Let us divide by this factor, and let H be the quotient. This quotient developed in powers of t may be written

$$H = H_0 + tH_1 + t^2 H_2 + \dots;$$

H_0 will be the first of the expressions

$$\frac{dW_k}{dy_1}\frac{dU'_1}{dy_2} - \frac{dW_k}{dy_2}\frac{dU'_1}{dy_1}$$

which will not vanish.

The equations $$\frac{dV}{dx_1} = \frac{dV}{dx_2} = 0$$

may be replaced by the following equations

$$H = 0 \qquad \frac{dW}{dy_1} = 0.$$

I shall prove that we may derive the y's from these equations in the form of series which are ordered in fractional and whole powers of t, which vanish with t and which have real coefficients.

In order to do this, according to statements presented in Nos. 32 and 33, it is sufficient to establish the fact that for t = 0, these equations have a real solution of odd order.

For t = 0, these equations may be reduced to

$$H_0 = 0, \qquad \frac{dW_0}{dy_1} = 0,$$

or /233

$$\frac{dW_k}{dy_1}\frac{dU'_1}{dy_2} - \frac{dW_k}{dy_2}\frac{dU'_1}{dy_1} = 0 \tag{2}$$

and

$$\pm\frac{dU'_1}{dy_1} + \frac{dU'_0}{dy_1} = 0. \tag{3}$$

Equation (2) indicates that W_k has a maximum or a minimum, if we assume that y_1 and y_2 are related by the relationship U'_1 = const.

For the present, if we assume that y_1 and y_2 are the coordinates of a point in a plane, the relationship U'_1 = const will represent an ellipse, because the quadratic form U_1 (and, consequently, the form U'_1) must be defined in order that V may have a maximum or a minimum. Due to the fact that an ellipse is a closed curve, the function W_2 must have at least a maximum and a minimum when the point y_1, y_2 describes this closed curve.

Therefore, whatever the constant value may be which is assigned to U'_1, equation (2) will have at least two roots, and two roots of odd order, because we have seen in No. 34 that a maximum or a minimum always corresponds to a root of odd order. At this point, where we have no more than one independent variable, the theorem presented in No. 34 is almost self-evident. Under this assumption, we may distinguish between two cases:

First case. U'_0 is not a power of U'_1. In this case, we do not have identically

$$\frac{dW_0}{dy_1}\frac{dU'_1}{dy_2} - \frac{dW_0}{dy_2}\frac{dU'_1}{dy_1} = 0.$$

We shall therefore have $W_k = W_0$, and

$$H_0 = \frac{dU'_0}{dy_1}\frac{dU'_1}{dy_2} - \frac{dU'_0}{dy_2}\frac{dU'_1}{dy_1} = 0.$$

Equation $H_0 = 0$ is then homogeneous in y_1 and y_1. No matter what the constant value is which is assigned to U'_1, it will provide us with the same values for the ratio $\frac{y_1}{y_2}$.

We may derive $\frac{y_1}{y_2}$ from equation (2) and, according to the preceding statements, we shall obtain at least two solutions of odd order. /234

Let $\frac{y_1}{y_2} = \frac{\alpha_1}{\alpha_2}$ be one of these solutions. Let us set $y_1 = \alpha_1 u$, $y_2 = \alpha_2 u$ and let us substitute in equation (3). We shall have

$$U'_0 = A u^p, \qquad U'_1 = B u^2$$

and equation (3) may be reduced to

$$A u^{p-2} \pm B = 0.$$

If p - 2 is odd, this equation will give us a real value for u.

If p - 2 is even, we may distinguish between two cases.

If A and B have the same sign, we shall take the lower sign

$$A u^{p-2} - B = 0.$$

If A and B have opposite signs, we shall take the upper sign

$$A u^{p-2} + B = 0,$$

and we shall have two real values for u.

In every case, these real solutions are simple.

Thus, equations (2) and (3) will always have solutions of odd order.

<u>Second case</u>. We have

$$U'_0 = A(U'_1)^{\frac{p}{2}}.$$

We shall begin by solving equation (3), which may be written as follows

$$\frac{p}{2} A (U'_1)^{\frac{p}{2}-1} \pm 1 = 0.$$

This equation provides us with the value of U_1'. This value is real and simple, but this is not sufficient because U_1' is a negative definite form. In order that the solution may be suitable, it is necessary that the value found for U_1' be negative; as a consequence, we shall choose the sign $\pm$.

The value of U_1' having thus been determined, we may assign this constant value to U_1', and in order to solve equation (3) we need only determine the maxima and minima of W_k. As we have seen, we shall derive at least two solutions of odd order. /235

We have therefore established the fact that equations (2) and (3) always have real solutions of odd order. The theorem presented at the beginning of this section has thus been proven.

332. Now let V be a function of n + 1 variables

$$x_1, \quad x_2, \quad \ldots, \quad x_n \quad \text{and} \quad z.$$

I shall assume the following:

1. V may be developed in powers of x and of z;

2. For

$$x_1 = x_2 = \ldots = x_n = 0,$$

we have the following, no matter what z may be

$$V = \frac{dV}{dx_1} = \frac{dV}{dx_2} = \ldots = \frac{dV}{dx_n} = 0.$$

3. Let us consider the group of terms of V which are second degree terms with respect to the x's. They represent a quadratic form which may be equated to the sum of n squares having positive or negative coefficients.

When z changes from positive to negative, I shall assume that two of these n coefficients change from positive to negative, and that the n - 2 other coefficients do not vanish.

Under these conditions, it may be stated that the equations

$$\frac{dV}{dx_1} = \frac{dV}{dx_2} = \ldots = \frac{dV}{dx_n} = 0 \tag{1}$$

have real solutions which differ from

$$x_1 = x_2 = \ldots = x_n = 0.$$

Let us develop V in powers of z and let us set

$$V = V_0 + V_1 z + V_2 z^2 + \ldots$$

Let U_0 and U_1 be the group of second degree terms of V_0 and V_1.

The group U_1 is a quadratic form which may be decomposed into a sum of n - 2 squares, because we know that, for z = 0, two of the coefficients which were in question above vanish. /236

Therefore, if we consider the discriminant of U_0, i.e., the functional determinant of

$$\frac{dU_0}{dx_1}, \quad \frac{dU_0}{dx_2}, \quad \ldots, \quad \frac{dU_0}{dx_n}$$

with respect to

$$x_1, \quad x_2, \quad \ldots, \quad x_n,$$

this determinant vanishes, as well as all of its minors of the first order. However, all of the second-order minors do not vanish, unless a third coefficient is zero, which we have not assumed.

We may also assume that a linear change in the variables has been performed, so that U_0 is restored to the form

$$U_0 = A_3 x_3^2 + A_4 x_4^2 + \ldots + A_n x_n^2$$

Consequently, the functional determinant of

$$\frac{dU_0}{dx_3}, \quad \frac{dU_0}{dx_4}, \quad \ldots, \quad \frac{dU_0}{dx_n}$$

with respect to

$$x_3, \quad x_4, \quad \ldots, \quad x_n$$

is not zero.

Let us then consider the equations

$$\frac{dV}{dx_3} = \frac{dV}{dx_4} = \ldots = \frac{dV}{dx_n} = 0 \tag{2}$$

which are n - 2 of equations (1). We may derive

$$x_3, \quad x_4, \quad \ldots, \quad x_n$$

in the form of series which are ordered according to powers of

$$z, \quad x_1, \quad x_2.$$

For this purpose, in view of the statements presented in No. 30, it is sufficient that the functional determinant of equations (2) with respect to

$$x_3, \quad x_4, \quad \ldots, \quad x_n$$

does not vanish when we set

$$z = x_1 = x_2 = x_3 = x_4 = \ldots = x_n = 0.$$

When we set $z = 0$ and when we limit ourselves to first-degree terms with respect to the x's, equations (2) may be reduced to /237

$$\frac{dU_0}{dx_3} = \frac{dU_0}{dx_4} = \ldots = \frac{dU_0}{dx_n} = 0$$

and we have just seen that the corresponding functional determinant is not zero.

Let us replace x_3, x_4, ..., x_n in V by their values derived from equations (2). We shall then be dealing with the conditions stipulated in the preceding section:

1. We have no more than three independent variables z, x_1 and x_2.

2. The function V may be developed in powers of these variables;

3. Equations (1) may be replaced by

$$\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2} = 0, \tag{3}$$

where the ∂'s represent the derivatives taken with respect to the x_2, x_4, ..., x_n's as functions of x_1 and of x_2 defined by equations (2).

In effect, we have

$$\frac{\partial V}{\partial x_1} = \frac{dV}{dx_1} + \frac{dV}{dx_3}\frac{dx_3}{dx_1} + \frac{dV}{dx_4}\frac{dx_4}{dx_1} + \ldots + \frac{dV}{dx_n}\frac{dx_n}{dx_1},$$

and, in view of equations (2), it follows from this that

$$\frac{\partial V}{\partial x_1} = \frac{dV}{dx_1},$$
$$\frac{\partial V}{\partial x_2} = \frac{dV}{dx_2}$$

4. For $z > 0$, V -- regarded as a function of x_1 and of x_2, -- has a maximum when these two variables are zero.

In order to illustrate this, we must try to find the second-degree terms with respect to x_1 and x_2 in V. Let

$$W_0 + zW_1 + z^2 W_2 + \dots$$

be these terms. In order to obtain

$$W_0 + zW_1$$

which are the only ones which interest me, I shall take the two terms /238

$$U_0 + zU_1,$$

and I shall neglect the other terms of V which cannot influence $W_0 + zW_1$.

I may derive the following from equations (2)

$$x_3, \quad x_4, \quad \dots, \quad x_n$$

in the form of series ordered in powers of x_1 and x_2. In these series, I shall only retain the terms which are of degree 1 with respect to x_1 and x_2, and of degree 0 with respect to z. The other terms may be neglected, because they do not influence

$$W_0 + zW_1.$$

Equations (2) may then be reduced to

$$2A_3 x_3 + z\frac{dU_1}{dx_3} = 0,$$

$$2A_4 x_4 + z\frac{dU_1}{dx_4} = 0,$$

$$\dots\dots\dots\dots\dots\dots$$

$$2A_n x_n + z\frac{dU_1}{dx_n} = 0.$$

If we substitute the values thus obtained in place of x_3, x_4, ..., x_n, in U_0, we shall find that U_0 is divisible by z^2. With respect to U_1, it may be reduced to

$$U_1^0 + zU_1^1 + z^2 U_1^2,$$

where U_1^0 is none other than the quantity which U_1 becomes when we cancel x_3, x_4, ..., x_n, and where U_1^1 and U_1^2 are two other quadratic forms with respect to the x's. We shall therefore have

$$U_0 = z^2 U_0^2; \qquad U_1 = U_1^0 + zU_1^1 + z^2 U_1^2$$

and

$$U_0 + zU_1 = zU_1^0 + z^2(U_0^2 + U_1^1) + z^3 U_1^2.$$

In order to calculate $W_0 + zW_1$, I may neglect the last two terms which may be divided by z^2 and z^3, and I shall simply have

$$W_0 + zW_1 = zU_1^0.$$

I shall demonstrate the fact that V has a maximum for $x_1 = x_2 = 0$ /239 and for z which is positive and which is very small. It is sufficient to illustrate this for $W_0 + zW_1$, i.e., for zU_1^0.

Finally, we must prove that U_1^0 is a negative definite form.

For this purpose, we shall write the quadratic form U_1 as follows

$$U_1 = U_1' + U_1'';$$

U_1' is a sum of two squares having coefficients whose sign I shall not predict. U_1'' depends only on the n - 2 variables

$$x_3, \quad x_4, \quad \ldots, \quad x_n.$$

This is always possible, according to the general properties of quadratic forms.

Let us consider the form

$$U_0 + zU_1 = zU_1' + (U_0 + zU_1'').$$

where z is assumed to be positive and very small. The form $U_0 + zU_1''$, which depends only on the n - 2 variables x_3, x_4, ..., x_n, may be equated to a sum of n - 2 squares having coefficients whose signs must be the same as those for A_3, A_4, ..., A_n, since -- due to the fact that z is very small -- this form differs very little from U_0. Therefore, they do not change sign when z makes a transition from positive to negative.

According to our hypotheses, when z makes the transition from positive to negative, n - 2 of our coefficients do not vanish, and, on the contrary, two coefficients make the transition from negative to positive.

These last two coefficients can only be the coefficients of U_1.

Therefore, U_1' is the sum of two squares having negative coefficients.

In order to have U_1^0, it is necessary to set the following in U_1'

$$x_3 = x_4 = \ldots = x_n = 0.$$

Then U_1'' vanishes, and U_1 may be reduced to U_1'.

Therefore, U_1^0 is a negative definite form.

q.e.d.

Therefore, V, regarded as a function of x_1 and x_2, is maximum for z which is positive and is very small, and for $x_1 = x_2 = 0$. /240

One will find in the same way -- or rather one will find at the same time -- that V is minimum for z which is negative and very small, and for $x_1 = x_2 = 0$.

As I have stated, we have thus returned to the conditions stipulated in the preceding section, and it may be assumed that the theorem presented at the beginning of this section has been substantiated.

Existence of Solutions of the Second Type

333. Let us return to the hypotheses given in No. 330. We have defined the function S_{mp}, which depends on μ, of the 2n variables

$$\begin{cases} X_1+\xi_1, & \ldots, \quad X_n+\xi_n, \\ Y_1+\eta_1, & Y_2+\eta_2, \quad \ldots, \quad Y_n+\eta_n. \end{cases} \tag{α}$$

The ξ_i's and the η_i's are the values of x_i and y_i for $t = 0$. The X_i's and the Y_i's are the values of x_i and y_i for $t = mpT$.

We sould like to study the solutions of the equations

$$\frac{dS_{mp}}{d(X_i+\xi_i)} = \frac{dS_{mp}}{d(Y_i+\eta_i)} = 0. \tag{1}$$

According to Nos. 321 and 322, these solutions correspond to periodic solutions of period mpT. We already know one of them, since a periodic solution of period T is at the same time periodic having the period mpT. I propose to show that there are others in addition.

First, however, I would like to illustrate the method which may be employed to regard S_{mp} as being dependent only on μ and on the 2n - 1 variables

$$\begin{cases} X_1+\xi_1, & X_2+\xi_2, \quad \ldots, \quad X_{n-1}+\xi_{n-1}, \\ Y_1+\eta_1, & Y_2+\eta_2, \quad \ldots, \quad Y_{n-1}+\eta_{n-1}, \quad Y_n+\eta_n. \end{cases} \tag{β}$$

For this purpose, we shall assume that

$$X_n+\xi_n = 0.$$

Let us now consider the equations

$$\frac{\partial S_{mp}}{\partial(X_i+\xi_i)} = \frac{\partial S_{mp}}{\partial(Y_i+\eta_i)} = 0. \tag{1'}$$

We shall employ the d's to represent the derivatives of S which is assumed to be a function of the variables (α), and shall employ the ∂'s

to represent the derivatives of this same function S which is assumed /241
to be a function of the variables (β).

I plan to show that equations (1) and (1') are equal.

Section No. 322 has provided us with the following

$$dS = \Sigma[(X_i - \xi_i)\,d(Y_i + \eta_i) - (Y_i - \eta_i - 2m_i\pi)\,d(X_i + \eta_i)].$$

Equations (1) may therefore be written

$$-(Y_i - \eta_i - 2m_i\pi) = X_i - \xi_i = 0$$
$$(i = 1, 2, \ldots, n),$$

and equations (1') may be written as follows

$$-(Y_i - \eta_i - 2m_i\pi) = X_i - \xi_i = 0$$
$$(i = 1, 2, \ldots, n-1),$$
$$X_n - \xi_n = 0.$$

In view of the energy equation, we have also

$$F(X_i, Y_i) = F(\xi_i, \eta_i + 2m_i\pi).$$

According to equations (1'), all of the X_i's equal the ξ_i's, and all of the Y_i's (except one) equal $\eta_i + 2m_i\pi$. The preceding identity may therefore be written as follows. For purposes of abbreviation, I shall write

$$F(\xi_1, \xi_2, \ldots, \xi_n;\ \eta_1 + 2m_1\pi,\ \eta_2 + 2m_2\pi,\ \ldots,\ \eta_{n-1} + 2m_{n-1}\pi,\ Y_n) = F(Y_n).$$

My identity may therefore be written in the following form

$$F[\eta_n + 2m_n\pi + (Y_n - \eta_n - 2m_n\pi)] - F(\eta_n + 2m_n\pi) = 0,$$

or, in view of the theorem of finite increases

$$(Y_n - \eta_n - 2m_n\pi)F'[\eta_n + 2m_n\pi + \theta(Y_n - \eta_n - 2m_n\pi)] = 0, \tag{2}$$

where θ is included between 0 and 1, and where F' is the derivative of F with respect to Y_n.

Let ξ_i^0 and η_i^0 be the values of ξ_i and η_i which correspond to the periodic solution of period T. The region under consideration only includes the immediate vicinity of the point $\mu = \mu_0$, $\xi_i = \xi_i^0$, $\eta_i = \eta_i^0$. Therefore, ξ_i and X_i will never deviate greatly from ξ_i^0, and η_i or $Y_i - 2m_i\pi$ will never deviate greatly from η_i^0. Therefore, the second factor F' of relationship (2) will never deviate greatly from its value for $\xi_i = \xi_i^0$, $\eta_i = \eta_i^0$, and in general this value will not be zero.

Therefore, the first factor of relationship (2) must vanish, and /242
we have

$$Y_n - \eta_n - 2m_n\pi = 0.$$

In other words, equations (1') entail equations (1). We may therefore regard S_{mp} as a function of the variables (β). When it is a maximum, considered as a function of the variables (β), it will also be a maximum as a function of the variables (α).

I have employed ξ_i^0 and η_i^0 to designate the values of ξ_i and of η_i which correspond to the periodic solution of period T. The corresponding values of $X_i + \xi_i$ and $Y_i + \eta_i$ will be $2\xi_i^0$ and $2\eta_i^0 + 2m_i mp\pi$ (if the periodic solution of period T changes y_i into $y_i + 2m_i\pi$, in conformance with the hypotheses formulated in No. 322). Let S_0 be the corresponding value of S_{mp}. Let us set

$$\mu = \mu_0 + \mu'; \qquad V = S_{mp} - S_0; \qquad X_i + \xi_i = 2\xi_i^0 + \xi_i',$$

$$Y_i + \eta_i = 2\eta_i^0 + 2m_i mpT + \eta_i'$$

and let us consider V as a function of μ', of the ξ''s, and of the η''s. The function V will be governed by the same conditions as the function V of the preceding section.

No matter what μ' may be, V and its first derivatives with respect to the ξ''s and to the η''s will vanish when

$$\xi_i' = \eta_i' = 0.$$

If we consider the group of second degree terms of V with respect to the ξ''s and the η''s, and if we regard it as one quadratic form which is decomposed into a sum of square terms, it may be seen that two of these coefficients of these square terms both make a transition from negative to positive, or both make a transition from positive to negative, when μ changes sign. The other coefficients do not vanish.

The expression

$$\frac{M_1}{\sqrt{-1}} \sin \frac{pm\alpha_1 T}{\sqrt{-1}}$$

changes sign, and the other expressions

$$\frac{M_k}{\sqrt{-1}} \sin \frac{pm\alpha_k T}{\sqrt{-1}}$$

do not vanish. The coefficient which I have designated as D in No. 323 no longer vanishes, and there is not another one because we have only /243 $2n - 1$ variables, the variables (β).

The conditions presented in the preceding section therefore hold, and we may state that the equations

$$\frac{dV}{d\xi'_i} = \frac{dV}{d\eta'_i} = 0$$

have other real solutions in addition to $\xi'_i = \eta'_i = 0$ or, which means the same thing, equations

$$\frac{dS_{mp}}{d(X_i + \xi_i)} = \frac{dS_{mp}}{d(Y_i + \eta_i)} = 0 \quad (1)$$

have other real solutions other than those corresponding to the periodic solution of period T.

The maxima of the function S_{mp}, or more generally the solutions of equations (1), correspond to periodic solutions of period mpT.

We must therefore conclude that our differential equations have periodic solutions of period mpT, which differ from the solution of period T, which is identical to that for $\mu = \mu_0$, and which differ only slightly for μ close to μ_0.

If attention is drawn to the preceding line of reasoning, we shall find that the periodic solution of period T need not correspond to a maximum of S_{mp}.

We shall therefore set $m = 1$.

It is not necessary that the solution of period T be stable. It is sufficient that one of the characteristic exponents α_1 equals

$$\frac{2k\pi\sqrt{-1}}{pT}$$

for $\mu = \mu_0$.

We therefore obtain the following result.

If the equations of dynamics have a periodic solution of period T, such that one of the characteristic exponents is close to

$$\frac{2k\pi\sqrt{-1}}{pT},$$

they will also have periodic solutions of period pT which differ very /244 little from the solution of period T, and which are identical to the latter when the characteristic exponent equals

$$\frac{2k\pi\sqrt{-1}}{pT}.$$

These are solutions of the second type.

Remarks

334. This entire line of reasoning assumes that S_{mp} is a uniform function of $X_i + \xi_i$, $Y_i + \eta_i$. Under this condition alone may it be stated that all the maxima of S_{mp} correspond to a periodic solution (see No. 321). This fact cannot be stressed enough. It is an obstacle which will be encountered frequently when we wish to derive the results of the theorem presented in No. 321.

Let us determine whether S_{mp} is a uniform function of these variables. We may assume that m = 1, which we have just illustrated. In addition, S_p is clearly a uniform function of the ξ_i's and the η_i's. It will also be a uniform function of the $X_i + \xi_i$'s and the $Y_i + \eta_i$'s, provided that the functional determinant of the $X_i + \xi_i$'s and the $Y_i + \eta_i$'s with respect to the ξ_i's and the η_i's does not vanish in the region under consideration. Due to the fact that this region may be reduced to the immediate vicinity of the values

$$\mu = \mu_0, \qquad \xi_i = \xi_i^0, \qquad \eta_i = \eta_i^0,$$

it will be sufficient that the functional determinant is not zero at this point. This functional determinant may be written as follows (assuming that n = 2, to formulate our ideas more clearly)

$$\begin{vmatrix} \frac{dX_1}{d\xi_1}+1 & \frac{dX_1}{d\eta_1} & \frac{dX_1}{d\xi_2} & \frac{dX_1}{d\eta_2} \\ \frac{dY_1}{d\xi_1} & \frac{dY_1}{d\eta_1}+1 & \frac{dY_1}{d\xi_2} & \frac{dY_1}{d\eta_2} \\ \frac{dX_2}{d\xi_1} & \frac{dX_2}{d\eta_1} & \frac{dX_2}{d\xi_2}+1 & \frac{dX_2}{d\eta_2} \\ \frac{dY_2}{d\xi_1} & \frac{dY_2}{d\eta_1} & \frac{dY_2}{d\xi_2} & \frac{dY_2}{d\eta_2}+1 \end{vmatrix}$$

It must therefore be verified that the equation in S /245

$$\begin{vmatrix} \frac{dX_1}{d\xi_1} - S & \frac{dX_1}{d\eta_1} & \frac{dX_1}{d\xi_2} & \frac{dX_1}{d\eta_2} \\ \frac{dY_1}{d\xi_1} & \frac{dY_1}{d\eta_1} - S & \frac{dY_1}{d\xi_2} & \frac{dY_1}{d\eta_2} \\ \frac{dX_2}{d\xi_1} & \frac{dX_2}{d\eta_1} & \frac{dX_2}{d\xi_2} - S & \frac{dX_2}{d\eta_2} \\ \frac{dY_2}{d\xi_1} & \frac{dY_2}{d\eta_1} & \frac{dY_2}{d\xi_2} & \frac{dY_2}{d\eta_2} - S \end{vmatrix} = 0$$

does not have a root which is equal to -1.

According to the statements presented in No. 60, the roots of this equation equal

$$e^{\alpha p T},$$

where the α's are characteristic exponents. We must therefore verify the fact that we do not have

$$\alpha = \frac{(2k+1)\pi\sqrt{-1}}{pT},$$

where k is an integer number. By hypothesis, the exponent α_1 equals

$$\frac{2k\pi\sqrt{-1}}{pT},$$

where k is an integer number, and the other exponents are not commensurable with $\frac{\pi\sqrt{-1}}{T}$. , in general.

The difficulty with which we are concerned will not therefore occur.

In order to avoid this, in No. 330, I assumed that

$$\alpha_1 = \frac{2k\pi\sqrt{-1}}{pT} \qquad \text{(k integer number)}$$

and not

$$\alpha_1 = \frac{k\pi\sqrt{-1}}{pT} \qquad \text{(k integer number)}$$

Special Cases

335. Let us say a few words about the simplest cases, and let us assume only two degrees of freedom.

Let us assume that the form which is similar to that which I have designated as U_0, in the analysis of No. 331, is homogeneous of the third degree only in x_1 and x_2. /246

The equation

$$\frac{dU_0}{dx_1}\frac{dU_1}{dx_2} - \frac{dU_0}{dx_2}\frac{dU_1}{dx_1} = 0 \quad (1)$$

always has real roots, as we have seen.

The theorem is self-evident here, since this equation is of the third degree in $\frac{x_1}{x_2}$. It may have one or three real roots. Let us first assume that it has only one in order to clarify our ideas.

If we then set

$$x_1 = a_1\rho\cos\varphi + b_1\rho\sin\varphi$$
$$x_2 = a_2\rho\cos\varphi + b_2\rho\sin\varphi,$$

choosing the coefficients a and b in such a way that U_1 is reduced to $-p^2$, the ratio

$$\frac{U_0}{U_1^{\frac{3}{2}}}$$ considered in No. 331

will only have a maximum and a minimum when ϕ varies from 0 to 2π. This maximum and this minimum, which are equal and which have opposite sign, will correspond to values of ϕ which are far removed from π.

We will then have

$$U_0 + zU_1 = \rho^3 f(\varphi) - z\rho^2.$$

The function $f(\phi)$ has a maximum and a minimum which are equal and which have opposite sign. The function $U_0 + zU_1$ then has:

For $z > 0$, a maximum for $\rho = 0$ and two minima.

For $z < 0$, a minimum for $\rho = 0$ and two maxima.

Employing the English term, I shall use the word minima to designate a point for which the first derivatives vanish, and where there is neither a maximum or a minimum.

The same will hold true for the function V, since -- if z is very small -- the terms $U_0 + zU_1$ alone will have an influence.

Therefore, no matter what z may be, the differential equations will have : /247

A solution of period T, of the first type, which is stable;

A solution of period pT, of the second type, which is stable for $z < 0$ and unstable for $z > 0$.

Let us now assume that equation (1) has three real roots.

The function $f(\phi)$ will have three maxima and three minima which are equal pairwise and have opposite signs.

In this case $U_0 + zU_1$, and consequently, V have:

For $z > 0$, a maximum for $\rho = 0$, and six minima;

For $z < 0$, a minimum for $\rho = 0$, six maxima.

No matter what z may be, the differential equations will therefore have:

A solution of period T, of the first type, which is stable;

Three solutions of period pT, of the second type. We shall see below that, from a certain point of view, none of these solutions are different.

Let us proceed to a case which is a little more complicated, and let us assume that U_0 is of the fourth degree.

In this case, equation (1) is of the fourth degree, and, since it always has at least two real roots according to No. 331, it will have two or four. We then no longer have

$$f(\varphi) = -f(\varphi + \pi)$$

but rather

$$f(\varphi) = f(\varphi + \pi).$$

Let us first assume that there are only two real roots.

The function $f(\phi)$ will then have a maximum and a minimum when ϕ varies from 0 to π, as well as when ϕ varies from π to 2π.

A distinction may be drawn between two cases, depending on the signs of this maximum and this minimum.

First case. The maximum and the minimum are positive.

The functions $U_0 + zU_1$ and V have:

For $z > 0$, a maximum for $\rho = 0$, two minima and two maxima.

For $z < 0$, a minimum for $\rho = 0$.

In addition to the solution of the first type which always exists, the differential equations have two solutions of the second type for $z > 0$, and do not have any for $z < 0$. Of these two solutions, one is stable and one is unstable. /248

Second case. The maximum is positive, and the minimum is negative.

The constants $U_0 + zU_1$ and V have:

For $z > 0$, a maximum for $\rho = 0$, two minima;

For $z < 0$, a minimuma for $\rho = 0$, two minima.

The differential equations always have an unstable solution of the second type, in addition to the solution of the first type which is stable.

Third case. The maximum itself is negative.

The differential equations then have:

For $z > 0$, a solution of the first type which is stable;

For $z < 0$, a solution of the first type which is stable, and two solutions of the second type of which one is stable and one is unstable.

We must now examine the case in which equation (1) has four real roots.

The equations then have:

For $z > 0$, a solution of the first type which is stable, h solutions of the second type which are unstable, and k solutions of the second type which are stable;

For $z < 0$, a solution of the first type which is stable, 2 - h solutions of the second type which are stable, and 2 - k solutions of the second type which are unstable.

The integer numbers h and k may take the following values, depending upon the signs of the maxima and the minima of $f(\phi)$:

$$h = k = 2; \quad h = 2, \ k = 1; \quad h = 2, \ k = 0; \quad h = 1, \ k = 0;$$
$$h = k = 0; \quad h = k = 1.$$

CHAPTER XXIX

DIFFERENT FORMS OF THE PRINCIPLE OF LEAST ACTION

336. Let

$$x_1, \quad x_2, \quad \ldots, \quad x_n$$
$$y_1, \quad y_2, \quad \ldots, \quad y_n$$

be a double series of variables, and let F be an arbitrary function of /249
these variables. Let us consider the integral

$$J = \int_{t_0}^{t_1} \left(-F + \Sigma y_i \frac{dx_i}{dt} \right) dt.$$

The variation of this integral may be written as follows.

$$\delta J = \int \left(-\delta F + \Sigma \delta y_i \frac{dx_i}{dt} - \Sigma y_i \frac{d\delta x_i}{dt} \right) dt.$$

In order that this variation may vanish, it is necessary that we have

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \qquad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \tag{1}$$

which provides us with the canonical equations, but this condition is not sufficient. If it is fulfilled, we have

$$\delta J = \Sigma [y_i \delta x_i]_{t=t_0}^{t=t_1}$$

and it is still necessary that the second term of this equation be zero. This is what occurs if we assume that the δx_i's are zero at the two limits -- i.e., that the initial and final values of the x_i's are given. Under these conditions, the integral J which I have designated as the action is minimum.

Let us perform the change in variables. Let x'_i, y'_i be the new variables, and let us assume that they have been chosen in such a way that

$$\Sigma y'_i \, dx'_i - \Sigma y_i \, dx_i = dS \tag{2}$$

is an exact differential. In this case, we have seen that the change /250
in variables does not change the canonical form of the equations, and this result is an immediate consequence of different propositions which will be presented below. Let

$$J' = \int_{t_0}^{t_1} \left(-F + \Sigma y'_i \frac{dx'_i}{dt} \right) dt.$$

We have

$$J' - J = \int \frac{dS}{dt}\, dt = S_1 - S_0,$$

where S_0 and S_1 are the values of the function S for $t = t_0$ and $t = t_1$.

We therefore have

$$\delta J' = \delta J + [\delta S]_{t=t_0}^{t=t_1}. \tag{3}$$

If the canonical equations (1) are satisfied, we have

$$\delta J = +[\Sigma y_i \delta x_i]_{t=t_0}^{t=t_1}, \tag{4}$$

and, consequently, in view of (2) and (3),

$$\delta J' = +[\Sigma y'_i \delta x'_i]_{t=t_0}^{t=t_1}. \tag{4'}$$

In the same way that relationship (4) is equivalent to equations (1), the relationship (4') is equivalent to the equations

$$\frac{dx'_i}{dt} = \frac{dF}{dy'_i}, \qquad \frac{dy'_i}{dt} = -\frac{dF}{dx'_i}. \tag{1'}$$

However, we have just seen that (4) is equivalent to (4'). Equations (1) are equivalent to equations (1'), which means -- as we already knew -- that the change in variables does not change the canonical form of the equations.

The action J' will be minimum when we assume that the initial and final values of the variables x'_i are given. Therefore, a new form of the principle of least action corresponds to each system of canonical variables.

The equations (1) entail the energy integral

$$F = h \tag{5}$$

where h is a constant.

Up to the present, we have assumed that the two limits t_0 and t_1 are given. What would take place if these limits are regarded as variables? Since F does not depend explicitly on time, we do not limit the conditions of generality by assuming that t_0 is constant, and by only increasing t_1 by δt_1. For example, let us assume that $t_0 = 0$ and that, after the variation, the variables, x_i and y_i, have the same values at the time $\frac{t}{t_1}(t_1 + \delta t_1)$ that they had at the time t before the variation. /251

Before the variation, we shall have

$$J = -ht_1 + \Sigma \int y_i \frac{dx_i}{dt} dt.$$

However,

$$\int y_i \frac{dx_i}{dt} dt = \int y_i dx_i$$

does not depend on time; its variation is therefore zero. We therefore simply have

$$\delta J = -h\,\delta t_1.$$

The derivative of the action J with respect to the upper integration limit t_1 therefore equals the energy constant h whose sign is changed.

If this constant is zero, the action J is still minimum, if we assume that the initial and final values of the variables x_i are given, and even when we do not assume that the initial and final values of the time t_0 and t_1 are given.

If we change F into F - h, J changes into

$$J + h(t_1 - t_0). \quad (6)$$

Since equations (1) do not change, this expression (6) is still minimum.

However, if we change F into F - h, the energy constant which was equal to h becomes zero. Consequently, expression (6) is minimum, even if we do not assume that t_1 and t_0 are given.

No matter what the variables x_i and y_i may be, the action J is minimum. It will therefore be minimum *a fortiori* if we impose a new condition upon it which is compatible with equations (1).

For example, let us impose the condition that the first series of equations (1) must be satisfied, i.e., the following must be satisfied /252

$$\frac{dx_i}{dt} = \frac{dF}{dy_i},$$

from which it follows that

$$J = \int_{t_0}^{t_1} \left(F + \Sigma y_i \frac{dy_i}{dF}\right) dt = \int_{t_0}^{t_1} H\, dt,$$

by setting

$$H = -F + \Sigma y_i \frac{dF}{dy_i}.$$

The action J, which is thus defined, is minimum.

This is the principle of least action written in its Hamiltonian form.

Let us now assume that

$$h = 0.$$

Therefore, we no longer regard the variables x_i and y_i as independents, but we impose the following condition upon them

$$F = 0.$$

This restriction, which is compatible with equations (1), will not impede the action J from being minimum.

We then have

$$J = \int \Sigma y_i \frac{dx_i}{dt} dt$$

and, since h is zero, this integral is minimum even when we do not assume that t_1 and t_0 are given.

Let us then impose the following conditions

$$\frac{dx_i}{dt} = \frac{dF}{dy_i},$$

from which we may derive the y_i's as a function of the $\frac{dx_i}{dt}$'s

$$y_i = \varphi_i\left(x_1, x_2, \ldots, x_n, \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt}\right)$$

or

$$y_i = \varphi_i\left(x_1, x_2, \ldots, x_n, \frac{dx_1}{dt}, \frac{dx_2}{dx_1}\frac{dx_1}{dt}, \frac{dx_3}{dx_1}\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dx_1}\frac{dx_1}{dt}\right). \quad (7)$$

In the place of the y_i's, let us substitute their values (7) in J and in the equation /253

$$F = 0.$$

We shall derive $\frac{dx_1}{dt}$ as a function of the x_k's and the $\frac{dx_k}{dx_1}$'s from this equation. We shall then substitute this value of $\frac{dx_1}{dt}$ in expressions (7) and in J. This last integral will take the following form

$$\int \Sigma y_i \frac{dx_i}{dx_1} dx_1 = \int \Phi \, dx_1,$$

where Φ is a function of the x_k's and of the derivatives $\frac{dx_k}{dx_1}$. This integral, which is thus written in a form independent of time, is still minimum. This is the principle of least action in its Maupertuis form.

If h were not zero, we would only have to change F into F - h.

337. Let us first examine the most important particular case.

Let us assume that we have

$$F = T - U,$$

where T is homogeneous of the second degree with respect to the variables y_i, while U is independent of these variables.

We then have

$$\Sigma y_i \frac{dF}{dy_i} = 2T, \qquad H = T + U.$$

According to the principle of Hamilton, the integral

$$\int_{t_0}^{t_1} (T + U)\, dt$$

must be minimum.

Let us determine what the principle of Maupertuis becomes. The energy equation may be written

$$T - U = h.$$

The Maupertuis action then has the following expression

$$\int (T + U + h)\, dt.$$

The equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i} = \frac{dT}{dy_i}$$

have their second terms which are linear and homogeneous with respect to the y_i's. Therefore, T is homogeneous of the second degree with respect to the $\frac{dx_i}{dt}$'s. Let $d\tau^2$ represent that which T becomes when $\frac{dx_i}{dt}$ is replaced by dx_i; we shall have /254

$$T = \frac{d\tau^2}{dt^2}$$

and $d\tau^2$ will be a form which is linear and homogeneous with respect to the n differentials dx_i. We may deduce the following from this

$$dt = \frac{d\tau}{\sqrt{T}} = \frac{d\tau}{\sqrt{U+h}}.$$

The Maupertuis action will then have the following expression

$$2\int d\tau \sqrt{U+h}.$$

338. For purposes of brevity, in order to be able to study other particular cases, let us set

$$x'_i = \frac{dx_i}{dt};$$

and let us derive the y_i's of the equations

$$x'_i = \frac{dF}{dy_i},$$

so as to take the x_i's and the x'_i's for new variables. Let us employ the ordinary d's to designate the derivatives taken with respect to the x_i's and to the y_i's, and let us employ round ∂'s to designate the derivatives taken with respect to the x_i's and the x'_i's.

We may readily obtain the well-known relationships

$$y_i = \frac{\partial H}{\partial x'_i}, \qquad \frac{\partial H}{\partial x_i} = -\frac{dF}{dx_i},$$

$$F = \Sigma x'_i \frac{\partial H}{\partial x'_i} - H$$

and we will see that equations (1) are equivalent to the Lagrange equations

$$\frac{d}{dt}\frac{\partial H}{\partial x'_i} = \frac{\partial H}{\partial x_i}.$$

Under this assumption, let us examine the case in which H has the /255 following form

$$H = H_0 + H_1 + H_2,$$

where H_0, H_1, H_2 are homogeneous, of degree 0, 1, 2, respectively, with respect to the variables x'_i.

We then have

$$\Sigma x'_i \frac{\partial H}{\partial x'_i} = 2H_2 + H_1,$$

$$F = H_2 - H_0$$

and the

$$y_i = \frac{\partial H_2}{\partial x'_i} + \frac{\partial H_1}{\partial x'_i}$$

are linear functions, but they are not homogeneous with respect to the x'_i's.

The Hamiltonian action retains the same form

$$\int H\,dt.$$

Let us determine what the Maupertuis action becomes.

Let h be the energy constant. The Maupertuis action will have the following expression

$$\int (H + h)\,dt$$

but it must be written in the form which is independent of time.

For this purpose, let us set

$$H_2 = \frac{d\tau^2}{dt^2},$$

and

$$H_1 = \frac{d\sigma}{dt}.$$

H_2 is nothing other than energy, and $d\tau^2$ is that which this energy becomes when x'_i is replaced by dx_i. In the same way, $d\sigma$ is that which H_1 becomes when x'_i is replaced by dx_i. It is therefore a form which is linear and homogeneous with respect to the differentials dx_i.

