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A COURSE IN CELESTIAL MECHANICS

VOLUME 2

by

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This is the second volume of "a course in celestial mechanics" and must be regarded as an immediate continuation of the first volume. This volume is concerned with the general theory of perturbative motion, the methods of evaluation of the perturbations of planets and comets and principles on the theory of the motion of the Moon.

This book may serve as a text-book for undergraduate and post-graduate students and being a sufficiently complete monograph on celestical mechanics, it may be of interest to all scientists working in this field.

Translation Editor's Note: The reader is advised to consult the foreign text for greater legibility of the graphic material.

PREFACE

Whereas the first volume of this course was entirely devoted to the two-body problem and to the methods for determining the orbitals, this volume is mainly given to the perturbation theory and its application in celestial mechanics.

In this book, it has been my aim to give beginners an orientation in modern celestial mechanics and to introduce them to the periodic literature. Secondly, I hope that this book will be accepted as a comprehensive and practical treatise on the solution of some of those problems which usually meet astronomers.

The material of this book is mainly based on the general and special courses I gave in the University of Leningrad during the past six years. The book is divided into four sections. The first section is devoted to the study of the general properties of motion of material points mutually interacting according to Newton's law of gravitation and, in particular, to the most important properties of perturbative motion. This section may be considered as an introduction to the ensuing sections, in which the actual determination of the perturbed coordinates is discussed.

The second section is given to the determination of the perturbed coordinates using methods based on the numerical intergration of differential equations. Here, as well as in volume I, I aim to give an exhuastive manual for carrying out these operations. In volume I, I did not deal with a detailed account of the theory of numerical integration of differential equations, I gave but a brief summary of this important subject in an appendix to volume I in order to fill as soon as possible one of the most important gaps in our literature. In volume II, I developed the subject to the necessary degree, and gave only pertinent examples. I preferred not to bother the reader by referring continuously to the above mentioned appendix and I found it better to expose all the theory of integration of the equations even at the cost of repeating a number of pages. This allowed me to exclude the above mentioned appendix from the new edition of volume I, which is being prepared at the present time.

After explaining in detail the methods of the numerical integration of the equations and illustrating this by means of several examples on how to use these methods, I considered in detail the application of the numerical integration methods to the study of the unperturbed motion. I then applied these methods to the evaluation of perturbations, and here I could not forego the illustration of examples, but the reader who studied in detail the preceeding two chapters will not find a need for these examples.

The third and fourth sections of the book are concerned with the analytic methods of evaluating perturbations. In these sections, I do not attempt to give a comprehensive account of the methods to be applied, as they are already found in special monographs which I am not trying to replace and thus restricted myself to the complete presentation of the main points of each of these important methods. Considerable space is devoted to the study of the motion of the Moon. This is not only one of the most well-developed areas of celestial mechanics, but is also considered not less important than the motion theory of planets. I do not need to mention that the theory of the motion of the Moon is used in star astronomy in the study of multiple stars. We do not forget that the work by Hill is considered to be one of the most important reference sources in celestial mechanics during the past decade.

Volume II is hence mainly devoted to the study of the methods of celestial mechanics. More details on the results, namely the comparison between theory and observation, will be given in volume III. There, more special methods (periodic orbitals, methods of Gulden and Brendel) and also the theory of the motion of stars will be given.

I tried to make the standard of presentation suitable for the self-study of the subject and I avoided complicated mathematical methods as much as possible. I hope that lecturers will be able to choose from this book various topics according to the standard and the degree of completeness they require.

I have carefully chosen notations for quantities, for which no standard notations exist and I would like to point out that the introduction of a complete set of notations is not only difficult, but also not always useful. For example, I used different notations in the motion theory of planets and in the theory of motion of the Moon. This should not cause the reader any trouble and should help him in reading the special literature.

I do not claim a complete and systematic bibliography and more details on the available bibliography could be found by investigating the well known literature.

Finally, I note that the chapters devoted to the determination of orbitals obtained from the various observations, which I had planned to include in volume II, were included in the new edition of volume I, thus allowing the contents of the present volume to be more homogeneous.

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PART ONE

PRINCIPLES OF THE THEORY OF PERTURBATIVE MOTION

CHAPTER I

THE n-BODY PROBLEM

1. The integrals of the n-body problem

Let us consider n material points of masses m_0, m_1, ... m_{n-1}. We denote by ξ_i, η_i and ζ_i the coordinates of point m_i relative to an arbitrary system of axes. Let Δ_{ij} be the distance between points m_i and m_j, so that

$$\Delta_{ij}^2 = (\xi_i - \xi_j)^2 + (\eta_i - \eta_j)^2 + (\zeta_i - \zeta_j)^2.$$

The force of gravitation by which point m_j acts on point m_i is equal to $k^2 m_i m_j \Delta_{ij}^{-2}$, where k^2 is the coefficient of gravity (vol. I, SS 3 and 6). The projections of this force on the coordinate axes are

$$k^2 m_i m_j \frac{\xi_j - \xi_i}{\Delta_{ij}^3}, \quad k^2 m_i m_j \frac{\eta_j - \eta_i}{\Delta_{ij}^3}, \quad k^2 m_i m_j \frac{\zeta_j - \zeta_i}{\Delta_{ij}^3}.$$

The equations of motion of point m_i are then given by

$$\begin{aligned} m_i \frac{d^2\xi_i}{dt^2} &= k^2 \sum_j m_i m_j \frac{\xi_j - \xi_i}{\Delta_{ij}^3}, \\ m_i \frac{d^2\eta_i}{dt^2} &= k^2 \sum_j m_i m_j \frac{\eta_j - \eta_i}{\Delta_{ij}^3}, \qquad (1) \\ m_i \frac{d^2\zeta_i}{dt^2} &= k^2 \sum_j m_i m_j \frac{\zeta_j - \zeta_i}{\Delta_{ij}^3}, \end{aligned}$$

where terms with $i = j$ should be dropped in the summation. Equation (1), with $i = 0, 1, \ldots, n-1$ form a system of the 6n-order, the integration of which gives a complete solution of the n-body problem. These equations may also be written as

$$m_i \frac{d^2\xi_i}{dt^2} = \frac{\partial U}{\partial \xi_i}, \quad m_i \frac{d^2\eta_i}{dt^2} = \frac{\partial U}{\partial \eta_i}, \quad m_i \frac{d^2\zeta_i}{dt^2} = \frac{\partial U}{\partial \zeta_i} \qquad (2)$$

once the force function

$$U = \frac{1}{2} k^2 \sum_i \sum_j \frac{m_i m_j}{\Delta_{ij}} \quad (i \neq j). \qquad (3)$$

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is introduced.