If we take the energy equation into account

$$H_2 = H_0 + h,$$

from which we have

$$dt = \frac{d\tau}{\sqrt{H_0 + h}}$$

the Maupertuis action will become /256

$$\int \left[2\, d\tau \sqrt{H_0 + h} + d\sigma \right].$$

The Maupertuis principle may therefore be applied to the case in which we are interested, as well as to that of absolute motion. However, there is one essential difference from the point of view of the following statements.

In all the problems which will be encountered, the energy T or H_2 is essentially positive; it is a quadratic, positive definite form. In the case of absolute motion (No. 337), the action

$$\int 2\, d\tau \sqrt{U + h}$$

is essentially positive. It does not change when the limits are interchanged. On the contrary, in actuality, the action is composed of two terms. The first

$$\int 2\, d\tau \sqrt{H_0 + h}$$

is always positive, and does not change when the limits are interchanged.

The second $\int d\sigma$ changes sign when the limits are interchanged, and it may therefore be positive or negative.

If we also note that in certain cases, the first term vanishes without the second term vanishing, we will find that the action is not always positive. This fact will cause a great deal of difficulty later on.

339. In order to show how the preceding considerations may be applied to relative motion, let us first consider the absolute motion of a system. Therefore, let

$$H = T + U$$

and let us assume that the position of this system is defined by n + 1 variables

$$x_1, \; x_2, \; \ldots, \; x_n, \; \omega,$$

where x_1, x_2, ..., x_n are sufficient to find the relative position of different points of the system, and ω is the orientation of the system in space.

If the system is isolated, U will depend only on $x_1, x_2, \ldots, x_n$. T will be a form which is quadratic and homogeneous with respect to $x'_1, x'_2, \ldots, x'_n, \omega'$ whose coefficients depend only on $x_1, x_2, \ldots, x_n$. /257

We will then have the equation

$$\frac{dT}{d\omega'} = p,$$

where p is a constant. This is the area integral.

Under this assumption, let J be the Hamiltonian action

$$J = \int_{t_0}^{t_1} H\,dt;$$

We shall have the following, if the equations of motion are satisfied

$$\delta J = \left[\sum \frac{dT}{dx'_i} \delta x_i + \frac{dT}{d\omega'} \delta\omega \right]_{t=t_0}^{t=t_1}.$$

The action will be minimum (or rather its first variation will be zero) if the initial and final values of the x_i's and of ω are assumed to be given -- i.e., if $\delta x_i = \delta\omega = 0$ for $t = t_0$ and for $t = t_1$.

Let us now assume that the initial and final values of these x_i's are given, but not those of ω. We shall have

$$\delta J = [p\,\delta\omega]_{t=t_0}^{t=t_1} = p[\delta\omega]_{t=t_0}^{t=t_1}.$$

Then let

$$H' = H - p\omega'$$

and

$$J' = \int H'\,dt,$$

and we shall obviously have

$$\delta J' = 0.$$

We may derive ω', which is a linear, nonhomogeneous function of the x'_i's, from the equation $\frac{dT}{d\omega'} = p$. It may also be seen that H' is a quadratic function which is not homogeneous with respect to the x'_i's.

H' therefore has the form $H_0 + H_1 + H_2$ which was studied in No. 338.

The integral J' will thus be minimum, even though the initial and final values of ω are not assumed to be given.

We have /258

$$J' = J - p(\omega_1 - \omega_0),$$

where ω_0 and ω_1 are the values of ω for $t = t_0$ and $t = t_1$.

340. Let us now assume that we have a system referred to moving axes and subjected to forces which depend only on the relative situation of the system with respect to the moving axes. In addition, let us assume that the axes rotate uniformly with a constant angular velocity ω'.

This problem may be directly related to the preceding one. We need only assign a very large moment of inertia to the moving axes, in such a way that its angular velocity remains constant.

For the absolute motion, we then have

$$H = T + U = T_1 + T_2 + U.$$

The function of the forces U depends only on the variables x_i which define the position of the system with respect to the moving axes. T_1, which is the energy of the system, depends on the x_i's, and is a quadratic form with respect to the x_i''s and to ω'. T_2, which is the energy of the moving axes, equals

$$\frac{I}{2}\omega'^2$$

and the moment of inertia I is very large.

We then have

$$p = \frac{dT_1}{d\omega'} + I\omega'$$

and

$$H' = H - p\omega' = (T_1 + T_2 + U) - \frac{dT_1}{d\omega'}\omega' - I\omega'^2$$

or

$$H' = T_1 + U - \frac{dT_1}{d\omega'}\omega' - \frac{I\omega'^2}{2}.$$

However,

$$I\omega' = p - \frac{dT_1}{d\omega'}.$$

Since I and p are very large with respect to $\frac{dT_1}{d\omega'}$, this equation gives us approximately the following /259

$$\omega' = \frac{p}{I}$$

and more exactly

$$\omega' = \frac{p}{I} - \frac{1}{I}\frac{dT_1}{d\omega'}.$$

In addition, we have

$$\frac{I\omega'^2}{2} = \frac{p^2}{2I} - \frac{p\frac{dT_1}{d\omega'}}{I} + \frac{1}{2I}\left(\frac{dT_1}{d\omega'}\right)^2.$$

We thus obtain

$$H' = T_1 + U - \frac{p^2}{2I} + \frac{1}{2I}\left(\frac{dT_1}{d\omega'}\right)^2.$$

In the second member, the term before the last is a constant. The last term is negligible, because I is very large.

Since we may add an arbitrary constant to H' without changing the Hamiltonian principle, we may set

$$H'' = T_1 + U$$

and we know that the integral

$$J'' = \int H'' dt$$

must be minimum (even though the initial and final values of ω are not given).

In the expression of H", ω' must be regarded as a given constant. H" is then a quadratic function, which is not homogeneous with respect to the x_i''s, having the form $H_0 + H_1 + H_2$.

For example, let a material point having the mass 1 move in a plane, whose coordinates with respect to the moving axes are ξ and η. We shall have

$$T_1 = \frac{(\xi' - \omega'\eta)^2 + (\eta' + \omega'\xi)^2}{2}.$$

We therefore have

$$H_2 = \frac{\xi'^2 + \eta'^2}{2}, \quad H_1 = \omega'(\xi\eta' - \xi'\eta), \quad H_0 = \frac{\omega'^2}{2}(\xi^2 + \eta^2) + U.$$

The integral

$$J = \int_{t_0}^{t_1} (H_2 + H_1 + H_0) dt$$

is then minimum, when we assume that the limits t_0 and t_1 are given, as well as the initial and the final values of ξ and η. /260

The energy integral may then be written

$$H_2 = H_0 + h$$

and we have seen that the integral

$$J' = \int (H_2 + H_1 + H_0 + h) dt = J + h(t_1 - t_0)$$

is minimum even though we do not assume that t_0 and t_1 are given.

We then obtain

$$J' = \int (2 H_2 + H_1) dt = \int [ds \sqrt{H_0 + h} + \omega'(\xi d\eta - \eta d\xi)]$$

by setting

$$ds^2 = d\xi^2 + d\eta^2.$$

This is the generalized principle of Maupertuis.

In the problems which we shall discuss, U will always be positive, and consequently J will always be essentially positive.

This will not always hold true for J'. If h is negative, we must assume that the point ξ, η is divided into sections in the region defined by the inequality

$$H_0 + h > 0.$$

The first term of the quantity under the sign $\int$ which is $ds\sqrt{H_0 + h}$ is essentially positive. This will not be true for the second term, which changes sign when we reverse the direction in which the trajectory is assumed to be traversed.

If the point ξ, η is very close to the border of the region in which it is confined, and if, consequently, H_0 + h is very small, the first term will be very small, and the second term is the one which will give the term its sign.

J' is therefore not essentially positive. This can also be seen by means of the following equation

$$J' = J + h(t_1 - t_0).$$

If h is negative, the first term J is positive and the second is negative.

Kinetic Focus

/261

341. Up to the present, when I have stated that a certain integral is minimum, I was employing abridged terminology which was incorrect and could not deceive anyone. I should say the first variation of this integral is zero; this condition is necessary in order that there be a minimum, but it is not sufficient.

We shall now try to determine the condition for which the integrals J and J', which we studied in the preceding sections and whose first variations are zero, are effectively minimum. This investigation is related to the difficult question of second variations and the excellent theory of kinetic focus.

Let us recall the principles of these theories.

Let $x_1, x_2, \ldots, x_n$ be the functions of t; let $x'_1, x'_2, \ldots, x'_n$ be their derivatives. Let us consider the integral

$$J = \int_{t_0}^{t_1} f(x_i, x'_i)\,dt,$$

whose first variation δJ is zero, assuming that the initial and final values of the x_i's are given.

In order that this integral may be minimum, a condition which I shall call condition (A) is necessary, but not sufficient.

The condition is that

$$f(x_i, x'_i + \varepsilon_i) - \Sigma \varepsilon_i \frac{df}{dx'_i},$$

regarded as a function of the ε_i's, is minimum.

Condition (A) is not sufficient, unless the integration limits are not very close. Except for this case, it is necessary to add another condition which I shall call condition (B). In order to explain this, I must first recall the definition of kinetic focus.

In order that

$$\delta J = 0,$$

it is necessary and sufficient that the x_i's satisfy n differential equations of the second order, which I shall call equations (C). /262

Let

$$x_i = \varphi_i(t)$$

be a solution of these equations.

Let us set the following for an infinitely close solution

$$x_i = \varphi_i(t) + \xi_i,$$

and let us formulate the variational equations, the linear equations of which satisfy the ξ_i's and which I shall call (D).

The general solution of these equations (D) will have the following form

$$\xi_i = \sum_{k=1}^{k=2n} A_k \xi_{ik} \qquad (i = 1, 2, \ldots, n).$$

The A_k's are 2n integration constants, and the ξ_{ik}'s are $2n^2$ functions of t, which are determined perfectly and which correspond to 2n particular solutions of the linear equations (D).

Under this assumption, let us state that the ξ_i's all vanish for two given times t = t', and t = t". We shall have 2n linear equations between which we may eliminate the 2n unknowns A_k.

We shall thus obtain the equation

$$\Delta(t', t'') = 0,$$

where Δ is the determinant

$$\Delta = \begin{vmatrix} \xi'_{1.1} & \xi'_{1.2} & \cdots & \xi'_{1.2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi'_{n.1} & \xi'_{n.2} & \cdots & \xi'_{n.2n} \\ \xi''_{1.1} & \xi''_{1.2} & \cdots & \xi''_{1.2n} \\ \cdots & \cdots & \cdots & \cdots \\ \xi''_{n.1} & \xi''_{n.2} & \cdots & \xi''_{n.2n} \end{vmatrix}$$

The quantities ξ'_{ik} and ξ''_{ik} represent that which the function ξ_{ik} becomes when t is replaced by t' and by t".

If the times t' and t" satisfy the equation $\Delta = 0$, we may say that these are two <u>conjugate times</u> and that the two points M' and M" in space having n dimensions, which have /263

$$\varphi_1(t'),\quad \varphi_2(t'),\quad \ldots,\quad \varphi_n(t'),$$
$$\varphi_1(t''),\quad \varphi_2(t''),\quad \ldots,\quad \varphi_n(t''),$$

respectively as coordinates, are two <u>conjugate points</u>.

In addition, if t" is the time conjugate to t' after t', which is the closest to t', we may state that M" is the <u>focus</u> of M'.

We may now state the following condition (B): There is no conjugate time of t_0 between t_0 and t_1.

In order that J be a minimum, it is necessary and sufficient that the conditions (A) and (B) be fulfilled.

A direct consequence may be inferred from this.

Let t_0, t_1, t_2, t_3 be four times.

Let M_0, M_1, M_2, M_3 be the corresponding points of the curve

$$x_1 = \varphi_1(t),\qquad x_2 = \varphi_2(t),\qquad \ldots,\qquad x_n = \varphi_n(t).$$

Let us assume that M_1 is the focus of M_0 and M_3 that of M_2.

If condition (A) is fulfilled, we may have

$$t_0 < t_1 < t_2 < t_3$$

or

$$t_0 < t_2 < t_1 < t_3$$

or

$$t_2 < t_3 < t_0 < t_1.$$

But we cannot have

$$t_0 < t_2 < t_3 < t_1 \;.$$

Otherwise, the integral $\displaystyle\int_{t_0}^{t_1-\varepsilon}$

must be minimum since condition (B) is fulfilled, and the integral

$$\int_{t_2}^{t_1-\varepsilon}$$

will not be minimum since the condition (B) will not be fulfilled for this term.

This is impossible, since we may vary the functions x_i between t_2 and $t_1 - \varepsilon$ without varying them between t_0 and t_2.

The geometric significance of the preceding statements may be readily seen. /264

A curve in space having n dimensions

$$x_i = \varphi_i(t)$$

representing a solution of the equations (c) can be called a trajectory, which I shall call (T).

The curve

$$x_i = \varphi_i + \xi_i$$

will represent an infinitely close trajectory.

If we draw one of these trajectories (T') which are infinitely close to (T) through the point M', and if this trajectory again intersects the trajectory (T) at M" (more precisely, the distance from M" to this trajectory will be an infinitely small quantity of higher order), the points M' and M" will be conjugate if, in addition, the point which follows (T') passes through M' and infinitely close to M" at the times t' and t".

342. In the case of the Hamiltonian principle, condition (A) is always fulfilled. In effect, we have

$$H = H_0 + H_1 + H_2,$$

and H_2 is a quadratic form which is homogeneous with respect to the x_i''s.

In all of these problems of dynamics, this quadratic form is definite and positive.

If we change x_i' into $x_i' + \varepsilon_i$, H_1 will change into

$$H_1(x_i') + \Sigma\varepsilon_i \frac{dH_1}{dx_i'}$$

and H_2 will change into

$$H_2(x'_i) + H_2(\varepsilon_i) + \Sigma\varepsilon_i \frac{dH_2}{dx'_i};$$

and in addition we have

$$\Sigma\varepsilon_i \frac{dH_0}{dx'_i} = 0.$$

Therefore, we have

$$H(x'_i + \varepsilon_i) = H_0 + H_1 + H_2 + \Sigma\varepsilon_i \frac{d(H_0 + H_1 + H_2)}{dx'_i} + H_2(\varepsilon_i),$$

from which we finally have /265

$$H(x'_i + \varepsilon_i) - \Sigma\varepsilon_i \frac{dH}{dx_i} = H + H_2(\varepsilon_i).$$

The first term corresponds to the function

$$f(x'_i + \varepsilon_i) - \Sigma\varepsilon_i \frac{df}{dx'_i}$$

since the quadratic form $H_2(\varepsilon_i)$ is positive definite, and we may see that the expression is minimum for $\varepsilon_i = 0$ -- i.e., that condition (A) is fulfilled.

343. Let us proceed to the case of the Maupertuis principle in absolute motion. The integral to be examined may then be written

$$\int d$$

where $d\tau^2$ is a positive definite, quadratic form with respect to the differentials dx_i.

For the time being, let us select x_1 as the independent variable. The integral becomes

$$\int \frac{d\tau}{dx_1} dx_1$$

where $\left(\frac{d\tau}{dx_1}\right)^2$ is a polynomial of the second order P which is not homogeneous (but essentially positive) with respect to the $\frac{dx_i}{dx_1}$'s. Therefore, let us set

$$\frac{d\tau}{dx_1} = \sqrt{P\left(\frac{dx_i}{dx_1}\right)}.$$

We must determine whether

$$\sqrt{P\left(\frac{dx_i}{dx_1}+\varepsilon_i\right)}-\Sigma\varepsilon_i\frac{d}{dx'_i}\sqrt{P(x'_i)}$$

is minimum for $\varepsilon_i = 0$. In other words, we must determine whether the second derivative, with respect to t, of the radical

$$\sqrt{P\left(\frac{dx_i}{dx_1}+\varepsilon_i t\right)}$$

is positive.

No matter what the $\frac{dx_i}{dx_1}$'s and the ε_i's may be, we shall have /266

$$P\left(\frac{dx_i}{dx_1}+\varepsilon_i t\right)=at^2+2bt+c,$$

where a, b, c are independent of t. The second derivative of the radical then equals

$$\frac{ac-b^2}{(at^2+2bt+c)^{\frac{3}{2}}}.$$

Since the polynomial P is essentially positive, this expression is also always positive, and condition (A) is always fulfilled.

344. Let us proceed to the Maupertuis principle in relative motion. We must then consider the integral

$$\int\left[ds\sqrt{H_0+h}+\omega'(\xi\,d\eta-\eta\,d\xi)\right],$$

or, choosing ξ as the independent variable, we have

$$\int d\xi\left[\sqrt{(H_0+h)(1+\eta'^2)}+\omega'(\xi\eta'-\eta)\right].$$

We must therefore determine whether the second derivative with respect to η' of

$$\sqrt{(H_0+h)(1+\eta'^2)}+\omega'(\xi\eta'-\eta)$$

is positive. This derivative is

$$\frac{\sqrt{H_0+h}}{(1+\eta'^2)^{\frac{3}{2}}}.$$

Condition (A) is therefore always fulfilled.

Thus, condition (A) is itself fulfilled in every case which we shall examine.

Maupertuis Focus

345. The kinetic focuses are not always the same, depending on whether Hamiltonian action or Maupertuis action is being considered. In order to clarify this point, let us assume only two degrees of freedom, and let x and y be the two variables which define the position of the system, and which we may regard as the coordinates of a point in a plane. /267

Let

$$x = f_1(t), \quad y = f_2(t)$$

be the equations of a trajectory (T) which will be a plane curve. Let us set

$$x = f_1(t) + \xi, \quad y = f_2(t) + \eta,$$

and, neglecting the squares of ξ and of η, let us formulate the variational equations. Since they are linear and of the fourth order, we shall have

$$\xi = a_1\xi_1 + a_2\xi_2 + a_3\xi_3 + a_4\xi_4,$$
$$\eta = a_1\eta_1 + a_2\eta_2 + a_3\eta_3 + a_4\eta_4,$$

where the a_i's are integration constants, and the ξ_i's and η_i's are functions of t.

The equation given in No. 341

$$\Delta(t', t'') = 0,$$

may then be written

$$\begin{vmatrix} \xi_1' & \xi_2' & \xi_3' & \xi_4' \\ \eta_1' & \eta_2' & \eta_3' & \eta_4' \\ \xi_1'' & \xi_2'' & \xi_3'' & \xi_4'' \\ \eta_1'' & \eta_2'' & \eta_3'' & \eta_4'' \end{vmatrix} = 0. \qquad (1)$$

It is this equation which defines the Hamiltonian focus.

It indicates that the point x, y, which describes the trajectory (T), and the point $x + \xi$, $y + \eta$, which describes the infinitely close trajectory (T'), occur at two different times, i.e., at the times t' and t", separated by an infinitely small distance of higher order.

However, these are not the conditions which the Maupertuis focuses must fulfill. Two points of the trajectory (T) -- i.e., the two points M' and M" which correspond to the times t' and t" -- must be separated by an infinitely small distance of higher order from the trajectory (T'). However, it is not necessary that the moving point which traverses (T') passes precisely at the time t" -- for example, infinitely close to M". On the other hand, the energy constant must have the same value for (T) and for (T'). This last condition is not imposed on Hamiltonian focuses.

One of the solutions of the variational equations is /268

$$\xi = f'_1(t), \quad \eta = f'_2(t).$$

We may therefore assume that

$$\xi'_1 = f'_1(t'), \quad \eta'_1 = f'_2(t'), \quad \xi''_1 = f'_1(t''), \quad \eta''_1 = f'_2(t'').$$

The two functions ξ_1 and η_1 are thus defined.

In addition, the difference between the energy constant relative to (T) and the energy constant relative to (T') is infinitely small. This is obviously a linear function of the four infinitely small constants a_1, a_2, a_3, a_4.

Without limiting the conditions of generality, we may assume that this difference is precisely equal to a_4.

The condition stipulating that the value of the energy constant be the same for T and (T') is then $a_4 = 0$, or

$$\xi = a_1 \xi_1 + a_2 \xi_2 + a_3 \xi_3,$$
$$\eta = a_1 \eta_1 + a_2 \eta_2 + a_3 \eta_3.$$

For $t = t'$, ξ and η must be zero, from which we have equations

$$a_1 \xi'_1 + a_2 \xi'_2 + a_3 \xi'_3 = 0,$$
$$a_1 \eta'_1 + a_2 \eta'_2 + a_3 \eta'_3 = 0.$$

In addition, the value of $x + \xi$, $y + \eta$ for $t = t'' + \varepsilon$ must be the same (up to quantities which are infinitely small of a higher degree) as that of x and y for $t = t''$, which may be written

$$(\varepsilon + a_1) \xi''_1 + a_2 \xi''_2 + a_3 \xi''_3 = 0,$$
$$(\varepsilon + a_1) \eta''_1 + a_2 \eta''_2 + a_3 \eta''_3 = 0,$$

from which we have, by elimination,

$$\begin{vmatrix} \xi'_2 & \xi'_3 & \xi'_1 & 0 \\ \eta'_2 & \eta'_3 & \eta'_1 & 0 \\ \xi''_2 & \xi''_3 & 0 & \xi''_1 \\ \eta''_2 & \eta''_3 & 0 & \eta''_1 \end{vmatrix} = 0. \qquad (2)$$

By developing the determinant, we obtain

$$\begin{vmatrix} \xi'_1\eta'_2 - \xi'_2\eta'_1 & \xi'_1\eta'_3 - \xi'_3\eta'_1 \\ \xi''_1\eta''_2 - \xi''_2\eta''_1 & \xi''_1\eta''_3 - \xi''_3\eta''_1 \end{vmatrix} = 0$$

and, setting /269

$$\frac{\xi_1\eta_2 - \eta_1\xi_2}{\xi_1\eta_3 - \xi_3\eta_1} = \zeta(t),$$

equation (2) becomes

$$\zeta(t') = \zeta(t''). \qquad (3)$$

Application to Periodic Solutions

346. If we are dealing with a periodic solution of period 2π, the functions $f_1(t)$ and $f_2(t)$ of the preceding section will be periodic of the period 2π. The same holds true for

$$\xi_1 = f'_1(t), \qquad \eta_1 = f'_2(t).$$

In addition, according to Chapter IV, the variational equations will have other particular solutions which will have the following form

$$\begin{aligned} \xi &= e^{\alpha t}\varphi_2(t), & \eta &= e^{\alpha t}\psi_2(t); \\ \xi &= e^{-\alpha t}\varphi_3(t), & \eta &= e^{-\alpha t}\psi_3(t); \\ \xi &= \varphi_4(t) + \beta t f'_1(t), & \eta &= \psi_4(t) + \beta t f'_2(t). \end{aligned}$$

In these equations, β is a constant, α and $-\alpha$ are the characteristic exponents, and the ϕ's and the ψ's are the periodic functions.

Let

$$F\left(x, y, \frac{dx}{dt}, \frac{dy}{dt}\right) = \text{const.}$$

be the energy equation. We must have

$$\frac{dF}{dx}\xi + \frac{dF}{dy}\eta + \frac{dF}{d\frac{dx}{dt}}\frac{d\xi}{dt} + \frac{dF}{d\frac{dy}{dt}}\frac{d\eta}{dt} = A,$$

where A is a constant. If we replace ξ and η by $e^{\alpha t}\phi_2$, $e^{\alpha t}\psi_2$ in this equation, the first term becomes a periodic function of t multiplied by $e^{\alpha t}$ and -- since it must be constant -- it is necessary that it be zero.

We shall therefore have

$$A = 0 .$$

This indicates that the two infinitely close trajectories which have the following equations /270

$$x = f_1(t), \quad y = f_2(t)$$

and

$$x = f_1(t) + e^{\alpha t}\varphi_2(t), \quad y = f_2(t) + e^{\alpha t}\psi_2(t)$$

correspond to the same value of the energy constant.

In the same way, we find that the same holds true for the trajectory which has the equation

$$x = f_1(t) + e^{-\alpha t}\varphi_3(t), \quad y = f_2(t) + e^{-\alpha t}\psi_3(t).$$

Nothing prevents us from setting

$$\xi_2 = e^{\alpha t}\varphi_2, \qquad \eta_2 = e^{\alpha t}\psi_2,$$
$$\xi_3 = e^{-\alpha t}\varphi_3, \qquad \eta_3 = e^{-\alpha t}\psi_3.$$

Then $\zeta(t)$ has the following form

$$\zeta(t) = e^{2\alpha t}G(t),$$

where G(t) is a periodic function.

Case of Stable Solutions

347. We must now distinguish between two cases:

1. The solution is stable and α^2 is negative. In this case ξ_2 and ξ_3, and η_2 and η_3 are imaginary and conjugate. The modulus of ζ and G is one. We shall formulate three hypotheses which we shall justify at a later point.

1. Let us first assume that G(t) never becomes either zero or infinite;

2. The function

$$t + \frac{1}{2\alpha}\log G(t) = \tau$$

which is essentially real also constantly increases;

3. In addition, let us assume that log G(t) is a periodic function.

Equation (3) may then be written as follows, employing τ' and τ'' to designate two values of τ which correspond to t' and to t":

$$\tau'' - \tau' = \frac{ki\pi}{\alpha} \quad \text{(k is an integer number)}$$

One single value of τ corresponds to each value of t, and one single value of t corresponds to each value of τ. We therefore cannot have k = 0 without t' = t". If we desire t" > t', it is necessary that k be positive. /271

By setting k = 1, we shall give the smallest value to t" - t'. We have

$$\tau'' - \tau' = \frac{i\pi}{\alpha}$$

and the point M" is then the focus of M'.

One factor must be pointed out.

In order that the preceding line of reasoning may be applicable, it is necessary that log G(t) be a periodic function. However, in general, all that we know is that G(t) is a periodic function, and as a result

$$\log G(t)$$

is increased by a multiple of $2i\pi$, for example, of $2ki\pi$, when t increases by 2π. Then

$$\log G(t) - ikt$$

is a periodic function.

Let us then set

$$G'(t) = G(t)e^{-ikt},$$

$$\alpha' = \alpha + \frac{ik}{2};$$

we have

$$\zeta(t) = e^{2\alpha t}G(t) = e^{2\alpha' t}G'(t).$$

We shall then no longer set

$$\tau = t + \frac{1}{2\alpha}\log G(t),$$

but rather

$$\tau = t + \frac{1}{2\alpha'} \log G'(t) .$$

Since log G(t) will be periodic, the preceding conclusions remain valid, and equation (3) will be written

$$\tau'' - \tau' = \frac{mi\pi}{\alpha'} \qquad \text{(m is an integer number)}$$

and, in addition, M" will be the focus of M' if /272

$$\tau'' - \tau' = \frac{i\pi}{\alpha'} .$$

348. One of our three hypotheses stating that log G(t) must be periodic has thus been proven. I may now state that the function τ must be constantly increasing, as we assumed.

Let us assume that this function has a maximum τ_0 for $t = t_0$. We may then find two times t'_1 and t''_1 such that the corresponding values τ'_1 and τ''_1 of the function τ are equal, and two other times t'_2 and t''_2 such that $\tau'_2 = \tau''_2$ and such that the five times which are very close to one another satisfy the following inequalities

$$t'_2 < t'_1 < t_0 < t''_1 < t''_2 .$$

Then t''_1 will be the focus of t'_1, t''_2 that of t'_2. We saw above that such inequalities are impossible when condition A is fulfilled.

I may now state that G(t) cannot vanish. We have

$$\zeta(t) = \frac{\xi_1\eta_2 - \xi_2\eta_1}{\xi_1\eta_3 - \xi_3\eta_1} .$$

The numerator and the denominator of $\zeta(t)$ are imaginary and conjugate. If one of them vanishes, the other also vanishes, so that the function $\zeta(t)$ cannot become either zero or infinite.

Thus, all of our hypotheses have been proven.

Unstable Solutions

349. Let us now assume that the unstable solution and α^2 are positive; in this case ξ_2, η_2, ξ_3, η_3, ζ, α, G are real.

For the same reason as given above, the function τ will be constantly increasing. However, two hypotheses are possible:

1. The quantity $\zeta(t)$ cannot vanish nor become infinite, and increases constantly from 0 to $+\infty$ when t increases from $-\infty$ to $+\infty$.

It then happens that no point of our periodic solution has a Maupertuis focus.

2. The quantity $\zeta(t)$ may vanish for $t = t_0$. It will also vanish /273 for $t = t_0 + 2\pi$, and since it cannot have either a maximum or a minimum it must become infinite in the interval. In the same way, if $\zeta(t)$ can become infinite, it must also be able to vanish.

In order to clarify our thoughts, let us assume that $\zeta(t)$ becomes infinite for

$$t = t_0, \quad t_1, \quad t_0 + 2\pi$$

and for values which differ from these by a multiple of 2π, and vanishes for

$$t = t'_0, \quad t'_1, \quad t'_0 + 2\pi.$$

I shall assume that

$$t_0 < t'_0 < t_1 < t'_1 < t_0 + 2\pi.$$

When t increases from t_0 to t_1, or from t_1 to t_2, or from t_2 to $t_0 + 2\pi$, $\zeta(t)$ increases constantly from $-\infty$ to $+\infty$.

The closed trajectory (T) which represents our periodic solution will therefore be divided into two arcs, whose extremities will correspond to the following values of t

$$t_0, \quad t_1, \quad t_0 + 2\pi.$$

Each of the points of one of the arcs will have its first focus on the following arc.

I may add that the points corresponding to the values of t

$$t_0, \quad t'_0, \quad t_1, \quad t'_1$$

coincide with their two focuses.

Let t'' be a value of t corresponding to an arbitrary point of (T), and let t''_n be the value of t which corresponds to its n^{th} focus. We shall have

$$\lim \frac{t''_n - t''}{n} = \frac{2\pi}{2}.$$

However, this is not all; we shall have

$$e^{2\alpha t''_n} G(t''_n) = e^{2\alpha t''} G(t'').$$

If n is very large and if G(t") is not infinite, since $t''_n - t''$ is very large and since we assume that α is positive, $G(t''_n)$ will be very small, so that if t" is, for example, included between t_0 and t_1, the difference /274

$$t''_{2n} - 2n\pi$$

will strive toward t'_0 when n increases indefinitely.

If n strives toward $-\infty$, this difference will strive toward t_0 or toward t_1, depending on whether t" will be included between t_0 and t'_0 or between t'_0 and t_1. I should add that the difference $t''_{2n} - 2n\pi$ is either constantly increasing or constantly decreasing with n.

The values t'_0, t'_1 correspond to the points where

$$\xi_1\eta_2 - \xi_2\eta_1 = 0$$

but $\xi_1\eta_2 - \xi_2\eta_1$ is a periodic function multiplied by $e^{\alpha t}$. However, a periodic function must vanish an even number of times in one period.

Consequently, the closed trajectory (T) will be divided by the points t_0, t_1, $t_0 + 2\pi$ into a certain number of arcs, and this number will always be even.

350. From the point of view in which we are interested, the unstable, periodic solutions may be divided into two categroies. However, it could be asked whether these two categories exist in actuality. It is therefore advantageous to cite some examples.

Let ρ and ω be the polar coordinates of a moving point in a plane. The equations of motion may be written

$$\frac{d^2\rho}{dt^2} = \left(\frac{d\omega}{dt}\right)^2 \rho + \frac{dU}{d\rho}, \qquad \rho^2 \frac{d^2\omega}{dt^2} + 2\rho \frac{d\rho}{dt}\frac{d\omega}{dt} = \frac{dU}{d\omega}. \tag{1}$$

For $\rho = 1$, let us assume that we have

$$U = 0, \quad \frac{dU}{d\omega} = 0, \quad \frac{dU}{d\rho} = -1, \quad \frac{d^2U}{d\rho^2} = \varphi(\omega) .$$

Equations (1) will have the solutions

$$\rho = 1, \qquad \omega = t$$

and this solution will correspond to a closed trajectory which will be a circumference.

Let us set /275

$$\rho = 1 + \zeta, \qquad \omega = t + v$$

and let us formulate the variational equations. They may be written

$$\frac{d^2\zeta}{dt^2} = \zeta + 2\frac{dv}{dt} + \zeta\varphi(t), \qquad \frac{d^2v}{dt^2} + 2\frac{d\zeta}{dt} = 0.$$

The second may be integrated immediately

$$\frac{dv}{dt} + 2\zeta = \text{const.};$$

but this constant must be zero if we want the energy constant to have the same value for the trajectory (T) and for the infinitely close trajectory.

Therefore, if we replace $\frac{dv}{dt}$ by -2ζ, the first variational equation will become

$$\frac{d^2\zeta}{dt^2} = \zeta[\varphi(t) - 3]. \tag{2}$$

Equation (2) which remains to be integrated is a linear equation having a periodic coefficient.

These equations were discussed in Sections 29 and 189 (see, in addition, Chapter IV, in various places).

It is known that they have two solutions of the following form:

$$\zeta = e^{\alpha t}G(t), \qquad \zeta = e^{-\alpha t}G_1(t)$$

where G and G_1 are periodic functions.

We are going to present examples for every case mentioned above. Let us first assume that ϕ may be reduced to a constant A (case of central forces).

If A < 3, we shall have a stable, periodic solution.

If A > 3, there will not be a Maupertuis focus on (T), and we shall have an unstable, periodic solution of the first category.

I must now show that we may also have periodic, unstable solutions of the second category.

The solution will be unstable and of the second category if G vanishes in such a way that the ratio

$$e^{2\alpha t}\frac{G}{G_1},$$

which corresponds to the function $\zeta(t)$ of the preceding sections can vanish, and consequently can become infinite. /276

We may obviously formulate a periodic function G which satisfies the following conditions:

1. It has two simple zeros and only two;

2. These zeros will also vanish

$$\frac{d^2G}{dt^2}+2\alpha\frac{dG}{dt}.$$

As a result, every time that

$$\zeta = e^{\alpha t}G$$

vanishes, its second derivative will also vanish in such a way that the ratio

$$\frac{1}{\zeta}\frac{d^2\zeta}{dt^2}=\varphi-3$$

remains finite.

One could obviously formulate a function G which satisfies these conditions. The periodic function ϕ formulated by means of this function G will correspond to an unstable, periodic solution of the second category.

As an example of function G satisfying this condition, we may set

$$G=\sin t-\frac{\alpha}{4}(\cos t-\cos 3t).$$

This function vanishes for $t = 0$ and $t = \pi$, and it does not have another zero if

$$\alpha<\frac{1}{\sqrt{5}}.$$

For $t = 0$ and for $t = \pi$, we have

$$\frac{d^2G}{dt^2}+2\alpha\frac{dG}{dt}=0.$$

In order that the ratio $\frac{G}{G_1}$ may vanish, it is not sufficient that G vanish; it is still necessary that G_1 does not vanish.

However, this is what occurs, because if G and G_1 vanished at the same time, the two solutions /277

$$\zeta = e^{\alpha t} G(t), \qquad \zeta = e^{-\alpha t} G_1(t)$$

could only differ by a constant factor (since they satisfy the same differential equation of the second order), and this is absurd.

351. One point to which I would like to draw attention is the fact that the unstable solutions of the first category and of the second category form two separate groups, so that we cannot pass from one to another continuously without passing through the intermediary of the stable solutions.

Let us first confine ourselves to the particular case given in the preceding section, and let us reconsider the equation

$$\frac{d^2\zeta}{dt^2} = \zeta(\varphi - 3). \tag{2}$$

Let us vary the function ϕ continuously, and let us determine whether we can pass directly from an unstable solution of the first category to an unstable solution of the second category. For this purpose, it is necessary that the function G, which is real, be first incapable of vanishing, and then be capable of vanishing. We would thus pass from the case in which the equation G = 0 has all its imaginary roots to the case in which it has real roots. At the time of passage, it would have a double root or, more generally, a multiple root on the order of 2m.

This zero, which would be on the order of 2m for G, would be on the order of 2m - 1 for $\frac{dG}{dt}$, on the order of 2m - 2 for $\frac{d^2G}{dt^2}$, so that the expression

$$\frac{\frac{d^2 G}{dt^2} + 2\alpha \frac{dG}{dt} + \alpha^2 G}{G}$$

would be come infinite, which is impossible since it equals ϕ - 3.

On the other hand, we may pass from a stable solution to an unstable solution of one or the other categories.

For a stable solution, G is imaginary. At the time when the solution becomes unstable, the imaginary part of G becomes identically zero. If /278 at this time the real part of G has zeros, we shall pass to an unstable solution of the second type; if this real part never vanishes, we shall pass to an unstable solution of the first type.

No difficulty is encountered in passing from the case in which the equation

$$\text{real part of } G = 0$$

has all imaginary roots to that in which this equation has real roots, provided that at the time of passage the imaginary part of G is not zero.

352. In order to clarify the preceding statements, I shall return to an example which is already familiar to us.

Let us return to the equation of Glyden, i.e., to equation (1) given in Number 178 (Volume II). We shall assign the number (3) to this equation, and we shall write it as follows

$$\frac{d^2x}{dt^2} = x(-q^2 + q_1 \cos 2t). \tag{3}$$

It can be seen that it has the same form as equation (2).

Just like equation (2), we have seen that this equation has two integrals having the following form

$$e^{\alpha t}G, \quad e^{-\alpha t}G_1,$$

which we have written in the notation given in No. 178 as follows

$$e^{iht}\varphi_1(t), \quad e^{-iht}\varphi_2(t).$$

The case of h real then corresponds to the case of stable solutions, and the case of h imaginary corresponds to that of unstable solutions.

We also considered two unusual integrals. The first is <u>even</u>

$$[F(0) = 1, \quad F'(0) = 0] \qquad F(t)$$

and the second is <u>uneven</u>

$$[f(0) = 0, \quad f'(0) = 1]$$

and we have obtained the following conditions /279

$$F(\pi)f'(\pi) - f(\pi)F'(\pi) = 1,$$
$$F(\pi) = f'(\pi) = \cos h\pi.$$

I shall now return to the figure presented on page 243 (Volume II) where, assuming that q and q_1 were the rectangular coordinates of one point, we separated the regions corresponding to stable solutions and those corresponding to unstable solutions. These latter regions are shaded.

These different regions are separated from each other by four analytic curves, whose equations I have presented on page 241 (Volume II).

Following are these equations:

$$F(\pi) = 1, \quad F'(\pi) = 0, \tag{α}$$

$$F(\pi) = 1, \quad f(\pi) = 0, \tag{β}$$

$$F(\pi) = -1, \quad F'(\pi) = 0, \tag{γ}$$

$$F(\pi) = -1, \quad f(\pi) = 0. \tag{δ}$$

To what category do the unstable solutions belong which correspond to our shaded regions? It is apparent that the unstable solutions corresponding to one of these regions are <u>all</u> of the same category. This is a direct result of the preceding statements.

At a point of one of the curves (β) and (δ), the function G may be reduced to f(t), and this function may vanish, since it is odd. Therefore, if a region is bounded by an arc of one of the curves, (β) and (δ), the corresponding solutions will belong to the second category.

However, this is not the case in all of our regions. Therefore, all of our unstable solutions belong to the second category.

Our example may be readily transformed in such a way that we have solutions of two categories. It is sufficient to replace q^2 by q, in such a way that this coefficient may become negative.

Our equation (3) may then be written

$$\frac{d^2x}{dt^2} = x(-q + q_1 \cos 2t). \tag{3'}$$

Let us always take q and q_1 as rectangular coordinates, and let us compile a figure similar to that shown on page 241. The portion of the figure located to the right of the q_1 axis on the side of the positive q's will be similar to the figure shown on page 241. But to the left of the q_1 axis, at the side of the negative q's, we shall have a shaded region which is bounded by a kind of parabola tangent to the axis of the q_1's. /280

The shaded regions on the right will correspond to solutions of the second category, as we have just seen. However, this will not hold true for the shaded region on the left.

To demonstrate this, it is sufficient to set $q_1 = 0$, from which we have

$$x = e^{t\sqrt{-q}}; \quad \alpha = \sqrt{-q}; \quad G = 1.$$

353. I have still only presented a discussion for a particular case. In order to extend it to the general case, I shall show that we always arrive at an equation having the same form as equation (2) in the preceding section.

Let us first consider the case of absolute motion. If U is the force potential and if x and y are the Cartesian coordinates of a point in a plane, the equations of motion may be written

$$x'' = \frac{dU}{dx}, \qquad y'' = \frac{dU}{dy}, \tag{1}$$

and the variational equations may be written

$$\begin{cases} \xi'' = \dfrac{d^2U}{dx^2}\xi + \dfrac{d^2U}{dx\,dy}\eta, \\ \eta'' = \dfrac{d^2U}{dx\,dy}\xi + \dfrac{d^2U}{dy^2}\eta. \end{cases} \tag{2}$$

For purposes of greater brevity, I shall employ accents to designate the derivations with respect to t. Thus, ξ'' represents $\frac{d\delta\xi}{dt^2}$ here, and no longer represents the value of ξ for t = t'', as was the case in No. 341.

The energy integral may be written

$$\frac{x'^2 + y'^2}{2} = U + h,$$

and the corresponding integral of (2) /281

$$x'\xi' + y'\eta' = \frac{dU}{dx}\xi + \frac{dU}{dy}\eta + \delta h \quad (\delta h \text{ is a constant}).$$

To apply the Maupertuis principle, we must assume that

$$\delta h = 0,$$

so that we shall have

$$x'\xi' + y'\eta' = \frac{dU}{dx}\xi + \frac{dU}{dy}\eta,$$

or

$$x'\xi' + y'\eta' = x''\xi + y''\eta. \tag{3}$$

Our equations (2) and (3) will then have three independent linear solutions which we have called in No. 345

$$\begin{cases} \xi_1 = x', & \eta_1 = y'. \\ \xi_2 & \eta_2 \\ \xi_3 & \eta_3 \end{cases} \tag{4}$$

Let us set

$$\theta = \xi y' - \eta x'. \tag{5}$$

If we then call θ_1, θ_2, θ_3 the three values of θ corresponding to the three solutions (4), we shall have $\theta_1 = 0$, and the function which we called $\zeta(t)$ in No. 345 will be nothing else than

$$\zeta(t) = \frac{\theta_2}{\theta_3}.$$

We may derive the following from equation (5)

$$\theta' = \xi' y' - \eta' x' + \xi y'' - \eta x'' \tag{6}$$

and

$$\theta'' = \xi y''' - \eta x''' + \xi'' y' - \eta'' x' + 2(\xi' y'' - \eta' x'').$$

However, x' and y' satisfy the equations (2), so that we have

$$x''' = \frac{d^2U}{dx^2} x' + \frac{d^2U}{dx\,dy} y',$$
$$y''' = \frac{d^2U}{dx\,dy} x' + \frac{d^2U}{dy^2} y'.$$

In the expression of θ'', let us replace the derivatives x''' and y''' by the values which have thus been found, and the derivatives ξ'' and η'' /282
by their values (2). We shall have

$$\theta'' - \theta \Delta U = 2(\xi' y'' - \eta' x''). \tag{7}$$

I shall designate the sum of the two second derivatives $\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2}$ by ΔU (or more briefly by Δ).

The following identity may be easily verified

$$\begin{aligned} &2(x'^2 + y'^2)(\xi' y'' - \eta' x'') \\ &\quad - 2(x'x'' + y'y'')(\xi y'' - \eta x'' + \xi' y' - \eta' x') + 2(x''^2 + y''^2)(\xi y' - \eta x') \\ &= 2(y''x' - y'x'')(\xi' x' + \eta' y' - \xi x'' - \eta y''), \end{aligned}$$

or, taking into account (5), (6), (7) and (3), we have

$$(x'^2 + y'^2)(\theta'' - \theta\Delta) - 2(x'x'' + y'y'')\theta' + 2(x''^2 + y''^2)\theta = 0. \tag{8}$$

This is the differential equation which defines the unknown function θ.

We shall set

$$\theta = \rho\sqrt{x'^2 + y'^2}$$

and our equation becomes

$$\frac{\varphi''}{\varphi} = \Delta - \frac{x'x''' + y'y''' + 3x''^2 + 3y''^2}{x'^2 + y'^2}, \tag{9}$$

an equation having the same form as equation (2) of the preceding section. The conclusions of the preceding section therefore remain in force. One periodic unstable solution is of the second category, or of the first category, depending on whether the function ϕ can vanish or not. We cannot pass directly from an unstable solution of the first category to an unstable solution of the second category, but can only pass through stable solutions.

354. Do the same results still remain valid in the case of relative motion?

The equations of motion then become

$$x'' - 2\omega y' = \frac{dU}{dx}, \qquad y'' + 2\omega x' = \frac{dU}{dy}, \tag{1'}$$

where ω designates the speed of rotation of moving axes.

The variational equations will be /283

$$\begin{cases} \xi'' - 2\omega\eta' = \dfrac{d^2U}{dx^2}\xi + \dfrac{d^2U}{dx\,dy}\eta, \\ \eta'' + 2\omega\xi' = \dfrac{d^2U}{dx\,dy}\xi + \dfrac{d^2U}{dy^2}\eta. \end{cases} \tag{2'}$$

Due to the fact that the energy equation is still valid, the same will hold true for

$$x'\xi' + y'\eta' = x''\xi + y''\eta. \tag{3}$$

Let us set

$$\theta = \xi y' - \eta x',$$

and equations (5) and (6) will continue to hold.

In addition, since x' and y' must satisfy equations (2'), we shall have

$$x''' - 2\omega y'' = \frac{d^2U}{dx^2}x' + \frac{d^2U}{dx\,dy}y',$$

$$y''' + 2\omega x'' = \frac{d^2U}{dx\,dy}x' + \frac{d^2U}{dy^2}y'.$$

Taking these equations into account, as well as equations (2'), and also taking into account equation (3), we may simplify the expression of θ'', and we again obtain the equation

$$\theta'' - \theta \Delta U = 2(\xi' y'' - \eta' x''). \tag{7}$$

Since the identity given in the preceding section is always valid, we shall obtain equations (8) and (9) again. Therefore, nothing needs to be changed in the conclusions given in the preceding section.

355. However, one new question arises.

The trajectory (T) is a closed curve. Up to the present, we have tried to determine whether an arc AB of this curve would correspond to an action which is smaller than any infinitely adjacent arc with the same end points.

However, we may also question whether this entire closed curve corresponds to an action which is smaller than every infinitely small closed curve.

Let us first assume that a point A of the curve (T) has its first focus B on the curve (T), so that the arc AB is smaller than the entire closed curve. /284

This is what occurs for unstable solutions of the first category. We have seen that the curve (T) may be divided into a certain even number of arcs for these solutions, and that every point on one of these arcs has its first focus on the following arc, so that -- starting from an arbitrary point -- its first focus will be encountered before the entire curve (T) has been traversed.

This also occurs for certain stable solutions. In the case of stable solutions, we have set (No. 347)

$$t + \frac{1}{2\alpha} \log G(t) = \tau$$

and we have seen that the τ of a point, and that of its first focus, differ by $\frac{i\pi}{\alpha}$. Therefore, if $\frac{\alpha}{i}$ is larger than $\frac{1}{2}$, the focus of a point will be encountered before (T) is completely traversed.

If this is the case, the action cannot be less for the curve (T) than it is for any infinitely adjacent curve.

Let ABCDEA be the curve (T), and let us assume that D is the focus of C. Since E is outside the focus of C, we may attach C to E by an arc CME which is very close to CDE, and which corresponds to a smaller action.

If I represent the action corresponding to the arc CME by (CME), we shall have

$$(CME) < (CDE)$$

and, consequently,

$$(ABCMEA) < (ABCDEA).$$

Let us now consider a stable solution, such that

$$\frac{\alpha}{i} > \frac{1}{2}.$$

It may be stated that the action will no longer be less for (T) than it is for any infinitely adjacent closed curve.

In order to clarify these ideas, I have compiled a figure, assuming that $\frac{\alpha}{i}$ ranges between $\frac{1}{4}$ and $\frac{1}{6}$, in such a way that the focus of a point is encountered before traversing (T) three times, and after traversing /285 (T) twice.

Let ABCDA be the curve (T). The focus F will be located between AB, and it will be encountered after traversing (T) twice.

Since B is located beyond this focus, we may attach A to B by an arc AEHNKHMEB, such that

$$(AEHNKHMEB) < (ABCABCAB).$$

Since the focus of A is not encountered by describing the arc AB without

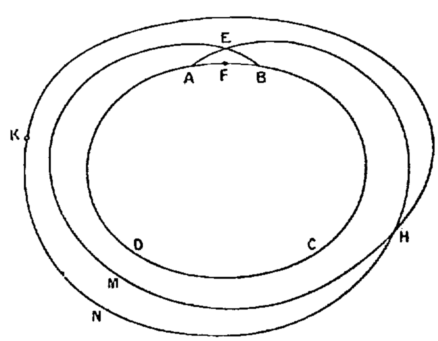

Figure 10

traversing (T), we shall have in addition

$$(AE + EB) > (AB),$$

from which we have the following by subtraction

$$(EHNKHME) < (ABCABCA)$$

or

$$(EHME) + (HNKH) < 2(ABCA).$$

We must therefore have either

$$(EHME) < (ABCA)$$

or

$$(HNKH) < (ABCA).$$

In every case, there is a closed curve which differs little from (T) and corresponds to a smaller action.

Therefore, in order that a closed curve may correspond to an action which is less than any infinitly adjacent closed curve, it is necessary /286 that this closed curve correspond to an unstable, periodic solution of the first category.

356. Is this condition sufficient? In order to determine this, let us study the asymptotic solutions corresponding to a similar unstable, periodic solution.

Let

$$x = \varphi_0(t , \qquad y = \psi_0(t)$$

be the equations of the periodic solution, and let

$$x = \varphi_0(t) + Ae^{\alpha t}\varphi_1(t) + A^2e^{2\alpha t}\varphi_2(t) + \dots,$$
$$y = \psi_0(t) + Ae^{\alpha t}\psi_1(t) + A^2e^{2\alpha t}\psi_2(t) + \dots$$

be the equations of the asymptotic solutions. The functions $\phi_i(t)$ and $\psi_i(t)$ will be periodic functions of t. We may also write the following setting $Ae^{\alpha t} = u$,

$$x = \varphi_0(t) + u\varphi_1(t) + \dots = \Phi(t, u),$$
$$y = \psi_0(t) + u\psi_1(t) + \dots = \Psi(t, u).$$

If u is sufficiently small, x and y will be uniform functions of t and u, which are periodic with respect to t of period 2π.

In addition, the functional determinant

$$\frac{\partial(x, y)}{\partial(t, u)} = \frac{d\Phi}{dt}\frac{d\Psi}{du} - \frac{d\Psi}{dt}\frac{d\Phi}{du}$$

will not vanish. For u = 0, this determinant may be reduced to

$$\varphi_0'(t)\psi_1(t) - \psi_0'(t)\varphi_1(t).$$

However, this expression is none other than the expression

$$\xi_1\eta_2 - \xi_2\eta_1$$

given in No. 345 divided by $e^{\alpha t}$. Therefore, it will not vanish if the unstable solution is of the first category.

Due to the fact that the functional determinant does not vanish for u = 0, it will not vanish for sufficiently small u either.

If u is sufficiently small, u, cos t and sin t will be uniform functions of x and y.

The equations of the asymptotic solutions may be written

$$\begin{cases} x = \Phi(t, \Lambda e^{\alpha t}) \\ y = \Psi(t, \Lambda e^{\alpha t}) \end{cases} \quad (1)$$

and it can be seen that the functional determinant /287

$$\frac{\partial(x, y)}{\partial(t, \Lambda)} = \frac{\partial(\Phi, \Psi)}{\partial(t, u)} e^{\alpha t}$$

cannot vanish. This means that the curves (1) do not have a double point, do not intersect each other, and do not intersect the trajectory (T) [this is the case if it is assumed that u is sufficiently small.

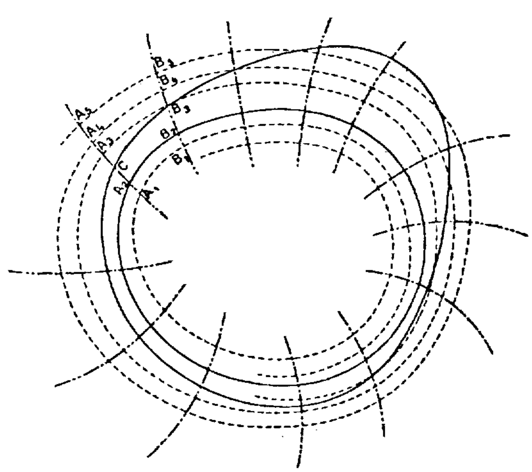

Figure 11

This would not be the case if the curves (1) were extended indefinitely in such a way that u becomes very large].

The curves (1) corresponding to the asymptotic solutions will therefore have the appearance of spirals passing around (T). This form is shown in the figure (2). The closed trajectory (T) is represented by a solid line, but I must point out that there are two curves shown by a solid line in the figure. Of these two curves, that which is located inside of the other represents (T).

The spiral curves (1) are represented by a dashed line ----.

I may note that there are two systems of asymptotic solutions corresponding to two characteristic exponents which are equal and have the opposite sign.

These asymptotic solutions of the second system will be spiral curves which are similar to curves (1), except that they turn in a different direction. They are not shown on the figure. /288

In the case of an unstable solution of the second category, curves (1) would have an entirely different form. They would intersect the closed trajectory (T) an infinite number of times, and the intersection points would form an infinite group having a finite number (even number) of boundary points. These boundary points would correspond to the values t_0, t_1 considered in No. 349.

357. Let us return to unstable solutions of the first category and to asymptotic solutions of the first system which are shown in figure (2). I shall establish the fact that the action is less for (T) than it is for an infinitely adjacent closed curve.

I shall consider an arbitrary closed curve which differs from (T) by an infinitely small amount. This curve, which I shall call (T'), is shown in figure (2) by a closed curve drawn with a solid line, outside of (T) and passing through the points C and B_3.

Let us confine ourselves to the case of absolute motion. In this case, we have the following well-known theorem.

Let A_1B_1, A_2B_2, ..., A_nB_n be a continuous series of trajectory arcs.

The end points of these arcs are located on two curves

$$A_1A_2\ldots A_n, \quad B_1B_2\ldots B_n.$$

If these two curves intersect the trajectories A_1B_1, A_2B_2, ..., A_nB_n orthogonally, we shall have

$$(A_1B_1)=(A_2B_2)=\ldots=(A_nB_n),$$

where the action corresponding to the arc A_1B_1 is always designated by (A_1B_1).

Let us therefore compile the orthogonal trajectories of the curves (1). These trajectories, which I shall call curves (2), will have the following differential equation

$$\left\{\begin{aligned} &\left(\frac{d\Phi^2}{dt^2}+\frac{d\Psi^2}{dt^2}\right)dt \\ &\quad+\left(\frac{d\Phi}{dt}\frac{d\Phi}{du}+\frac{d\Psi}{dt}\frac{d\Psi}{du}\right)(du+\alpha u\,dt)+\left(\frac{d\Phi^2}{du^2}+\frac{d\Psi^2}{du^2}\right)\alpha u\,du=0. \end{aligned}\right. \tag{3}$$

One curve (2), and only one, passes through each point of the plane, provided that u is small enough. This could only not be true if the coefficients dt and du were zero at the same time, which could only occur if the functional determinant of Φ and Ψ with respect to t and u vanished. We have seen that this was not the case. /289

The curves (2) are shown on the figure (2) by a dot-dash line -··-··-·· .

Let $A_1A_2, \ldots, A_5$, $B_1B_2, \ldots, B_5$ be two of the infinitely adjacent curves. They intersect the arc A_2B_2 on (T), the arcs A_1B_1, A_3B_3, A_4B_4, A_5B_5, on the curves (1) and the arc CB_3 on (T').

For my purpose, it is sufficient for me to establish the fact that the action of (CB_3) is larger than for the corresponding arc A_2B_2 of (T).

In effect, we have

$$(A_2B_2)=(A_3B_3)$$

and, in the infinitely small, curvilinear, rectangular triangle A_3CB_3, we have

$$(CB_3)>(A_3B_3).$$

We therefore have

$$(CB_3)>(A_2B_2),$$

and, consequently,

$$\text{action of } (T') > \text{action of } (T).$$

q.e.d.

358. We must now determine whether the same result is still obtained for relative motion.

The irreversibility of the equations constitutes a great difference from the preceding case. The action for an arbitrary arc AB is no longer the same as for the same arc traversed in a different direction. If an arbitrary curve satisfies differential equations, this will not hold true for the same curve traversed in a different direction.

Finally, the orthogonal trajectories of the curves (1) will no longer have the basic property which I discussed in the preceding section. However, there are other curves which I shall define, and which have this property. This is sufficient for the result of the preceding section to remain valid.

In No. 340, we obtained the following for the expression of the /290
action

$$J' = \int \left[ds\sqrt{H_0 + h} + \omega'(\xi\, d\eta - \eta\, d\xi) \right].$$

For purposes of simplification, I shall set $\sqrt{H_0 + h} = F$. I shall no longer designate the coordinates by ξ and η, but rather by x and y, in order to approximate the notation employed in the preceding sections. And I shall no longer designate the angular velocity by ω', but rather by ω, removing the accent which has become useless. I shall then have

$$J' = \int \left[F\sqrt{dx^2 + dy^2} + \omega(x\, dy - y\, dx) \right],$$

from which we have

$$\delta J' = \int \Big[\delta F\, ds + F\, \frac{dx\, \delta dx + dy\, \delta dy}{ds} + \omega(\delta x\, dy - \delta y\, dx) + \omega(x\, \delta dy - y\, \delta dx) \Big]$$

or, integrating by parts,

$$\left\{ \begin{aligned} \delta J' = \int & \left[\delta F\, ds + 2\omega(\delta x\, dy - \delta y\, dx) - \delta x\, d\left(\frac{F\, dx}{ds}\right) - \delta y\, d\left(\frac{F\, dy}{ds}\right) \right] \\ & + \left[F\, \frac{dx\, \delta x + dy\, \delta y}{ds} + \omega(x\, \delta y - y\, \delta x) \right]_0^1. \end{aligned} \right. \qquad (4)$$

The definitive expression of $\delta J'$ therefore includes two parts: a definite integral which must be taken between the same limits as the integral J', and a known part which I have placed between two brackets (according to common usage) with the indices 0 and 1. This notation indicates that we must calculate the expression between the brackets for the two integration limits, and must then take the difference.

Let us now assume that the expression included under the sign $\int$ in the second term of (4) is set equal to zero. We shall obtain differential equations which will be precisely the equations of motion, and which will be satisfied by all of our trajectories, particularly the curves (1).

These equations may be obtained in an infinite number of ways, because δx and δy are two entirely arbitrary functions.

We may first assume that $\delta x = 0$, from which we have $\delta F = \frac{dF}{dy}\,\delta y$. Dividing by $\delta y\, ds$, we may then write our equation as follows

$$\frac{dF}{dy} - 2\omega\frac{dx}{ds} = \frac{dF}{ds}\frac{dy}{ds} + F\frac{d^2y}{ds^2}. \tag{6}$$

If, on the contrary, we had assumed that $\delta y = 0$, we would obtain /291

$$\frac{dF}{dx} + 2\omega\frac{dy}{ds} = \frac{dF}{ds}\frac{dx}{ds} + F\frac{d^2x}{ds^2}.$$

These two equations are equivalent, as could readily be determined beforehand. If they are added after having multiplied them by $\frac{dy}{ds}$ and $\frac{dx}{ds}$, respectively, and if the following relationships are taken into account

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1; \qquad \frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} = 0,$$

we arrive at an identity.

If we therefore consider the curves (1), they will satisfy equation (6). If we take this equation into account, relationship (4) becomes

$$\delta J' = \left[F\frac{dx\,\delta x + dy\,\delta y}{ds} + \omega(x\,\delta y - y\,\delta x)\right]_0^1.$$

Let A_1B_1, A_2B_2, ..., A_nB_n be a continuous series of arcs pertaining to the curves (1), whose end points $A_1A_2 \ldots A_n$, $B_1B_2 \ldots B_n$ form two continuous curves C and C'.

Let $A_iB_i, A_{i+1}B_{i+1}$ be two of these arcs which differ from each other by an infinitely small amount. Let x, y be the coordinates of the point A_i, $x + \delta x$, and let $y + \delta y$ be the coordinates of the infinitely adjacent point A_{i+1}.

Let J' be the action relative to the arc A_iB_i and $J' + \delta J'$ be the action relative to the arc $A_{i+1}B_{i+1}$.

If α is the angle which the tangent to the curve A_iB_i [which is a curve (1)] makes with the axis of the x's, and if the two curves C and C' satisfy the differential equation

$$F(\cos\alpha\,\delta x + \sin\alpha\,\delta y) + \omega(x\,\delta y - y\,\delta x) = 0, \tag{7}$$

we shall have

$$\delta J' = 0$$

and, consequently,

$$(A_1B_1) = (A_2B_2) = \ldots = (A_nB_n).$$

The curves defined by equation (7) may therefore play the role which the orthogonal trajectories of the curves (1) played in the preceding section.

We may therefore consider figure (2) again, and we may assume that the curves shown by the dot-dash line no longer represent these orthogonal trajectories, but rather the curves defined by equation (7). There will be nothing to change in the proof. /292

However, one point is no longer clear. In the infinitely small, rectangular triangle A_3CB_3, I have

$$(CB_3) > (A_3B_3).$$

The triangle is no longer rectangular, and in addition I have changed the definition of the action. Does the inequality still exist?

It may be readily seen that this equality equals conditions (a) of No. 341, and we have seen in No. 344 that they are fulfilled. The inequality therefore holds, and our proof remains valid.

To sum up, <u>in order that a closed curve corresponds to an action which is less than any infinitely adjacent closed curve, it is necessary and sufficient that this closed curve corresponds to an unstable, periodic solution of the first category</u>.

359. We must make a few remarks regarding the classification of unstable solutions into two categories.

From another point of view, the unstable, periodic solutions may be divided into two classes. Those of the first class are those for which the characteristic exponents α is real, so that $e^{\alpha t}$ is real and positive, where T is the period.

The solutions of the second class are those for which this exponent α has $\frac{i\pi}{T}$ as an imaginary part, so that $e^{\alpha T}$ is real and negative.

In the preceding statements, we only considered unstable solutions of the first class. Let us see whether those of the second class may also be divided into two categories.

We may set $$\alpha = \alpha' + \frac{i\pi}{T},$$

where α' is real. We may then set, just as in No. 346"

$$\xi_2 = e^{\alpha' t}\varphi_2, \qquad \eta_2 = e^{\alpha' t}\psi_2,$$
$$\xi_3 = e^{-\alpha' t}\varphi_3, \qquad \eta_3 = e^{-\alpha' t}\psi_3,$$

where ϕ_2, ψ_2, ϕ_3 and ψ_3 are functions of t which change sign when t changes into t + T. These functions will be real. /293

We then have

$$\zeta(t) = e^{2\alpha' t}\frac{\xi_1\psi_2 - \eta_1\varphi_2}{\xi_1\psi_3 - \eta_1\varphi_3} = e^{2\alpha' t}G(t).$$

The numerator and the denominator of G are functions of t which change sign when t changes into t + T.

It is therefore certain that these two functions vanish, and consequently that the same holds true for

$$\xi_1\eta_2 - \eta_1\xi_2, \quad \xi_1\eta_3 - \eta_1\xi_3.$$

These last two functions satisfy the same linear differential equation of the second order, whose coefficients are periodic functions of t which have not become infinite. The coefficient of the second derivative may be reduced to a constant. These two functions cannot become zero at the same time, because if two integrals of the same linear equation become zero at the same time, they could only differ by a constant factor. However, $\zeta(t)$ is not a constant.

The numerator and the denominator of $\zeta(t)$ therefore both become zero, and do not become zero at the same time. Therefore $\zeta(t)$ [and consequently G(t)] may vanish and become infinite.

All of the unstable solutions in question therefore belong to the second category. Apart from this, there is nothing to be changed in the preceding statements.

CHAPTER XXX

FORMULATION OF SOLUTIONS OF THE SECOND TYPE

360. We shall now demonstrate the manner in which periodic solutions of the second type may be effectively formulated. /294

Let

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \tag{1}$$

be a system of canonical equations. Let us assume that they have a periodic solution of the first type

$$x_i = \varphi_i(t), \quad y_i = \psi_i(t). \tag{2}$$

We plan to study periodic solutions of the second type which proceed from the solution of the first type (2).

The analysis may be simplified, at least for purposes of discussion, if equations (1) are reduced to a suitable form by a series of changes in the variables.

We shall assume that there are only two degrees of freedom. When t increases by one period, y_1 and y_2 will increase, respectively, by

$$2k_1\pi, \quad 2k_2\pi,$$

where k_1 and k_2 are integer numbers.

I may first assume that $k_1 = 0$, because, if this were not the case, I could cause k_1 to vanish by the change in variables given in No. 202.

I may then assume that the periodic solution (2) may be reduced to

$$x_1 = 0, \quad x_2 = 0, \quad y_1 = 0,$$

because, if this were not true, I could perform the change in variables presented in No. 208.

Under this assumption, we shall see how the determination of periodic solutions of the second type is related either to the analysis given /295 in No. 274, or to the analysis presented in No. 44.

361. Let us recall the results obtained in Nos. 273 to 277. Let the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i} \quad (i = 1, 2, \ldots, n) \tag{1}$$

include a parameter λ, and let us assume that they have one periodic solution

$$x_i = \varphi_i(t), \qquad y_i = \psi_i(t), \tag{2}$$

of period T_0, corresponding to the value C_0 of the energy constant and corresponding to $\lambda = 0$. Equations (1) will be formally satisfied by series having the following form; these series will develop in powers of the quantities

$$\lambda, \quad A_k e^{\alpha_k t}, \quad A'_k e^{-\alpha_k t} \qquad (k = 1, 2, \ldots, n-1).$$

The coefficients will be periodic functions of t + h, depending on the energy constant C. The period T will also depend on C and the products $A_k A'_k$. It may be reduced to T_0 for

$$C = C_0, \qquad A_k A'_k = 0, \qquad \lambda = 0.$$

The exponents α_k are constants which may be developed in powers of λ and the products $A_k A'_k$, and in addition they depend on C. They may be reduced to the characteristic exponents of the solution (2) for

$$C = C_0, \qquad A_k A'_k = 0, \qquad \lambda = 0.$$

The A_k's, the A'_k's and h's are integration constants.

When studying asymptotic solutions, we assumed that the α_k's were real, and we made one of the two constants A vanish.

In order to apply the same results to a study of periodic solutions of the second type, we shall assume, on the contrary, that the exponents α_k are purely imaginary.

I shall assume only two degrees of freedom, which allows me to remove the index k which has become useless. /296

In order that we may obtain periodic solutions, it is necessary that the exponent α be commensurable with $\frac{2\pi}{T}$. If our series were convergent, this condition would be sufficient. However, they are divergent, and only satisfy equations (2) from the formal point of view. A more detailed discussion is therefore necessary. A method similar to that employed in Nos. 211 and 218 could be applied. We would thus obtain series which would have the same relationship with those given in Nos. 273 and 277 as the series of M. Bohlin have with those given in Nos. 125 and 127. By an indirect method, we would thus fall back on the periodic solutions of the second type. However, I prefer to proceed in a different manner.

Effective Formulation of the Solutions

362. By performing the changes in variables presented in Nos. 209, 210, 273, 274, which are always applicable when we have a system of canonical equations having a periodic solution, we may change our equations to the form of the equations presented in No. 274. In this section, we have formulated the following equations (page 95)

$$\frac{dx'_i}{dt} = \frac{dF'}{dy'_i}, \quad \frac{dy'_i}{dt} = -\frac{dF'}{dx'_i}, \tag{3}$$

$$F' = F'_0 + \varepsilon F'_1 + \varepsilon^2 F'_2 + \ldots,$$

where F'_p is a whole polynomial in x'_1, y'_1, x'_2, which will be homogeneous of degree p + 2 if it is assumed that x'_1 and y'_1 are of the first order, and x'_2 is of the second order. The coefficients of this polynomial are periodic functions of y'_2 whose period is 2π.

Just as in Section No. 274, we shall remove the accents which have become useless and shall write F, F_p, x_i, y_i instead of F', F'_p, x'_i, y'_i.

We then assume the following (see pages 97, 98, 99)

$$F_0 = Hx_2 + 2Bx_1y_1,$$

where H and B are constants. I could also set H = 1, but I shall not do this.

Just as on page 99, let us then set /297

$$x_1 = e^v\sqrt{u}, \quad y_1 = e^{-v}\sqrt{u};$$

The equations will retain the canonical form, and we shall have

$$F_0 = Hx_2 + 2Bu;$$

The other terms F_1, F_2, ..., will be periodic of period 2π, both with respect to i_v and with respect to y_2.

Our equations will then have the form which is similar to that which we have studied several times, and in particular, in Nos. 13, 42, 125, etc., where the parameter ε plays the role of the parameter μ. Therefore, we may employ the procedure given in No. 44 for these equations.

However, there is one obstacle. The Hessian of F_0 with respect to x_2 and u is zero, which is precisely one of the exceptions given in No. 44.

This fact compels me to assume that F depends on a certain parameter

λ, and we shall carry out the development at the same time in powers of λ and of ε. We saw in Chapter XXVIII that when studying periodic solutions of the second type it is always convenient to introduce a similar parameter, because the property of being reduced to a solution of the first type for $\lambda = 0$, and of differing from it for $\lambda \gtrless 0$, characterizes solutions of the second type.

To facilitate the discussion, instead of an arbitrary parameter, I shall introduce two parameters, which I shall call λ and μ.

We shall therefore assume that the different coefficients of F may be developed in powers of two parameters λ and μ, and that for $\mu = \lambda = 0$, H and 2B may be reduced to -1 and to -in, where n is a commensurable, real number.

I shall assume that λ and μ may be developed in increasing powers of ε, in the following form

$$\lambda = \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \ldots; \qquad \mu = \mu_1 \varepsilon + \mu_2 \varepsilon^2 + \ldots,$$

where λ_1, λ_2, ..., are constants which I shall provisionally leave undetermined, but I reserve the right to determine them in the computations which follow.

Under this assumption, let us follow the computation presented in No. 44 step by step. We shall set /298

$$\left\{\begin{aligned} x_2 &= \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \ldots, \\ y_2 &= \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \ldots, \\ u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots, \\ v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots. \end{aligned}\right. \tag{4}$$

These formulas are similar to the formulas (2) of No. 44.

The ξ_k's, η_k's, u_k's, and the v_k's are therefore periodic functions of t; ξ_0 and u_0 are constants, and we have

$$\eta_0 = t, \qquad v_0 = nt + \varpi,$$

where ϖ is an integration constant which I shall determine more completely below.

Instead of λ, μ, x_2, y_2, u and v, let us substitute their expansions in powers of ε in F. Then, F may be equally developed in powers of ε, and we shall have

$$F = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \ldots.$$

I would like to point out that Φ_k is homogeneous of degree k + 2, if we assume that ξ_p and u_p are of degree p + 2, η_p and v_p are of the degree p, λ_p and μ_p is of degree p.

It is therefore a whole polynomial with respect to

$$\xi_p, \quad u_p, \quad \eta_p, \quad v_p, \quad \lambda_p, \quad \mu_p \qquad (p > 0),$$

and with respect to

$$\sqrt{u_0}\, e^{v_0}, \qquad \sqrt{u_0}\, e^{-v_0}.$$

These last two quantities may be assumed to be on the order of 1. Finally, the coefficients of this polynomial are periodic functions of η_0 whose period is 2π.

In addition, we shall obtain

$$\Phi_k = \Theta_k - \xi_k - inu_k + \lambda_k H_0 \xi_0 + 2 B_0 \mu_k u_0,$$

where H_0 and B_0 are the values of $\frac{dH}{d\lambda}$ and $\frac{dB}{d\mu}$ for $\lambda = \mu = 0$. (We may assume that we have $\frac{dH}{d\mu} = \frac{dB}{d\lambda} = 0$ for $\lambda = \mu = 0$.) In addition, Θ_k depends only on

$$\xi_p, \quad \eta_p, \quad u_p, \quad v_p, \quad \lambda_p, \quad \mu_p \qquad (p \leq k-1).$$

Our differential equations may then be written /299

$$\frac{d\xi_k}{dt} = \frac{d\Phi_k}{d\eta_0}, \qquad \frac{d\eta_k}{dt} = -\frac{d\Phi_k}{d\xi_0}, \qquad \frac{du_k}{dt} = \frac{d\Phi_k}{dv_0}, \qquad \frac{dv_k}{dt} = -\frac{d\Phi_k}{du_0}. \tag{5}$$

For k = 0, they may be reduced to

$$\frac{d\xi_0}{dt} = \frac{du_0}{dt} = 0; \qquad \frac{d\eta_0}{dt} = 1; \qquad \frac{dv_0}{dt} = in.$$

They demonstrate the fact that ξ_0 and u_0 are constants, and that

$$\eta_0 = t, \qquad v_0 = int + \varpi,$$

where ϖ is a constant which must be determined.

We may advantageously add other equations having a similar form to equations (4) and (5), which are only transformations of them.

Let us develop x_1 and y_1 in powers of ε, and let

$$\begin{cases} x_1 = \xi_0' + \varepsilon\xi_1' + \varepsilon^2\xi_2' + \dots, \\ y_1 = \eta_0' + \varepsilon\eta_1' + \varepsilon^2\eta_2' + \dots \end{cases} \tag{4'}$$

The expansions (4') may be directly concluded from the two last expansions (4).

We then find that Φ_k is a whole polynomial with respect to the quantities

$$\xi_p,\quad \eta_p,\quad \xi'_p,\quad \eta'_p,\quad \lambda_p,\quad \mu_p \quad \text{(writing } \eta_0 \text{ separately)}, \tag{6}$$

and that this polynomial is homogeneous of degree k + 2, if we assume that

$$\begin{aligned} \xi_p &\text{ is of degree } \quad p+2, \\ \xi'_p, \eta'_p &\text{ is of degree } \quad p+1, \\ \eta_p, \lambda_p, \mu_p &\text{ is of degree } \quad p. \end{aligned}$$

We then have the following equations

$$\frac{d\xi'_k}{dt} = \frac{d\Phi_k}{d\eta'_0}, \qquad \frac{d\eta'_k}{dt} = -\frac{d\Phi_k}{d\xi_0}, \tag{5'}$$

which are equivalent to the last two equations (5).

We may note that $\frac{d\Phi_k}{d\eta_0}, \frac{d\Phi_k}{d\xi_0}, \frac{d\Phi_k}{d\xi'_0}, \frac{d\Phi_k}{d\eta'_0}$ are polynomials having the same form as Φ_k with respect to the quantities (6). Using the same convention employed above regarding the degrees, we find that they are homogeneous, the first of the order k + 2, the second of order k, and the last two of /300
the order k + 1.

We have

$$\xi'_0 = \sqrt{u_0}\,e^{nit+\varpi}, \qquad \eta'_0 = \sqrt{u_0}\,e^{-(nit+\varpi)}.$$

Let us replace ξ'_0, η'_0 by these values, and at the same time let us replace η_0 by t, in equations (5) and (5'), in which it must be assumed that we have set k = 1, and let us employ them to determine ξ_1, η_1, u_1, v_1, ξ'_1 η'_1.

We thus have the six following equations

$$\left\{\begin{aligned} &\frac{d\eta_1}{dt} = -\frac{d\Phi_1}{d\xi_0} = -\frac{d\Theta_1}{d\xi_0} - \lambda_1 H_0, \qquad \frac{d\xi_1}{dt} = \frac{d\Theta_1}{d\eta_0}, \\ &\frac{du_1}{dt} = \frac{d\Theta_1}{dv_0}, \qquad \frac{dv_1}{dt} = -\frac{d\Theta_1}{du_0} - 2\mu_1 B_0, \\ &-\frac{d\xi'_1}{dt} = \frac{d\Theta_1}{d\eta'_0} - in\frac{du_1}{d\eta'_0} + 2B_0\mu_1\frac{du_0}{d\eta'_0} = \frac{d\Theta_1}{d\eta'_0} - in\xi'_1 + 2B_0\mu_1\xi'_0, \\ &-\frac{d\eta'_1}{dt} = -\frac{d\Theta_1}{d\xi'_0} + in\frac{du_1}{d\xi'_0} - 2B_0\mu_1\frac{du_0}{d\xi'_0} = -\frac{d\Theta_1}{d\xi'_0} + in\eta'_1 - 2B_0\mu_1\eta'_0. \end{aligned}\right. \tag{7}$$

Let us first consider the second of these equations. The second term is a homogeneous, whole polynomial of the third degree with respect to

$$\sqrt{\xi_0},\ \xi'_0,\ \eta'_0,$$

whose coefficients are periodic functions of $\eta_0 = t$, of period 2π.

Since n is commensurable, our second term will also be a periodic function of t on which it depends in two ways: by means of η_0 which equals t, and by means of ξ'_0 and η'_0 which are functions of $nt + \varpi$.

The period will be a multiple of 2π, i.e., it will equal as many multiples of 2π as there are units in the denominator of n.

Our second term can therefore be developed in Fourier series in the following form

$$\Sigma A e^{i[pt+q(nt+\varpi)]}, \tag{8}$$

where p and q are integer numbers. However, q does not exceed 3 in absolute value, since our second term is a polynomial of the third degree.

As a result, in general the mean value of the second term is zero. This mean value will be obtained by retaining the terms which are independent of t in the series (8), i.e., /301

$$p + qn = 0.$$

I have stated that $|q|$ can not exceed 3. I would like to add that, due to the fact that our second term is a whole and homogeneous polynomial of degree 3 in ξ_0, ξ'_0 and η'_0, it is assumed that ξ_0 is of degree 2 or a second term cannot contain ξ'_0 and η'_0 except in an uneven power -- i.e., q must be odd and can only take one of the values ± 1 or ± 3.

Therefore, we can only have

$$p + qn = 0$$

if the denominator of n equals 1 or 3.

We shall exclude the first hypothesis which would make n a whole number, but two cases remain for our consideration:

1. The denominator of n does not equal 3. In this case, due to the fact that the mean value of the second term is zero, the equation will immediately provide us with ξ_1 by simple quadrature. Then ξ_1 is determined up to a constant which I shall call γ_1, and this constant remains undetermined up to a new order. It should be pointed out that the same holds true for ϖ.