Since we are dealing with a system on which no external force acts, the general equations of mechanics allow us to derive for the system 6 integrals of motion of the centre of mass, and 3 integrals of area. These integrals can be easily obtained from equation (1). Substituting i in each of equations (1) where i = 0 to i = n-1 to obtain

$$\sum_i m_i \ddot{\xi}_i = 0, \quad \sum_i m_i \ddot{\eta}_i = 0, \quad \sum_i m_i \ddot{\zeta}_i = 0,$$

where the dots denote differentiations with respect to time.

Then

$$\sum_i m_i \dot{\xi}_i = \alpha_1, \quad \sum_i m_i \dot{\eta}_i = \beta_1, \quad \sum_i m_i \dot{\zeta}_i = \gamma_1 \qquad (4)$$

$$\sum_i m_i \xi_i = \alpha_1 t + \alpha_2, \quad \sum_i m_i \eta_i = \beta_1 t + \beta_2, \quad \sum_i m_i \zeta_i = \gamma_1 t + \gamma_2 \qquad (5)$$

where α_1, β_1, γ_1, α_2, β_2 and γ_2 are arbitrary constants. Equations (4) and (5) show that the centre of mass of the system moves with a linear and uniform velocity. These relations are called the <u>integrals of motion of the centre of mass</u>.

In order to derive the integrals of area, let us consider the following relations, which can be easily obtained from equations (1):

$$\sum_i m_i (\eta_i \ddot{\zeta}_i - \zeta_i \ddot{\eta}_i) = 0$$

$$\sum_i m_i (\zeta_i \ddot{\xi}_i - \xi_i \ddot{\zeta}_i) = 0$$

$$\sum_i m_i (\xi_i \ddot{\eta}_i - \eta_i \ddot{\xi}_i) = 0.$$

Integrating these equations and denoting by C_1, C_2 and C_3 the new arbitrary constants, we obtain the <u>integrals of area</u>

$$\sum_i m_i(\eta_i\dot{\zeta}_i - \zeta_i\dot{\eta}_i) = C_1$$
$$\sum_i m_i(\zeta_i\dot{\xi}_i - \xi_i\dot{\zeta}_i) = C_2 \qquad (6)$$
$$\sum_i m_i(\xi_i\dot{\eta}_i - \eta_i\dot{\xi}_i) = C_3.$$

The left-hand side of equations (6) are the projections on the coordinate axes of the angular momentum of the system, which equals the sum of angular momenta of all the points. In this manner, equations (6) show that the magnitude and direction of the angular momentum of the system are conserved. The plane perpendicular to the angular momentum of the system is given by the following equation

$$C_1(\xi - \xi^0) + C_2(\eta - \eta^0) + C_3(\zeta - \zeta^0) = 0, \qquad (7)$$

and called Laplace's invariable plane.

Each of the terms of the sums in the left hand side of equation (6) can be interpreted as the projection of the areal velocity of a given point m on the respective coordinate plane, multiplied by the mass of the particle. In other words, the left hand side of each of these equations is the sum of the mass projections of areal velocities of all particles of the system on one of the coordinate planes. If the sum of mass areal velocities is projected on an arbitrarily chosen plane, the projection will evidently be equal to

$$\sqrt{C_1^2 + C_2^2 + C_3^2}\cos\beta,$$

where β is the angle between the normal to this plane and the angular momentum of the system. From here, it follows that the invariable plane may be defined as the plane for which the sum of mass projections of the areal velocities of all the points of the system is maximum.

The remaining integral of motion owes its existence to the situation that the force function, given by equation (3), does not depend explicity on time. To obtain this integral, we multiply equations (2) by $\dot{\xi}_i$, $\dot{\eta}_i$ and $\dot{\zeta}_i$ respectively, and by adding the resulting equations, we obtain

$$\sum_i m_i(\dot{\xi}_i\ddot{\xi}_i + \dot{\eta}_i\ddot{\eta}_i + \dot{\zeta}_i\ddot{\zeta}_i) = \frac{dU}{dt}.$$

The integration of this equation directly leads to the following integral of kinetic energy

$$\frac{1}{2}\sum m_i(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2) = U + h, \tag{8}$$

where h is a new constant. The quantity

$$T = \frac{1}{2}\sum m_i(\dot{\xi}_i^2 + \dot{\eta}_i^2 + \dot{\zeta}_i^2)$$

is actually the kinetic energy of the system. The potential energy is equal to - U. Then, the total energy is

$$M = T - U$$

and the integral (8), assuming H = h, expresses the law of conservation of energy for the system under consideration.

We conclude that the general theorems of mechanics lead for the n-body problem to ten integrals of motion. Various attempts to find other integrals have not been successful. In the year 1887, Bruns has shown that any first integral, algebraic with respect to the coordinates ξ_i, η_i and ζ_i and their derivatives $\dot{\xi}_i$, $\dot{\eta}_i$ and $\dot{\zeta}_i$ should be a consequent of the ten integrals obtained above even for three body problem. In the year 1889, Poincare also found that for three body problems there exists no other first integrals that could be definitely expressed by single valued analytic functions. We shall not stop on the proof of these theorems. Knowing that they exist, we shall not search for other integrals of motion, other than the above mentioned ten. If other integrals of motion were obtainable, they would be too complicated to be of practical use.

2. Reduction of the n-body problem to the integration of a system of order 6n-12 and two quadratures

The 6 integrals of motion of the centre of mass, the 3 integrals of area and the integral of kinetic energy found in the previous section reduce the order ofsystem (1) by ten units. We can reduce the order of the system by one more unit when we make use of the time independence of the gravitational forces. For this purpose, we have to exclude from the equations of motion the increment dt. The system of equations obtained in this manner is then integrated, while time is determined by evaluation of a quadrature. The problem is then reduced to the solution of a system of the order 6n-11 and one quadrature.

We can again reduce the order of the system by one more unit by making use of the property of the forces that they depend only on the interpoint distances. To do this, we introduce the generalized coordinates in the following way. We draw a straight line passing through one of the points of the system and fix its direction. We then draw a plane through this line passing by another point in the system. We denote by φ the azimuthal angle of this plane with respect to an arbitrary direction and let this angle be one of the generalized coordinates. The motion of the whole system will be defined by the coordinate φ and the coordinates which define the position of the system relative to the plane. We then prove that the coordinate is cyclic, i.e. that the Lagrange equations of the system involve only the time derivative φ but not the coordinate φ itself. We choose the z-axis in the direction of the fixed straightline and let the x-axis be in the rotating plane. We denote by x_k , y_k and z_k the coordinates of point

m_k in the rotating coordinate system. The components of velocity of point m_k will then be:

$$\dot{x}_k - y_k\dot{\varphi}, \quad \dot{y}_k + x_k\dot{\varphi}, \quad \dot{z}_k.$$

The kinetic energy of the system will be given by

$$T = \frac{1}{2}\sum_k m_k[(\dot{x}_k - y_k\dot{\varphi})^2 + (\dot{y}_k + x_k\dot{\varphi})^2 + \dot{z}_k^2],$$

and will thus depend on $\dot{\varphi}$ but not on φ. Similarly, the Lagrangian of the system

$$L = T + U,$$

will not depend on φ, so that the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0$$

will yield

$$\frac{\partial L}{\partial \dot{\varphi}} = \text{const.}$$

This equation allows us to exclude φ from the remaining equations of motion. Integrating the system of equations obtained in this manner, we will be able to know the motion of the material points relative to the rotating plane. From the last equation, we obtain an expression for $\dot{\varphi}$ as a function of the other coordinates of the system. Solving one quadrature, we obtain the value of the angle φ which defines the position of the rotating plane.