2. The denominator of n equals 3. In order that the equation may be integrated, it is necessary that the value of the second term be zero.

For this purpose, we shall employ the constant $\overline{\omega}$.

Let $[\Theta_1]$ be the mean value of Θ_1. We should point out that we have

$$-n\left[\frac{d\Theta_1}{d\eta_0}\right]=\frac{d[\Theta_1]}{d\overline{\omega}};$$

and we shall therefore determine $\overline{\omega}$ by the equation

$$\frac{d[\Theta_1]}{d\overline{\omega}}=0, \tag{9}$$

and a quadrature will then provide us with ξ_1, up to a constant γ_1.

Let us now take the first equation (7). The same line of reasoning may be pursued for this equation. However, since $\frac{d\Theta_1}{d\xi_0}$ is no longer a polynomial of the third degree, but rather of the first degree, and since n is not an integer number, we shall be certain that the mean value of $\frac{d\Theta_1}{d\xi_0}$ is zero. /302

It is therefore sufficient for us to take $\lambda_1 = 0$ in order that the second term may have a mean value of zero, and in order that η_1 may be determined to a constant δ_1.

Let us now pass on to the last two equations (7). They may be written as follows

$$-\frac{d\xi_1'}{dt}+in\xi_1'=\frac{d\Theta_1}{d\eta_0'}+2B_0\mu_1\xi_0',$$
$$-\frac{d\eta_1'}{dt}-in\eta_1'=-\frac{d\Theta_1}{d\xi_0'}-2B_0\mu_1\eta_0'.$$

The second terms are the known periodic functions of t. In order that integration may be possible, it is therefore sufficient that the second term of the first equation does not include a term containing e^{int}, and that the second term of the second equation does not include a term containing e^{-int}.

This double condition could be discussed more readily by considering these third and fourth equations (7) which are equivalent to the last two, and which may be written

$$\frac{du_1}{dt}=\frac{d\Theta_1}{dv_0}, \quad \frac{dv_1}{dt}=-\frac{d\Theta_1}{du_0}-2\mu_1 B_0.$$

It is necessary that the mean values of the second terms be zero.

With respect to the first of these equations, the condition is fulfilled by itself, and we have

$$\left[\frac{d\theta'_1}{dv_0}\right] = \frac{d[\theta_1]}{d\varpi}.$$

This latter expression is zero because of equation (9), if the denominator of n equals 3, and in the opposite case because $[\Theta_1]$ is identically zero.

The second condition may be written

$$\frac{d[\theta_1]}{du_0} = -2\mu_1 B_0.$$

If the denominator of n equals 3, it will provide us with μ_1.

On the other hand, if the denominator does not equal 3, it will provide us with $\mu_1 = 0$, because $[\Theta_1]$ is identically zero.

Thus, we may see that ξ_1, η_1, ξ'_1, η'_1 are periodic functions of t and of $\overline{\omega}$. They may therefore be developed in Fourier series in the following form /303

$$\Sigma A e^{i(pt+qnt+q\varpi)}.$$

However, we may add a few words more. We must deal with equations having the following form

$$\frac{d\xi}{dt} = X = \Sigma A e^{i(pt+qnt+q\varpi)}, \qquad \frac{d\eta}{dt} + in\eta = Y = \Sigma B e^{i(pt+qnt+q\varpi)},$$

and we shall derive the following

$$\xi = \sum \frac{A}{i(p+qn)} e^{i(pt+qnt+q\varpi)} + \gamma,$$
$$\eta = \sum \frac{B}{i(p+qn+n)} e^{i(pt+qnt+q\varpi)} + \gamma' e^{-int},$$

where γ and γ' are integration constants.

Therefore, if X and Y are whole and homogeneous polynomials with respect to

$$\sqrt{\xi_0}, \quad \sqrt{u_0}e^{i(nt+\varpi)}, \quad \sqrt{u_0}e^{-i(nt+\varpi)}$$

the same will hold true for ξ and η, unless it is assumed that the constants γ and γ' are zero. If it is not assumed that these constants are zero, ξ and η will still be whole polynomials, but not homogeneous.

Let us apply these principles to the quantities which we have just computed. Due to the fact that

$$\frac{d\Theta_1}{d\xi_0}, \quad \frac{d\Theta_1}{d\eta_0}, \quad \frac{d\Theta_1}{d\xi_0'}, \quad \frac{d\Theta_1}{d\eta_0'}$$

are polynomials which, according to the convention which we have employed regarding degrees, are of the following degrees, respectively

$$1, \quad 3, \quad 2, \quad 2,$$

the same will hold true for

$$\eta_1, \quad \xi_1, \quad \eta_1', \quad \xi_1'.$$

When we substitute the values of these quantities which are, respectively, of degrees 1, 3, 2, 2, instead of these quantities in Θ_2, it may be seen that Θ_2 becomes a polynomial of the fourth degree, and that /304

$$\frac{d\Theta_2}{d\xi_0}, \quad \frac{d\Theta_2}{d\eta_0}, \quad \frac{d\Theta_2}{d\xi_0'}, \quad \frac{d\Theta_2}{d\eta_0'}$$

will be polynomials of the following degrees, respectively

$$2, \quad 4, \quad 3, \quad 3.$$

We may therefore formulate a generalization of this result.

Equations (5) and (5') enable us to compute the unknowns ξ_k, η_k, ξ_k', η_k' from place to place. This would only be prevented if the mean value of the second term of one of the equations (5) were different from zero.

Let us assume that this does not occur. It may be stated that

$$\xi_k, \quad \eta_k, \quad \xi_k', \quad \eta_k'$$

will be polynomials of the following degrees

$$k+2, \quad k, \quad k+1, \quad k+1$$

with respect to

$$\sqrt{\xi_0}, \quad \sqrt{u_0}\, e^{i(nt+\varpi)}, \quad \sqrt{u_0}\, e^{-i(nt+\varpi)}, \tag{10}$$

where the coefficients of these polynomials are themselves periodic functions of t of period 2π.

Let us assume that this is valid for every value of the index which is less than k.

We know that Θ_k is a whole polynomial of degree k + 2 with respect to

$$\xi_q, \quad \eta_q, \quad \xi_q', \quad \eta_q' \qquad (q < k) \tag{11}$$

assuming that these quantities are of degree q + 2, q, q + 1, q + 1, respectively. If we substitute polynomials whose degree, with respect to the quantities (10), is precisely q + 2, q, q + 1, q + 1, in place of these quantities (11), it is apparent that the result of the substitution will be a polynomial of degree k + 2 with respect the the quantities (10).

Therefore, Θ_k is a polynomial of degree k + 2 with respect to the quantities (10), and for the same reason

$$\frac{d\Theta_k}{d\xi_0},\ \frac{d\Theta_k}{d\eta_0},\ \frac{d\Theta_k}{d\xi'_0},\ \frac{d\Theta_k}{d\eta'_0}$$

will be polynomials of the following degrees /305

$$k,\ k+2,\ k+1,\ k+1$$

with respect to the same quantities.

The same holds true for the second terms of the first, second, fifth, and sixth equations (7). Consequently, by repeating the previous line of reasoning, we should readily see that the same holds true for

$$\eta_k,\ \xi_k,\ \eta'_k,\ \xi'_k.$$

q.e.d.

The integration of equations (7) has introduced four new integration constants. They provide us with information concerning ξ_1, η_1, ξ'_1, η'_1, up to the following terms

$$\gamma_1,\ \delta_1,\ \gamma'_1 e^{+i(nt+\varpi)},\ \delta'_1 e^{-i(nt+\varpi)},$$

containing the four arbitrary constants

$$\gamma_1,\ \delta_1,\ \gamma'_1,\ \delta'_1.$$

We shall retain only one of these constants and we shall set

$$\gamma_1 = \delta_1 = 0, \qquad \delta'_1 = -\gamma'_1.$$

Under this assumption, let us try to determine

$$\xi_2,\ \eta_2,\ \xi'_2,\ \eta'_2,$$

by means of equations (5) and (5') and by setting k = 2.

It is necessary that the second term of the first equation (5) has a mean value of zero. This mean value equals

$$\left[\frac{d\Theta_2}{d\eta_0}\right].$$

and we always employ the brackets to represent the mean value of a function. We must therefore have

$$\left[\frac{d\Theta_2}{d\eta_0}\right] = 0. \qquad (9')$$

Let us assume that Θ_2 is developed in Fourier series in the following form

$$\Sigma A e^{i(pt+qnt+q\varpi)}.$$

Since θ_2 is a polynomial of the fourth degree, q could not exceed 4 in absolute value. Consequently, if the denominator of n is larger than 4, $[\Theta_2]$ will be identically zero, and the condition (9') will be fulfilled by itself. The constant ϖ will remain undetermined. /306

If the denominator of n equals 2 or 4, the condition (9') will determine ϖ.

If the denominator of n equals 3, the constant $\overline{\omega}$ has already been determined by condition (9), and condition (9') will enable us to determine the constant γ'_1.

Let us calculate the terms depending on this constant γ'_1 in Θ_2.

We obviously will obtain

$$\gamma'_1\left(\frac{d\Theta_1}{d\xi'_0} e^{i(nt+\varpi)} - \frac{d\Theta_1}{d\eta'_0} e^{-i(nt+\varpi)}\right),$$

i.e.,

$$\frac{\gamma'_1}{i\sqrt{u_0}} \frac{d\Theta_1}{d\varpi}.$$

The mean value of this will be

$$\frac{\gamma'_1}{i\sqrt{u_0}} \frac{d[\Theta_1]}{d\varpi}.$$

The condition (9') may therefore be written { if it is noted that $-n\left[\frac{d^2\Theta_1}{d\eta_0\, d\varpi}\right] = \frac{d^2[\Theta_1]}{d\varpi^2}$ }

$$\frac{\gamma'_1}{i\sqrt{u_0}} \frac{d^2[\Theta_1]}{d\varpi^2} + H = 0,$$

where H depends on $\overline{\omega}$, but not on γ'_1.

If the denominator of n does not equal 3, $[\Theta_1]$ is zero and condition (9') is independent of γ'_1. Therefore, if this denominator equals 2 or 4, equation (9') will depend on $\overline{\omega}$ and not on γ'_1 and will determine $\overline{\omega}$.

If the denominator equals 3, condition (9') depends on γ'_1 and will determine γ'_1 (it will provide us with $\gamma'_1 = 0$).

In any case, having thus determined ξ_2, let us try to calculate η_2 by means of the second equation (5). We shall employ λ_2 in such a way that the second term has a mean value of zero.

We should point out that λ_2 will not be zero in general, and /307

$$\frac{d[\Theta_2]}{d\xi_0}$$

will not be zero in general, because, due to the fact that Θ_2 is a polynomial of degree 4, it will include a term containing ξ_0^2 which is independent of the ξ'_k's and η'_k's. The coefficient of this term will be a periodic function of t of period 2π, and the mean value will not be zero in general.

Let us proceed to equations (5') or, which is the same thing, to the last two equations (5). The second terms of these last two equations must have a mean value of zero.

We must have

$$\left[\frac{d\Theta_2}{du_0}\right] = -2\mu_2 B_0,$$

which determines μ_2. However,

$$u_0 \frac{d\Theta_2}{du_0} = \eta'_0 \frac{d\Theta_2}{d\xi'_0} + \xi'_0 \frac{d\Theta_2}{d\eta'_0}$$

is a polynomial of the fourth order. F_2 therefore includes terms containing $x_1^2 y_1^2$, and consequently $u_2 \dfrac{d\Theta_2}{du_0}$ includes a term containing

$$u_0^2 = (\sqrt{u_0}e^{i(nt+\varpi)})^2(\sqrt{u_0}e^{-i(nt+\varpi)})^2.$$

The coefficient of this term is a periodic function of t, whose mean value is not zero in general. Therefore, in general $\left[\dfrac{d\Theta_2}{du_0}\right]$ and, consequently, μ_2 are not zero. This is the same line of reasoning as is employed for λ_2.

We must then have

$$\left[\frac{d\Theta_2}{d\varpi_0}\right] = 0. \tag{12}$$

However, it may be stated that this condition is fulfilled by itself.

We have the energy integral F = const., from which we may deduce the series of equations

$$\Phi_0 = \text{const.}, \quad \Phi_1 = \text{const.}, \quad \Phi_2 = \text{const.}, \quad \ldots.$$

Let us consider the third of these equations

$$\Phi_2 = \theta_2 - \xi_2 - inu_2 + \lambda_2 H_0 \xi_0 + 2 B_0 \mu_2 u_0 = \text{const.}$$

This equation may replace the fourth equations (5) and, when λ_2, μ_2, ξ_2, η_2 and v_2 have been determined by means of the first three equations (5), it will determine u_2 without any integration. We may therefore be assured that the determination of u_2 is possible, and, consequently, that the condition (12) is fulfilled. /308

We will have thus determined ξ_2, η_2, ξ'_2, η'_2 up to the following terms

$$\gamma_2, \quad \delta_2, \quad \gamma'_2 e^{i(nt+\varpi)}, \quad \delta'_2 e^{-i(nt+\varpi)},$$

depending on the four arbitrary constants. We shall retain only one of these constants, and we shall set

$$\gamma_2 = \delta_2 = 0, \qquad \delta'_2 = -\gamma'_2.$$

363. The calculation will be continued in the same way. The ability of equations (5) to be integrated requires the following conditions

$$\left[\frac{d\theta_k}{d\eta_0}\right] = 0, \quad \left[\frac{d\theta_k}{dv_0}\right] = 0; \quad \left[\frac{d\theta_k}{d\xi_0}\right] + \lambda_k H_0 = 0, \quad \left[\frac{d\theta_k}{du_0}\right] + 2\mu_k B_0 = 0.$$

The last two conditions will determine λ_k and μ_k. The second will be a consequence of the first, according to what we have learned with respect to condition (12). We must then study the first.

Expression $\frac{d\Theta_k}{d\eta_0}$ is a polynomial of order k + 2. If it is developed in Fourier series

$$\Sigma A e^{i(pt+qnt+q\varpi)},$$

the integer number q cannot exceed k + 2 in absolute value. If k + 2 is smaller than the denominator of n, we could not have

$$p + qn = 0$$

and the mean value of our expression will be zero. The condition

$$\left[\frac{d\theta_k}{d\eta_0}\right] = 0 \qquad (13)$$

will therefore be fulfilled by itself.

We have introduced the following arbitrary constants:

$$\varpi, \gamma'_1, \gamma'_2, \ldots \tag{14}$$

and Θ_k may depend on /309

$$\varpi, \gamma'_1, \gamma'_2, \ldots, \gamma'_{k-1}.$$

Let us determine the form of this dependence. Let us assume that we are considering the expansion

$$F = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \ldots \tag{15}$$

and that in this expansion we replace the ξ's, the η's, the ξ''s, and the η''s by their values. The different terms of the expansion will then depend on the constants (14). In this expansion (15), let us cancel all the constants γ', retaining only $\overline{\omega}$. We will thus obtain a new expansion

$$\Phi'_0 + \varepsilon\Phi'_1 + \varepsilon^2\Phi'_2 + \ldots. \tag{16}$$

In the expansion (16), let us now replace the constant $\overline{\omega}$ by the expansion

$$\varpi + \varepsilon\varpi_1 + \varepsilon^2\varpi_2 + \ldots,$$

where $\overline{\omega}_1$, $\overline{\omega}_2$ are new constants. Each term in the expansion (16) may be developed in its turn in powers of ε. When this expansion is ordered anew in powers of ε, we obtain a new expansion

$$\Phi''_0 + \varepsilon\Phi''_1 + \varepsilon^2\Phi''_2 + \ldots. \tag{17}$$

This expansion must be identical to the expansion (15), under the condition that the constants $\overline{\omega}$ are replaced by the suitably chosen functions of the constants γ'_k.

It may be readily seen that Φ''_k depends only on

$$\varpi, \varpi_1, \ldots, \varpi_{k-1}$$

and that Φ_k depends only on

$$\varpi, \gamma'_1, \ldots, \gamma'_{k-1}.$$

We may conclude from this that $\overline{\omega}_k$ depends only on

$$\gamma'_1, \gamma'_2, \ldots, \gamma'_k$$

and γ'_k on

$$\varpi_1, \varpi_2, \ldots, \varpi_k.$$

It may be readily seen that /310

$$\Phi''_k = \Sigma A D \Phi'_m \varpi_1^{\alpha_1} \varpi_2^{\alpha_2} \ldots \varpi_k^{\alpha_k},$$

where A is a numerical coefficient and where $D\Phi'_m$ is a derivative of Φ'_m with respect to $\overline{\omega}$. The order of this derivative equals

$$\alpha_1 + \alpha_2 + \ldots + \alpha_k$$

and we then have

$$k = m + \alpha_1 + 2\alpha_2 + \ldots + k\alpha_k.$$

Since m is at least equal to 1, and since Φ_0 does not depend on $\overline{\omega}$,' it may be seen that α_k is zero, which we already knew.

Let us consider an arbitrary term where α_k, α_{k-1}, ..., α_{h+1} are zero, but where α_h is not zero. We must have

$$m \leq k - h.$$

If the denominator of n is larger than k - h + 2, the mean value of $D\Phi_m$ will be zero. This means that those terms of Φ''_k which depend on $\overline{\omega}_h$ have a mean value of zero.

An important result may be concluded from this concerning the mean value of Φ''_k, and consequently the mean value of Θ_k.

If the denominator of n equals k + 2, $[\Theta_k]$ will depend only on $\overline{\omega}$.

If the denominator of n equals k + 1, $[\Theta_k]$ will depend on $\overline{\omega}$ and $\overline{\omega}_1$.

If the denominator of n equals k, $[\Theta_k]$ will depend on $\overline{\omega}$, $\overline{\omega}_1$ and $\overline{\omega}_2$.

If the denominator of n equals k - 1, $[\Theta_k]$ will depend on $\overline{\omega}$, $\overline{\omega}_1$, $\overline{\omega}_2$ and $\overline{\omega}_3$.

The statements which I have just made concerning $[\Theta_k]$ also apply to $\left[\frac{d\Theta_k}{d\eta_0}\right]$.

Therefore, if the denominator of n equals k + 2, relationship (13), which will only include $\overline{\omega}$, will determine $\overline{\omega}$.

If the denominator equals k + 1, relationship (13) will contain $\overline{\omega}$ and $\overline{\omega}_1$. However, $\overline{\omega}$ will have been previously determined by the relationship

$$\left[\frac{d\Theta_{k-1}}{d\eta_0}\right] = 0.$$

Relationship (13) will therefore determine $\overline{\omega}_1$ and, consequently, γ_1.

If the denominator equals k, relationship (13) will contain $\overline{\omega}$, $\overline{\omega}_1$ and $\overline{\omega}_2$. However, $\overline{\omega}$ and $\overline{\omega}_1$ will have been previously determined by relationships having the same form as (13). Therefore, (13) will determine $\overline{\omega}_2$ and consequently γ_2. This process will then be continued. /311

Discussion

364. The solution which we have obtained still includes the following arbitrary constants

$$\varepsilon, \quad \xi_0, \quad u_0.$$

With respect to the parameters λ and μ, we have obtained them from their expansions in increasing powers of ε, and we have successively calculated the coefficients of these expansions. These coefficients λ_k and μ_k depend on the two constants ξ_0 and u_0; these coefficients were calculated by means of the following equations

$$\left[\frac{d\Theta_k}{d\xi_0}\right] + \lambda_k H_0 = \left[\frac{d\Theta_k}{du_0}\right] + 2\mu_k B_0 = 0.$$

where Θ_k, $\frac{d\Theta_k}{d\xi_0}$ and $u_0 \frac{d\Theta_k}{du_0}$ are whole polynomials in

$$\xi_0, \quad \sqrt{u_0}\, e^{\pm i(nt+\varpi)}.$$

Let us set

$$\Theta_k = \Sigma P \xi_0'^{h_1} \eta_0'^{h_2} \xi_0^{h_3} = \Sigma Q,$$

where P is a whole polynomial with respect to

$$\xi_1, \quad \xi_2, \quad \ldots, \quad \xi'_1, \quad \xi'_2, \quad \ldots, \quad \eta'_1, \quad \eta'_2, \quad \ldots \tag{18}$$

whose coefficients are periodic functions of η_0.

We then have

$$\xi_0 \frac{d\Theta_k}{d\xi_0} = \Sigma h_3 Q, \qquad u_0 \frac{d\Theta_k}{du_0} = \Sigma \left(\frac{h_1 + h_2}{2}\right) Q.$$

Let us then replace the quantities (18) by their expansions, and let us set

$$P = \Sigma B \xi_0'^{b_1} \eta_0'^{b_2} \xi_0^{b_3},$$

where B is a periodic function of t of the period 2π. We obtain the following from this

$$\Theta_k = \Sigma B \xi_0^{b_1+h_1} \eta_0^{b_2+h_2} \xi_0^{h_3+h_3} = \Sigma R,$$

$$\xi_0 \frac{d\Theta_k}{d\xi_0} = \Sigma h_3 R, \qquad u_0 \frac{d\Theta_k}{du_0} = \Sigma \frac{h_1+h_2}{2} R.$$

We shall obtain /312

$$\xi_0 \left[\frac{d\Theta_k}{d\xi_0}\right], \quad u_0 \left[\frac{d\Theta_k}{du_0}\right]$$

while retaining the terms which are independent of t in the expansions. The different terms of R contain the following exponentials as factors

$$e^{ipt} \times e^{i(nt+\varpi)(b_1+h_1-b_2-h_2)}.$$

In order that this term may be independent of t, it is necessary that

$$p + n(b_1 + h_1 - b_2 - h_2) = 0,$$

which illustrates the fact that $b_1 + h_1 - b_2 - h_2$ must be divisible by the denominator of n. Therefore we have

$$b_1 + h_1 + b_2 + h_2 > b_1 + h_1 - b_2 - h_2 > \text{denominator of } n \geqq 2,$$

which indicates that R is divisible by u_0, since u_0 is included with the exponent $\frac{1}{2}$ $(b_1 + h_1 + b_2 + h_2)$.

There would only be an exception to this if we had

$$b_1 + h_1 = b_2 + h_2;$$

but we would then have either

$$b_1 + h_1 \geqq 1,$$
$$b_1 + h_1 + b_2 + h_2 \geqq 2,$$

in such a way that R would be divisible by u_0, or

$$b_1 = h_1 = b_2 = h_2 = 0,$$

from which we have

$$\frac{h_1 + h_2}{2} = 0.$$

However, the corresponding terms would not then appear in $u_0 \left[\frac{d\Theta_k}{du_0}\right]$.

In the same way R will always be divisible by ξ_0, unless $h_3 = 0$, in which case the term would not be included in $\xi_0 \left[\frac{d\Theta_k}{d\xi_0}\right]$.

Therefore, to sum up we have

$$\left[\frac{d\theta_k}{du_0}\right], \quad \left[\frac{d\theta_k}{d\xi_0}\right]$$

and, consequently, λ_k and μ_k are whole polynomials of ξ_0 and $\sqrt{u_0}$. Therefore λ and μ are series which may be developed in powers of /313

$$\varepsilon, \quad \xi_0, \quad \sqrt{u_0},$$

but these three constants do not enter arbitrarily.

Let us recall the method which we employed to introduce the auxiliary constant ε, which only served to simplify the discussion. For this purpose, let us again consider the notation given in No. 274, and on page 95. We have set

$$x_1 = \varepsilon x'_1, \quad y_1 = \varepsilon y'_1, \quad x_2 = \varepsilon^2 x'_2, \quad y_2 = y'_2.$$

Therefore, our equations do not cease to be satisfied when we change

$$\varepsilon, \quad x'_1, \quad y'_1, \quad x'_2$$

into

$$\varepsilon k^{-1}, \quad x'_1 k, \quad y'_1 k, \quad x'_2 k^2$$

and when the parameters λ and μ retain their initial values.

We then remove the accents which have become useless, and we develop x'_1, y'_1, x'_2, y'_2, which we shall hereafter designate by the letters x_1, y_1, x_2, y_2, in powers of ε. We thus obtained the expansions

$$\begin{cases} \xi_0 + \varepsilon\xi_1 + \varepsilon^2\xi_2 + \ldots, \\ \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \ldots, \\ \xi'_0 + \varepsilon\xi'_1 + \varepsilon^2\xi'_2 + \ldots, \\ \eta'_0 + \varepsilon\eta'_1 + \varepsilon^2\eta'_2 + \ldots. \end{cases} \tag{19}$$

We shall not cease to satisfy the equations if we change ε into $\frac{\varepsilon}{k}$, and if we multiply the four expansions (19), respectively by

$$k^2, \quad 1, \quad k, \quad k,$$

or, which is the same thing, if we change

$$\xi_p, \quad \eta_p, \quad \xi'_p, \quad \eta'_p$$

into

$$\xi_p k^{2-p}, \quad \eta_p k^{-p}, \quad \xi'_p k^{1-p}, \quad \eta'_p k^{1-p}.$$

By means of this change, we must again obtain expansions which are identical to the expansions (19), but with different values of the constants ξ_0 and u_0. However, it may be seen that ξ_0 and u_0 are changed into $k^2\xi_0$ and k^2u_0 by means of this change. /314

Therefore

$$\xi_p,\quad \eta_p,\quad \xi'_p,\quad \eta'_p$$

change into

$$\xi_p k^{2-p},\quad \eta_p k^{-p},\quad \xi'_p k^{1-p},\quad \eta'_p k^{1-p}$$

when ξ_0 and u_0 change into $k^2\xi_0$ and k^2u_0.

In other words, if the four expansions (19) are multiplied respectively by ε^2, 1, ε, ε, the four products thus obtained may be developed in powers of

$$\varepsilon^2\xi_0,\quad \varepsilon\sqrt{u_0}.$$

The same must be true of λ and μ, which did not have to change when ε, ξ_0, u_0 were changed into $\frac{\varepsilon}{k}$, $k^2\xi_0$, k^2u_0.

Therefore, let us assume that λ and μ are expressed as functions of $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$. It is apparent that we shall thus have relationships from which we may derive $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$ inversely as functions of λ and μ.

365. Let k + 2 be the denominator of n. The constant ϖ will then be determined by the equation

$$\left[\frac{d\Theta_k}{d\eta_0}\right] = 0.$$

There is only an exception to this in the case of k + 2 = 2, where ϖ is determined by

$$\left[\frac{d\Theta_2}{d\eta_0}\right] = 0.$$

The expression $\frac{d\Theta_k}{d\eta_0}$ is a whole polynomial of degree k + 2 with respect to

$$e^{\pm i(nt+\varpi)}.$$

Therefore, each of these terms contains factors having the form

$$e^{\pm iq(nt+\varpi)}.$$

Only terms which are independent of t will remain in the mean value $\left[\frac{d\Theta_k}{d\eta_0}\right]$, and we have seen that q must be divisible by the denominator of n, i.e., by k + 2. /315

Therefore, our expression has the following form

$$ae^{i\varpi(k+2)} + b + ce^{-i\varpi(k+2)}.$$

I shall now show that the coefficient b is zero.

For this purpose, I shall employ the following method. Let us calculate

$$\begin{matrix} \xi_0, & \xi_1, & \dots, & \xi_{k-1}, \\ \eta_0, & \eta_1, & \dots, & \eta_{k-1}, \\ \xi'_0, & \xi'_1, & \dots, & \xi'_{k-1}, \\ \eta'_0, & \eta'_1, & \dots, & \eta'_{k-1}, \end{matrix}$$

by the procedure presented above. However, when computing ξ_k, I shall retain an arbitrary value for $\bar{\omega}$, instead of assigning a value which cancels $\left[\frac{d\Theta_k}{d\eta_0}\right]$ to $\bar{\omega}$. Then the following equation

$$\frac{d\xi_k}{dt} = \frac{d\theta_k}{d\eta_0}$$

will allow me to compute ξ_k. However, instead of being a periodic function of t, ξ_k will be a periodic function of t in addition to a non-periodic term

$$t\left[\frac{d\theta_k}{d\eta_0}\right].$$

We have another method of calculating

$$\begin{matrix} \xi_0, & \xi_1, & \dots, & \xi_k, \\ \eta_0, & \eta_1, & \dots, & \eta_{k-1}, \\ \dots, & \dots, & \dots, & \dots, \end{matrix}$$

and, consequently, this term $t\left[\frac{d\Theta_k}{d\eta_0}\right]$. This method consists of again performing the calculation presented in No. 274.

We shall determine S_0, S_1, ..., by means of equations (2) on page 100.

No difficulty will be entailed in calculating S_0, S_1, ..., S_{k-1}, but we shall encounter some difficulty when calculating S_k by the equation

$$\frac{dS_k}{dy_2} + 2B\frac{dS_k}{dv} = \Phi + C_k.$$

In effect, the second term represents a group of terms having the following form /316

$$A e^{i m_1 y_2 + m_2 v},$$

where m_1 and m_2 are integer numbers. Nothing impedes us from performing integration, provided that we do not have

$$i m_1 + 2 m_2 B = 0.$$

Since 2B equals in, where n is a commensurable number whose denominator equals k + 2, the second term of our equation will include terms satisfying this condition. As a result, S_k will not be a periodic function of y_2 and v, but may equal

$$T_k + y_2 U_k,$$

where T_k and U_k are periodic.

Having thus determined the function S and having obtained the approximation to terms of the order ε^{k+1}, we may employ the procedure given in No. 275 and may thus determine x_1, y_1, x_2, y_2.

These two computational methods must lead to the same result. Therefore, let us set

$$\Sigma = S_0 + \varepsilon S_1 + \ldots + \varepsilon^k S_k.$$

Let us compile the equations (see page 102)

$$x_2 = \frac{d\Sigma}{dy_2}, \quad u = \frac{d\Sigma}{dv}, \quad n_1 t + \varpi_1 = \frac{d\Sigma}{d\alpha_0}, \quad n_2 t + \varpi_2 = \frac{d\Sigma}{d\beta_0},$$

$$n_1 = -\frac{dC}{d\alpha_0}, \quad n_2 = -\frac{dC}{d\beta_0}$$

and let us derive x_2 from them as a function of t. The value of x_2 which is thus obtained must equal

$$\xi_0 + \varepsilon\xi_1 + \ldots + \varepsilon^k \xi_k$$

up to terms of the order ε^{k+1}.

We are interested in calculating ξ_k, particularly that of the secular term

$$t\left[\frac{d\theta_k}{d\eta_0}\right].$$

This secular term can only come from the secular term of S_k, which equals $y_2 U_k$.

We thus have the following, up to terms of the order ε^{k+1} (equating /317 the secular terms in the equation $x_2 = \frac{d\Sigma}{dy_2}$)

$$\varepsilon^k t \left[\frac{d\Theta_k}{d\eta_0}\right] = \varepsilon^k y_2 \frac{dU_k}{dy_2}. \tag{20}$$

In the first approximation -- i.e., up to terms of the order ε -- we have (see page 102)

$$x_2 = \alpha_0 = \xi_0, \quad u = \beta_0 = u_0, \quad n_1 t + \varpi_1 = y_2 = \eta_0 = t,$$
$$n_1 = 1; \quad n_2 = in; \quad n_2 t + \varpi_2 = v = v_0 = i(nt + \varpi).$$

We shall therefore commit an error of the order ε^{k+1} if, in the second term of (20), we replace

$$\alpha_0, \quad \beta_0, \quad y_2, \quad v$$

by

$$\xi_0, \quad u_0, \quad t, \quad i(nt + \varpi).$$

We shall therefore obtain $\left[\frac{d\Theta_k}{d\eta_0}\right]$ by making the same substitution in $\frac{dU_k}{dy_2}$. However, U_k only includes terms containing

$$im_1 y_2 + m_2 v,$$

where

$$im_1 + 2m_2 B = 0.$$

We therefore have

$$\frac{dU_k}{dy_2} = -2B \frac{dU_k}{dv}.$$

However, U_k is a periodic function of y_2 and iv. Therefore, $\frac{dU_k}{dv}$ cannot contain a term which is independent of v. Therefore $\left[\frac{d\Theta_k}{d\eta_0}\right]$ does not contain a term which is independent of ϖ.

q.e.d.

In order to clarify the preceding calculation, I would like to make one more remark. The mean motions n_1 and n_2 are given by

$$n_1 = -\frac{dC}{d\alpha_0}, \quad n_2 = -\frac{dC}{d\beta_0}.$$

In general, they depend on ε, and they are only reduced to 1 and *in* for $\varepsilon = 0$,

However, we are here employing two parameters λ and μ which may be replaced by the arbitrary functions of ε, or, if it is preferred, we may /318 employ an infinite number of constants $\lambda_1, \lambda_2, \ldots, \mu_1, \mu_2, \ldots$. We may then employ these constants in such a way that n_1 and n_2 remain equal to 1 and to in, no matter what ε may be.

366. In order to determine $\overline{\omega}$, we therefore have an equation of the following form

$$a e^{(k+2)i\varpi} + c e^{-(k+2)i\varpi} = 0,$$

where a and c are conjugate and imaginary. In general, a and c are not zero, otherwise $\overline{\omega}$ could only be determined to the following approximation.

The equation will provide us with the following series of real values for $\overline{\omega}$

$$\varpi_0, \quad \varpi_0 + \frac{\pi}{k+2}, \quad \varpi_0 + \frac{2\pi}{k+2}, \quad \varpi_0 + \frac{3\pi}{k+2}, \quad \ldots.$$

It is apparent that we do not have two values which are actually different when we change $\overline{\omega}$ into $\overline{\omega} + 2\pi$, but we have more than this. It may be stated that the two values

$$\varpi_0, \quad \varpi_0 + \frac{2\pi}{k+2}$$

do not correspond to two periodic solutions which are actually different.

Since t is not explicitly included in our equations, by changing t into t + h we may transform an arbitrary periodic solution into another solution which is not essentially different.

Therefore, let us change t into $t + 2h\pi$, where h is an integer number.

Then η_0 changes into $\eta_0 + 2h\pi$ and $v_0 = i(nt + \overline{\omega})$ into

$$i(nt + 2nh\pi + \varpi).$$

Since all of our functions are periodic, of the period 2π, in η_0 and iv, we shall not change our solution in any way by subtracting two multiples of 2π from η_0 and $\frac{v_0}{i}$, respectively, for example $2h\pi$ and $2h'\pi$. Then η_0 will again become η_0 and v_0 will change into

$$i(nt + 2nh\pi + \varpi - 2h'\pi).$$

In other words, we will have changed $\overline{\omega}$ into /319

$$\varpi + 2\pi(nh - h').$$

However, we may always choose the integer numbers h and h' in such a way that

$$nh - h' = \frac{1}{k+2}.$$

We therefore do not obtain a solution which is actually new by changing $\bar{\omega}$ into $\bar{\omega} + \frac{2\pi}{k+2}$.

q.e.d.

We therefore have only two solutions which are actually different, corresponding to the two following values of $\bar{\omega}$

$$\varpi_0, \quad \varpi_0 + \frac{\pi}{k+2}.$$

We must now determine the constants $\varepsilon^2\xi_0$ and $\varepsilon^2 u_0$. For this purpose, we shall employ equations which relate these two constants to λ and μ. In the questions which are customarily discussed, there is only one arbitrary parameter, and we have introduced two in order to facilitate the discussion. It is therefore convenient to assume that λ and μ are related by one relationship -- for example, $\lambda = \mu$.

The expansion of λ and that of μ in powers of $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$ begins in general with terms containing $\varepsilon^2\xi_0$ and $\varepsilon^2 u_0$ (if we disregard the case in which the denominator of n equals 3).

If we therefore assume that $\mu = \lambda$, we shall derive $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$ from this which may be developed in powers of $\sqrt{\lambda}$. Either the coefficients of the expansion in powers of $\sqrt{\lambda}$ will be real, or, on the contrary, the coefficients of the expansion in powers of $\sqrt{-\lambda}$ will be the ones which are real.

In the first case, the problem will have two real solutions for $\lambda > 0$ and will not have any for $\lambda < 0$. In the second case, the opposite will hold true.

In order to determine which of these two cases is valid, let us examine the equation which relates μ to u_0, restricting ourselves to terms containing ε^2. We shall have /320

$$\lambda = \mu = -\frac{\varepsilon^2}{2B_0}\left[\frac{d\Theta_2}{du_0}\right]; \qquad \lambda = -\frac{\varepsilon^2}{H_0}\left[\frac{d\Theta_2}{d\xi_0}\right]. \tag{21}$$

I may first observe that $\left[\frac{d\Theta_2}{du_0}\right]$ and $\left[\frac{d\Theta_2}{d\xi_0}\right]$ are not only independent of t but also of $\bar{\omega}$. There is only one exception for

$$k + 2 = 2, 3 \text{ or } 4.$$

This is due to the fact that, for $k + 2 > 4$, terms having the following form

$$e^{i(pt + qnt + q\varpi)}$$

which may be included in the second term in one of the equations (21) can only be independent of t if

$$q = 0,$$

since $|q|$ cannot exceed 4 and since qn must be an integer number.

Thus, the second terms of equations (21) are linear and homogeneous functions of ξ_0 and u_0. The coefficients of these linear functions are absolute constants which are independent of $\overline{\omega}$.

However, u_0 must be positive; otherwise $\sqrt{u_0}$ would be imaginary. The equations (21) added to inequality $u_0 > 0$ will determine the sign of λ.

I need only point out that this sign does not depend on $\overline{\omega}$, since equations (21) do not depend on it. We have seen that the equation which determines $\overline{\omega}$ has two solutions which are actually different

$$\varpi = \varpi_0, \qquad \varpi = \varpi_0 + \frac{\pi}{k+2}.$$

In conformance with the preceding statements, a periodic solution which will be real if the sign of λ is suitably chosen corresponds to each of them. The choice of this sign does not depend on $\overline{\omega}$, and these two solutions will both be real for $\lambda > 0$ and will both be imaginary for $\lambda < 0$, or the opposite will hold true.

It first appears that two periodic solutions correspond to each solution of the equation for $\overline{\omega}$, since two systems of values for the unknowns /321 $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$ are obtained from the relationships between λ, μ, $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$. This is not the case, however. Without restricting the conditions of generality, we may assume that $\sqrt{u_0}$ is positive, because we do not change our formulas in any way by changing $\sqrt{u_0}$ into $-\sqrt{u_0}$, and $\overline{\omega}$ into $\overline{\omega} + \pi$.

Out of our two systems of values, there is only one for which $\sqrt{u_0}$ is positive.

Therefore, we have:

Two real, periodic solutions of the second type for $\lambda > 0$ (or for $\lambda < 0$).

No solution of the second type for $\lambda < 0$ (or for $\lambda > 0$).

Let us again employ the notation given in Chapter XXVIII and, in particular, that given in No. 331.

U_1 may be reduced to ρ^2, and corresponds to the term containing $x_1 y_1$ which appears in Θ_0.

U_0 may be reduced to a constant factor multiplied by ρ^4, corresponding to terms coming from $\left[\frac{d\Theta_2}{du_0}\right]$ and $\left[\frac{d\Theta_2}{d\xi_0}\right]$.

The first term of W which may not be reduced to a power U_1 has the following form

$$\rho^{k+2}[A\cos(k+2)\varphi + B]$$

and comes from Θ_{k+2}.

The function whose maxima and minima we must study, and which must play the role of the function

$$U_0 + zU_1 = \rho^3 f(\varphi \quad z\rho^2$$

studied on page 247, will have the following form

$$A\rho^{k+2}\cos(k+2)\varphi + P\rho^4 - z\rho^2,$$

where P is a whole polynomial in ρ^2 with constant coefficients.

We have disregarded the particular cases in which the denominator of n equals 2, 3 or 4.

Discussion of Particular Cases

/322

367. Let us assume that this denominator equals 4.

Then $[\Theta_2]$, $\left[\frac{d\Theta_2}{d\xi_0}\right]$, $\left[\frac{d\Theta_2}{du_0}\right]$ will no longer be independent of ϖ, and they will include terms containing $e^{\pm 4i\varpi}$.

The equation for ϖ will always yield two different solutions

$$\varpi = \varpi_0, \quad \varpi = \varpi_0 + \frac{\pi}{4}$$

which will provide us with two periodic solutions. Due to the fact that only the sign of λ may depend on ϖ, the following cases may occur:

Two real solutions of the second type for $\lambda > 0$; zero solution for $\lambda < 0$;

One real solution of the second type for $\lambda > 0$; one solution for

$\lambda < 0$;

Zero real solution of the second type for $\lambda > 0$; two solutions for $\lambda < 0$.

The function $U_0 + zU_1$ given on page 247 becomes

$$\rho^4(A \cos 4\varphi + B) - z\rho^2.$$

Let us now assume that the denominator of n equals 3.

The expansion of μ in powers of ε then begins with a term containing $\varepsilon\sqrt{u_0}$, so that if we set $\mu = \lambda$, we shall obtain $\varepsilon^2\xi_0$ and $\varepsilon\sqrt{u_0}$ in series which may be developed in powers of λ, and no longer of $\sqrt{\lambda}$.

The sign of $\sqrt{u_0}$ will depend on $\overline{\varpi}$, and if it is positive for $\overline{\varpi} = \varpi_0$ it will be negative for $\overline{\varpi} = \overline{\varpi}_0 + \frac{\pi}{3}$.

Therefore, if it is always convenient for us to assume that $\sqrt{u_0}$ is mainly positive, we shall readily find that we have:

A real solution of the second type for $\lambda > 0$ and a real solution of the second type for $\lambda < 0$.

The function $U_0 + zU_1$ given on page 247 becomes

$$A\rho^3 \cos 3\varphi - z\rho^2.$$

Finally, if the denominator of n equals 2, $[\Theta_2]$, $\left[\frac{d\Theta_2}{d\xi_0}\right]$, $\left[\frac{d\Theta_2}{du_0}\right]$, include terms containing $e^{\pm 4i\varpi}$, $e^{\pm 2i\varpi}$. /323

The equation for $\overline{\varpi}$ takes the form

$$A \cos(4\varpi + B) + A' \cos(2\varpi + B') = 0$$

and it has eight solutions

$$\varpi_0, \quad \varpi_0 + \frac{\pi}{2}, \quad \varpi_0 + \pi, \quad \varpi_0 + \frac{3\pi}{2},$$

$$\varpi_1, \quad \varpi_1 + \frac{\pi}{2}, \quad \varpi_1 + \pi, \quad \varpi_1 + \frac{3\pi}{2}.$$

Of the two terms $\overline{\varpi}_0$ and $\overline{\varpi}_1$, at least one is real.

The following hypotheses are possible: (4, 0), (3, 1), (2, 2), (1, 3), (0, 4), (2, 0), (1, 1), (0, 2).

The first number between the parenthesis represents the number of periodic solutions for $\lambda > 0$, and the second is the same number for $\lambda < 0$.

The function given on page 247 becomes

$$A\rho^4 \cos 4\varphi + B\rho^4 \cos 2\varphi + C\rho^4 \sin 2\varphi + D\rho^4 - z\rho^2.$$

Application to Equations of No. 13

368. Let us return to the canonical equations of dynamics:

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}. \tag{1}$$

Just as in No. 13, No. 42, No. 125, etc., I shall assume that F is a periodic function of the y's, which may be developed in powers of a parameter μ in the following form

$$F = F_0 + \mu F_1 + \dots$$

and that F_0 depends only on the x's.

We saw in No. 42 that these equations have an infinite number of solutions of the first type

$$x_i = \varphi_i(t), \quad y_i = \psi_i(t) \tag{2}$$

where the functions ϕ_i and ψ_i may be developed in increasing powers of μ. /324

Let us consider one of these solutions (2).

Let T be the period, and let α be one of the characteristic exponents. There will be two of them, which are different from zero, which are equal and have opposite signs, where we may assume two degrees of freedom.

We saw in Chapter IV that α depends on μ, and may be developed in powers of $\sqrt{\mu}$. When μ varies continuously, the same will hold true for α. For $\mu = \mu_0$, let us assume that αT is commensurable with $2i\pi$ and equal to $2ni\pi$.

We may conclude from this that, for μ which is close to μ_0, there are solutions of the second type, which are derived from (2) and whose period is (k + 2)T, where k + 2 designates the denominator of n.

If we put aside the cases in which k + 2 equals 2, 3, or 4, we have seen that two of these solutions exist when λ (here $\mu - \mu_0$) has a certain sign, and that they do not exist when λ (here $\mu - \mu_0$) has the opposite sign.

I have stated that the cases in which k + 2 = 2, 3, 4 have been disregarded, and I may do this without causing any inconvenience. The following

$$\frac{\alpha T}{2 i\pi} = n$$

may be developed in powers of $\sqrt{\mu}$, and vanishes with $\sqrt{\mu}$. For small values of μ, n is therefore very small, and its denominator is definitely larger than 4.

We therefore have two hypotheses:

Either the solutions of the second type occur only for $\mu > \mu_0$, or they occur for $\mu < \mu_0$.

Which of these two hypotheses is valid?

Everything depends on the sign of a certain term Q, which depends itself on the coefficients of u_0 and ξ_0 in

$$\left[\frac{d\theta_2}{d\xi_0}\right], \quad \left[\frac{d\theta_2}{du_0}\right].$$

In order to determine this sign, we shall not need to formulate this term, and the following considerations will suffice.

369. Let us first take a simple case, which will be that presented /325 in No. 199. Let us set

$$F = x_2 + x_1^2 + \mu \cos y_1$$

with the canonical equations

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i},$$

which yields

$$\frac{dx_2}{dt} = 0, \quad \frac{dy_2}{dt} = -1, \quad \frac{dx_1}{dt} = \mu \sin y_1, \quad \frac{dy_1}{dt} = -2x_1. \tag{1}$$

The function S of Jacobi may be written

$$S = x_2^0 y_2 + \int \sqrt{C - \mu \cos y_1}\, dy_1$$

with two constants x_2^0 and C. We may derive the following from this

$$\left\{\begin{aligned} x_2 &= x_2^0, & y_2 &= -t + y_2^0, \\ x_1 &= \sqrt{C - \mu \cos y_1}, & A - t &= \int \frac{dy_1}{2\sqrt{C - \mu \cos y_1}}, \end{aligned}\right. \tag{2}$$

where A and y_2^0 are two new integration constants.

It may be seen that the elliptical integral is introduced

$$\int \frac{dy_1}{2\sqrt{C - \mu \cos y_1}}. \tag{3}$$

This integral has a real period, which is the integral taken between 0 and 2π, if $|C| > |\mu|$, and two times the integral taken between

$$\pm \arc\cos\left|\frac{C}{\mu}\right|$$

if $|C| < |\mu|$.

Let us call ω this real period.

A periodic solution corresponds to each value of ω which is commensurable with 2π. However, we must distinguish between two cases.

If $|C| > |\mu|$, y_1 and y_2 increase by a multiple of 2π during one period. The corresponding periodic solutions are solutions of the first type.

If $|C| < |\mu|$, y_2 increases by a multiple of 2π during one period, and y_1 returns to its original value. The corresponding solutions are solutions of the second type. /326

This discussion must be supplemented by two unusual periodic solutions which must be regarded as solutions of the first type. Let us set $\mu > 0$, and these solutions will then be

$$\begin{cases} x_2 = x_2^0, & y_2 = -t + y_2^0, \quad C = \mu, \quad x_1 = 0, \quad y_1 = 0, \\ x_2 = x_2^0, & y_2 = -t + y_2^0, \quad C = -\mu, \quad x_1 = 0, \quad y_1 = \pi. \end{cases} \tag{4}$$

I have stated that it must be assumed that these latter solutions are of the first type, and that the solutions corresponding to $|C| < |\mu|$ must be regarded as solutions of the second type.

Let us assign to C a value which is a little higher than $-\mu$, and let us set

$$C = (\varepsilon - 1)\mu,$$

where ε is very small. y_1 will not be able to deviate very greatly from π. We shall approximately have

$$C - \mu \cos y_1 = \mu\left[\varepsilon - \frac{(\pi - y_1)^2}{2}\right],$$

and the period ω will be equal to

$$\frac{\pi}{\sqrt{2\mu}},$$

from which we may draw the following conclusions. Let α be an arbitrary number which is commensurable with 2π. There is a series of periodic solutions such that $|C| < |\mu|$ and that $\omega = \alpha$. If $\sqrt{\mu}$ is very close to $\frac{\pi}{\sqrt{2\alpha}}$, C will be very close to $-\mu$, and for

$$\sqrt{\mu} = \frac{\pi}{\sqrt{2\alpha}},$$

these periodic solutions will coincide with the second solution (4) which is of the first type. We may now recognize the characteristic property of solutions of the second type.

It may be seen that the second solution (4) -- i.e., that of the two solutions (4) which is stable -- gives rise to solutions of the second type, as was explained in Chapter XXVIII.

If the other solutions of the first type -- those which are such that $|C| > |\mu|$ -- do not produce solutions of the second type, this is due to /327 the very particular form of the equations (1). (For these solutions, the characteristic exponents are always zero.)

Let us first consider solutions of the first type, such that $|C| > |\mu|$.

Let us set $C = C_0 + \varepsilon$. The period ω, i.e., the integral (3) taken between 0 and 2π, may be developed in powers of ε and of μ, and the known terms may be reduced to

$$\frac{\pi}{\sqrt{C_0}}.$$

Let us assign an arbitrary commensurable value to $\sqrt{C_0}$. We shall have a periodic solution every time that we have

$$\omega = \frac{\pi}{\sqrt{C_0}}.$$

The equation is satisfied for $\varepsilon = \mu = 0$, and we may derive ε and, consequently, C from this equation, in series which develop in powers of μ. The equations (2) will then give us x_1 and y_1 developed in powers of μ. These are the expansions of Chapter III.

Let us pass to the second type, such that $|C| < |\mu|$. Let us set $C = \varepsilon\mu$. We shall have

$$\omega = \frac{1}{\sqrt{\mu}} \int \frac{dy_1}{2\sqrt{\varepsilon - \cos y_1}}.$$

It can be seen that $\omega\sqrt{\mu}$ is only a function of ε. On the other hand,

$$\frac{x_1}{\sqrt{\mu}} = \sqrt{\varepsilon - \cos y_1}, \qquad (A - t)\sqrt{\mu} = \int \frac{dy_1}{2\sqrt{\varepsilon - \cos y_1}},$$

which indicates to us that sin y_1, cos y_1 and $\frac{x_1}{\sqrt{\mu}}$ are functions of $(A - t)\sqrt{\mu}$ and of ε, which are doubl periodic with respect to $(A - t)\sqrt{\mu}$. They are also functions of $(A - t)\sqrt{\mu}$ and of $\omega\sqrt{\mu}$, since ε is a function of $\omega\sqrt{\mu}$. Therefore, if we assign a constant value which is commensurable with 2π to ω, we shall obtain a series of periodic solutions. For these solutions /328

$$\cos y_1, \quad \sin y_1 \text{ and } \frac{x_1}{\sqrt{\mu}}$$

may be developed in Fourier series according to the sines and cosines of the multiples of $\frac{2\pi t}{T}$, where T is the smallest common multiple of ω and 2π. The function of μ is an arbitrary coefficient of the expansion, and it is this function which I would like to study.

For this purpose, we must first study the relationship between ε and $\omega\sqrt{\mu}$.

We may vary ε from -1 to +1. For $\varepsilon = -1$, we have

$$\omega\sqrt{\mu} = \frac{\pi}{\sqrt{2}}.$$

For $\varepsilon = +1$, we have $\omega\sqrt{\mu} = \infty$. Therefore, when ε varies from -1 to +1, $\omega\sqrt{\mu}$ increases from $\frac{\pi}{\sqrt{2}}$ to $+\infty$.

Therefore, there is only a periodic solution corresponding to a given value of ω, which is commensurable with 2π, if

$$\sqrt{\mu} > \frac{\pi}{\omega\sqrt{2}}.$$

The coefficients of the Fourier expansion are therefore functions of μ, which are real for

$$\sqrt{\mu} > \frac{\pi}{\omega\sqrt{2}}$$

and imaginary for

$$\sqrt{\mu} < \frac{\pi}{\omega\sqrt{2}}.$$

It is apparent that the same line of reasoning would lead to the same result if, instead of

$$F = x_2 + x_1^2 + \mu \cos y_1,$$

we had set

$$F = F_0 + \mu[F_1],$$

where F_0 depends only on x_1 and x_2, and $[F_1]$ depends only on x_1, x_2 and y_1. The solutions of the second type would still have been real for $\mu > \mu_0$. /329

370. In the general case, the quantity Q, which was in question at the end of No. 368 and whose sign we shall try to determine, obviously depends on μ. If μ is sufficiently small, the first term of the expansion will provide its sign.

Let us determine the function S by the Bohlin method, and let us set

$$S = S_0 + \sqrt{\mu} S_1 + \mu S_2 + \dots .$$

If μ is small enough, it will obviously be the first two terms

$$S_0 + \sqrt{\mu} S_1$$

which will be the most important. If we set

$$F = F_0 + \mu F_1 + \mu^2 F_2 + \dots ,$$

we have seen in Chapter XIX that S_0 and S_1 depend neither on F_2 or $F_1 - [F_1]$, but only on F_0 and $[F_1]$, where the mean value of F_1 is designated by $[F_1]$.

Let us again take the quantity Q from No. 368. The first term of its expansion will only depend on S_0 and S_1, and consequently on F_0 and $[F_1]$. The same would hold true if we had set

$$F = F_0 + \mu[F_1],$$

which is, consequently, the same as in the preceding section.

In the preceding section we found that solutions of the second type exist only for

$$\mu > \mu_0.$$

This conclusion still holds in the general case, provided that μ_0 is sufficiently small.

What is the value of μ_0 for which this conclusion would no longer hold?

Let us again consider the notation given in No. 361, which is that of No. 275. The exponent α which appears there may be developed in powers of the product AA'.

It may be reduced to the characteristic exponent for $AA' = 0$.

Since we assume that the solution of the first type is stable and /330 α is imaginary, A and A' are imaginary and conjugate, and the product AA' is positive.

For small values of μ, α decreases when AA' increases. If the reverse were true, solutions of the second type would exist only for $\mu < \mu_0$.

The desired value of μ_0 is therefore that for which α ceases to decrease when AA' increases. It is therefore that which cancels the derivative of α with respect to AA'.

CHAPTER XXXI

PROPERTIES OF SOLUTIONS OF THE SECOND TYPE

Solutions of the Second Type and the Principle of Least Action /331

371. I cannot pass over the relationships between the theory of solutions of the second type and the principle of least action in silence. I wrote Chapter XXIX just for these relationships. However, in order to understand them some preliminary remarks are still necessary.

Let us assume two degrees of freedom. Let x_1 and x_2 be the two variables of the first series, which may be regarded as the coordinates of a point in a plane. The plane curves which satisfy our differential equations will comprise what I have designated as trajectories.

Let M be an arbitrary point in the plane. Let us consider the group of trajectories emanating from the point M, and let E be their envelope. Let F be the n^{th} kinetic focus of M on the trajectory (T). This trajectory will touch the envelope E at the point F, according to the definition of kinetic focuses. I would like to recall that the n^{th} focus of M, or its focus of the order n, is the n^{th} point of intersection of T with the infinitely adjacent trajectory passing through M. However, the conditions of this contact may vary. It may happen that F is not a singular point of the curve E, and that the contact is of the first order. This is the most general case.

Let

$$x_1 = \varphi(x_2)$$
$$x_1 = \varphi(x_2) + \psi(x_2)$$

be the equations of the trajectory (T) and of a trajectory (T') which is very close, emanating from the point M.

Let z_1 and z_2 be the coordinates of the point M, and let u_1 and u_2 be the coordinates of F. Since (T) passes through M and F, and since (T') passes through M, we shall have /332

$$z_1 = \varphi(z_2), \qquad u_1 = \varphi(u_2), \qquad \psi(z_2) = 0.$$

Due to the fact that the trajectory (T') is very close to (T), the function ψ will be very small. I may call α the angle at which two trajectories intersect the point M. It is this angle which will define the trajectory (T'), and the function ψ will depend on the angle α. It will be very small if, as we have assumed, this angle α is itself very small, and it will vanish with α.

The value of $\psi'(z_2)$ (designating the derivative of ψ by ψ') will have the same sign as α. With respect to $\psi'(u_2)$ [if we assume that α is very small and if the system of coordinates has been defined in such a way that the function $\phi(x_2)$ is uniform, which is always possible], it has the same sign as α, if F is a focus of even order, and it has the opposite sign if F is a focus of odd order.

One characteristic of the case in which we are interested is the fact that $\psi(u_2)$ is of the same order as α^2, and always of the same sign.

For example, let us assume that $\psi(u_2)$ is positive.

If the sign of α is such that $\psi'(u_2)$ is positive, the trajectory (T') will intersect (T) at a point F' which is close to the point F, and not as far away from M as the point F (assuming that $u_2 > z_2$). In this case, (T') touches E before F', while (T) touches E after F'. According to a well-known line of reasoning, the action is larger (at least in absolute motion) when we pass from M to F' proceeding along (T') than it is when we pass from M to F' proceeding along (T).

If the sign of α is such that $\psi'(u_2)$ is negative, (T) intersects (T) at a point F' which is farther away from M than F. In this case, (T') touches E after F', and (T) touches E before F'. When we pass from M to F', the action is greater along (T) than it is along (T').

The results would be just the opposite if $\psi(u_2)$ were negative. However, in any case, among the trajectories (T') adjacent to (T) there are some which intersect (T) close to F and beyond F, and others which intersect (T) close to F and just short of F.

In this case, we may say that F is an <u>ordinary focus</u>.

It cannot happen that F is an ordinary point of E, and that the contact is of a higher order than the first.

Let us develop $\psi(x_2)$ in powers of α, and let us set /333

$$\psi(x_2) = \alpha\psi_1(x_2) + \alpha^2\psi_2(x_2) + \dots$$

The condition under which there would be a contact of higher order would be

$$\psi_1'(u_2) = 0.$$

But we already have

$$\psi_1(u_2) = 0$$

and the function $\psi_1(x_2)$ satisfies a linear, differential equation of the second order, whose coefficients are finite and given functions of x_2. The coefficient of the second derivative is reduced to unity.

If the integral $\psi_1(x_2)$ vanishes, as well as its first derivative, for $x_2 = u_2$, it would be identically zero, which is absurd.

Therefore, there is never a contact of higher order.

However, it may happen that F is a cusp of the curve E. Either the cusp point is on the side of M, so that a moving point proceeding from M to F will encounter M with the cusp point directed at M, or the cusp point is turned the opposite way so that the moving point encounters M with the cusp point turned away from M. In the first case, I shall state that F is a pointed focus, and in the second case I shall state that F is a taloned focus.

In one and the other case, $\psi(u_2)$ is on the order of α^3. In this case, the pointed focus has the sign of α, if P is a focus of odd order, and it has the opposite sign of α if F is a focus of even order. The opposite is true in the case of a taloned focus.

In the case of a pointed focus, all the trajectories (T') intersect (T) at a point F' which is close to F and beyond F. Proceeding from M to F', the action is greater along (T) than it is along (T').

In the case of a taloned focus, all the trajectories (T') intersect (T) at a point F' which is close to F and just short of F. Proceeding from M to F', the action is greater along (T') than it is along (T).

Let F' be a point of (T) which is sufficiently close to F. In the case of a pointed focus, I may join M with F' by a trajectory (T'), if F' is beyond F. In the case of a taloned focus, I may join M with F' if F' is just short of F.

It could finally be the case that F is a singular point of E which is more complicated than an ordinary cusp. I would then state that it is a singular focus. /334

I would only like to note that we cannot pass from a pointed focus to a taloned focus except through a singular focus, because at the time of passage $\psi(u_2)$ must be of the order α^4.

372. Let us now consider an arbitrary periodic solution. It will correspond to a closed trajectory (T). Let α be the characteristic exponent and T be the period. In Chapter XXIX we saw how to determine successive kinetic focuses (No. 347).

Let us assume that α equals $\frac{2in\pi}{T}$, where n is a commensurable number whose numerator is p. In this case, the application of the rule given in No. 347 shows that each point of (T) coincides with its $2p^{th}$ focus.

If, just as in No. 347, we take a unit of time such that the period T equals 2π, we have $\alpha = in$. If we designate the value of the function τ at the point M by τ_0, and if $\tau_1, \tau_2, \ldots, \tau_{2p}$ are the values of this function τ at the first, second, ..., up to the $2p^{th}$ focus of M, according to the rule given in No. 347, we shall have the following

$$\tau_1 - \tau_0 = \frac{i\pi}{\alpha}, \quad \tau_2 - \tau_0 = \frac{2i\pi}{\alpha}, \quad \ldots, \quad \tau_{2p} - \tau_0 = \frac{2pi\pi}{\alpha} = \frac{2p\pi}{n}.$$

If p is the numerator of n, it can be seen that $\tau_{2p} - \tau_0$ is a multiple of 2π, i.e., that M and its $2p^{th}$ focus coincide.

The trajectory emanating from the point M which is infinitely close to (T) will therefore pass through the point M again after having gone around the closed trajectory (T) k + 2 times, if k + 2 is the denominator of n.

The point M is therefore its $2p^{th}$ focus. However, we may wish to know what category of focuses it belongs to, from the point of view of the classification presented in the preceding section.

Let us adopt a system of coordinates which are similar to the polar coordinates, so that the equation for the closed trajectory (T) is

$$\rho = 1$$

and so that ω varies from 0 to 2π when one passes around this closed trajectory. The curves ρ = const. are then closed curves which form an envelope around each other in the same way as concentric circles. The curves ω = const. form a bundle of divergent curves which intersect all the curves ρ = const., in such a way that the curve $\omega = a + 2\pi$ coincides with the curve $\omega = a$. /335

Then let ω_0 be the value of ω which corresponds to the point of departure M. The value of ω which will correspond to this same point M, regarded as the $2p^{th}$ focus of the point of departure, will be

$$\omega_0 + 2(k+2)\pi.$$

Let

$$\rho = 1 + \psi(\omega)$$

be the equation of a trajectory (T') which is close to (T) and passes through M. The function $\psi(\omega)$ will correspond to the function $\psi(x_2)$ given in the preceding section. We shall have $\psi(\omega_0) = 0$, and we must now discuss the sign of

$$\psi[\omega_0 + 2(k+2)\pi].$$

We must therefore formulate the function $\psi(\omega)$, and for this purpose we need only apply the principles of Chapter VII, or the principles given in No. 274. For example, if we apply the latter principles, we shall obtain the following. The function $\psi(\omega)$ may be developed in powers of the two quantities

$$A e^{\alpha\omega}, \quad A' e^{-\alpha\omega}.$$

The coefficients of the expansion are periodic functions of the period 2π; A and A' are two integration constants. With respect to α, it is a constant which may be developed in powers of the product AA'

$$\alpha = \alpha_0 + \alpha_1(AA') + \alpha_2(AA')^2 + \dots.$$

The term α_0 equals the characteristic exponent of (T), i.e., it equals *in*.

If (T') differs very little from (T), the two constants A and A' are very small. They are on the order of the angle which I called α in the preceding section, and which must not be confused with the exponent which I have designated by the same symbol in the present section. /336

If we take the approximation up to the third order inclusively with respect to A and A', $\psi(\omega)$ will be reduced to a polynomial of the third order with respect to these two constants, and I may then write

$$\psi(\omega) = A e^{\alpha\omega}\sigma + A' e^{-\alpha\omega}\sigma' + f(A e^{\alpha\omega}, A' e^{-\alpha\omega})$$

where f is a whole polynomial with respect to $Ae^{\alpha\omega}$, and $A'e^{-\alpha\omega}$ only includes terms of the second and third degree. The coefficients of the polynomial f, just the same as σ and σ', are periodic functions of the period 2π.

Under this assumption, since α equals α_0, up to terms of the second order, and since it equals $\alpha_0 + \alpha_1$ (AA') up to terms of the fourth order, we may write the following, neglecting all terms of the fourth order with respect to A and A':

$$\psi(\omega) = A \sigma e^{\omega(\alpha_0+\alpha_1 AA')} + A'\sigma' e^{-\omega(\alpha_0+\alpha_1 AA')} + f(A e^{\alpha_0\omega}, A' e^{-\alpha_0\omega})$$

or

$$\begin{aligned}\psi(\omega) = {} & A e^{\alpha_0\omega}\sigma + A' e^{-\alpha_0\omega}\sigma' \\ & + \alpha_1 \omega AA'(A e^{\alpha_0\omega}\sigma - A' e^{-\alpha_0\omega}\sigma') + f(A e^{\alpha_0\omega}, A' e^{-\alpha_0\omega}).\end{aligned}$$

When ω increases by $(2k + 4)\pi$, the coefficients of f, as well as σ and σ', do not change. The same holds true for $e^{\alpha_0\omega}$, since $\frac{\alpha_0}{i} = n$ has $k + 2$ for the denominator. Therefore, the same still holds true for

$$A e^{\alpha_0\omega}\sigma, \quad A' e^{-\alpha_0\omega}\sigma', \quad f(A e^{\alpha_0\omega}, A' e^{-\alpha_0\omega}).$$

We finally have

$$\psi(\omega + 2k\pi + 4\pi) - \psi(\omega) = (2k+2)\pi\alpha_1 AA'(A e^{\alpha_0\omega}\sigma - A' e^{-\alpha_0\omega}\sigma').$$

However, $\psi(\omega_0)$ is zero. The term whose sign we must determine is therefore

$$(2k+2)\pi\alpha_1 AA'(A e^{\alpha_0\omega_0}\sigma_0 - A' e^{-\alpha_0\omega_0}\sigma'_0).$$

I shall employ σ_0 and σ'_0 to designate the values of σ and σ' for $\omega = \omega_0$.

I should first point out that this term is of the third order which, according to the preceding section, indicates to us that our focuses will in general be pointed focuses or taloned focuses. It may now be stated that this term always has the same sign, and that its coefficient cannot vanish. /337

The two constants A and A' are related by the following relationship

$$\psi(\omega_0) = 0$$

which may be written as follows, since A and A' are infinitely small quantities

$$A e^{\alpha_0\omega_0}\sigma_0 + A' e^{-\alpha_0\omega_0}\sigma'_0 = 0. \quad (1)$$

In addition, α_0 is purely imaginary, and σ_0 and σ'_0 are imaginary and conjugate. The same holds true for A and A'.

The product AA' is therefore positive, and cannot vanish, since A and A' cannot be zero at the same time.

In addition, we cannot have

$$A e^{\alpha_0\omega_0}\sigma_0 - A' e^{-\alpha_0\omega_0}\sigma'_0 = 0, \quad (2)$$

because the equations (1) and (2) would entail the following

$$\sigma_0 = \sigma'_0 = 0.$$

However, these equations are impossible. They would mean that all trajectories close to (T) would pass through the point M, which is clearly false.

Therefore, our term $\psi(\omega_0 + 2k\pi + 4\pi)$ always has the same sign. Our focuses are therefore all pointed focuses, or are all taloned focuses. Everything depends on the sign of α_1.

373. We have disregarded the case in which α_1 would be zero, an unusual case in which all the focuses would be singular, and that in which k + 2 would equal 2, 3, or 4. Following is the reason for this.

We saw in the computations performed in Chapter VII that the following small divisors are introduced

$$\gamma\sqrt{-1} + \Sigma\alpha\beta - \alpha_i$$

(see No. 104, Volume I, page 338).

The calculation is finished, and secular terms occur if one of these divisors vanishes.

It may be readily stated that if k + 2 equals 2, 3, or 4, we are thus finished with the calculation of terms of the first three orders, which are those which we had to take into account. If, on the other hand, k + 2 > 4, we will only be finished with the calculation of the terms of higher order, which are not included in the preceding analysis. /338

374. For example, let us assume that all the focuses are pointed. Let M be an arbitrary point of (T); this point will be the $2p^{th}$ focus with respect to itself. Let M' be a point located a little beyond the point M in the direction in which the trajectory (T) and the trajectories close to (T') are traversed. I may draw a trajectory (T') emanating from point (M), which will deviate very little from (T), which will pass around (T) k + 2 times, which will finally end at the point M', and which will have 2p + 1 points of intersection with (T), counting the intersection points M and M'.

Due to the fact that the focus is a pointed focus, the trajectories (T') which are close to (T) will all intersect (T) again beyond the focus. We may therefore draw the trajectory (T') which satisfies the conditions I have just discussed, provided that the distance MM' is smaller than δ. It is apparent that the upper limit, which must not exceed the distance MM', depends upon the position of M on (T). However, it never vanishes, since there is not a singular focus. It is therefore sufficient for me to set δ equal to the smallest value which this upper limit can take on, and I shall assume that δ is a constant.

Therefore, if the distance MM' is smaller than δ, we may draw a trajectory (T') satisfying our conditions. We may even draw two of them, one intersecting (T) at M at a positive angle, and the other intersecting it at a negative angle.

Under this assumption, let us assume that our differential canonical equations depend on the parameter λ. For $\lambda = 0$, the closed trajectory (T) has $\alpha_0 = in$ as the characteristic exponent. Let us assume that, for $\lambda > 0$, the characteristic exponent divided by i is larger than n, and that for $\lambda < 0$, on the other hand, it is smaller than n.

For $\lambda \gtrless 0$, the point M will no longer be its own $2p^{th}$ focus. Its $2p^{th}$ focus will be located a little short of M for $\lambda > 0$, and beyond M for $\lambda < 0$. Let F be this focus. The distance MF will naturally depend on the position of M on (T). I shall designate ε as the largest value of this distance. It is apparent that ε will be a continuous function of λ, and that it will vanish with λ. We should point out, that for $\lambda \gtrless 0$, the focus F is always beyond M, or always a little short of it, according to the principles given in No. 347, depending on the value of the characteristic exponent. The distance MF can never vanish. /339

Let F' be a point located a little beyond F. We may connect M with F' by a trajectory (T'), provided that the distance FF' is less than a certain quantity δ'. It is apparent that δ' is a continuous function of λ, and that it may be reduced to δ for $\lambda = 0$.

Let us set $\lambda > 0$, in such a way that M is beyond F. We may have M play the role of F', and we may connect M to itself by a trajectory (T'), provided that the distance MF is smaller than δ', or provided that

$$\varepsilon < \delta'.$$

For $\lambda = 0$, ε is zero, and $\delta' = \delta > 0$. Therefore, we may take λ small enough so that the inequality is satisfied.

We may then connect the point M to itself through a trajectory (T') deviating a little from (+), passing around (T) k + 2 times, and intersecting (T) 2p + 1 times.

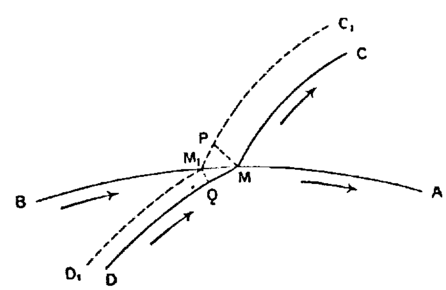

Figure 12

In the figure, BA represents an arc of (T) on which M is located. MC is an arc of (T') starting from M and DM is another arc of this same trajectory bordering upon M. The arrows indicate the direction in which the trajectories are traversed.

The point M may also be connected to itself not through one trajectory, but through two (T'). For one, as the figure indicates, the angle CMA is positive, so that CM is above MA. For the other, the angle CMA /340 would be negative.

The trajectory (T') must not be regarded as a closed trajectory. It leaves the point M to return to the point M, but the direction of the tangent is not the same at the point of departure as it is at the point of arrival, so that the arcs MC and DM do not join each other.

The trajectory (T'), thus proceeding from M to M with a hooked angle at M, will form what may be called a loop. If the same construction is followed for the points M of (T), we shall obtain a series of loops. We shall obtain two of them, the first corresponding to the case in which the angle CMA is positive, and the second corresponding to the case in which this angle is negative. These two series are separated from each other, and the passage from one to another may only be made if the angle CMA is infinitely small.

The trajectory (T'), which is infinitely close to (T), would pass through the focus F, according to the definition of focuses. However, since it must end at the point M, the points M and F would coincide, and this cannot happen according to the principles presented in No. 347.

Therefore, if all of the focuses are pointed, we have two series of loops for $\lambda > 0$, and we have no more for $\lambda < 0$.

If all the focuses were taloned, the same line of reasoning could be repeated. We would find that there are two series of loops for $\lambda < 0$, and that there are no more for $\lambda > 0$.

375. Let us consider one of the series of loops defined in the preceding section. The action calculated along one of these loops will vary with the position of the point M; it will have at least one maximum or one minimum.

If the action is maximum or minimum, it may be stated that the two arcs MC and CD coincide, so that the trajectory (T') is closed and corresponds to a periodic solution of the second type.

For example, let us assume that the trajectory (T') corresponds to the minimum of the action, and that the angle CMA is larger than the angle BMD, just as in the figure. Let us then take a point M_1 to the left of M and infinitely close to M, and let us construct a loop (T_1') which /341
differs by an infinitely small amount from the loop (T'), having its hooked point at M_1. Let M_1C_1 and M_1D_1 be two arcs of this loop.

From M and from M_1 I may draw two normals MP and M_1Q on M_1C_1 and MD.

According to a well-known theorem, the action along (T') from the point M up to the point Q will equal the action along (T_1') from the point

P to M_1. We shall therefore have

$$\text{action}(T_1') = \text{action}(T') + \text{action}(M_1P) - \text{action}(MQ)$$

or

$$\text{action}(T_1') = \text{action}(T') + \text{action}(MM_1)(\cos CMA - \cos BMQ),$$

or finally

$$\text{action}(T_1') < \text{action}(T'),$$

which is absurd, since (T') was assumed to correspond to the minimum of the action.

If we set

$$CMA < BMD,$$

we would arrive at the same absurd result placing M_1 to the right of M.

We must therefore assume that

$$CMA = BMD,$$

i.e., that the two arcs coincide.

The same line of reasoning may be applied to the case of the maximum.

Each series of loops therefore contains at least two closed trajectories.

Each of these closed trajectories passes around (T) k + 2 times, and intersects (T) at 2p points. For p of these points, the angle similar to CMA is positive, and for the other p points, it is negative. Due to the fact that the curve (T') is closed, it must intersect (T) as many times in one direction as in the other direction.

Therefore, it may be assumed that this closed trajectory consists of 2p types of loops, because we may regard any arbitrary one of our 2p points of intersection as the hooked point. For p of these types, the loop thus defined would belong to the first series, and for the other p types, it would belong to the second series.

Among the loops of each series, there are therefore not two, but at least 2p of them, which may be reduced to closed trajectories. However, /342
one thus obtains not 4p, but only two different closed trajectories.

The fact that there are not more of them is, in general, not the result of the preceding line of reasoning, but may be concluded from the principles presented in the preceding chapter.

The trajectory (T') thus defined will have $\frac{1}{2}(k + 1)p$ double points, if k is odd, and $\frac{1}{2}(k + 2)p$ double points if k is even. This is valid for small values of λ, and it remains valid no matter how large λ may be as long as (T') exists. The number of double points could only vary if two branches of the curve (T') were tangent to each other. However, two trajectories cannot be tangent to each other without coinciding.

For the same reason, no matter how large λ may be, as long as the two trajectories (T) and (T') exist, they will intersect at 2p points.

376. The entire line of reasoning presented in the preceding section assumes that we are dealing with absolute motion.

If this line of reasoning is extended to the case of relative motion, difficulties will be encountered which are not insurmountable, but which I shall not try to surmount at this point.

To begin with, we must modify the construction employed in the preceding section. Instead of drawing MP and M_1Q normal to M_1C_1 and MD, we must proceed as follows. In order to construct MP, for example, we should construct a circle which is infinitely small and which satisfies the following conditions. It intersects M_1C_1 at P and touches the line MP at this point. The line connecting M to the center must have a given direction, and the ratio of the line length to the radius must be given. The line MP thus constructed has the same properties as the normal in absolute motion. Unfortunately, in certain cases this construction entails certain difficulties.

In addition, the action (MM_1) is not always positive. If it became zero, this line of reasoning would still have a defect. The maximum or the minimum could be reached at the point M, so that the action (MM_1) is zero, and this could occur without the necessity of the arcs MC and DM coinciding. /343

Our line of reasoning therefore only applies to the case of relative motion, if the action is positive along (T).

In any case, one of the conclusions is still valid. The closed trajectory (T') always exists, since -- if the line of reasoning given in the preceding section is lacking -- the same does not hold true for the line of reasoning given in Chapters XXVIII and XXX. In addition, (T') intersects (T) at 2p points, and has $\frac{p}{2}(k + 1)$ or $\frac{p}{2}(k + 2)$ double points.

This is valid for small values of λ, but it cannot be concluded any longer that this is valid no matter what λ may be, because two trajectories may be tangent without coinciding, provided that they are traversed in the opposite direction.

377. Let us assume that there are only two degrees of freedom, two of the characteristic exponents are zero, and the two others are equal and have opposite signs.

The equation which has the following as roots

$$e^{\pm\alpha T}$$

is an equation of the second order whose coefficients are real (T represents the period and α represents one of the characteristic exponents).

Its roots are therefore real or imaginary and conjugate.

If they are real and positive, the α's are real, and the periodic solution is unstable.

If they are imaginary, the α's are imaginary and conjugate. Since the product equals +1, the α's are purely imaginary, and the periodic solution is stable.

If they are real and negative, the α's are imaginary but complex, with the imaginary part equalling $\frac{i\pi}{T}$. The periodic solution is still unstable.

They cannot be real and have opposite signs, since the product equals +1.

There are therefore two kinds of unstable solutions, corresponding to the following two hypotheses /344

$$e^{\alpha T} > 0, \quad e^{\alpha T} < 0.$$

The passage from stable solutions to unstable solutions of the first type occurs for the value

$$\alpha = 0.$$

The passage from stable solutions to unstable solutions of the second type occurs for the value

$$\alpha = \frac{i\pi}{T}.$$

378. Let us first study the passage to unstable solutions of the first type. At the moment of passage, we have

$$e^{\alpha T} = 1.$$

Let us again consider the terms β_k and ψ_k defined in Chapter III, and let us consider the equation

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1} - S & \frac{d\psi_1}{d\beta_2} & \frac{d\psi_1}{d\beta_3} & \frac{d\psi_1}{d\beta_4} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} - S & \frac{d\psi_2}{d\beta_3} & \frac{d\psi_2}{d\beta_4} \\ \frac{d\psi_3}{d\beta_1} & \frac{d\psi_3}{d\beta_2} & \frac{d\psi_3}{d\beta_3} - S & \frac{d\psi_3}{d\beta_4} \\ \frac{d\psi_4}{d\beta_1} & \frac{d\psi_4}{d\beta_2} & \frac{d\psi_4}{d\beta_3} & \frac{d\psi_4}{d\beta_4} - S \end{vmatrix} = 0. \tag{1}$$

This equation has the following roots

$$0, \quad 0, \quad e^{\alpha T} - 1, \quad e^{-\alpha T} - 1.$$

At the time of passage, the four roots become zero.