The above procedure for the exclusion of the angle φ from the equations of motion has been named "method of elimination of nodes" by Jacobi in his study of the three-body problem.

The integration of the system (1) is finally reduced to the integration of a system of 6n-12 equations and two quadratures. We are not going to do this reduction here since it is only necessary for the complete solution of the n-body problem[1]. We shall only consider the reduction of the order of the system by (units, using integrals (4) and (5), which is of practical importance.

3. <u>Equations of relative motion</u>

Let us return back to equations (1) which describe the motion of n bodies relative to an arbitrary system of fixed axes. Let us make use of the integrals of motion, given by equations (4) and (5) to exclude from equations (1) three arbitrary coordinates, say ξ_0, η_0 and ζ_0 as well as their derivatives with respect to time. For this purpose, we introduce the new coordinates x_i, y_i and z_i so that

$$\xi_i = \xi_0 + x_i, \quad \eta_i = \eta_0 + y_i, \quad \zeta_i = \zeta_0 + z_i \qquad (i = 1, 2, \ldots, n-1)$$

These are the coordinates of point m_i relative to three axes passing by point m_0 parallel to thefixed axes. Noting that $x_0 = y_0 = z_0 = 0$, we rewrite equations (1) as follows

$$\begin{aligned}
\frac{d^2\xi_i}{dt^2} &= k^2 \sum_{j \neq i} m_j \frac{\xi_j - \xi_i}{\Delta_{ij}^3} = \\
&= k^2 \left(m_0 \frac{x_0 - x_i}{\Delta_{0i}^3} + m_1 \frac{x_1 - x_i}{\Delta_{1i}^3} + \cdots + m_{n-1} \frac{x_{n-1} - x_i}{\Delta_{n-1,i}^3} \right) = \\
&= -k^2 m_0 \frac{x_i}{\Delta_{0i}^3} + k^2 \sum{}' m_j \frac{x_j - x_i}{\Delta_{ij}^3} \\
\frac{d^2\xi_0}{dt^2} &= k^2 \sum_{j \neq 0} m_j \frac{\xi_j - \xi_0}{\Delta_{0j}^3} = k^2 m_i \frac{x_i}{\Delta_{0i}^3} + k^2 \sum{}' m_j \frac{x_j}{\Delta_{0j}^3},
\end{aligned}$$

---

[1] Details on thedifferent methods of reduction of the order of the system of differential equations mentioned above can be found in the article: E.T. Whittaker, Prinziplen der Storungstheorie und allgemeine Theorie der Bahnkurven in dynamischen Problem, Enzyklopadie der mathem. Wiesenschaften, Bd. VI_2, 512-556, and also in the book "Analytic dynamics" by the same author where a whole chapter is devoted to the above problem. The reduction of the n-body problem to a system of 16 n-12 equations has been done by T.L. Bennett (Messenger of Math. (2), 644, 1901).

Where $\sum'$ denotes a summation over j in which terms with j = 0 and j = i are dropped. Introducing the notation $\Delta_{oj} = r_j$, we obtain the following equations for the motion of point m_i relative to point m_o

$$\begin{aligned}
\frac{d^2x_i}{dt^2} &= -k^2(m_0+m_i)\frac{x_i}{r_i^3} + k^2\sum' m_j\left(\frac{x_j-x_i}{\Delta_{ij}^3}-\frac{x_j}{r_j^3}\right)\\
\frac{d^2y_i}{dt^2} &= -k^2(m_0+m_i)\frac{y_i}{r_i^3} + k^2\sum' m_j\left(\frac{y_j-y_i}{\Delta_{ij}^3}-\frac{y_j}{r_j^3}\right)\\
\frac{d^2z_i}{dt^2} &= -k^2(m_0+m_i)\frac{z_i}{r_i^3} + k^2\sum' m_j\left(\frac{z_j-z_i}{\Delta_{ij}^3}-\frac{z_j}{r_j^3}\right),\\
&(i = 1, 2, \ldots, n-1).
\end{aligned} \tag{9}$$

where

$$\Delta_{ij}^2 = (x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2,$$
$$r_i^2 = x_i^2+y_i^2+z_i^2.$$

Introducing the notation

$$R_i = k^2\sum' m_j\left(\frac{1}{\Delta_{ij}}-\frac{x_ix_j+y_iy_j+z_iz_j}{r_j^3}\right). \tag{10}$$

we rewrite equations (9) as follows

$$\begin{aligned}
\frac{d^2x_i}{dt^2} + k^2(m_0+m_i)\frac{x_i}{r_i^3} &= \frac{\partial R_i}{\partial x_i}\\
\frac{d^2y_i}{dt^2} + k^2(m_0+m_i)\frac{y_i}{r_i^3} &= \frac{\partial R_i}{\partial y_i}\\
\frac{d^2z_i}{dt^2} + k^2(m_0+m_i)\frac{z_i}{r_i^3} &= \frac{\partial R_i}{\partial z_i}.
\end{aligned} \tag{11}$$

Let us assume that the masses of all points except points m_o and m_i equal zero. In thiscase R_i = 0 and equations (11) turn out to be the well known Kepler problem of two interacting bodies (vol. I, Ch. II). In Astronomy, the Kepler problem is usually referred to as the <u>nonperturbative</u> motion, and all deviations from it are called <u>perturbative</u>. For this reason, the functions R_i will be referred to as <u>perturbation</u>

functions. The derivatives of these functions with respect to the coordinates of m_i are equal to the components of the relative acceleration aquired by the body m_i from its interaction with the rest of the particles of the relative acceleration acquired by the body m_i from its interaction with the rest of the particles of the system, except the central body m_o.

When we solve the (6n-6)-order system, given by equations (11), we have a complete information about the motion of all points of the system relative to point m_o. After that, it is easy to obtain the absolute motion of all bodies of the system. Actually, equations (5) lead to

$$
\begin{aligned}
M\xi_0 + \sum_1^{n-1} m_i x_i &= \alpha_1 t + \alpha_2 \\
M\eta_0 + \sum_1^{n-1} m_i y_i &= \beta_1 t + \beta_2 \\
M\zeta_0 + \sum_1^{n-1} m_i z_i &= \gamma_1 t + \gamma_2,
\end{aligned} \tag{12}
$$

where

$$M = m_0 + \sum_1^{n-1} m_i.$$

From these equations, we can find the values of ξ_0, η_0 and ζ_0.