Before studying the simple case in which we are dealing with equations of dynamics with two degrees of freedom, and in which we assume that the function F does not depend explicitly on time and that, consequently, the equations have the energy integral F = const., it is advantageous to consider for a moment a case which is even simpler.

Let F be an arbitrary function of x, y and t, which is periodic of period T with respect to t. Let us consider the canonical equations /345

$$\frac{dx}{dt} = \frac{dF}{dy}, \quad \frac{dy}{dt} = -\frac{dF}{dx}; \tag{2}$$

These are the equations of dynamics with only one degree of freedom. However, due to the fact that F depends on t, they do not have the energy equation F = const.

Let us assume that these equations (2) have a periodic solution of period T. The characteristic exponents will be provided by the following equation which is similar to (1)

$$\begin{vmatrix} \frac{d\psi_1}{d\beta_1} - S & \frac{d\psi_1}{d\beta_2} \\ \frac{d\psi_2}{d\beta_1} & \frac{d\psi_2}{d\beta_2} - S \end{vmatrix} = 0 \tag{3}$$

which has the following roots

$$e^{\alpha T} - 1, \quad e^{-\alpha T} - 1.$$

These roots all become zero at the moment of passage.

Let us assume that F depends on a certain parameter μ and that, for $\lambda = 0$, the two roots of the equation (3) are zero. The functions ψ_1 and ψ_2 will depend not only on β_1 and β_2, but also on μ. We shall assume that F may be developed in powers of μ, and that consequently ψ_1 and ψ_2 may be developed in powers of β_1, β_2 and μ.

The periodic solutions will be provided by the following equations

$$\psi_1 = 0, \qquad \psi_2 = 0. \tag{4}$$

For $\mu = 0$, $\beta_1 = \beta_2 = 0$, the functional determinant of the ψ's with respect to the β's is zero. However, in general the four derivatives $\frac{d\psi_i}{d\beta_k}$ will not vanish at the same time. For example, let us assume

$$\frac{d\psi_1}{d\beta_1} \lesseqgtr 0,$$

and we shall derive β_1 in series developed in powers of β_2 and μ from the first equation (4), and we shall substitute it in the second equation (1). /346
Let

$$\Psi(\beta_2, \mu) = 0 \tag{5}$$

be the result of the substitution. Our functional determinant being zero, we shall have

$$\frac{d\Psi}{d\beta_2} = 0.$$

However, we may distinguish between two cases:

1. The derivative $\frac{d\Psi}{d\mu}$ is not zero, or, in other words, the functional determinant of ψ_1 and ψ_2 with respect to β_1 and μ is not zero.

In this case, if we assume that β_2 and μ are the coordinates of a point in a plane, the curve represented by equation (5) will have an ordinary point at the origin, where the tangent will be the line $\mu = 0$.

In general, the second derivative

$$\frac{d^2\Psi}{d\beta_2^2}$$

will not be zero, i.e., the origin will not be a point of inflection of the curve (5).

If we intersect the line $\mu = \mu_0$, where μ_0 is a rather small constant, we may have two points of intersection for this line and the curve (5) in the vicinity of the origin, or we may not have any, depending on the sign of μ_0.

For example, if this curve is above its tangent, we shall have two intersections for $\mu_0 > 0$, and consequently two periodic solutions, and for $\mu_0 < 0$ we shall not have any.

We have thus seen two periodic solutions approach each other, coincide, and then disappear.

Let us consider the two points of intersection of the line $\mu = \mu_0$ with the curve (5). They will correspond to two consecutive roots of the equation (5) and, consequently, to two values having opposite signs of the derivative $\frac{d\Psi}{d\beta_2}$, and therefore to two values of opposite signs of the functional determinant of the ψ_i's with respect to the β's, that is, of the product /347

$$(e^{\alpha T}-1)(e^{-\alpha T}-1) = 2 - e^{\alpha T} - e^{-\alpha T},$$

i.e., of α^2.

Therefore, <u>one of the two periodic solutions which coincides then to disappear is always stable, and the other is unstable</u>.

2. The derivative $\frac{d\Psi}{d\mu} = 0$, or in other words the functional determinant of ψ_1 and ψ_2 with respect to β_1 and μ, is zero.

The curve (5) then has a singular point at the origin which, in general, will be an ordinary, double point.

Two branches of the curve intersect at the origin, and the line $\mu = \mu_0$ will always meet the curve at two points. We shall therefore have two periodic solutions, no matter what the sign of μ_0 may be.

The two branches of the curve determine four regions in the vicinty of the origin. In two of these regions which are opposite the peak Ψ will be positive; in the other two regions, it will be negative.

Let OP_1, OP_2, OP_3, OP_4 be the four half-branches which converge at the origin. OP_1 will be the extension of OP_3 and OP_2 will be the extension of OP_4. OP_1 and OP_2 will correspond to $\mu_0 > 0$; OP_3 and OP_4 will correspond to $\mu_0 < 0$. The function Ψ will be positive for the angles P_1OP_2, P_3OP_4, and negative for the angles P_2OP_3, P_1OP_4.

We have just seen that the stability depends on the sign of the derivative $\frac{d\Psi}{d\beta_2}$. For example, when we pass over OP_1, Ψ will change from negative to positive. The derivative will be positive, and the solution will be stable, for example. It will also be stable when we pass over OP_4, and unstable when we pass over OP_2 or OP_3.

The periodic solutions corresponding to OP_1 are stable, and they form an analytical sequence with respect to those which correspond to OP_3 and which are unstable.

Conversely, those which correspond to OP_2 and which are unstable are the analytical sequence with respect to those which correspond to OP_4, and which are stable.

We thus have two analytical series of periodic solutions which coincide for $\mu = 0$, and at this instant of time <u>the two series exchange their stability.</u>

We have just studied the two simplest cases, but there may be a multitude of other cases corresponding to different singularities which the /348 curve (5) may have at the origin.

However, no matter what these singularities may be, we shall observe an <u>even</u> p + q number of half-branches emanating from the origin, i.e., p for $\mu > 0$ and q for $\mu < 0$. Let us assume that a small circle about the origin encounters them in the following order

$$OP_1, \quad OP_2, \quad \ldots, \quad OP_{p+q}.$$

Let

$$OP_1, \quad OP_2, \quad \ldots, \quad OP_p \tag{6}$$

be those which correspond to $\mu > 0$ and let

$$OP_{p+1}, \quad OP_{p+2}, \quad \ldots, \quad OP_{p+q} \tag{7}$$

be those which correspond to $\mu < 0$.

Then the half-branches (6) will correspond alternately to periodic stable solutions and to unstable solutions. For purposes of brevity, I may state that these half-branches are alternately stable or unstable.

The same holds true for the half-branches (7).

In addition, OP_p and OP_{p+1} are both stable or both unstable.

Consequently, the same holds true for OP_{p+q} and OP_1.

Therefore, let p' and p" be the number of stable half-branches and the number of unstable half-branches for $\mu > 0$, so that we have

$$p' + p'' = p.$$

Let q' and q" be the corresponding numbers for $\mu < 0$, so that $q' + q'' = q$. There are therefore only three possible hypotheses

$$p' = p'', \qquad q' = q'',$$
$$p' = p'' + 1, \qquad q' = q'' + 1,$$
$$p' = p'' - 1, \qquad q' = q'' - 1.$$

In any case, we have

$$p' - p'' = q' - q''.$$

Let us assume that p does not equal q, and, for example, that $p > q$, in such a way that a certain number of periodic solutions disappears when we pass from $\mu > 0$ to $\mu < 0$. It may be seen that this number is always even, and in addition as many stable solutions as unstable solutions would always disappear, according to the preceding equation. /349

Let us now assume that we have an analytical series of periodic solutions and that, for $\mu = 0$, we pass from stability to instability, or vice-versa (in such a way that the exponent α vanishes). Then q' and p'' (for example) are at least equal to 1. Therefore, $p' + q''$ is at least equal to 2. It follows from this that we shall have at least another analytical series of real, periodic solutions which intersect the first for $\mu = 0$.

Therefore, if, for a certain value of μ, a periodic solution loses stability or acquires it (in such a way that the exponent α is zero) it will coincide with another periodic solution, with which it will have exchanged its stability.

379. Let us now return to the case which I was first going to discuss -- that in which the time does not enter explicitly in the equations, where, consequently, we have the energy integral $F = C$, where finally there are two degrees of freedom.

I shall pursue the same line of reasoning as was the case in No. 317, and I shall assume that the period of the periodic solution, which is T for the solution which corresponds to $\mu = 0$, $\beta_i = 0$, equals $T + \tau$, and differs very little from T for adjacent periodic solutions. I shall write the following equations

$$\psi_1 = 0, \quad \psi_2 = 0, \quad \psi_3 = 0, \quad F = C_0, \quad \beta_1 = 0, \tag{1}$$

which include the following variables

$$\beta_1, \; \beta_2, \; \beta_3, \; \beta_4, \; \mu, \; \tau.$$

According to our hypotheses, the functional determinant of the ψ's with respect to the β's must vanish, as well as all its minors of the first order. However, the minors of the second order will not all be zero at the same time, in general.

Therefore let us set $\beta_1 = 0$ in equations (1), and let us consider the functional determinant Δ of /350

$$\psi_1,\ \psi_2,\ \psi_3,\ F$$

with respect to

$$\beta_2,\ \beta_3,\ \beta_4,\ \tau.$$

This determinant vanishes when the β's, μ's and τ's vanish, but in general the minors of the first order will not vanish.

Let us consider the functional determinants of F and of two of the four functions ψ with respect to τ, and with respect to two of the four variables β. Can they all be zero at the same time?

According to the theory of determinants this could only happen if the following were true:

1. All the minors of the two first orders of the determinants of the ψ's with respect to the τ's were zero at the same time, which does not occur, in general, and which we shall not assume.

2. The derivatives of F were all zero at the same time. We saw in No. 64 that they must be zero all along the periodic solution. We shall no longer assume this.

3. The derivatives of the ψ's and of F with respect to τ were all zero at the same time. The following values

$$\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$$

would not correspond to a periodic solution strictly speaking, but to a position of equilibrium (see No. 68).

We shall no longer assume this.

We may therefore always assume that all the minors of the first order of Δ are not zero.

Let us then eliminate four of our unknowns β and τ among the equations (1).

For example, let us eliminate β_1, β_3, β_4, τ; we shall still have an equation of the following form

$$\Psi(\beta_2, \mu) = 0.$$

Due to the fact that this equation has exactly the same form as equation (5) of the preceding section, it will be handled in the same way, and we shall arrive at the same results:

1. When periodic solutions disappear after having coincided, an even number, and as many stable as unstable solutions, always disappear.

2. When a periodic solution loses or acquires stability when we vary μ continuously (in such a way that α vanishes), we may always be certain /351 that at the moment of passage another real, periodic solution of the same period coincides with it.

380. Let us proceed to the second case, that in which

$$\alpha = \frac{i\pi}{T}.$$

Due to the fact that none of the characteristic exponents vanishes for

$$\mu = 0,$$

except the two which are always zero, there is no periodic solution of period T which coincides with the first for

$$\mu = 0.$$

On the other hand, according to principles presented in Chapter XXVIII, there are periodic solutions of the second type, of period 2T, which coincide with the given solution whose period is T for $\mu = 0$.

What may we say regarding their stability? For $\mu > 0$, for example, we shall have a stable solution of period T which will become unstable for $\mu < 0$.

For $\mu > 0$, let p' and p" be the number of stable solutions and the number of unstable solutions which have the period 2T, without having the period T. Let q' and q" be the corresponding numbers for $\mu < 0$.

Let us then consider all the solutions of period 2T, whether they have have the period T or not. Applying the principles presented in No. 378 to them, I find that I may postulate the following three hypotheses regarding these four numbers:

$$p' + 1 = p'', \qquad q' = q'' + 1,$$
$$p' = p'', \qquad q' = q'' + 2,$$
$$2 + p' = p'', \qquad q' = q''.$$

However, if we refer to the principles given in Chapter XXVIII, we shall find that these four numbers cannot take all values which are compatible with the three hypotheses. The simplest and most frequent cases are investigated in No. 335.

381. In Volume XXI of Acta Mathematica, M. G. H. Darwin studied certain periodic solutions in detail. He discusses the hypotheses given in No. 9, and considers a perturbing planet which he calls Jupiter, and to which he attributes a mass which is ten times smaller than that of the Sun. This fictitious planet describes a circular orbit around the Sun, and a small perturbed planet having zero mass moves in the plane of this orbit.

He has thus acknowledged the existence of certain periodic solutions which are again included in those which I have called solutions of the first type, and which he has studied in detail. These orbits are referred to moving axes, turning around the Sun with the same angular velocity as Jupiter. These orbits are closed curves, in relative motion with respect to these moving axes.

M. Darwin has called the first class of periodic orbits the class of planets A. The orbit is a closed curve encircling the Sun, but not encircling Jupiter. The orbit is stable when the Jacobi constant is larger than 39, and unstable in the opposite case. The instability corresponds to a characteristic exponent having $\frac{i\pi}{T}$ as the imaginary part.

For values of the Jacobi constant which are close to 39, there are therefore periodic solutions of the second type whose period is double.

The corresponding orbit will be a closed curve with a double point passing around the Sun twice. The two loops of this curve differ very little from each other, and both differ very little from a circle.

We shall study these solutions of the second type in greater detail at a later point.

M. Darwin also obtained oscillating satellites which he called a and b, and are those which we discussed in No. 52. They are always unstable.

Finally, he obtained satellites which, strictly speaking, with respect to the system of moving axes under consideration, describe closed curves encircling Jupiter, but not encircling the Sun. /353

For C = 40 (C is the Jacobi constant), we have only one satellite A which is stable. For C = 39.5, the satellite A becomes unstable with a real exponent α. However, we have two new satellites B and C, the second of which is stable, and the first of which is unstable with a real exponent α. For C = 39, we obtain the same result. For C = 38.5, the satellite C becomes unstable with a complex exponent α (whose imaginary part is $\frac{i\pi}{T}$). Finally, for C = 38, we obtain the same result.

We must therefore consider three passages:

1. The passage of satellite A from stability to instability;

2. The appearance of the satellites B and C;

3. The passage of satellite C from stability to instability;

The last two passages do not entail any difficulties.

Two periodic solutions B and C will appear simultaneously which differ very little from each other. One is stable and the other is unstable; the exponent α is real for the unstable solution. This conforms with the conclusions reached in No. 378.

The passage of the satellite C from stability to instability no longer presents any difficulties, because the exponent α is complex in the case of instability. The conditions presented in No. 380 therefore hold. We therefore have periodic solutions of the second type corresponding to closed curves which pass around Jupiter twice.

382. On the other hand, the passage of satellite A from stability to instability entails great difficulty, because the exponent α is real in the case of instability. According to No. 378, we should therefore have exchange of stability, with other periodic solutions corresponding to closed curves passing around Jupiter only once. This would not seem to result from the calculations of Darwin.

We are naturally led to think that the unstable satellites A discovered by Darwin do not represent the analytical extension of the stable satellites A.

Other considerations lead to the same result.

The stable satellites A have ordinary closed curves for orbits; the unstable satellites A have orbits in the form of a figure eight. /354

How may we pass from one case to another? This may only be done by a curve having a cusp, but the velocity must be zero at the cusp and, for reasons of symmetry, this cusp could only be located on the axis of the x's. It could not be between the Sun and Jupiter. In Figure 1, Darwin gives the curves of zero velocity. For $C > 40, 18$, these curves intersect the axis of the x's between the Sun and Jupiter, but this no longer holds for $C < 40, 18$, and the passage occurs between $C = 40$ and $C = 39.5$.

We are left with the hypothesis that the cusp is located beyond Jupiter, but this is no longer satisfactory. Let us compare the two orbits corresponding to $C = 40$ and to $C = 39.5$. The first intersects the axis of the x's twice at a right angle, once beyond Jupiter and once just short of it. Let P and Q be the two intersection points. In the same way, the second orbit (if we disregard the double point) intersects the axis of the x's twice at a right angle, once beyond Jupiter, and once just short of Jupiter. Let P' and Q' be the two intersection points. Let us consider the intersection point P or P' which is beyond Jupiter, and let us determine the sign of $\frac{dy}{dt}$. We shall see that this sign is positive for one orbit or the other. However, $\frac{dy}{dt}$ would have to change sign when passing through the cusp.

The point P, the hypothetical cusp, and the point P' cannot therefore be regarded as the analytical extension of each other. We must then assume that at a given moment an exchange has occurred between the two intersection points of the orbit of the satellite A' and of the x-axis, that which is located on the right passing to the left, and vice versa. Nothing in the behavior of the curves constructed by M. Darwin justifies such an assumption.

Therefore, I may conclude that the unstable satellites A are not the analytical extension of the stable satellites A. But when do the satellites A become stable?

I can only formulate hypotheses on this point and, in order to do otherwise, it would be necessary to reconsider the mechanical quadratures of M. Darwin. However, if we examine the behavior of the curves, it appears that at a certain time the orbit of the satellite A must pass through Jupiter, and that it then becomes what M. Darwin has called an oscillating satellite. /355

383. Let us study the planets A in greater detail, and the passage of these planets from stability to instability.

The orbits of these planets correspond to what we have designated as periodic solutions of the first type (No. 40). The orbit with a double point, which passes around the Sun twice and which differs very little from that of the planet A at the moment when the orbit of this planet has just become unstable, corresponds to which we have designated as periodic solutions of the second type (47).

If we apply the procedure by which we deduced periodic solutions of the second type from those of the first type to solutions of the first type, we shall obtain solutions of the second type exactly.

In solutions of the second type, the mean anomalistic motions, which differ very little from the mean motions strictly speaking, are in a commensurable ratio. We must therefore consider the case in which, for our solution of the second type (and, consequently, for the planet A at the time of passing from stability to unstability), the ratio of the mean motions is close to a simple commensurable number. Since the orbit must pass around the Sun twice, this ratio will be close to a multiple of $\frac{1}{2}$.

In other words, at the moment of passage, the term which M. Darwin has called nT must be close to a multiple of π.

In effect, this is what occurs. The tables of M. Darwin provide us with the following

$$C = 40 \quad A \text{ stable}, \quad nT = 154^\circ,$$
$$C = 39{,}5 \quad A \text{ stable}, \quad nT = 165^\circ,$$
$$C = 39 \quad A \text{ unstable}, nT = 177^\circ,$$
$$C = 38{,}5 \quad A \text{ unstable}, nT = 191^\circ.$$

It can be seen that the passage must be made around nT = 170°, and this number is close to 180°. /356

The mean motion of the planet A is therefore almost three times that of Jupiter.

We could consider applying the principles presented in Chapter XXX to a study of these solutions of the second type, but several difficulties would be encountered because we would be dealing with an exception. It would be better to resume this study directly.

384. Let us again consider the notation given in No. 313, and let us set the following, just as in this section

$$x_1 = L - G, \qquad x_2 = L + G,$$
$$2y_1 = l - g + t, \qquad 2y_2 = l' + g - t,$$
$$F' = R + G = F_0 + \mu F_1 + \ldots,$$
$$F_0 = \frac{2}{(x_1 + x_2)^2} + \frac{x_2 - x_1}{2}.$$

The term L must have the same sign as G (see page 201, *in fine*), and the eccentricity must be very small. Since x_1 is on the order of the square of the eccentricity, this variable will also be very small.

Since we only wish to determine the number of periodic solutions and their stability, we shall be content with an approximation.

We shall therefore neglect $\mu^2 F_2$ and the following terms. In the term μF_1, we only take into account secular terms and terms with a very long period, and we shall neglect the powers which are higher than x_1. We shall have

$$F_1 = a + bx_1 + cx_1\cos\omega,$$

where a, b, c are functions only of x_2, and where $cx_1\cos\omega$ is the very long period term which has been retained.

The very long period terms are terms with $\ell + 3g - 3t$, i.e., terms with $2y_2 - y_1$. We therefore have

$$\omega = 4y_2 - 2y_1.$$

We then have

$$F' = \frac{2}{(x_1 + x_2)^2} + \frac{x_2 - x_1}{2} + \mu(a + bx_1 + cx_1\cos\omega)$$

and we may apply the method of Delaunay.

The canonical equations have the integral /357

$$x_2 + 2x_1 = k,$$

from which we have

$$F' = \frac{2}{(k - x_1)^2} + \frac{k}{2} - \frac{3x_1}{2} + \mu(a + bx_1 + cx_1\cos\omega).$$

With the approximation which has been adopted, we may replace a, b, c by

$$a_0 - 2x_1 a'_0, \quad b_0, \quad c_0,$$

designating that which a, $\frac{da}{dx_2}$, b, c, become by a_0, a'_0, b_0, c_0 when we replace x_2 by k. Thus,

$$\alpha = a_0, \quad \beta = b_0 - 2a'_0, \quad \gamma = c_0$$

designate the constants which depend on k, and we have

$$F' = \frac{2}{(k - x_1)^2} + \frac{k}{2} - \frac{3x_1}{2} + \mu(\alpha + \beta x_1 + \gamma x_1\cos\omega).$$

Let us assume that k is a constant, $\sqrt{x_1}\cos\frac{\omega}{2}$, $\sqrt{x_1}\sin\frac{\omega}{2}$ are rectangular coordinates of a point in a plane, and let us compile the curve

$$F' = C,$$

where C designates a second constant.

This curve also depends on the two constants k and C. If it has a double point, this double point will correspond to a periodic solution, which will be stable if the two tangents to the double point are imaginary, and unstable if the two tangents are real.

We should note that the curve is symmetrical with respect to the two axes of the coordinates and that the two double points, which are symmetrical to each other with respect to the origin, do not correspond to two periodic solutions which are actually different.

The double points may only be located on one of the axes of the coordinates, so that they will be obtained by setting

$$\omega = 0, \qquad \omega = \pi.$$

If we set

$$C = \frac{2}{k^2} + \frac{k}{2} + \mu\alpha,$$

the curve F' = C passes through the origin and has a double point. The tangents to the double point are given by the equation /358

$$\frac{4}{k^3} - \frac{3}{2} + \mu\beta + \mu\gamma \cos\omega = 0.$$

Therefore, if

$$\frac{4}{k^3} - \frac{3}{2} + \mu\beta > \mu\gamma \tag{1}$$

the tangents are imaginary. If

$$\mu\gamma > \frac{4}{k^3} - \frac{3}{2} + \mu\beta > -\mu\gamma, \tag{2}$$

the tangents are real. Finally, if

$$-\mu\gamma > \frac{4}{k^3} - \frac{3}{2} + \mu\beta, \tag{3}$$

the tangents are again imaginary.

The coefficient β is positive. I wrote the preceding inequalities also assuming that γ is positive. If γ were negative, we would only have to change ω into $\omega + \pi$.

The double point at the origin corresponds to the solution of the first type, i.e., to planet A of M. Darwin. It may be seen that this solution is stable when the inequalities (1) or (3) hold, and is unstable when the inequalities (2) hold.

Let us now study the double points which may be located on the line $\omega = 0$.

If we set $\omega = 0$, the function F' becomes

$$F' = \frac{2}{(k - x_1)^2} + \frac{k}{2} - \frac{3x_1}{2} + \mu x + \mu x_1(\beta + \gamma) = C. \tag{4}$$

Keeping k constant, if we vary x_1 from 0 to k, we find that the maxima and minima of F' are given by the equation

$$\frac{4}{(k - x_1)^3} - \frac{3}{2} + \mu(\beta + \gamma) = 0, \tag{5}$$

which has a solution if the inequality (3) holds, and does not have a solution in the opposite case.

Therefore, if the inequality (3) does not hold, the function F' is constantly decreasing if it holds. The function F' first increases, reaching a maximum, and then decreases.

This maximum corresponds to a double point located on the line $\omega = 0$, /359 or rather to two double points which are symmetrical with respect to the origin.

However, we must determine how we may obtain these double points for a given value of the constant C. Equation (5) provides us with x_1 as a function of k. We must deduce x_1 from it as a function of C.

However, equations (4) and (5) may be written

$$F' = C, \qquad \frac{dF'}{dx_1} = 0$$

from which we have

$$\frac{dC}{dx_1} = \frac{dF'}{dx_1} + \frac{dF'}{dk}\frac{dk}{dx_1} = \frac{dF'}{dk}\frac{dk}{dx_1},$$

$$\frac{d^2F'}{dx_1^2} + \frac{d^2F'}{dk\,dx_1}\frac{dk}{dx_1} = 0.$$

Neglecting terms containing μ, we have

$$\frac{dF'}{dk} + \frac{dF'}{dx_1} = -1$$

from which we have

$$\frac{dF'}{dk} = -1,$$

$$\frac{d^2F'}{dk\,dx_1} = \frac{d^2F'}{dx_1^2} = 0; \qquad \frac{d^2F'}{dx_1^2} = \frac{12}{(k - x_1)^4} = 12\left(\frac{3}{8}\right)$$

and

$$\frac{dk}{dx_1} = 1; \qquad \frac{dC}{dx_1} = -1.$$

It results from this that x_1 is a constantly decreasing function of C.

For a value of C, we have only a maximum at the most, i.e., we have at the most two double points which are symmetrical to each other with respect to the origin on the line $\omega = 0$.

Let C_0 be the value of C which satisfies the double equality

$$C_0 = \frac{2}{k^2} + \frac{k}{2} + \mu\alpha,$$

$$\frac{4}{k^3} - \frac{3}{2} + \mu(\beta + \gamma) = 0.$$

We shall see that, for $C > C_0$, there will not be a double point on the line $\omega = 0$ and that, for $C < C_0$, there will be two of them. /360

The same discussion may be applied to the case of double points located on the line $\omega = \pi$. The values of x_1 will be given by the equation

$$\frac{4}{(k - x_1)^3} - \frac{3}{2} + \mu(\beta - \gamma) = 0 \qquad (5')$$

which has a solution if the inequalities (2) or (3) hold.

If C_1 is the value of C which satisfies the double equality

$$C_1 = \frac{2}{k^2} + \frac{k}{2} + \mu\alpha,$$

$$\frac{4}{k^3} - \frac{3}{2} + \mu(\beta - \gamma) = 0,$$

the condition for which there are two double points on the line

$$\omega = \pi,$$

is $C < C_1$.

We would like to point out that $C_1 > C_0$, that C_0 is the value of C for which one passes from inequality (2) to inequality (3), and that C_1 is the one for which we may pass from inequality (1) to inequality (2).

When compiling the curves, we would readily find that the tangents are real for the double points located on $\omega = 0$, and that they are imaginary for the double points located on $\omega = \pi$.

We may therefore sum up our results as follows:

First case

$$C > C_1.$$

The inequality (1) holds.

The solution of the first type (planet A) is stable.

There is no solution of the second type (orbit with double point).

Second case

$$C_1 > C > C_0.$$

The inequalities (2) hold.

The solution of the first type becomes unstable.

There is a solution of the second type which is stable.

/361

Third case

$$C < C_0.$$

Inequality (3) holds.

The solution of the first type is stable.

There are two solutions of the second type, one of which is stable and one of which is unstable. The first corresponds to the two double points located on the line $\omega = \pi$, and the second corresponds to the two double points located on the line $\omega = 0$.

These conclusions are valid, provided that μ is sufficiently small. Is the value adopted by M. Darwin, $\mu = \frac{1}{10}$, sufficiently small?

I have not verified this, but it seems very likely.

It is therefore likely that M. Darwin would have obtained stable orbits if he had continued his study of the planets A for values of C smaller than 38.

CHAPTER XXXII

PERIODIC SOLUTIONS OF THE SECOND TYPE

385. Let us again consider the equations of No. 13 /362

$$\frac{dx_i}{dt} = \frac{dF}{dy_i}, \quad \frac{dy_i}{dt} = -\frac{dF}{dx_i}, \quad F = F_0 + \mu F_1 + \dots \tag{1}$$

with p degrees of freedom. According to the statements given in No. 42, these equations will have periodic solutions such that, when t increases by the period T, the variables $y_1, y_2, \dots, y_p$ increase respectively by

$$2k_1\pi, \quad 2k_2\pi, \quad \dots, \quad 2k_p\pi.$$

The integer numbers $k_1, k_2, \dots, k_p$ may be arbitrary.

However, this is only valid if the hessian of F_0 with respect to the x's is not zero. The proof presented in No. 42 is invalid when this hessian is zero, particularly when F_0 does not depend on all the variables x.

This is precisely what occurs in the three-body problem. I would like to recall that y_1, y_2; y_3, y_4; y_5, y_6 represent, respectively, the mean longitudes of the planets, of the perihelions and of the nodes, and that F_0 depends only on the two first variables x_1 and x_2 which are proportional to the square roots of the major axes.

Let us consider a periodic solution. According to the stipulated conventions, one solution will be assumed to be periodic, provided that the differences of the y's increase by multiples of 2π, when t increases by one period. In actuality, F only depends on these differences.

Let

$$2k_1\pi, \quad 2k_2\pi, \quad 2k_3\pi, \quad 2k_4\pi, \quad 2k_5\pi$$

be the quantities by which the following increase /363

$$y_1 - y_6, \quad y_2 - y_6, \quad y_3 - y_6, \quad y_4 - y_6, \quad y_5 - y_6,$$

when t increases by a period.

In Chapter III we could only establish the fact that there are periodic solutions corresponding to arbitrary values of k_1 and k_2, assuming that k_3, k_4 and k_5 are zero.

It may be inquired whether in this case, just as in the general case, there are periodic solutions corresponding to arbitrary values of the five integer numbers k, solutions which I would like to designate as solutions of the second type.

386. Do these solutions of the second type exist? It is a temptation to answer in the affirmative, based upon reasons of continuity and considering the fact that the form of the function F needs to be modified very little in order to obtain canonical equations to which the line of reasoning pursued in No. 42 applies.

However, one difficulty is entailed. What happens to these solutions when we cancel the term we have designated as μ and which is proportional to the disturbing masses?

If the disturbing masses are zero, the two planets obey the laws of Kepler. The perihelions and the nodes are fixed, and it would appear that the numbers k_3, k_4 and k_5 can have no other value than zero.

This difficulty may be resolved as follows. If the masses are infinitely small, the two planets will obey the laws of Kepler, unless their distance itself becomes infinitely small at certain times.

Let us assume that the two planets, which are very far away from each other, both describe a Keplerian ellipse. It could happen that these two ellipses will meet, or will pass very close to each other, in such a way that the distance between the two planets becomes very small at a certain time. At this time, their mutual perturbing action could become significant, and the two orbits could undergo large perturbations. The planets, moving /364 away from each other again, would then describe Keplerian ellipses again. However, these new ellipses will differ greatly from the old ellipses. The perihelions and the nodes will undergo considerable variations.

I would like to employ the word collision to designate this phenomenon, although it is not a collision in the true sense of the word, since the two planets do not come in contact and since the difference between them need only be rather small in order to have considerable attraction, in spite of the smallness of the masses.

However that may be, if we take these orbits with collisions into account, it is no longer valid to state that the perihelions and the nodes are fixed for $\mu = 0$, and that consequently the numbers k_3, k_4 and k_5 must be zero.

We must thus conclude that solutions of the second type exist and that, if we make μ strive to zero, they will tend to be reduced to orbits with a series of collisions. However, this rough sketch is not sufficient, and a more detailed examination is necessary.

387. Let us first consider the effect of the collision. Let E and E' be the ellipses described by the first planet before and after the collision; let E_1 and E_1' be the ellipses described by the second planet. It is apparent that these four ellipses must intersect at the same point, in such a way that the two planets describing these four orbits will pass through the encounter point at the time of the collision.

As long as the distance between them is considerable, the two planets describe curves which differ very little from an ellipse. During the very short period of time when the distance between them is very small, they describe orbits which are very different from an ellipse. These orbits may be reduced to small arcs of curves C having a radius of curvature which is very small; these arcs differ very little from arcs of a hyperbola. At the limit, the very short time of the collision may be reduced to an instant. The small arcs C may be reduced to a point, and the orbit, being reduced to two arcs of an ellipse, has a hooked point.

In order to define the orbits E, E', E_1, E_1', it is necessary to know the magnitude and direction of the velocities of the two planets P and P_1 before and after the collision. What are the relationships between these velocities? I would first like to note that the velocity of the center of gravity of the two bodies P and P_1 must be the same before and after the collision. The magnitude and direction must also be the same before and after the collision. /365

We must now consider the relative velocity of P with respect to P_1; the magnitude of this velocity must be the same before and after the collision, but it may differ in direction.

Following is the rule for determining the direction of this velocity after the collision.

Let us consider moving axes whose origin is at P_1, and let us consider a line AB which represents, in magnitude and direction, the relative velocity of P with respect to P_1 before the collision. This line AB must pass through the point P_1, since the body which moves at the velocity which it represents must collide with the point P_1, which is fixed with respect to our moving axes. However, this is only valid at the limit. This is only valid because we regard the masses, on the one hand, and the distance at which the mutual attraction of P and P_1 begins to be manifested, on the other hand -- i.e., that which could be designated as the radius of action -- as infinitely small quantities. It would therefore be more exact to say that the distance δ from P to the line AB is an infinitely small quantity of the same order as the radius of action.

Let A'B' be the line which represents the relative velocity of P with respect to P_1 after the collision. In terms of magnitude, A'B' equals AB, and the distance from P_1 to A'B' equals δ.

We finally have the rule for determining the direction of A'B'. The point P_1 and the two lines AB and A'B' are in the same plane (up to quantities which are infinitely small of higher order). The angle of AB and of A'B' may be determined as follows. The tangent of half of this angle is proportional to δ and to the square of the length of AB.

It may thus be seen that the direction of A'B' may be arbitrary.

The only two conditions which must be imposed upon our four velocities are the following: permanence of the velocity of the center of gravity in magnitude and direction; permanence of the relative velocity in magnitude alone. These conditions may be given as follows:

The energy and the area constants must not be changed by the collision.

388. Let us try to compile the orbits with collisions which are the limits toward which the solutions of the second type tend when μ strives to zero. /366

I would first like to point out that at least two collisions must be assumed in order that such an orbit may be periodic. Let us assume that two consecutive collisions never occur at the same point. Let E and E_1 be the ellipses described by the planets P and P_1 in the interval of two consecutive collisions. These two ellipses must intersect at two points and, since they have a common focus, they are in the same plane, unless the two intersection points and the focus are on a straight line.

Let us assume that we are dealing with an exception. Let Q and Q' be the two intersection points of the ellipses E and E'_1 which I shall assume are not in the same plane. These two points are on a straight line with the focus F; let E and E'_1 be the ellipses described by the two planets after the collision. They will pass through the point Q, where the collision has just been produced, and they will not be in the same plane in general. Their planes will intersect along the line FQ, so that their second intersection point (which must exist if two consecutive collisions never occur at the same point) will be located on this line FQ. I would like to add that the two ellipses E and E_1 will have the same parameter. Due to the fact that the points F, Q, and Q' are on a straight line, the inverse of the parameter of the ellipse E or of the ellipse E_1 will be $\frac{1}{2FQ} + \frac{1}{2FQ'}$.

Under this assumption, we shall employ the following procedure. For purposes of clarity, let us assume four collisions; let Q_1, Q_2, Q_3, Q_4 be the points where the four collisions occur.

We may specify these four points arbitrarily, provided that they are located on the same line passing through F.

We must construct two ellipses E and E_1 which intersect at Q_1 and Q_2, two ellipses E' and E_1' which intersect at Q_2 and Q_3, two others E'' and E_1'' which intersect at Q_3 and Q_4, and finally two others E''' and E_1''' which intersect at Q_4 and Q_1.

The orbit of P is composed of arcs pertaining to the four ellipses E, E', E'', E''', and the orbit of P_1 is composed of arcs pertaining to the four ellipses E_1, E_1', E_1'', E_1'''.

We shall specify the energy and area constants arbitrarily. These /367 constants must be the same for the interval between the first two collisions (orbits E and E_1) for the following interval, and for all the other intervals. According to the statements presented in the preceding section, this is the only condition which must be fulfilled.

In order to compile E and E_1, we shall proceed as follows. Let us consider the motion of three bodies. Since we assume $\mu = 0$, this motion is Keplerian, and the central body may be regarded as being fixed at F. We know the total energy of the system. The two planets P and P_1 must leave the point Q_1 simultaneously in order to arrive at the point Q_2 simultaneously. When P and P_1 go from Q_1 to Q_2, the true longitude of P increases by $(2m + 1)\pi$, and that of P_1 increases by $(2m_1 + 1)\pi$. We may still specify the two integer numbers m and m_1 arbitrarily. The problem has then been completely determined. It should be pointed out that the inclination of the orbits does not intervene. In order to resolve this, we may assume planar motion. The problem can always be resolved. We need only apply the principle of Maupertuis, and Maupertuis action, which is essentially positive, always has a minimum.

We must now determine the planes of the two ellipses. We know the area constants. We therefore know the invariable plane which passes through the line FQ_1Q_2. The areal velocity of the system is represented by a vector perpendicular to the invariable plane, whose magnitude and direction we know. It is the geometric sum of the areal velocities of the two planets, represented by two vectors whose magnitude we know, since they equal, respectively, mp and m_1p, where m and m_1 are the masses of the two planets and p is the common parameter of the two ellipses E and E_1. We may therefore compile the directions of these two base vectors which are perpendicular to the plane of E and to the plane of E_1, respectively.

The terms E' and E_1', E'' and E_1'', ..., may be determined in the same way.

389. Let us now assume that all of the successive collisions occur at the same point Q. The period will be divided into as many intervals as there will be collisions. Let us consider one of these intervals during which the two planets describe the two ellipses E and E_1. As in the preceding section, we will specify the energy constant and the area constant which must be the same for all the intervals. We must construct E and E_1. /368

Let us assume that during the interval under consideration the planet P has performed m complete revolutions, and that the planet P_1 has completed m_1 complete revolutions. We can arbitrarily specify the two whole numbers m and m_1. Since we know these two whole numbers, we know the ratio of the major axes. Since we know, on the other hand, the energy constant, we also know the major axes themselves.

On the other hand, we know the area constant. Consequently, we know the vector which represents the areal velocity of the system. This vector can be decomposed an infinite number of ways into two base vectors which represent the areal velocities of P and P_1. We shall arbitrarily specify this decomposition. If we know the two base vectors, we know the planes of the two ellipses and their parameters. The orientation of each of these ellipses in its plane remains to be determined. We will determine it by passing the ellipse through the point Q.

Summarizing, we can arbitrarily specify:

1. The point Q and the number of intervals;

2. For all the intervals, the area constant and the energy constant;

3. For <u>each</u> interval, the whole numbers m and m_1 and the decomposition of the areolar vector.

In order to make the problem tractable, these arbitrary numbers must satisfy certain inequalities which I will not describe.

390. Let us disregard the exceptional case where all the collisions take place along the same line or at the same point, and let us consider the case of motion in a plane. Let Q_1, Q_2, ..., be the points where the successive collisions take place. We will arbitrarily specify the energy constant and the area constant which must be the same for all the intervals.

Let us consider one of the intervals, for example, the one where the two planets pass from Q_1 to Q_2. We will arbitrarily specify the <u>magnitude</u> of the radius vectors FQ_1 and FQ_2, but not the angle between these two radius vectors, nor the duration of the interval.

We know that in this interval the difference in longitude of the two /369 planets has increased by $2m\pi$. Let us arbitrarily specify the whole number m.

Since we know this whole number, the two lengths FQ_1 and FQ_2, as well as the two energy constants and the area constants, we have everything needed to determine the orbits E and E_1. This means that the principle of Maupertuis must be applied. However, the Hamiltonian action must be defined as was done in No. 339 and the Maupertuis action must be derived according to the procedure of Nos. 336 and 337. Unfortunately, this Maupertuis action is not always positive and therefore one is not certain that it always has a minimum.

Summarizing, we can arbitrarily specify:

1. The number of intervals and the lengths FQ_1, FQ_2, ...;

2. The area constants and the energy constants;

3. For each interval, the whole number m.

The collision orbits obtained in this way are all planar. Among the periodic orbits of the second kind which reduce to these collision orbits for $\mu = 0$, there are certainly some which are planar. It is also possible that there are some which are not planar for $\mu > 0$, and only become so at the limit.

391. Let us now see how one may demonstrate the existence of periodic solutions of the second kind which, in the limit, reduce to the collision orbits which we constructed above.

Let us now consider one of the collision orbits and let t_0 be a time before the first collision and t_1 a time between the first and the second collisions. In the same way, let t_2 be a time between the second and the third collisions. For the discussion I will assume that there are three collisions. I will call T the period in such a way that at the time $t_0 + T$ the three bodies appear in the same configuration as was the case at the time t_0.

As the variables, I will take the major axes, the inclinations and the eccentricities, and the differences of the mean longitudes, the longitudes of the perihelia and the nodes. In all, there are eleven variables. The orbit is regarded as periodic if the three bodies have the same relative configuration at the end of the period.

Let x_1^0, x_2^0, ..., x_{11}^0 be the values of these variables at the instant t_0 for the collision orbit under discussion and consequently for $\mu = 0$. /370
Let x_i^1 be the values of these variables at the time t_1 for this same collision orbit, x_i^2 their values at the time t_2, and x_i^3 their values at the time $t_0 + T$. One will have

$$x_i^3 = x_i^0 + 2m_i\pi$$

where m_i is a whole number which must be zero for the major axes, the eccentricities and the inclinations.

Let us now consider an orbit which is slightly different from the collision orbit. Let us assign a very small value to μ, but different from zero. In this new orbit, our variables will have the values $x_i^0 + \beta_i^0$ at the time t_0, $x_i^1 + \beta_i^1$ at the time t_1, $x_i^2 + \beta_i^2$ at the time t_2 and finally $x_i^3 + \beta_i^3$ at the time $t_0 + T + \tau$.

The condition for which the solution is periodic with period $T + \tau$ is

$$\beta_i^3 = \beta_i^0.$$

Assuming $\mu = 0$, in order that a collision occurs between the time t_0 and the time t_1, the variables β_i^0 must satisfy two conditions.

Let

$$f_1(\beta_i^0) = f_2(\beta_i^0) = 0$$

be these two conditions.

Let us set

$$f_1(\beta_i^0) = \gamma_1^0\mu, \quad f_2(\beta_i^0) = \gamma_2^0\mu; \quad \beta_k^0 = \gamma_k^0 \quad (k = 3, 4, \ldots, 11);$$

it can be seen that the β_i^0's are holomorphic functions of the γ_i^0's and of μ. By applying the principles of Chapter II, it can be shown that the same holds for the β_i^1's.

In order that there be a collision between the times t_1 and t_2 (assuming that $\mu = 0$), two conditions are necessary, which I may write as follows

$$f_1(\beta_i^1) = f_2(\beta_i^1) = 0. \tag{1}$$

Replacing the β_i^1's in relationships (1) by their values as a function of the γ_i^0's and of μ, and then setting $\mu = 0$, I obtain

$$\theta_1(\gamma_i^0) = \theta_2(\gamma_i^0) = 0.$$

Let us then set

$$\theta_1(\gamma_i^0) = \gamma_1^1\mu, \quad \theta_2(\gamma_i^0) = \gamma_2^1\mu; \quad \beta_k^1 = \gamma_k^1 \quad (k = 3, \ldots, 11).$$

I find that the β_i^1's and the β_i^2's are holomorphic functions of the γ_i^1's /371 and of μ. The same holds true for the γ_i^0's, and consequently for the β_i^0's.

Finally, in order that there be a collision between the times t_2 and $t_0 + T\tau$, two conditions are necessary which I may write as follows

$$f_1''(\beta_i^2) = f_2''(\beta_i^2) = 0.$$

If the β_i^2's are replaced by their values as a function of the γ_i^1's and of μ, and if we then set $\mu = 0$, they become

$$\gamma_1(\gamma_i^1) = \gamma_2(\gamma_i^1) = 0.$$

I may set

$$\gamma_1(\gamma_i^1) = \gamma_1^2\mu, \quad \gamma_2(\gamma_i^1) = \gamma_2^2\mu, \quad \beta_k^2 = \gamma_k^2 \quad (k = 3, \ldots, 11)$$

and I then find that the β_i^0's, the β_i^1's, and the β_i^2's are holomorphic functions of the γ_i^2's and of μ. In the same way, the β_i^3's are holomorphic functions of the γ_i^2's, of μ, and of τ.

The relationships $\beta_i^3 = \beta_i^0$ are therefore equations whose two terms are holomorphic with respect to the γ_i^2's, μ, and τ. These equations could be discussed in the same manner as in Chapter III. The existence of solutions of the second type could then be demonstrated.

I do not believe that this is necessary, because these solutions deviate too much from the orbits traversed in actuality by celestial bodies.

CHAPTER XXXIII

DOUBLY ASYMPTOTIC SOLUTIONS

Different Methods of Geometric Representation

392. In order to study doubly asymptotic solutions, we shall confine /372 ourselves to a very special case, that of Section No. 9: Zero mass of the perturbed planet; circular orbit of the perturbing planet; zero inclinations. The three-body problem then has the well-known integral called the Jacobi integral.

Returning to No. 299 devoted to this problem from No. 9, we must distinguish between several cases. We saw on page 159 that we must have the following inequality

$$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \frac{n^2}{2}(\xi^2 + \eta^2) = V + \frac{n^2}{2}(\xi^2 + \eta^2) > -h. \quad (1)$$

We then distinguished between the case in which m_1 is much smaller than m_2, and in which -h is sufficiently large (page 160). We saw that the following curve

$$V + \frac{n^2}{2}(\xi^2 + \eta^2) = -h \quad (2)$$

may be broken down into three closed branches which we have called C_1, C_2 and C_3. Therefore, in view of the inequality (1), the point ξ, η must always remain inside of C_1, or always inside of C_2, or always outside of C_3 (ξ, η are the rectangular coordinates of the perturbed planet with respect to the moving axes).

We shall assume below that the value of the constant -h is large enough for curve (2) to be broken down into three closed branches, and that the point ξ, η always remains inside of C_2. In this way, the distance r_2 from the perturbed planet to the central body may vanish, but this is not true /373 for the distance r_1 between the two planets.

This hypothesis corresponds to the following hypothesis, which we formulated on pages 199 and 200 -- i.e., the curve F = C has the form shown in Figure 9, and the point x_1, x_2 remains on the utilizable arc AB.

We shall employ the notation given in No. 313, and we shall introduce the Keplerian variables L, G, l, g. However, these Keplerian variables may be defined in two ways. Just as in No. 9, we could relate the perturbed body to the center of gravity of the perturbing body and of the central body, and we could consider the oscillating ellipse described around this center of gravity. However, it is preferable to refer the perturbed body to the central body itself, and to consider the oscillating ellipse described around this central body.

These two procedures are equally legitimate. We saw in No. 11 that the body B may be related to the body A, and the body C may be related to the center of gravity of A and of B. It is apparent that we could also refer C to A, and B to the center of gravity of A and C. If A represents the central body, B the perturbing body, and C the perturbed body, it can be seen that the first solution is that which was adopted in No. 9. It may also be seen that in the second solution, which we shall adopt from this point on, the two bodies B and C are both related to the central body, since -- due to the fact that the mass of C is zero -- the center of gravity of A and C is at A.

We then have

$$F' = R + G = \frac{\sqrt{1-\mu}}{2L^2} + G + \frac{\mu\sqrt{1-\mu}}{r_1} - \frac{\mu}{2\sqrt{1-\mu}}(r_1^2 - 1 - r_2^2)$$

where μ and $1 - \mu$ designate the masses of the perturbing body and of the central body, r_1 designates the distance between the two planets, 1 designates the constant distance from the perturbing body to the central body, and r_2 designates the distance of the perturbed body to the central body.

Just as in No. 313, we shall set

$$x_1 = L - G, \qquad x_2 = L + G,$$
$$2y_1 = l - g + t, \qquad 2y_2 = l + g - t;$$
$$F' = F_0 + \mu F_1, \qquad F_0 = \frac{1}{2L^2} + G = \frac{2}{(x_1 + x_2)^2} + \frac{x_2 - x_1}{2};$$
$$\mu F_1 = \frac{\sqrt{1-\mu} - 1}{2L^2} + \frac{\mu\sqrt{1-\mu}}{r_1} - \frac{\mu}{2\sqrt{1-\mu}}(r_1^2 - 1 - r_2^2).$$

I would like to stress the following important point. It can be seen that the function F_1 always remains finite in the region from which the point ξ, η cannot leave. /374

We shall employ the method of representation given on page 200, and we shall represent the configuration of the system by the point in space whose rectangular coordinates are

$$X = \frac{\sqrt{x_2}\cos y_2}{\sqrt{x_2 + 4x_1 - 2\sqrt{x_1}\cos y_1}}, \qquad Y = \frac{\sqrt{x_2}\sin y_2}{\sqrt{x_2 + 4x_1 - 2\sqrt{x_1}\cos y_1}},$$
$$Z = \frac{2\sqrt{x_1}\sin y_1}{\sqrt{x_2 + 4x_1 - 2\sqrt{x_1}\cos y_1}}.$$

It can be seen that, when the ratio $\frac{x_1}{x_2}$ is constant, the point X, Y, Z describes a torus. This torus may be reduced to the Z axis when this ratio is infinite, and may be reduced to the circle

$$Z = 0, \quad X^2 + Y^2 = 1,$$

when this ratio is zero.

The derivatives $\frac{dF_1}{dx_1}$ and $\frac{dF_1}{dx_2}$ remain finite in the region under consideration, just as does the function F_1 itself, except when x_1 or x_2 is very small. This would not be true for the derivatives $\frac{dF_1}{dy_1}$, $\frac{dF_1}{dy_2}$ which could become infinite for $r_2 = 0$. As a result,

$$-n_1 = \frac{dF'}{dx_1}, \quad -n_2 = \frac{dF'}{dx_2}$$

differ very little from $\frac{dF_0}{dx_1}$ and $\frac{dF_0}{dx_2}$. We saw on page 201 that, in terms of the hypothesis with which we are dealing, $\frac{dF_0}{dx_2}$ and consequently n_2 cannot vanish because the energy constant C (the constant C given in No. 313 may be readily reduced to the constant h given in No. 299) is larger than $\frac{3}{2}$ (on page 201, we must set $\frac{3}{2}$ instead of $\frac{3}{4}$ everywhere).

If x_2 is not very small, we shall therefore have

$$n_2 > 0,$$

because $\frac{dF_2}{dx_2}$ can only become infinite for $x_2 = 0$, from which it follows that /375
y_2 is always increasing, except for very small x_2.

Let M be a point X, Y, Z, such that $y_2 = 0$. On the half-plane we shall have

$$Y = 0, \quad X > 0.$$

When x_1, x_2, y_1, y_2 vary in conformance with differential equations, the point X, Y, Z will describe a certain trajectory. When y_2, which increases constantly, reaches the value 2π, the point X, Y, Z -- which has moved to M_1 -- will again be located on the half-plane Y = 0, X > 0.

The point M_1 is then the consequent of M, according to the definition given in No. 305. Since y_2 is always increasing, every point on the half-plane has a consequent and an antecedent. There is only an exception for very small x_2 -- i.e., for points on the half-plane which are very far from the origin, or very close to the Z axis.

We shall have an integral invariant, in terms of the meaning attributed to this word in No. 305. Let us try to formulate this invariant.

Due to the fact that the equations are canonical equations, they have the following integral invariant

$$\int dx_1\, dx_2\, dy_1\, dy_2.$$

Let us set $z = \frac{x_2}{x_1}$, and let us select F', z, y_1, y_2 as new variables. The invariant will become

$$-\int \frac{x_1^2\, dF'\, dz\, dy_1\, dy_2}{x_1 \frac{dF'}{dx_1} + x_2 \frac{dF'}{dx_2}} = \int \frac{x_1^2\, dF'\, dz\, dy_1\, dy_2}{x_1 n_1 + x_2 n_2}.$$

We may deduce the following triple invariant from this quadruple invariant (due to the existence of the integral F' = C)

$$\int \frac{x_1^2\, dz\, dy_1\, dy_2}{x_1 n_1 + x_2 n_2}.$$

In this triple integral, we assume that $x_1, x_2, n_1 = -\frac{dF'}{dx_1}$, $n_2 = -\frac{dF'}{dx_2}$ are replaced as functions of z, y_1, y_2 by means of the equations

$$x_2 = x_1 z, \qquad F' = C.$$

Let us now take the variables X, Y, Z, and let us employ Δ to designate the Jacobian of X, Y, Z, with respect to z, y_1, y_2. The invariant will become /376

$$\int \frac{x_1^2\, dX\, dY\, dZ}{(x_1 n_1 + x_2 n_2)\Delta}.$$

Let us set

$$R = \frac{\sqrt{z}}{\sqrt{z+4} - 2\cos y_1}, \qquad Z = \frac{2\sin y_1}{\sqrt{z+4} - 2\cos y_1},$$

from which it follows that

$$X = R\cos y^2, \qquad Y = R\sin y_2.$$

Let us again set

$$D = [(R-1)^2 + Z^2][(R+1)^2 + Z^2].$$

A simple calculation provides the following

$$\Delta = \frac{RD}{8\sqrt{z(z+4)}}.$$

Our invariant may therefore be written

$$\int \frac{8x_1^2\sqrt{z(z+4)}\,dX\,dY\,dZ}{(x_1 n_1 + x_2 n_2)RD}.$$

The principles presented in No. 305 enable us to deduce the following invariant, in the sense of No. 305

$$\int \frac{8x_1^2\sqrt{z(z+4)}}{D}\,\frac{n_2}{x_1 n_1 + x_2 n_2}\,dX\,dZ.$$

n_2 and R play the role which Ω and ρ played in the analysis of No. 305.

The tern under the sign $\int$ is essentially positive, except for very small x_2 -- i.e., for points of the half-plane which are very far from the origin, or very close to the Z axis.

393. This fact (that a point will no longer have a consequent if it is too far, or if it is too close, to the Z axis) could cause some difficulty, and it would be advantageous to avoid this difficulty by whatever method.

We could employ the statements presented in No. 311, and we could replace our half-plane by a simply connected curve on a surface. We shall choose this curve on a surface in the following way.

If x_2 is very small, the eccentricity is very small, and the two planets turn in the opposite direction. The principles presented in No. 40 are applicable, and we may affirm the existence of a periodic solution of the /377 first type which will clearly satisfy the following conditions: The quantities

$$\sqrt{x_2}\cos y_2, \quad \sqrt{x_2}\sin y_2, \quad x_1, \quad \cos y_1, \quad \sin y_1$$

are periodic functions of the time t. These functions depend on μ and on the energy constant C. They may be developed in powers of μ; the period T also depends on μ and on C. The angle y_1 increases by 2π when t increases by a period. Finally, $\sqrt{x_2}\cos y_2$ and $\sqrt{x_2}\sin y_2$ are divisible by μ, so that we have $x_2 = 0$ for $\mu = 0$.

With our method of representation, this periodic solution, which I have called σ, is represented by a closed curve K. Since x_2 is very small when μ is very small, this curve is displaced very little from the Z axis.

It may be stated that it is displaced from it very little, in the same way that a circle having a very large radius is displaced very little from a straight line. Every point on the K curve is either very far from the origin or very close to the Z axis.

Under this assumption, our curve on a surface S would have the curve K for the perimeter, and it would be displaced very little from the half-plane Y = 0, X > 0, except in the immediate vicinity of the curve K. It would be very easy to conclude this determination in such a way that every point on this surface would have a consequent on this surface itself. For this purpose, if I designate an arbitrary trajectory by (T) -- i.e., one of the curves defined in our method of representation by differential equations -- it would be sufficient that the surface S was not tangent at any point to any of the trajectories (T).

However, there is still another method, which does not basically differ from the first method. If we reflect on this a little, we will find that this difficulty is similar to that in Chapter XII. We must therefore perform the change in variables similar to that performed in No. 145.

Let us first set

$$\xi_2 = \sqrt{2x_2}\cos y_2, \qquad \eta_2 = \sqrt{2x_2}\sin y_2,$$

and we then have

$$S = \xi_2'\eta_2 + x_1' y_1 + \mu S_1$$

where S_1 is a function of ξ'_2, η_2, x'_1, y_1. Let us then set /378

$$\begin{cases} \xi_2 = \dfrac{dS}{d\eta_2} = \xi_2' + \mu\dfrac{dS_1}{d\eta_2}; & \eta_2' = \dfrac{dS}{d\xi_2'} = \eta_2 + \mu\dfrac{dS_1}{d\xi_2'}; \\ x_1 = \dfrac{dS}{dy_1} = x_1' + \mu\dfrac{dS_1}{dy_1}; & y_1' = \dfrac{dS}{dx_1'} = y_1 + \mu\dfrac{dS_1}{dx_1'}, \end{cases} \tag{1}$$

and finally

$$\xi_2' = \sqrt{2x_2'}\cos y_2', \qquad \eta_2' = \sqrt{2x_2'}\sin y_2'.$$

I should first point out that the canonical form of the equations will not be changed when I pass from the variables x_1, y_1, x_2, y_2, to x_1, y_1, ξ_2, η_2, then to x'_1, y'_1, ξ'_2, η'_2, and finally to x'_1, y'_1, x'_2, y'_2.

I must now choose the function S_1.

I know that F' is a holomorphic function of $\sqrt{2x_1}\cos y_1$, $\sqrt{2x_1}\sin y_1$, $\sqrt{2x_2}\cos y_2$, $\sqrt{2x_2}\sin y_2$ in the region under consideration. I would like it to remain a holomorphic function of the new variables

$$\sqrt{2x'_i}\cos y'_i, \quad \sqrt{2x'_i}\sin y'_i.$$

For this purpose, I would like the old variables $\sqrt{2x_i}\,{\cos \atop \sin}\,y_i$ to be holomorphic functions of the new variables $\sqrt{2x'_i}\,{\cos \atop \sin}\,y'_i$ and of μ.

To do this, we need only assume that S_1 is a holomorphic function of

$$\sqrt{2x'_1}\cos y_1, \quad \sqrt{2x'_1}\sin y_1, \quad \xi'_2, \quad \eta_2, \quad \mu$$

and is divisible by x'_1.

For our periodic solution σ, I would like to have

$$\xi'_2 = \eta'_2 = 0, \qquad x'_1 = x^0_1 = \text{const.}$$

Therefore, let

$$\xi_2 = A, \qquad \eta_2 = B, \qquad x_1 = C$$

be the equations of the periodic solution. A, B, C are functions of y_1 which are periodic of the period 2π and may be developed in powers of μ.

Then $C - \frac{dA}{dy_1}B$ will also be a periodic function of y_1. Let x^0_1 be its mean value. We may obtain another periodic function α such that

$$C - \frac{dA}{dy_1}B = x^0_1 + \frac{d\alpha}{dy_1}.$$

We shall no longer assume that, for $x'_1 = x^1_0$, the function μS_1 may be reduced to /379

$$\alpha - B\xi'_2 + A\eta_2. \tag{2}$$

This will be sufficient for the equations of the periodic solution to be reduced with the new variables to

$$\xi'_2 = \eta'_2 = 0, \qquad x'_1 = x^0_1.$$

It is clearly possible to obtain a function μS_1 which may be developed in powers of $\sqrt{2x'_1}\,{\cos \atop \sin}\,y_1$, which may be divisible by x'_1, and which at the same time may be reduced to expression (2) for $x'_1 = x^0_1$.

Let us adopt the new variables x'_1, y'_1, x'_2, y'_2.

The function F', which was holomorphic with respect to $\sqrt{2x_1}\,{\cos \atop \sin}\,y_1$, $\sqrt{2x_2}\,{\cos \atop \sin}\,y_2$, will also be holomorphic with respect to $\sqrt{2x'_1}\,{\cos \atop \sin}\,y'_1$, $\sqrt{2x'_2}\,{\cos \atop \sin}\,y'_2$. In addition, since one of the solutions of the differential equations is

$$\xi'_2 = \eta'_2 = 0, \qquad x'_1 = x_1^0,$$

we must have the following relationships for $\xi'_2 = \eta'_2 = 0$, $x'_1 = x_1^0$

$$\frac{dF'}{d\xi'_2} = \frac{dF'}{d\eta'_2} = \frac{dF'}{dy'_1} = 0. \tag{3}$$

For small values of ξ'_2 and η'_2, F' may be developed in powers of ξ'_2 and η'_2. In view of relationships (3), for $x'_1 = x_1^0$, the terms of the first degree in this expansion will vanish, and the terms of zero degree will be reduced to a constant which is independent of μ_1.

This constant can be nothing else than the energy constant C, so that the conditions $\xi'_2 = \eta'_2 = 0$, $x'_1 = x_1^0$ may be replaced by the following conditions

$$\xi'_2 = \eta'_2 = 0, \qquad F' = C.$$

Thus, for $F' = C$, the terms of the first degree in ξ'_2 and η'_2 will vanish in the expansion of F'.

The difficulty arises from the fact that F' and F_1 include terms of the first degree in /380

$$\xi_2 = \sqrt{2x_2}\cos y_2, \qquad \eta_2 = \sqrt{2x_2}\sin y_2$$

and that, consequently, the derivative $\frac{dF_1}{dx_2}$ includes terms $\frac{1}{\sqrt{x_2}}$ which become infinite for $x_2 = 0$.

This difficulty no longer exists now. We no longer have terms of the first degree in ξ'_2, η'_2. Therefore the derivative $\frac{dF_1}{dx'_2}$ remains finite, even for $x'_2 = 0$, and $\frac{dF'}{dx'_2}$, which differs very little from $\frac{dF_0}{dx'_2}$, always retains the same sign. Therefore, with our new variables which only differ from the old variables by very small quantities on the order of μ, we shall constantly have

$$\frac{dx'_2}{dt} > 0.$$

With our new variables, let us formulate a convention which is similar to that given in the preceding section, and let us represent the configuration of the system by the point in space whose coordinates are

$$X = \frac{\sqrt{x'_2}\cos y'_2}{\sqrt{x'_2} + 4x'_1 - 2\sqrt{x'_1}\cos y'_1}, \quad Y = \frac{\sqrt{x'_2}\sin y'_2}{\sqrt{x'_2} + 4x'_1 - 2\sqrt{x'_1}\cos y'_1},$$

$$Z = \frac{2\sqrt{x'_1}\sin y'_1}{\sqrt{x'_2} + 4x'_1 - 2\sqrt{x'_1}\cos y'_1}.$$

Everything which we have stated still holds. However, since $\frac{dx'_2}{dt}$ can never vanish, <u>every point on the half-plane</u>, without exception, <u>will have a consequent</u>.

It may now be stated that the integral invariant is always positive. There can only be some question of doubt for the denominator which, with the same variables, was $x_1n_1 + x_2n_2$ and which now would be

$$-\left(x'_1 \frac{dF'}{dx'_1} + x'_2 \frac{dF'}{dx'_2}\right),$$

which -- assuming that F' is a function of the following four variables

$$\xi'_i = \sqrt{2x'_i}\cos y'_i, \quad \eta'_i = \sqrt{2x'_i}\sin y'_i,$$

may be written /381

$$-\frac{1}{2}\left(\xi'_1 \frac{dF'}{d\xi'_1} + \eta'_1 \frac{dF'}{d\eta'_1} + \xi'_2 \frac{dF'}{d\xi'_2} + \eta'_2 \frac{dF'}{d\eta'_2}\right).$$

In this form, it may be readily seen that the denominator is holomorphic with respect to the ξ''s, the η''s, and μ. However, for $\mu = 0$, F' may be reduced to

$$\frac{2}{(x'_1 + x'_2)^2} + \frac{x'_2 - x'_1}{2}$$

and it may readily be shown that the denominator is always positive. It will still be positive for small values of μ.

394. In the following statements, we shall adopt the variables defined in the preceding section. We shall remove the accents which have become useless, and we shall write F, x_i and y_i in place of F', x'_i and y'_i. We then have the integral invariant (in the sense of No. 305)

$$J = \int \frac{8x_1^2\sqrt{z(z+4)}}{D} \frac{\frac{dF}{dx_2}}{x_1\frac{dF}{dx_1} + x_2\frac{dF}{dx_2}} dX\, dZ$$

from which we have

$$D = [(X-1)^2 + Z^2][(X+1)^2 + Z^2].$$

I would first like to note that this integral invariant, which is always positive, remains finite when it is extended over the entire half-plane.

If $\sqrt{(X - 1)^2 + Z^2}$ is an infinitely small quantity of the first order, the numerator $x_1^2 \sqrt{z(z + 4)}$ is an infinitely small quantity of the second order, and the same holds true for D. If $\sqrt{(X - 1)^2 + Z^2}$ is an infinitely large quantity of the first order, the numerator remains finite, while D is very large of the fourth order. All of the other quantities remain finite.

I shall call J_0 the value of the invariant J extended over the entire half-plane.

The periodic solutions and the trajectory curves which represent them are characterized by the fact that these curves intersect the half-plane at points whose successive consequents are finite in number. For example, let us refer to No. 312 and, in particular, to Figure 7 shown in page 195.

In this figure, the closed trajectory which represents a periodic /382
solution intersects the half-plane at five points M_0, M_1, M_2, M_3, M_4, each of which is the consequent of the others. For purposes of brevity, I shall call such a system a <u>system of periodic points or a periodic system</u>.

Two systems of asymptotic solutions correspond to each unstable, periodic solution. These solutions are represented by trajectories (in the sense of No. 312), and the total group of these trajectories forms what I have designated as asymptotic surfaces. The intersection of an asymptotic surface with the half-plane will be called an <u>asymptotic curve</u>. Just as we saw in Figure 7, page 195, four branches of asymptotic curves (MA, MB, MP, MQ) -- each two of which are located in the extension of the other -- lead to each of the points M_i of an unstable periodic system.

There is an infinite number of asymptotic curves, because there is an infinite number of unstable, periodic solutions and, consequently, an infinite number of systems of unstable periodic points, even if we confine ourselves to solutions of the first type which we defined in Nos. 42 and 44.

A distinction may be drawn between asymptotic curves of the first family and of the second family, depending on whether the corresponding characteristic exponent is positive or negative. Curves of the first family are characterized by the following property: The n^{th} antecedent of an arbitrary point is very close to a periodic point if n is very large. For curves of the second family, it would be the n^{th} consequent, and not the n^{th} antecedent, which would be very close to a periodic point.

On the figure shown on page 195, the curves MA and MP belong to the first family, and the curves MB and MQ belong to the second family.

These asymptotic curves may be regarded as invariant curves in the sense of Chapter XXVII, under the condition that one of the two following conventions is employed. Let us again consider the figure shown on page 195, and we shall find the curve M_0A_0 which has M_1A_1, M_2A_2, M_3A_3, M_4A_4, M_0A_5 for successive consequents. If we consider the five curves M_0A_0, M_1A_1, M_2A_2, M_3A_3, M_4A_4, this total group will clearly constitute an invariant curve. If we only consider the consequents in groups of 5, and if we designate the $5p^{th}$ consequent, which it has been called up to the present, as the p^{th} consequent, it is apparent that only the curve $M_0A_0A_5$ under consideration will be an invariant curve. /383

Two curves of the same family cannot intersect. These two curves will end at the same periodic point -- for example, the point M_0. These two curves will coincide (since M_0A_0 with its extension M_0P_0 is the only curve of the first family which passes through M_0), and we must determine whether an asymptotic curve can have a double point. The question has been answered in the negative (No. 309, page 186).

Or, these two curves will lead to two periodic points of the same periodic system -- for example, to the two points M_0 and M_1. If two curves, which would then be M_0A_0 and M_1A_1, had a point in common Q, the $5p^{th}$ antecedent of Q would have to be very close to M_0 for very large p, because Q would belong to M_0A_0, and it would have to be very close to M_1 at the same time because Q would belong to M_1A_1. This is absurd.

Or, finally the two curves would lead to two points belonging to two different periodic systems. For example, let us assume that the two curves belong to the first family, and that Q is their point of intersection.

For very large n, the n^{th} antecendent of Q would have to be very close to one of the points of the first periodic system and one of the points of the second system at the same time. This is also impossible.

Conversely, there is no reason that two asymptotic curves of different families cannot intersect.

Let S and S' be two unstable periodic solutions, let T and T' be the corresponding closed trajectories, and let P and P' be the corresponding periodic systems.

Let Σ and Σ' be two asymptotic surfaces which pass through T and T', respectively, and which intersect the half-plane along two asymptotic curves C and C' -- one belonging to the first family, and the other belonging to the second family.

What will happen if C and C' have a point in common Q? The two surfaces Σ and Σ' will intersect along a trajectory τ, which will correspond to a special solution σ. The trajectory τ will belong to two asympototic surfaces, so that for $t = -\infty$ it will closely approach T, and for $t = +\infty$ it will closely approach T'. For very large n, the n^{th} antecedent of Q will be very close to one of the points of system P and its n^{th} consequent will be /384 very close to one of the points of system P'.

The solution σ is therefore doubly asymptotic.

There is nothing absurd in any of these results.

We must distinguish between two cases, however. The two solutions S and S' coincide, so that τ first closely approaches T = T', then recedes farther away from it, and again closely approaches this same trajectory T = T'. I could then state that the solution σ is homoclinous. Or, S differs from S', and T differs from T'; I may then state that σ is heteroclinous.

The existence of homoclinous solutions will be demonstrated very shortly. The existence of heteroclinous solutions remains doubtful, at least in the case of the three-body problem.

Homoclinous Solutions

395. At the end of No. 312, we found that "the arcs A_0A_5 and B_0B_5 intersect". However, the arc A_0A_5 belongs to the curve $M_0A_0A_5$ which is an asymptotic curve of the first family, and the arc B_0B_5 is part of the curve M_3B_0 which belongs to the second family.

The line of reasoning is general, and we must conclude that the two asymptotic surfaces which pass through the same closed trajectory must always intersect beyond this trajectory. The asymptotic curves of the first family which lead to the points of a periodic system always intersect the curves of the second family, which lead to these same points.

In other words, on each asymptotic surface there is at least one doubly asymptotic, homoclinous solution. We shall see very shortly that there is an infinite number of them, but we shall now show that there are at least two of them.

For this purpose, let us turn to the figure shown on page 195. Following the line of reasoning in Nos. 308 and 312, we find that the integral invariant J extended over the quadrilateral $A_0B_0A_5B_5$ must be zero. It is for this reason that this curvilinear quadrilateral cannot be convex, and that the opposite sides A_0A_5 and B_0B_5 must intersect. Let Q be one of the /385 intersection points of these two arcs. We should note that the point B_0 was chosen arbitrarily on the asymptotic curve MA_0. If we place the point A_0 at the point Q itself, this point A_0 will also be located on the curve M_3B_0 and will coincide with the point B_0. If the two points A_0 and B_0 coincide, the same will hold true for their five consequents A_5 and B_5.

The quadrilateral $A_0B_0A_5B_5$ will therefore be reduced to the firgure formed by two arcs of a curve having the same end points. This figure cannot be convex, since the integral invariant extended over the quadrilateral must be zero. Therefore the two arcs A_0A_5 and B_0B_5 must have points in common, other than their end points.

There will therefore be at least two different intersection points (a point and an arbitrary consequent of it are not regarded as being different).

There will therefore always be at least two doubly asymptotic solutions.

Let us assume that the points A_0 and B_0 coincide, and let us extend the arcs A_0A_5 and B_0B_5 up to the first point at which they touch C_0. We will have thus determined an area which will be convex this time (from the point of view of *Analysis situs*) and which will be bounded by two arcs which are a part of the two arcs A_0A_5 and B_0B_5, respectively, having the same end points -- i.e., $A_0 = B_0$ and C_0.

Let α_0 be this area, and let α_n be its n^{th} consequent. The area α_n -- like α_0 -- will obviously be convex and bounded by two arcs of a curve -- one belonging to the first family, and the other belonging to the second family.

The integral J will have the same value for α_0 and α_n. Let j be this value. Since the value J_0 of the integral invariant for the entire half-plane is finite, following the line of reasoning presented in No. 291, we will find that, if

$$n > p\frac{J_0}{j},$$

the area α_0 will have a part in common, at least with p of the areas

$$\alpha_1, \alpha_2, \ldots, \alpha_n.$$

Since n cannot be taken arbitrarily large, I may stipulate the following result:

Among the areas α_n, there is an infinite number of them which have a part in common with α_0.

How may it happen that α_0 has a part in common with α_n? /386

The area α_0 cannot be entirely within α_n, since the integral invariant has the same value for the two areas. For the same reason, the area α_n cannot be entirely within α_0. Neither can the two areas coincide. If one part of an asymptotic curve (for example, belonging to the first family) coincided with its n^{th} consequent, the same would hold true for its p^{th} antecedent, no matter how large p may be. However, if p is large, this p^{th} antecedent is very close to the periodic points, and the principles formulated in Chapter VII will demonstrate that this coincidence does not occur.

We must therefore assume that the perimeter of α_0 intersects that of α_n. However, the perimeter of α_0 is composed of an arc $A_0H_0C_0$ belonging to the curve $M_0A_0A_5$ of the first family, and of an arc

$$B_0K_0C_0 = A_0K_0C_0$$

belonging to the curve $M_3B_5B_0$ of the second family.

In the same way, the perimeter of α_n will be composed of the arc $A_nM_nC_n$, the n^{th} consequent of $A_0H_0C_0$, which will belong to the same asymptotic curve as $A_0H_0C_0$ -- i.e., to a curve of the same family -- and it will also be composed of the arc $A_nK_nC_n$, the n^{th} consequent of $A_0K_0C_0$, which will belong to the same asymptotic curve as $A_0K_0C_0$ -- i.e., to a curve of the second family.

Due to the fact that two curves of the same family cannot intersect, it is necessary that $A_0H_0C_0$ intersect $A_nK_nC_n$, or that $A_0K_0C_0$ intersects $A_nH_nC_n$. However, if the two arcs $A_0K_0C_0$ and $A_nH_nC_n$ intersect, their n^{th} antecendents $A_{-n}K_{-n}C_{-n}$ and $A_0H_0C_0$ will equally intersect. It is therefore necessary that $A_0H_0C_0$ intersect the n^{th} consequent, or the n^{th} antecedent, of $A_0K_0C_0$.

However, the arc $A_0K_0C_0$, all of its antecedents, and all of its consequents will belong to the same invariant curve of the second family, which was shown in the figure on page 195 by the total group of curves M_3B_0, M_1B_3, M_4B_1, M_2B_4, M_0B_2.

The arc $A_0H_0C_0$ is therefore intersected an infinite number of times by this group of curves.

The two surfaces Σ and Σ' which passed through the closed trajectory T therefore have an infinite number of other intersection curves.

Therefore, on the surface Σ there is an infinite number of double asymptotic, homoclinous solutions. /387

q.e.d.

396. Let $A_0H_0C_0$ be an arbitrary arc of our asymptotic curve of the first family, and let us assume that this arc intersects an asymptotic curve of the second family at two end points A_0 and C_0. It may be stated that there will always be other points of intersection with the curve of the second family between these two points A_0 and C_0.

Let $A_0K_0C_0$ be the arc of the curve of the second family which unites these two points A_0 and C_0.

Either the two arcs $A_0H_0C_0$ and $A_0K_0C_0$ have points in common other than their end points, in which case the theorem has been proven.

Or, these two arcs do not have a point in common other than their end points A_0 and C_0. The two arcs then bound an area α_0 which is similar to that which we considered at the end of the preceding section. The same line of reasoning may then be applied, and we may conclude that the arc $A_0H_0C_0$ intersects the curve of the second family an infinite number of times.

Therefore, there is an infinite number of other points on an asymptotic curve of the first family, between two arbitrary points of intersection with the curve of the second family.

On an arbitrary asymptotic surface, between two doubly asymptotic arbitrary solutions, there is an infinity of other solutions.

We may not yet conclude that the doubly asymptotic solutions are everywhere dense on the asymptotic surface, but this seems very likely.

The points of intersection of two asymptotic curves may be divided into two categories. The asymptotic curve may be traversed in two opposite directions. We assume that this direction is positive, if we proceed from a point to its consequent. Let A be a point of intersection of the two curves, and let BAB', CAC' be two asymptotic curve arcs intersecting at A. Let us assume that BAB' belongs to the first family, and CAC' belongs to the second family, and that -- when following the curves in the positive direction -- one proceeds from A to B', and from A to C'. Depending upon whether the direc- /388 tion AB' is to the right or the left of AC', the intersection point A will belong to the first or to the second category.

Under this assumption, let $A_0H_0C_0$ be an arc of the first family, intersected at A_0 and C_0 by an arc $A_0K_0C_0$ of the second family. No matter what category A_0 and C_0 belong to, the group of two arcs $A_0H_0C_0K_0A_0$ will form a closed curve. If the two arcs have no other point in common except their end points, this closed curve does not have a double point and defines an area α_0. If the two arcs had points in common other than their end points, and if, for example, the two arcs $A_0H_0D_0H_0'C_0$, $A_0K_0D_0K_0'C_0$ intersect at D_0, we may replace the points A_0 and C_0 by the points A_0 and D_0 located between A_0 and C_0, and the arcs $A_0H_0C_0$, $A_0K_0C_0$ by the two arcs $A_0H_0D_0$ and $A_0K_0D_0$. This may be continued until we arrive at two arcs which have no point in common other than their end points.

Let us assume that the two arcs define an area α_0. According to the statements we have just presented, the arc $A_0H_0C_0$ must intersect the asymptotic curve of the second family an infinite number of times. Therefore, the curve of the second family must penetrate within α_0 an infinite number of times, and it must leave it an infinite number of times. It may penetrate it or leave it only by intersecting $A_0H_0C_0$, because it cannot intersect $A_0K_0C_0$ which also forms a part of the curve of the second family. It is apparent that points through which it will penetrate into the area, and the points through which it will leave the area, will not belong to the same category.

Therefore, between two arbitrary intersection points of two curves, there is an infinity of other points belonging to the first category, and an infinity of other points belonging to the second category.

Let us employ (1), (2), (3), ..., to designate the successive points at which the curve of the second family and the arc $A_0H_0C_0$ meet, taken in the order in which they are encountered proceeding along the curve of the second family in the positive direction. They will belong to two categories in succession. Let us study the order in which they are encountered proceeding along the arc $A_0H_0C_0$.