We can find four integrals of motion for the system (11) which correspond to the integrals (6) and (8) of the absolute motion. The first of equations (6) gives

$$\sum_1^{n-1} m_i[(\eta_0 + y_i)(\dot\zeta_0 + \dot z_i) - (\zeta_0 + z_i)(\dot\eta_0 + \dot y_i)] + m_0(\eta_0\dot\zeta_0 - \zeta_0\dot\eta_0) = C_1$$

or, in other words

$$
\begin{aligned}
M(\eta_0\dot\zeta_0 - \zeta_0\dot\eta_0) + \dot\zeta_0 \sum m_i y_i - \zeta_0 \sum m_i \dot y_i + {} \\
+ \eta_0 \sum m_i \dot z_i - \dot\eta_0 \sum m_i z_i + \sum m_i (y_i \dot z_i - z_i \dot y_i) = C_1
\end{aligned}
$$

We determine the values of ξ_o, η_o and ζ_o from equations (12). Substituting here these values, we obtain the following integrals of equation (11)

$$
\begin{aligned}
&M\sum m_i(y_i\dot{z}_i - z_i\dot{y}_i) + \sum m_i z_i \sum m_i \dot{y}_i - \sum m_i y_i \sum m_i \dot{z}_i = C_1' \\
&M\sum m_i(z_i\dot{x}_i - x_i\dot{z}_i) + \sum m_i x_i \sum m_i \dot{z}_i - \sum m_i z_i \sum m_i \dot{x}_i = C_2' \\
&M\sum m_i(x_i\dot{y}_i - y_i\dot{x}_i) + \sum m_i y_i \sum m_i \dot{x}_i - \sum m_i x_i \sum m_i \dot{y}_i = C_3'.
\end{aligned}
\qquad (13)
$$

where

$$
\begin{aligned}
C_1' &= MC_1 + \beta_1\gamma_2 - \beta_2\gamma_1 \\
C_2' &= MC_2 + \gamma_1\alpha_2 - \gamma_2\alpha_1 \\
C_3' &= MC_3 + \alpha_1\beta_2 - \alpha_2\beta_1.
\end{aligned}
$$

carrying out similar transformations on the integral of kinetic energy given by equation (8), we obtain another integral for the equations of relative motion.

Equations (11) are widely used in celestral mechanics, especially in the study of the motion of planets and comets. In these cases, the central body m_o is usually taken to be the sun. With this choice, each term of the perturbation function (10) is proportional to the mass m_j of one of the planets. Therefore, the right-hand sides of equations (11) are so small that their influence can be treated as a perturbation.

4. A second form for the equations of relative motion.

The form of equations of relative motion obtained in the previous section is not always convenient since these equations involve a particular perturbation function R_i for each body. Sometimes a different form of equations of relative motion is used, which is based on the following choice of relative coordinates:

(1) Draw three coordinate axes through the first point m_o, parallel to the fixed axes and define the position of point m_1 in this

system by the coordinates x_1, y_1 and z_1.

(2) Draw three other axes through the centre of gravity G_1 of points m_o and m_1 parallel to the previous axes and define the position of point m_2 relative to these axes by the coordinates x_2, y_2 and z_2.

(3) Define the position of point m_3 by the coordinates x_3, y_3 and z_3 relative to a system of axes parallel to the previous ones and having their origin at the centre of gravity of points m_o, m_1 and m_2; and so on.

In this way, every subsequent point m_{i+1} is related to the centre of gravity G_i of all previous points m_o, m_1, ... m_i.

Let X_i, Y_i and Z_i be the coordinates of point G_i, so that

$$M_iX_i = m_0\xi_0 + m_1\xi_1 + \ldots + m_i\xi_i,$$

where

$$M_i = m_0 + m_1 + \ldots + m_i.$$

By definition

$$x_i = \xi_i - X_{i-1},$$

so that

$$\begin{aligned} M_{i-1}x_i &= M_{i-1}\xi_i - (m_0\xi_0 + m_1\xi_1 + \ldots + m_{i-1}\xi_{i-1}) = \\ &= m_0(\xi_i - \xi_0) + m_1(\xi_i - \xi_1) + \ldots + m_{i-1}(\xi_i - \xi_{i-1}). \end{aligned} \tag{14}$$

In order to express the old coordinates, ξ, η and ζ, in terms of the new ones, x, y and z, we note that

$$M_iX_i - M_{i-1}X_{i-1} = m_i\xi_i$$

or

$$M_iX_i - M_{i-1}X_{i-1} = (M_i - M_{i-1})\xi_i. \tag{15}$$

Hence,

$$(M_i - M_{i-1})\xi_i = M_i(\xi_{i+1} - x_{i+1}) - M_{i-1}(\xi_i - x_i).$$

so that

$$\xi_{i+1} - \xi_i = x_{i+1} - M_{i-1} M_i^{-1} x_i .$$

Adding these equations term by term in successive order of index, we obtain

$$\begin{aligned} &\xi_1 - \xi_0 = x_1, \\ &\xi_{i+1} - \xi_0 = x_{i+1} + \frac{m_i x_i}{M_i} + \frac{m_{i-1} x_{i-1}}{M_{i-1}} + \dots + \frac{m_1 x_1}{M_1}. \end{aligned} \tag{16}$$

We now write the differential equations of motion in terms of the new variables. Differentiating equation (14) twice, and using equations (2), we obtain

$$M_{i-1} \frac{d^2 x_i}{dt^2} = \frac{M_{i-1}}{m_i} \frac{\partial U}{\partial \xi_i} - \frac{\partial U}{\partial \xi_0} - \frac{\partial U}{\partial \xi_1} - \dots - \frac{\partial U}{\partial \xi_{i-1}}.$$

On the other hand, the following relations follow from equations (14)

$$\begin{aligned} \frac{\partial U}{\partial \xi_0} &= \frac{\partial U}{\partial x_1} - \frac{m_0}{M_1}\frac{\partial U}{\partial x_2} - \frac{m_0}{M_2}\frac{\partial U}{\partial x_3} - \dots - \frac{m_0}{M_{n-2}}\frac{\partial U}{\partial x_{n-1}} \\ \frac{\partial U}{\partial \xi_1} &= \frac{\partial U}{\partial x_1} - \frac{m_1}{M_1}\frac{\partial U}{\partial x_2} - \frac{m_1}{M_2}\frac{\partial U}{\partial x_3} - \dots - \frac{m_1}{M_{n-2}}\frac{\partial U}{\partial x_{n-1}} \\ \frac{\partial U}{\partial \xi_2} &= \frac{\partial U}{\partial x_2} - \frac{m_2}{M_2}\frac{\partial U}{\partial x_3} - \dots - \frac{m_2}{M_{n-2}}\frac{\partial U}{\partial x_{n-1}} \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{\partial U}{\partial \xi_{n-1}} &= \frac{\partial U}{\partial x_{n-1}}, \end{aligned}$$