This order cannot be completely arbitrary, and certain successions are excluded -- for example, the following: /389

$$\begin{array}{llll} (2m), & (2p), & (2m+1), & (2p+1) \\ (2m+1) & (2p), & (2m), & (2p+1) \\ (2m), & (2p+1), & (2m+1), & (2p) \\ (2m), & (2p), & (2m-1), & (2p-1) \end{array}$$

as well as the same inverse successions, and the similar successions where 2m + 1 and 2p + 1 are replaced by 2m - 1 and 2 p - 1.

397. When we try to represent the figure formed by these two curves and their intersections in a finite number, each of which corresponds to a doubly asymptotic solution, these intersections form a type of trellis, tissue, or grid with infinitely serrated mesh. Neither of the two curves must ever cut across itself again, but it must bend back upon itself in a very complex manner in order to cut across all of the meshes in the grid an infinite number of times.

The complexity of this figure will be striking, and I shall not even try to draw it. Nothing is more suitable for providing us with an idea of the complex nature of the three-body problem, and of all the problems of dynamics in general, where there is no uniform integral and where the Bohlin series are divergent.

Different hypotheses are possible.

1. We may assume that the group of points of two asymptotic curves E_0, or the group of points in the vicinity of which there is an infinite number of points belonging to E_0 -- i.e., the group E'_0, the "derivative of E_0" -- occupies the entire half-plane. We would then have to conclude that instability of the solar system exists.

2. We may assume that the group E'_0 has a finite area and occupies a finite region of the half-plane, but does not occupy it completely. Either one part of this half-plane remains outside of the meshes of our grid, or a "gap" remains within one of these meshes. For example, let U_0 be one of these meshes bounded by two or more asymptotic curve arcs of the two families. Let us compile its successive consequents, and let us apply the procedure /390
presented in No. 291. Just as on page 145, let us formulate the following

$$U_\alpha,\ U'_0,\ U'_\beta,\ U''_0,\ U''_\gamma,\ \ldots,\ E.$$

If it is finite, the area E will represent one of the gaps which we just mentioned. It would appear that we may apply the line of reasoning employed in No. 294, and may conclude that this area must coincide with one of its consequents. However, this group E could be composed of a region of finite area and of a group located outside of this region, whose total area would be zero. According to page 151, we may only conclude that E_λ (the λth consequent of E) includes E, and that the group $E_\lambda - E$ has area zero. In the same way, the groups $E - E_{-\lambda}$, $E_{-\lambda} - E_{-2\lambda}$, ..., $E_{-n\lambda} - E_{-(n+1)\lambda}$ will have area zero (by area of a group, we mean the value of the integral J extended over this group). On the other hand $E_{-(n+1)\lambda}$ is a part of $E_{-n\lambda}$. When n increases indefinitely, $E_{-n\lambda}$ tends toward a group ε including every point which is part of all the groups $E_{-n\lambda}$ at the same time. The area of this group ε is finite and equals that of E. Finally, ε coincides with its λth consequent.

3. Finally, we may assume that the group E'_0 has area zero.

It would then be similar to those "perfect groups which are not condensed in any interval".

398. We may represent the different intersection points of the two curves in the following way. Let x be a variable which varies from $-\infty$ to $+\infty$, when the asymptotic curve of the first family M_0A_0 is followed, from the point M_0 up to infinity, and which increases by unity when we pass from one point to its fifth consequent -- from A_0 to A_5, for example (to clarify this point, we shall assume that we are dealing with the conditions of the figure

shown on page 195). Let y be another variable which varies from $+\infty$ to $-\infty$ when the curve of the second family M_3B_5 is followed from the point M_3 up to infinity, and which increases by unity when we pass from a point to its fifth consequent.

The different intersection points of the two curves are characterized by two values of x and y, and each of them may be represented by the point on a plane whose rectangular coordinates are x and y.

We shall thus have an infinite number of representative points of the doubly asymptotic solutions in the plane. An infinite number of other points may be deduced from each of these points. If the point x, y corresponds to an intersection of the two curves, the same will hold true for the points /391

$$x+1, y+1; \quad x+2, y+2; \quad \ldots; \quad x+n, \quad y+n,$$

where n is a positive or negative whole number. In order to determine all the representative points, it is sufficient to know all those which are included in the region $0 < x < 1$, or in the region $0 < y < 1$.

We would also like to note that the order in which the projections of these representative points will occur on the x axis will have no relationship with the order in which their projections will occur on the y axis. This results in the following.

Let us consider several doubly asymptotic solutions. For t which is negative and very large, they will all be very close to the periodic solution, and they will appear in a certain order -- some of them will be closer to, and others will be farther from, the periodic solution.

All of them will then recede appreciably from the periodic solution, and -- for t which is positive and very large -- they will all again be very close to it. <u>However, they will then appear in an entirely different order</u>. Out of two solutions, if the first is closer than the second to the periodic solution for $t = -\infty$, it may happen that for $t = +\infty$ the first is farther away than the second from the periodic solution, but the opposite could also occur.

We have pointed this out in order to illustrate the great complexity of the three-body problem, and to show how many different transcendents out of all those which we know must be considered in order to solve it.

Heteroclinous Solutions

399. Do heteroclinous solutions exist?

As far as we can determine, if there is one of them, there is an infinite number of them.

Let M_0 be a point belonging to a periodic system. Let M_0A_0 and M_0B_0 /392 be two asymptotic curves bordering upon this point M_0 -- one belonging to the first family, and the other belonging to the second family. We have just seen how these curves intersect, so that the doubly asymptotic, homoclinous solutions may be determined.

Now let M'_0 be a point belonging to another periodic solution. Let $M'_0A'_0$, $M'_0B'_0$ be two asymptotic curves, $M'A'_0$ belongs to the first family, and $M'B'_0$ belongs to the second family.

Let us assume that $M'_0A'_0$ intersects M_0B_0 at Q_0. This intersection will correspond to a doubly asymptotic, heteroclinous solution.

However, if these two curves intersect at Q_0, they will also intersect at an infinite number of points Q_n, the consequents of Q_0.

I shall state this precisely. For example, I shall assume that the periodic system of which M_0 is a part is composed of five points M_0, M_1, M_2, M_3, M_4. Then the fifth consequent of an arbitrary point of the curve M_0B_0 will still be located on this curve, and in general -- if Q_0 is on this curve -- the same will hold true for its n^{th} consequent Q_n, provided that n is a multiple of five.

In the same way, let us assume that the periodic system of which M'_0 is a part is composed of seven points. Then, if Q_0 is on the curve $M'_0A'_0$, the same will hold true for its n^{th} consequent Q_n, provided that n is a multiple of 7.

Therefore, if the two curves have an intersection at Q , they will still have an intersection at Q_n, provided that n is a multiple of 35.

Let $Q_0H_0Q_n$ be an arc of M_0B_0, and let $Q_0K_0Q_n$ be an arc of $M'_0A'_0$. Due to the fact that these two arcs have the same end points, together they will form a closed curve. We may pursue the same line of reasoning as in No. 396 for this closed curve. We shall find that, if the two arcs have no other point in common except their end points, this closed curve does not have a double point, and defines an area which is similar to the area α_0 given in Nos. 395 and 396. If the two arcs have points in common other than their end points, we may obtain two other arcs which are part of the two arcs $Q_0H_0Q_n$, $Q_0K_0Q_n$ which have only their end points in common and which define an area similar to α_0.

The same line of reasoning as was employed in Nos. 395 and 396 may be used for this area α_0, and we will find that an infinite number of other points may be obtained on each of the two curves, between two arbitrary /393 points of intersection with the other curve.

This line of reasoning shows that if there is one heteroclinous solution, there is an infinite number of them.

400. If there is a heteroclinous solution, the grid of which we spoke in No. 397 must be still more complicated. Instead of a single curve M_0A_0 bending back upon itself without ever cutting across itself, and intersecting the other curve M_0B_0 an infinite number of times, we shall have two curves M_0A_0, $M'_0A'_0$ which must intersect M_0B_0 an infinite number of times without ever cutting across each other.

In No. 397, we defined the group E'_0 with respect to the point M_0 and to the asymptotic curves M_0A_0, M_0B_0. We may also define a similar group with respect to the point M'_0 and to two asymptotic curves $M'_0A'_0$, $M'_0B'_0$.

If there is no heteroclinous solution, these two groups must be outside of each other; therefore, they cannot occupy the half-plane. If, on the contrary, there is a heteroclinous solution, these two groups will coincide. It may be seen that the existence of such a solution -- if it could be established -- would provide an argument against stability.

In Chapter XIII we studied the series of Newcomb and Lindstedt, and we showed in No. 149 that these series cannot converge for every value of the constants which they contain. However, one question remains in doubt. Could these series converge for certain values of these constants and, for example, could it happen that the convergence occurs when the ratio $\frac{n_1}{n_2}$ is the square root of a commensurable number which is not a perfect square (see Volume II, page 104, *in fine*).

However, if a heteroclinous solution does exist, the answer to this question must be in the negative. Let us assume that for certain values of the ratio $\frac{n_1}{n_2}$ the series of Newcomb and Lindstedt converge, and let us return to our method of representation. The solutions of the differential equations which would correspond to this value of $\frac{n_1}{n_2}$ could be represented by certain trajectory curves. The group of these curves would form a surface, having the same connections as the torus, and this surface would intersect our half-plane proceeding along a certain closed curve C. /394

The group E'_0 which we just mentioned would have to be completely outside of this curve, or completely inside of it.

Let M_0 and M'_0 be two points belonging to two different systems. If M_0 is within the curve C and M'_0 is outside of this curve, the group E'_0 with respect to M_0 would have to be entirely within it, while the group E'_0 with respect to M'_0 would have to be entirely outside of it.

These two groups could not have any point in common, and no doubly asymptotic, heteroclinous solution could exist, proceeding from M_0 to M'_0.

If we admit the hypothesis advanced in Volume II, page 104, which I have just presented -- i.e., if the convergence occurs for an infinite number of values of the ratio $\frac{n_1}{n_2}$, for example, for those whose square is commensurable -- there would be an infinite number of curves C which would separate the points belonging to different periodic systems. This hypothesis is incompatible with the existence of heteroclinous solutions (at least if the two points M_0 and M'_0 which we are considering, or the corresponding periodic solutions, correspond to two different values of the number $\frac{n_1}{n_2}$.)

Comparison with No. 225

401. Before trying to present examples of heteroclinous solutions, we shall return to the example of No. 225, where the existence of doubly asymptotic, homoclinous solutions may be illustrated.

We set

$$-F = p + q^2 - 2\mu \sin^2\frac{y}{2} - \mu\varepsilon\varphi(y)\cos x$$

(p, x; q, y) are the two pairs of conjugate variables.

We then formulated the function S of Jacobi, and we developed it in powers of ε /395

$$S = S_0 + S_1\varepsilon + S_2\varepsilon^2 + \ldots.$$

Let us consider the second term, neglecting ε^2, and let us write

$$S = S_0 + S_1\varepsilon.$$

We then obtain

$$S_0 = A_0 x + \sqrt{2\mu}\int\sqrt{h + \sin^2\frac{y}{2}}\,dy,$$

or, assigning the value zero to the constants A_0 and h,

$$S_0 = \pm 2\sqrt{2\mu}\cos\frac{y}{2};$$

and we then obtain

$$S_1 = \text{real part } \psi e^{ix},$$

where ψ is a function of y defined by the equation

$$i\psi + 2\sqrt{2\mu}\sqrt{h + \sin^2\frac{y}{2}}\,\frac{d\psi}{dy} = \mu\varphi(y).$$

We set

$$\operatorname{tang}\frac{y}{4} = t,$$

and assuming that

$$h = 0, \qquad \varphi(y) = \sin y, \qquad \alpha = \frac{t}{2\sqrt{2\mu}},$$

we obtained (pages 464 and 465, Volume II) two values of ψ corresponding to the two asymptotic curves of the two families. One of these values is

$$\psi = \sqrt{2\mu}\,\frac{t}{1+t^2} + it^{-2\alpha}\int_t^\infty \frac{t^{2\alpha}\,dt}{1+t^2},$$

and the other is

$$\psi' = \sqrt{2\mu}\,\frac{t}{1+t^2} - it^{-2\alpha}\int_0^t \frac{t^{2\alpha}\,dt}{1+t^2}.$$

The equations of the two asymptotic surfaces will then be

$$p = \varepsilon\frac{d}{dx}\ \text{real part}\ [\psi e^{ix}];$$

$$q = \sqrt{2\mu}\sin\frac{y}{2} + \varepsilon\frac{d}{dy}\ \text{real part}\ [\psi e^{ix}];$$

and /396

$$p = \varepsilon\frac{d}{dx}\ \text{real part}\ [\psi' e^{ix}];$$

$$q = \sqrt{2\mu}\sin\frac{y}{2} + \varepsilon\frac{d}{dy}\ \text{real part}\ [\psi' e^{ix}]$$

In order to obtain the doubly asymptotic solutions, we must determine the intersection of these two asymptotic surfaces. It will be sufficient for us to equate the two values of p and the two values of q.

Let us set

$$J = \int_0^\infty \frac{t^{2\alpha}\,dt}{1+t^2},$$

$$u = 2\log t.$$

We shall obtain

$$\frac{d}{dx}\ \text{real part}\ [J\,ie^{-\alpha u+ix}] = 0,$$

$$\frac{d}{dy}\ \text{real part}\ [J\,ie^{-\alpha u+ix}] = 0,$$

or, setting $J = \rho e^{i\omega}$,

$$x - \frac{u}{2\sqrt{2\mu}} + \omega = K\pi + \frac{\pi}{2}$$

where K is a whole number.

This is the equation of doubly asymptotic solutions.

In reality, this equation provides us with two different solutions, one corresponding to even values of K, and the other corresponding to odd values of K.

402. We may be surprised at not obtaining more than two doubly asymptotic solutions, when we know that there is an infinite number of them.

The following approximations should provide us with no more than a finite number of doubly asymptotic solutions. How may this paradox be explained?

In the preceding sections we saw that the different doubly asymptotic solutions correspond in an infinite number to different intersections of a certain arc $A_0H_0C_0$ with the different consequents of another arc $A_0K_0C_0$.

Let us assume that the first of its consequents which encounters $A_0H_0C_0$ is the consequent of order N. The number N will clearly depend on the con- /397
stant ε, and the smaller the constant is, the larger it will be. It will become infinite when ε is zero.

If we develop in powers of ε and stop at an arbitrary term in the expansion, it is as though we regarded ε as being infinitely small.

The arc $A_0H_0C_0$ no longer encounters the consequents of infinitely large order of the other arc $A_0K_0C_0$, and for this reason we have not analyzed the majority of the doubly asymptotic solutions.

Examples of Heteroclinous Solutions

403. Let us try to generalize, and let us set

$$F = F_0 + \varepsilon F_1.$$

F_0 is a function of p, q and y, and F_1 is a function of p, q, x and y. These two functions are periodic, both in x and y.

Let us consider the curves

$$F_0 = \text{const.} \tag{1}$$

in which we regard p as a parameter, and q and y are regarded as the coordinates of a point.

Out of these curves, those which must draw our attention are the ones having double points. These double points correspond to periodic solutions of the canonical equations when we assume that ε is zero and that F may be reduced to F_0.

We have a double infinity of curves (1) whose general equation is

$$F_0 = h.$$

and which depend on two parameters p and h.

I have just stated that the most interesting ones are those which have a double point, especially in the case in which some of these curves have two or more double points. It is in this case that we shall encounter heteroclinous solutions. /398

Just as in No. 225, let us try to formulate the function S of Jacobi, and let us set

$$S = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \dots.$$

The function S_0 may be formulated immediately. We shall have

$$\frac{dS_0}{dx} = p, \qquad \frac{dS_0}{dy} = q, \qquad S_0 = px + \int q\, dy,$$

where q is a function of y defined by equation (1) and depending on two parameters p and h.

We then obtain

$$\frac{dF_0}{dp}\frac{dS_1}{dx} + \frac{dF_0}{dq}\frac{dS_1}{dy} + F_1 = 0. \qquad (2)$$

We regard p as a constant in $\frac{dF_0}{dp}$, $\frac{dF_0}{dq}$ and F_1, and we replace q by its value obtained from equation (1). Equation (2) is therefore a linear equation with respect to the derivatives of S_1, whose coefficients are the given functions of x and y, which depend in addition on the parameters h and p.

Since F_1 is periodic in x, I shall set

$$F_1 = \Sigma \Phi_m e^{imx},$$

where Φ_m only depends on y, just like the derivatives of F_0.

In the same way, I shall set

$$S_1 = \Sigma \psi_m e^{imx}$$

and the function ψ_m will be given by the equation

$$im \frac{dF_0}{dp} \psi_m + \frac{dF_0}{dq} \frac{d\psi_m}{dy} + \Phi_m = 0 \tag{3}$$

whose coefficients are the given functions of y.

This equation may clearly be integrated by quadratures.

Let us try to determine our asymptotic surfaces in this way. We must first choose the constants h and p so that the curve (1) has a double point. In addition, I shall assume that these constants are such that two real values of q correspond to each value of y (this is what occurs in the example presented in No. 225).

These two values of q are periodic functions of y, which become equal /399 to each other at the double point -- for example, for $y = y_0$.

Just as in No. 225, we may also assume that these two values of q are the analytical extension of each other.

The function q then seems to us to be uniform in y and periodic of period 4π such as the function $\sin \frac{y}{2}$.

This uniform function will take the same value for $y = y_0$ and $y = y_0 + 2\pi$.

If we had several double points, instead of one, we could still regard q as a uniform function of y of period 4π, if the number of double points were odd. On the other hand, if this number were even, we would have two values for q which would not be interchanged when y increased by 2π, and which could consequently be regarded as two different uniform functions of y, having 2π for the period.

In order to formulate our ideas more clearly, we shall assume that we have two double points corresponding to the values y_0 and y_1 of y.

As a result, for $y = y_0$ and for $y = y_1$, equation (1) must have a double root, since the two values of q coincide, and consequently $\frac{dF_0}{dq}$ must vanish.

Equation (3) is a linear equation with a second term, whose integration is similar to the integration of an equation without a second term, and consequently similar to the integration of the following equation

$$\frac{dF_0}{dp}\theta + \frac{dF_0}{dq}\frac{d\theta}{dy} = 0 \tag{4}$$

from which we have

$$\theta = e^{-\int \frac{dy \frac{dF_0}{dp}}{\frac{dF_0}{dq}}}.$$

The function θ thus defined is a holomorphic function of y for all real values of this variable, except for the values $y = y_0$, $y = y_1$, which correspond to the double points. For these values, the function θ -- which plays a role similar to that of $t = \tan \frac{y}{4}$ in No. 226 -- becomes zero or infinite.

We then obtain /400

$$\psi_m = \theta^{im} \int \frac{\theta^{-im} \Phi_m \, dy}{\frac{dF_0}{dp}} + C_m \theta^{im}$$

where C_m is an integration constant, from which we have

$$S_1 = \Sigma \theta^{im} e^{imx} \int \frac{\theta^{-im} \Phi_m \, dy}{\frac{dF_0}{dp}} + \Sigma C_m \theta^{im} e^{imx}.$$

In order to obtain equations of asymptotic surfaces, we may write

$$p = \frac{dS}{dx}; \qquad q = \frac{dS}{dy}$$

assigning suitable values to the integration constants.

Let us first neglect ε. We shall set $S = S_0$, and we shall assign the values corresponding to the curve which has two double points to the constants h and $p = p_0$.

With this approximation, the differential equations have the following as periodic solutions

$$p = p_0, \quad q = q_0, \quad y = y_0, \tag{5}$$
$$p = p_0, \quad q = q_1, \quad y = y_1, \tag{6}$$

where y_0, q_0; y_1, q_1 are the coordinates of the two double points.

In order to represent our asymptotic surfaces, we may take a point in four-dimensional space, whose coordinates are

$$(p+a)\cos x, \quad (p+a)\sin x, \quad (q+b)\cos y, \quad (q+b)\sin y,$$

where a and b are two positive constants which are large enough that we need only consider positive values of $p + a$ and $q + b$.

Equations (5) and (6) then represent two closed curves of this four-dimensional space, corresponding to the two periodic solutions.

Two asymptotic surfaces pass through each of these curves -- one belonging to the first family, and the other belonging to the second family.

However, with the degree of approximation employed -- i.e., neglecting ε -- these four asymptotic surfaces coincide pairwise. /401

The equations of the asymptotic surfaces will be

$$p = p_0, \qquad F_0 = h.$$

As we have seen, the equation $F_0 = h$ has two roots which coincide for $y = y_0$ and for $y = y_1$, which are not interchanged when y increases by 2π, and which are periodic in y of period 2π. Let q' and q'' be these two roots. The equations of our asymptotic surfaces thus become

$$\begin{cases} p = p_0, & q = q', \\ p = p_0, & q = q''. \end{cases} \tag{7}$$

In order to determine the significance of these equations more precisely, let us distinguish between the different layers of our surfaces. We have four asymptotic surfaces. Each of them passes through one of the curves (5) or (6), and is divided into two layers by this curve, which I shall designate by the following notation:

The surface of the first family passing through the curve (5) will be divided into two layers N_1 and N'_1.

The surface of the second family passing through the curve (5) will be divided into two layers N_2 and N'_2.

The surface of the first family passing through the curve (6) will be divided into two layers N_3 and N'_3.

The surface of the second family passing through the curve (6) will be divided into two layers N_4 and N'_4.

With the degree of approximation employed, these layers will have the following equation

$$
\begin{array}{llll}
N_1; & p = p_0, \quad q = q', \quad y > y_0; & N'_1; & p = p_0, \quad q = q', \quad y < y_0; \\
N_2; & p = p_0, \quad q = q'', \quad y > y_0; & N'_2; & p = p_0, \quad q = q'', \quad y < y_0; \\
N_3; & p = p_0, \quad q = q'', \quad y > y_1; & N'_3; & p = p_0, \quad q = q'', \quad y < y_1; \\
N_4; & p = p_0, \quad q = q', \quad y > y_1; & N'_4; & p = p_0, \quad q = q', \quad y < y_1.
\end{array}
$$

It can be seen that the two surfaces $N_1 + N'_1$ and $N_4 + N'_4$ coincide with this degree of approximation, just like the two surfaces $N_2 + N'_2$ and $N_3 + N'_3$.

Let us proceed to the following approximation, and let us set /402

$$S = S_0 + \varepsilon S_1.$$

In order to define S_1, we must choose the constants C_m.

For the layers N_1 and N'_1, we must choose these constants so that the functions ψ_m have a regular behavior for $q = q'$, $y = y_0$. We need only refer to the analysis given on page 466, Volume II, in order to understand that this condition is sufficient for completely determining its constants. I shall call $S_{1.1}$ the function S_1 which is thus determined.

For the layers N_2 and N'_2, we shall choose the C_m's so that the ψ_m's are regular for $q = q''$, $y = y_0$, and we shall call $S_{1.2}$ the function S_1 which is thus determined.

For the layers N_2 and N'_2, we shall choose the C_m's so that the ψ_m's are regular for $q = q''$, $y = y_1$. For the layers N_4 and N'_4, the ψ_m's must be regular for $q = q'$, $y = y_1$. We shall designate the two functions S_1 which are thus determined by $S_{1.3}$ and $S_{1.4}$.

The equations of our four surfaces thus become

$$
\left\{
\begin{array}{lll}
N_1 + N'_1; & p = p_0 + \varepsilon \dfrac{dS_{1.1}}{dx}; & q = q' + \varepsilon \dfrac{dS_{1.1}}{dy}; \\
N_2 + N'_2; & p = p_0 + \varepsilon \dfrac{dS_{1.2}}{dx}; & q = q'' + \varepsilon \dfrac{dS_{1.2}}{dy}; \\
N_3 + N'_3; & p = p_0 + \varepsilon \dfrac{dS_{1.3}}{dx}; & q = q'' + \varepsilon \dfrac{dS_{1.3}}{dy}; \\
N_4 + N'_4; & p = p_0 + \varepsilon \dfrac{dS_{1.4}}{dx}; & q = q' + \varepsilon \dfrac{dS_{1.4}}{dy}.
\end{array}
\right. \tag{8}
$$

However, we should note that the function $S_{1.1}$, for example, has a regular behavior for $y = y_0$, and has an irregular behavior for $y = y_1$. As a result, our equations cease to be valid, even as a first approximation, after the value y_1 is exceeded.

In order to provide a better illustration of this, I shall confine myself to the following remarks.

Let y' and y" be two values of y such that

$$y_0 < y' < y_1 < y''.$$

Let M_0 be the point of our asymptotic curve which corresponds to the value y'. Let M_n be its n^{th} consequent. I shall assume that n is chosen large enough that the corresponding value of y is larger than y". /403

The value which must be assigned to n clearly depends on ε, and it increases indefinitely when ε strives to zero.

In general, the following are the values of y for which our equations may serve as the first approximation:

$$N_1 \text{ et } N_3;\ y_1 > y > y_0; \qquad N'_1 \text{ et } N'_3;\ y_0 > y > y_1 - 2\pi.$$
$$N_2 \text{ et } N_4;\ y_0 + 2\pi > y > y_1; \quad N'_2 \text{ et } N'_4;\ y_1 > y > y_0.$$

For example, if the surfaces N_1 and N'_4 coincide, the intersection will correspond to a heteroclinous, doubly asymptotic solution which will be very close to the periodic solution (5) for t = -∞, and very close to the periodic solution (6) for t = +∞.

In order to determine this intersection, let us compare the equations of N_1 and N'_4

$$p = p_0 + \varepsilon \frac{dS_{1.1}}{dx}, \qquad p = p_0 + \varepsilon \frac{dS_{1.4}}{dx},$$

and the intersection will clearly be given by

$$\frac{d(S_{1.1} - S_{1.4})}{dx} = 0. \tag{9}$$

$S_{1.1} - S_{1.4}$ is a function of x and y, which may be developed in positive and negative whole powers of

$$\theta^i e^{ix}.$$

The fact that it is a periodic function of x is important to us. It therefore has at least a maximum and a minimum. Equation (9) therefore has at least two solutions, which means that there are at least two heteroclinous solutions.

In the same way, it could be shown that there are two solutions corresponding to the intersections of the surfaces N_4 and N'_2, two corresponding to the surfaces N_2 and N'_3, and two corresponding to the surfaces N_3 and N'_1.

The preceding analysis does not yield the homoclinous solutions.

404. For example, let us set

$$F_0 = -p - q^2 + 2\mu \sin^2 \frac{y-y_0}{2} \sin^2 \frac{y-y_1}{2},$$
$$F_1 = \mu \cos x \sin(y-y_0) \sin(y-y_1).$$

The periodic solutions (5) and (6) toward which the heteroclinous solutions /404 strive for $t = -\infty$ and $t = +\infty$ are then

$$x = t, \quad p = q = 0, \quad y = y_0,$$
$$x = t, \quad p = q = 0, \quad y = y_1.$$

It will be noted that, for $\mu = 0$, F may be reduced to $-p - q^2$. Therefore, for $\mu = 0$, the function F depends only on variables of the first series p and q, and does not depend on variables of the second series x and y. The function F therefore has the form considered in Nos. 13, 125, etc.

Nevertheless, we shall not be content with this example, which proves that the canonical equations having the form considered in No. 13 can have heteroclinous solutions.

The two solutions (5) and (6) both correspond to the same value of the quantities $\frac{dx}{dt}$ and $\frac{dy}{dt}$ -- i.e.,

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 0.$$

However, these quantities $\frac{dx}{dt}$, $\frac{dy}{dt}$ are nothing else than the numbers which were called n_1 and n_2 above.

Therefore, we find that doubly asymptotic solutions exist, which come infinitely close to two different periodic solutions for $t = -\infty$ and $t = +\infty$. However, these two periodic solutions correspond to the same values of the numbers n_1 and n_2.

Therefore, I shall formulate another example, in which we shall deal with equations having the same form as those presented up to No. 13, and which have doubly asymptotic solutions coming arbitrarily close to two periodic solutions which are not only different, but correspond to different values of the ratio $\frac{n_1}{n_2}$.

Unfortunately, I would like to show that these solutions exist for values of μ which are close to 1, but I still am not able to establish the fact that they also exist for small values of μ.

405. We shall take two pairs of conjugate variables

$$\xi_1, \ \eta_1; \ \xi_2, \ \eta_2$$

or

$$x_1,\ y_1;\ x_2,\ y_2,$$

by setting

$$\xi_i = \sqrt{2x_i}\cos y_i; \qquad \eta_i = \sqrt{2x_i}\sin y_i.$$

This change in variables does not alter the canonical form of the equations. We shall set

$$F = F_0(1-\mu) + \mu F_1.$$

We shall assume that F_0 is a holomorphic function of x_1 and x_2, independent of y_1 and y_2, and that for $x_1 = \frac{\alpha^2}{2}$, $x_2 = \frac{1}{2}$, we have

$$\frac{dF_0}{dx_2} = 0, \qquad \frac{dF_0}{dx_1} = -1,$$

We shall also assume that for $x_2 = \frac{1}{2}$, $x_1 = \frac{\alpha^2}{2}$ we have

$$\frac{dF_0}{dx_2} = -1, \qquad \frac{dF_0}{dx_1} = 0.$$

I shall assume that $\alpha < 1$ holds for the quantity α.

It follows from these hypotheses that, if we set $\mu = 0$, from which we have $F = F_0$, our equations will have two special periodic solutions.

The first solution, which I shall call σ, may be written

$$x_1 = \frac{\alpha^2}{2}, \qquad x_2 = \frac{1}{2}, \qquad y_1 = t, \qquad y_2 = 0,$$
$$\xi_1 = \alpha\cos t; \qquad \eta_1 = \alpha\sin t, \qquad \xi_2 = 1, \qquad \eta_2 = 0.$$

The second solution, which I shall call σ', may be written

$$x_1 = \frac{1}{2}, \qquad x_2 = \frac{\alpha^2}{2}, \qquad y_1 = 0, \qquad y_2 = t,$$
$$\xi_1 = 1, \qquad \eta_1 = 0, \qquad \xi_2 = \alpha\cos t, \qquad \eta_2 = \alpha\sin t.$$

The first corresponds to $n_1 = 1$, $n_2 = 0$, and the second corresponds to $n_1 = 0$, $n_2 = 1$. These two periodic solutions do not correspond to the same value of the ratio $\frac{n_1}{n_2}$.

In order to define F_1, I shall set

$$\xi_1 = 1 - r\cos\omega, \qquad \xi_2 = 1 - r\sin\omega,$$

assigning a value which is essentially positive to the variable r.

I shall then assume that (due to the fact that ρ is a positive, very small quantity) we have the following for $r > \rho$ /406

$$F_1 = -\frac{\eta_1^2 + \eta_2^2}{2} - \frac{(r-1)^2}{2} + \varepsilon \frac{\psi(\omega)}{r^2}, \tag{1}$$

where $\psi(\omega)$ is a function of ω, which is regular for every real value of ω, periodic with the period 2π, and finally which vanishes with its derivative for $\omega = 0$ and for $\omega = \frac{\omega}{2}$.

Since the function (1) would be infinite for $r = 0$ -- i.e., for $\xi_1 = \xi_2 = 1$ -- I shall assume that for $r \leq \rho$, the function F_1 takes on arbitrary values, in such a way that it nevertheless remains finite and continuous, as well as its derivatives of the two first orders.

It may be readily verified that for $\mu = 1$ -- i.e., for $F = F_1$ -- our equations still have two periodic solutions σ and σ'. For the first of these solutions, we have $\omega = 0$, and for the second we have $\omega = \frac{\pi}{2}$.

It may be immediately concluded that <u>for every value of μ</u>, our equations will have these two periodic solutions.

406. We shall now integrate our equations in the case of $\mu = 1$ (assuming at least that r constantly remains $> \rho$).

If we first assumed that $\varepsilon = 0$, we would be dealing with the problem of central forces, and the integration would be immediately possible. This is hardly true in the general case.

The Jacobi method leads to the partial differential equation

$$\frac{1}{2}\left(\frac{dS}{dr}\right)^2 + \frac{1}{2r^2}\left(\frac{dS}{d\omega}\right)^2 + \frac{(r-1)^2}{2} - \varepsilon\frac{\psi(\omega)}{r^2} = h,$$

where h is a constant. Let us set

$$\frac{1}{2}\left(\frac{dS}{d\omega}\right)^2 - \varepsilon\psi(\omega) = k,$$

where k is a second constant, and we shall have

$$S = \sqrt{2}\int\sqrt{h + \frac{k}{r^2} - \frac{(r-1)^2}{2}}\,dr + \sqrt{2}\int\sqrt{k + \varepsilon\psi}\,d\omega.$$

The general solution of our equations is therefore /407

$$(\xi_1-1)\eta_1+(\xi_2-1)\eta_2=\sqrt{2hr^2-2k-r^2(r-1)^2},$$
$$(\xi_1-1)\eta_2-(\xi_2-1)\eta_1=\sqrt{2}\sqrt{k+\varepsilon\psi}.$$

$$\int \frac{r\,dr}{\sqrt{2hr^2-2k-r^2(r-1)^2}}=h'+t, \tag{2}$$

$$\int_0^{\omega}\frac{d\omega}{\sqrt{2k+2\varepsilon\psi}}+\int\frac{2\,dr}{r^2\sqrt{2hr^2-2k-r^2(r-1)^2}}=k', \tag{3}$$

where h' and k' are two new constants.

We shall obtain our two periodic solutions σ and σ', assigning the following particular values to the constants

$$k=0,\qquad h=\frac{\alpha^2}{2},\qquad k'\sqrt{2k}=0,$$
$$k=0,\qquad h=\frac{\alpha^2}{2},\qquad k'\sqrt{2k}=\frac{\pi}{2}.$$

Let us assume that we would like to employ equation (2) to define r as a function of h' + t. If we assign values which are close to zero and $\frac{\alpha^2}{2}$ to the constants k and h, r will then be a periodic function of t + h'. We shall set

$$u=n(t+h'),$$

where the number n is chosen in such a way that r is a periodic function of u with the period 2π. This number n, which is a type of mean motion, will naturally depend on the constants h and k.

In the same way, $\frac{dr}{dt}$ will be a periodic function of u.

For k = 0, we simply have

$$r=1+\sqrt{2h}\cos u.$$

407. We therefore have two periodic solutions σ and σ' which will be represented by two closed curves, if we may regard the ξ's and the η's as the coordinates of a point in four-dimensional space. Two asymptotic surfaces pass through each of these curves -- one belonging to the first family, the other belonging to the second family. We shall see that the /408
four surfaces coincide pairwise, as occurred in No. 403 (equation 7), when ε is neglected.

In order to obtain the equations of these surfaces, it is sufficient to assign the values zero and $\frac{\alpha^2}{2}$ to k and h. We thus have

$$(\xi_1 - 1)\eta_1 + (\xi_2 - 1)\eta_2 = r\sqrt{\alpha^2 - (r-1)^2},$$
$$(\xi_1 - 1)\eta_2 - (\xi_2 - 1)\eta_1 = \pm\sqrt{2\varepsilon\psi}.$$

These are the equations of asymptotic surfaces for $\mu = 1$. It may be seen that we only obtain two of these surfaces, corresponding to the double sign of the second radical.

We shall assume that the function $\varepsilon\psi$, which vanishes for $\omega = 0$ and $\omega = \frac{\pi}{2}$, is positive for every other value of ω.

We shall now try to formulate the equations of asymptotic surfaces for values of μ which are close to 1.

We have

$$F = F_1 + (1-\mu)(F_0 - F_1);$$

where F_0 and F_1 are holomorphic functions of the ξ's and the η's, and consequently of r, ω, $\frac{dr}{dt}$ and $\frac{d\omega}{dt}$.

The equations of our surfaces may be written

$$(\xi_1 - 1)\eta_1 + (\xi_2 - 1)\eta_2 = r\frac{dS}{dr},$$
$$(\xi_1 - 1)\eta_2 - (\xi_2 - 1)\eta_1 = \frac{dS}{d\omega},$$

where S is a function of r and ω, satisfying the partial differential equation

$$F = \text{const.},$$

where we have replaced $\frac{dr}{dt}$ and $\frac{d\omega}{dt}$ by $\frac{dS}{dr}$ and $\frac{1}{r^2}\frac{dS}{d\omega}$.

Let us develop S in powers of $1 - \mu$

$$S = S_0 + (1-\mu)S_1 + (1-\mu)^2 S_2 + \ldots,$$

and we shall have, as the first approximation, /409

$$(\xi_1 - 1)\eta_1 + (\xi_2 - 1)\eta_2 = r\frac{dS_0}{dr} + (1-\mu)r\frac{dS_1}{dr},$$
$$(\xi_1 - 1)\eta_2 - (\xi_2 - 1)\eta_1 = \frac{dS_0}{d\omega} + (1-\mu)\frac{dS_1}{d\omega}.$$

for the equations of our asymptotic surfaces.

We have already obtained

$$\frac{dS_0}{dr} = \sqrt{\alpha^2 - (r-1)^2}, \qquad \frac{dS_0}{d\omega} = \pm\sqrt{2\varepsilon\psi}.$$

We must now determine S_1. For this purpose, we have the equation

$$\frac{dS_0}{dr}\frac{dS_1}{dr} + \frac{1}{r^2}\frac{dS_0}{d\omega}\frac{dS_1}{d\omega} = F_1 - F_0.$$

In the second term, $\frac{dr}{dt}$ and $\frac{d\omega}{dt}$ must be replaced by $\frac{dS_0}{dr} = \sqrt{\alpha^2 - (r-1)^2}$ and by $\frac{1}{r^2}\frac{dS_0}{d\omega} = \frac{\pm\sqrt{2\varepsilon\psi}}{r^2}$. This second term is therefore a known function of r and ω.

The equation becomes

$$r^2\sqrt{\alpha^2 - (r-1)^2}\frac{dS_1}{dr} \pm \sqrt{2\varepsilon\psi}\frac{dS_1}{d\omega} = r^2(F_1 - F_0).$$

Let us set

$$v = \int \frac{dr}{r^2\sqrt{\alpha^2 - (r-1)^2}}.$$

It may be seen that r and $\sqrt{\alpha^2 - (r-1)^2}$ are periodic functions of v, and we may regard S as a function of v and ω.

Our equation then becomes

$$\frac{dS_1}{dv} \pm \sqrt{2\varepsilon\psi}\frac{dS_1}{d\omega} = r^2(F_1 - F_0.$$

The second term is a known function of v and ω, which is periodic with respect to v.

This equation thus has exactly the same form as equation (2) given in No. 403, where v plays the role of x, and ω plays the role of y.

It will be handled in the same way. The procedures presented in No. 403 will be employed to determine the four functions $S_{1.1}$, $S_{1.2}$, $S_{1.3}$, $S_{1.4}$ corresponding to four asymptotic surfaces.

Just as in No. 403, it will be found that these asymptotic surfaces /410 coincide, and consequently heteroclinous solutions exist.

However, this has only been established for values of μ which are close to 1. I do not know whether this is still valid for small values of μ.

The result is therefore incomplete. However, I hope that the reader will pardon the length of this digression, because the question which I have posed, rather than solved, seems to be directly related to the question of stability, as I indicated in No. 400.

END OF THE THIRD AND LAST VOLUME

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