so that

$$\begin{aligned} \frac{M_{i-1}}{m_i}\frac{\partial U}{\partial \xi_i} &= \frac{M_{i-1}}{m_i}\frac{\partial U}{\partial x_i} - \frac{M_{i-1}}{M_i}\frac{\partial U}{\partial x_{i+1}} - \dots - \frac{M_{i-1}}{M_{n-2}}\frac{\partial U}{\partial x_{n-1}}, \\ \sum_0^{i-1}\frac{\partial U}{\partial \xi_k} &= \frac{M_{i-1}}{M_{i-1}}\frac{\partial U}{\partial x_i} - \frac{M_{i-1}}{M_i}\frac{\partial U}{\partial x_{i+1}} - \dots - \frac{M_{i-1}}{M_{n-2}}\frac{\partial U}{\partial x_{n-1}}, \end{aligned}$$

Finally, we obtain the following differential equations for the relative motion of the system

$$\mu_i \frac{d^2x_i}{dt^2} = \frac{\partial U}{\partial x_i}, \quad \mu_i \frac{d^2y_i}{dt^2} = \frac{\partial U}{\partial y_i}, \quad \mu_i \frac{d^2z_i}{dt^2} = \frac{\partial U}{\partial z_i}, \tag{17}$$

where

$$\mu_i = \frac{m_i M_{i-1}}{M_i}.$$

Equations (16) allow us to express the force function U in terms of the new variables. Equations (17) are successfully applied in the study of the motion of satellites and systems of multiple stars. In order to study the motion of the Moon, for example, it is convenient to choose point m_o to be the Earth, point m_1 the Moon and point m_2 the Sun. Denoting by x_1 , y_1 and z_1 the coordinates of the Moon relative to the Earth, and by x_2 , y_2 and z_2 the coordinates of the Sun relative to the centre of mass of the Moon and Earth, and putting

$$\mu_1 = \frac{m_1 m_0}{m_0 + m_1}, \quad \mu_2 = \frac{m_2(m_0 + m_1)}{m_0 + m_1 + m_2},$$

we obtain the following equations of motion

$$\begin{aligned}
\mu_1 \frac{d^2x_1}{dt^2} &= \frac{\partial U}{\partial x_1}; & \mu_2 \frac{d^2x_2}{dt^2} &= \frac{\partial U}{\partial x_2} \\
\mu_1 \frac{d^2y_1}{dt^2} &= \frac{\partial U}{\partial y_1}; & \mu_2 \frac{d^2y_2}{dt^2} &= \frac{\partial U}{\partial y_2} \\
\mu_1 \frac{d^2z_1}{dt^2} &= \frac{\partial U}{\partial z_1}; & \mu_2 \frac{d^2z_2}{dt^2} &= \frac{\partial U}{\partial z_2},
\end{aligned} \tag{18}$$

where

$$U = k^2\left(\frac{m_1 m_2}{\Delta_{12}} + \frac{m_0 m_1}{\Delta_{01}} + \frac{m_0 m_2}{\Delta_{02}}\right).$$

It has been already pointed out (vol. I, § 7) that the motion of the sun relative to the centre of gravity of the system earth-moon is approximately elliptical. Therefore, the coordinates x_2 , y_2 and z_2

can approximately be found by solving a two-body problem. This approximation essentially simplifies the solution of the system [18] .

We finally show how to express the kinetic energy in terms of the new coordinates. We substitute $x_i + X_{i-1}$ for ξ_i in equation (15) and obtain

$$(M_i - M_{i-1})\, x_i = M_i (X_i - X_{i-1}).$$

Squaring both sides of this equation and of equation (15) yields

$$(M_i - M_{i-1})^2 x_i^2 = M_i^2 (X_i - X_{i-1})^2$$
$$(M_i - M_{i-1})^2 \xi_i^2 = (M_i X_i - M_{i-1} X_{i-1})^2.$$

Eliminating here the product $X_i X_{i-1}$, we get

$$m_i \left(\xi_i^2 - \frac{M_{i-1}}{M_i} x_i^2 \right) = M_i X_i^2 - M_{i-1} X_{i-1}^2 .$$

Summing this equation from i = 1 to i = n-1, we obtain

$$\sum_1^{n-1} m_i \xi_i^2 = \sum_1^{n-1} \frac{m_i M_{i-1}}{M_i} x_i^2 + M_{n-1} X_{n-1}^2 - M_0 X_0^2 ,$$

or, since $M_o = m_o$ and $X_o = \xi_o$,

$$\sum_0^{n-1} m_i \xi_i^2 = \sum_1^{n-1} \mu_i x_i^2 + M_{n-1} X_{n-1}^2 .$$

Adding this equation term by term to the corresponding equations for the other coordinates, we find

$$\sum_0^{n-1} m_i (\xi_i^2 + \eta_i^2 + \zeta_i^2) = \sum_1^{n-1} \mu_i (x_i^2 + y_i^2 + z_i^2) + M_{n-1} (X_{n-1}^2 + Y_{n-1}^2 + Z_{n-1}^2).$$

This relation has been obtained owing to the linear relations between the coordinates ξ, η and ζ and the coordinates x,y and z. Hence, a similar relation holds for the derivatives of these coordinates, such that

$$2T = \sum_1^{n-1} \mu_i (\dot x_i^2 + \dot y_i^2 + \dot z_i^2) + M_{n-1} (\dot X_{n-1}^2 + \dot Y_{n-1}^2 + \dot Z_{n-1}^2).$$

We note that equations (17) have the same form as equations (2). Therefore, we may obtain from equations (17) the integrals of real and kinetic energy by replacing the masses m_i by μ_i in the corresponding integrals obtained for the absolute motion.

The most general linear transformations of coordinates, which preserve the form of the integrals of motion in the three-body problem are given by Hopfner(1).

5. <u>Jacobi's formula</u>

The force function for the n-body problem, given by equation (3), is a homogeneous function of coordinates. Jacobi made use of this property to show that the kinetic energy integral may be obtained in a very simple form.

Since U is a homogeneous function of ξ_i, η_i and ζ_i of the $(-1)^{th}$ order, then

$$\sum_0^{n-1}\left(\xi_i\frac{\partial U}{\partial \xi_i}+\eta_i\frac{\partial U}{\partial \eta_i}+\zeta_i\frac{\partial U}{\partial \zeta_i}\right)=-U.$$

Therefore, multiplying both sides of equations (2)respectively by ξ_i, η_i and ζ_i and adding, we find

$$\sum_0^{n-1} m_i(\xi_i\ddot{\xi}_i+\eta_i\ddot{\eta}_i+\zeta_i\ddot{\zeta}_i)=-U.$$

Adding this equation to the kinetic energy integral (8) yields

$$\sum_0^{n-1} m_i(\xi_i\ddot{\xi}_i+\eta_i\ddot{\eta}_i+\zeta_i\ddot{\zeta}_i+\dot{\xi}_i^2+\dot{\eta}_i^2+\dot{\zeta}_i^2)=U+2h,$$

or

$$\frac{d}{dt}\sum_0^{n-1} m_i(\xi_i\dot{\xi}_i+\eta_i\dot{\eta}_i+\zeta_i\dot{\zeta}_i)=U+2h.$$

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(1) Hopfner, Uber eine Verallgemeinerung der relativen kanonischen Koordinaten von Jacobi, Astr. Nachr., 195, 1913, 257-262.

or, finally

$$\frac{d^2}{dt^2}\sum_{0}^{n-1} m_i(\xi_i^2+\eta_i^2+\zeta_i^2) = 2U + 4h \qquad (19)$$

The sum involved here

$$J = \sum_{0}^{n-1} m_i(\xi_i^2+\eta_i^2+\zeta_i^2)$$

is the polar moment of inertia of our system. It is well known that J can be expressed in terms of the squares of the interpoint distances as well as the quantity

$$J_0 = M(X^2+Y^2+Z^2),$$

where M is the sum of all masses m_i, and X, Y and Z are the coordinates of the centre of mass of the system. Making use of the following identity

$$\sum_i m_i \sum_i m_i \xi_i^2 - (\sum_i m_i \xi_i)^2 = \sum_{ij} m_i m_j (\xi_i^2 + \xi_j^2 - 2\xi_i \xi_j),$$

where each combination of the symbols i and j in the right-hand side appears only once. Adding this identity to two similar identities for the variables η and ζ, and noting that

$$MX = \sum m_i \xi_i, \quad MY = \sum m_i \eta_i, \quad MZ = \sum m_i \zeta_i, \qquad (20)$$

we obtain

$$MJ - MJ_0 = \sum_{i,j} m_i m_j \Delta_{ij}^2$$

Hence, using the integrals (5), characterizing the motion of the centre of gravity of the system, we obtain

$$MJ = (\alpha_1 t + \alpha_2)^2 + (\beta_1 t + \beta_2)^2 + (\gamma_1 t + \gamma_2)^2 + \sum_{i,j} m_i m_j \Delta_{ij}^2 .$$

Substituting this expression for J in equation (19) and denoting by h' the new constant, we obtain

$$\frac{d^2R}{dt^2} = 2U + 4h', \qquad (21)$$

where

$$R = \frac{1}{M}\sum_{i,j} m_i m_j \Delta_{ij}^2.$$

This formula is known as Jacobi's formula. A particular case of this formula was obtained by Lagrange (1772) for the three-body problem. However, the general case was obtained by Jacobi in 1842.

The following results of equation (21) was obtained by Jacobi and was probably the first application of the qualitative methods in celestial mechanics. Let us integrate equation (21) from 0 to t and obtain

$$\frac{dR}{dt} \geq R'_0 + (2\alpha + 4h')t,$$

where R'_0 stands for $\frac{dR}{dt}$ at $t = 0$, and α for the lower limit of the function U. Evidently α can be set equal zero. Another integration in the same limits yields.

$$R \geq R_0 + R'_0 t + (\alpha + 2h')t^2$$

This inequality shows that the motion of the system is stable only if $h' < 0$. Actually, if $h' \geq 0$, then $\alpha + 2h' > 0$ and the right-hand side of this inequality indefinitely increases as $t \rightarrow \infty$. In this case, at least one of the mutual distances Δ_{ij} should tend to infinity.

In the two-body case, equation (21) becomes

$$\frac{d^2 r^2}{dt^2} = 2k^2(m_1 + m_2)\left(\frac{1}{r} - \frac{1}{a}\right), \qquad (22)$$

where r is the distance between the two bodies and a is the semimajor axis of the relative orbital. If $a < 0$, the relative motion proceeds through a hyperbola so that $r \rightarrow \infty$ when $t \rightarrow \infty$.

6. Laplace's invariable plane

It was shown in § 1 that when n material points are entirely under the action of their mutual gravitation, there exists a plane which conserves its direction in space. The plane is determined by equation (7), where the coefficients C_1, C_2 and C_3 are given by equations (6). However, equations (6) are not of practical use since they require the knowledge of the absolute motion of all points of the system. Also, the integrals of area of the relative motion, given in the form of equations (13) are not useful since they involve the quantities $\alpha_1, \alpha_2, \beta_1, \ldots\ldots$ which characterize the absolute motion of the centre of mass.

We shall see now that the direction of the invariable plane may be found in terms of only the relative coordinates and velocities of the points of the system. The reason for this is very simple. Equations (6) or (13) determine the values of the quantities C_1, C_2 and C_3. However, it is sufficient to the ratios $C_1 : C_2 : C_3$ in order to define the position of the invariable plane.

Let us introduce a new coordinate system having its origin at the centre of mass of points m_o, m_1, ... m_{n-1} and the directions of its axis in the space fixed. The coordinates of point m_i in this system are denoted by x_i, y_i and z_i, and the absolute coordinates of the centre of mass by X, Y and Z.. Using equations (5) and (17), we obtain

$$MX = \alpha_1 t + \alpha_2, \quad MY = \beta_1 t + \beta_2, \quad MZ = \gamma_1 t + \gamma_2,$$

so that

$$\ddot{X} = \ddot{Y} = \ddot{Z} = 0$$

We now express the old coordinates in equations (1) in terms of the new coordinates using the following relations

$$\xi_i = X + x_i, \quad \eta_i = Y + y_i, \quad \zeta_i = Z + z_i$$

We then obtain the following equations for the motion of point m_i relative to the centre of mass of the system

$$\left.\begin{aligned} m_i \ddot{x}_i &= k^2 m_i \sum_j m_j \frac{x_j - x_i}{\Delta_{ij}^3} \\ m_i \ddot{y}_i &= k^2 m_i \sum_j m_j \frac{y_j - y_i}{\Delta_{ij}^3} \\ m_i \ddot{z}_i &= k^2 m_i \sum_j m_j \frac{z_j - z_i}{\Delta_{ij}^3}. \end{aligned}\right\} \qquad (23)$$

Since these equations have the same form as equations (1), we can immediately write for them the integrals of area

$$\begin{aligned} \Sigma m_i (y_i \dot{z}_i - z_i \dot{y}_i) &= C_1'' \\ \Sigma m_i (z_i \dot{x}_i - x_i \dot{z}_i) &= C_2'' \\ \Sigma m_i (x_i \dot{y}_i - y_i \dot{x}_i) &= C_3''. \end{aligned}$$

The constants C_1'' , C_2'' and C_3'' fix the position of the Laplace's plane passing through the arbitrary point X^0 , y^0 and z^0 such that its equation is

$$C_1''(x - x^0) + C_2''(y - y^0) + C_3''(z - z^0) = 0,$$

Thus, to determine the position of Laplace's plane, it is sufficient to know the coordinates x_i , y_i and z_i and the components of velocity $\dot{x}_i$, $\dot{y}_i$ and $\dot{z}_i$ for all points of the system at any time.

In order to study the motion of the bodies in the solar system, it is more natural to choose the Laplace plane as a basis rather than use the ecliptics of a given epoch (1750.0, 1850.0, or 1900.0). However, the application of the Laplace plane is met with certain difficulties. The position of this plane depends on the masses of the planets which are only approximately known. Consequently, it is only possible to approximately know the position of the Laplace plane. When new and more exact determination of the masses of planets are made one must cansequently change the basic plane. Another difficulty comes from the fact that the sun and planets are not material points. The angular momentum, defined by the quantities C_1'' , C_2'' and C_3'' may be changed by the values of angular momenta acquired by individual bodies of the system, for example during tidal processes.

Relative to the ecliptic and equinox 1850.0 the position of the invariable plane is given by the elements

$$\Omega = 106^\circ 14', \quad i = 1^\circ 35' 19''.$$

As expected, this plane slightly differs from the plane of the orbital of Jupiter and is situated between this planc and the orbital plane of Saturn.

CHAPTER II

THE EQUATIONS OF MOTION EXPRESSED IN POLAR COORDINATES

1. The equations of motion expressed in cylindrical coordinates

It is well known that rectangular coordinates are used side by side with polar coordinates for the determination of the positions of material points. In the following, we express the equations of motion in different polar coordinates. We start by the simplest case of a stationary spherical polar coordinate system.

Let us consider the motion of a material point P, which we shall call-planet, relative to other point S, which we shall call-sun. We choose the origin of the coordinate system at point S, and call the plane xy the ecliptic. We denote by r and ρ respectively, the radius vector of point P and its projection in the plane xy, and by v the longitude of this plane as measured from the x-axis. In this case

$$x = \rho \cos v, \quad y = \rho \sin v,$$
$$r^2 = \rho^2 + z^2.$$

In order to express the equations of motion of point P in terms of the polar coordinates ρ , v and z, it is best to start with the Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k.$$

Let us put $q_1 = \rho$, $q_2 = v$ and $q_3 = z$, and note that the kinetic energy is given in terms of these coordinates by

$$T = \frac{1}{2} m(\dot{\rho}^2 + \rho^2 \dot{v}^2 + \dot{z}^2),$$

where m is the mass of point P. Let us denote by P, T and Z the components of accleration of point P in the direction of the projection of the radius vector in the plane xy, in the direction perpendicular to this projection in the plane and on the z-axis. We thus obtain

$$Q_1 = mP, \quad Q_2 = m\rho T, \quad Q_3 = mZ.$$

In this manner the equations of motion will be given by

$$\begin{aligned} \frac{d^2\rho}{dt^2} - \rho\left(\frac{dv}{dt}\right)^2 &= P \\ \frac{d}{dt}\left(\rho^2 \frac{dv}{dt}\right) &= \rho T \\ \frac{d^2 z}{dt^2} &= Z. \end{aligned} \qquad (1)$$

When the force function is given by mU, then the previous equations may be written as

$$\begin{aligned} \frac{d^2\rho}{dt^2} - \rho\left(\frac{dv}{dt}\right)^2 &= \frac{\partial U}{\partial \rho} \\ \frac{d}{dt}\left(\rho^2 \frac{dv}{dt}\right) &= \frac{\partial U}{\partial v} \\ \frac{d^2 z}{dt^2} &= \frac{\partial U}{\partial z} \end{aligned} \qquad (2)$$

Equations (1) have been used by several authors to study the motion of the Moon. In this case, instead of coordinate z, the following quantity is introduced

$$s = \frac{z}{\rho},$$

which represents the tangent of the MOON"S LATITUDE.

If the perturbation of motion of point P is taken into account, then

$$U = \frac{k^2}{r} + R, \qquad (3)$$

Here the first term corresponds to the attraction by the sun and the second term to the perturbation function. We notice here that the coefficient k^2 must be replaced by $k^2(1 + m)$ if mass m of planet P cannot be neglected in comparison with the mass of the sun, which is assumed to be equal to unity. Substituting equation (3) into equations (2), we obtain

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{dv}{dt}\right)^2 + k^2\frac{\rho}{r^3} = \frac{\partial R}{\partial \rho}$$

$$\frac{d}{dt}\left(\rho^2\frac{dv}{dt}\right) = \frac{\partial R}{\partial v} \qquad (4)$$

$$\frac{d^2z}{dt^2} + k^2\frac{z}{r^3} = \frac{\partial R}{\partial z}.$$

These equations are applied in the calculation of the perturbation of the planets and comets using the methods of numerical integration of differential equations.

8. The Clepo-Laplace equations

Let us consider the case when the perturbation function R in equations (4) vanishes. In this case, the unperturbed motion takes place in the invariable plane passing by S. Choosing this plane as the xy-plane, we set Z = 0 and ρ = r. Then, the equations of motion read

$$\frac{d^2r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 + \frac{k^2}{r^2} = 0$$

$$\frac{d}{dt}\left(r^2\frac{dv}{dt}\right) = 0 \qquad (1')$$

The general solution of these equation is given (vol. 1, Ch. II) by the well-known formulae

$$E - e\sin E = k a^{-3/2}(t - t_0)$$

$$\mathrm{tg}\,\frac{v - v_0}{2} = \sqrt{\frac{1+e}{1-e}}\,\mathrm{tg}\,\frac{E}{2}$$

$$r = \frac{a(1-e^2)}{1 + e\cos(v - v_0)},$$

where a, e, v_0 and t_0 are arbitrary constants. The inspection of these formulae indicates that it is easier to express r and t by functions of v rather than express r and v by functions of t. It is therefore advisable to choose the longitude v as the independent variable in equations (4) instead of the time t. Assuming that the perturbed motion is slightly different from the unperturbed, we expect that this replacement will also simplify the solution in the case of perturbed motion.

When the radius vector r is expressed by a function of v, it satisfies a rather difficult equation, obtained by excluding t from' equations (4'). On the other hand, the inverse quantity

$$u = \frac{1}{r} = \frac{1}{a(1-e^2)}\,[1 + e\cos(v - v_0)]$$

satisfies a very simple equation, namely

$$\frac{d^2u}{dv^2} + u = \frac{1}{a(1-e^2)}\,.$$

Taking this into consideration, let us rearrange the equations of motion given in the previous section, choosing

$$u = \frac{1}{\rho}\,, \qquad s = \frac{z}{\rho}$$

and t as the unknown quantities and the longitude v as the independent variable. Assuming

$$\rho^2\frac{dv}{dt} = H,$$

so that

$$\frac{dv}{dt} = Hu^2,$$

we easily replace the derivatives with respect to t by derivatives with respect to v. Since

$$\begin{aligned}\frac{d^2\rho}{dt^2} = \frac{d}{dt}\left(\frac{d\rho}{dt}\right) &= Hu^2\frac{d}{dv}\left(Hu^2\frac{d\rho}{dv}\right) = \\ &= -Hu^2\frac{d}{dv}\left(H\frac{du}{dv}\right) = \\ &= -H^2u^2\frac{d^2u}{dv^2} - Hu^2\frac{dH}{dv}\frac{du}{dv}\,,\end{aligned}$$

the first of equations (1) becomes

$$H^2u^2\left(\frac{d^2u}{dv^2}+u\right)+H\frac{dH}{dv}u^2\frac{du}{dv}=-P \qquad (4')$$

The second of these equations gives

$$H\frac{dH}{dv}=u^{-3}T. \qquad (5)$$

Substituting into the third of these equations $z = su^{-1}$, and noting that

$$\frac{d^2(su^{-1})}{dt^2}=Hu^2\frac{d}{dv}\left(Hu^2\frac{d(su^{-1})}{dv}\right)=$$
$$=H^2u^2\left(u\frac{d^2s}{dv^2}-s\frac{d^2u}{dv^2}\right)+H\frac{dH}{dv}u^2\left(u\frac{ds}{dv}-s\frac{du}{dv}\right),$$

we obtain

$$H^2u^3\left(\frac{d^2s}{dv^2}+s\right)+H\frac{dH}{dv}u^3\frac{ds}{dv}=Z-Ps,$$

where the second derivative with respect to u is eliminated using equation (4"). Applying equation (5) to exclude the derivatives of H, we finally obtain the equations of motion in the form

$$\left.\begin{aligned}
&\frac{d^2u}{dv^2}+u=H^{-2}u^{-2}\left(-P-Tu^{-1}\frac{du}{dv}\right)\\
&\frac{d^2s}{dv^2}+s=H^{-2}u^{-3}\left(-Ps-T\frac{ds}{dv}+Z\right)\\
&H\frac{dH}{dv}=Tu^{-3}.
\end{aligned}\right\} \qquad (6)$$

Integrating these equations, we express the quantities u, s and H in terms of functions of v. We still have to determine the time t. For this purpose, we use the following equation

$$\frac{dt}{dv}=H^{-1}u^{-2}. \qquad (7)$$

It is easy to exclude the auxiliary function H from the above equations. Actually, it follows from equation (5) that

$$H^2 = h^2 + 2\int Tu^{-3} dv,$$

where h is a constant of integration. Then equations (6) and (7) may be replaced by

$$\left.\begin{aligned} &\left(h^2 + 2\int Tu^{-3} dv\right)\left(\frac{d^2u}{dv^2} + u\right) = u^{-2}\left(-P - Tu^{-1}\frac{du}{dv}\right) \\ &\left(h^2 + 2\int Tu^{-3} dv\right)\left(\frac{d^2s}{dv^2} + s\right) = u^{-3}\left(-Ps - T\frac{ds}{dv} + Z\right) \\ &\frac{dt}{dv} = u^{-2}\left(h^2 + 2\int Tu^{-3} dv\right)^{-\frac{1}{2}}. \end{aligned}\right\} \qquad (8)$$

We finally rewrite the equations obtained for the case when the force function U is present. In this case

$$P = \frac{\partial U}{\partial \rho}, \quad T = \frac{1}{\rho}\frac{\partial U}{\partial v}, \quad Z = \frac{\partial U}{\partial z}.$$

It follows from the euqality

$$U(u, v, s) = U\left(\frac{1}{\rho}, v, \frac{z}{\rho}\right)$$

that

$$\frac{\partial U}{\partial \rho} = -u^2\frac{\partial U}{\partial u} - su\frac{\partial U}{\partial s}, \quad \frac{\partial U}{\partial z} = u\frac{\partial U}{\partial s}.$$

Hence, we condlue that

$$\left.\begin{aligned} &\left(h^2 + 2\int u^{-2}\frac{\partial U}{\partial v} dv\right)\left(\frac{d^2u}{dv^2} + u\right) = \frac{\partial U}{\partial u} - u^{-2}\frac{du}{dv}\frac{\partial U}{\partial v} + su^{-1}\frac{\partial U}{\partial s} \\ &\left(h^2 + 2\int u^{-2}\frac{\partial U}{\partial v} dv\right)\left(\frac{d^2s}{dv^2} + s\right) = su^{-1}\frac{\partial U}{\partial u} - u^{-2}\frac{ds}{dv}\frac{\partial U}{\partial v} + u^{-2}(1 + s^2)\frac{\partial U}{\partial s} \\ &\frac{dt}{dv} = u^{-2}\left(h^2 + 2\int u^{-2}\frac{\partial U}{\partial v} dv\right)^{-\frac{1}{2}} \end{aligned}\right\} \qquad (9)$$

These equations were first obtained by Laplace, although the principal idea of using the longitude as the independent variable was due to Clepo. Clepo derived equation (9) for the particular case of s = 0. He applied them to study the perturbation produced by the Sun on the motion of the moon assuming that the moon is moving in an ecliptic plane.

Equations (6) were widely used by Adams in his contributions to the theory, of the moon's motion.

9. Application of the Clepo-Laplace equations to the study of motion in a resisting system.

Let us assume that a planet moving around the sun is subject to a resistance of magnitude $\alpha\, mVr^{-2}$, where m is the mass of the planet, V its velocity, r its distance from the sun and α is a small coefficient constant. Let the direction of this resistance be along the tangent to the trajectory of the planet and opposite to the direction of its motion. The motion of the planet is evidently in a plane. Choosing this plane to be the xy plane, we rewirte equations (6) as follows

$$\frac{d^2u}{dv^2} + u = H^{-2}u^{-2}\left(-P - Tu^{-1}\frac{du}{dv}\right),$$
$$H\frac{dH}{dv} = Tu^{-3}, \qquad \frac{dv}{dt} = Hu^2, \tag{10}$$

where

$$r = u^{-1}$$

Let us evaluate the components of acceleration caused by the resistance of the medium in the direction of the radius vector and along the perpendicular to the radius vector in the orbital plane. The cosines of the angles between these directions and the positive direction of the tangent to the orbit are respectively equal to

$$rV^{-1} \quad \text{and} \quad rvV^{-1}$$

where

$$V = \sqrt{\dot{r}^2 + r^2\dot{v}^2}.$$

Then, the components of acceleration will be

$$-\alpha r^{-2}\dot r = +\alpha H u^2 \frac{du}{dv}$$

and

$$-\alpha r^{-1}\dot v = -\alpha H u^3.$$

Consequently

$$P = -k^2(1+m)u^2 + \alpha H u^2 \frac{du}{dv}$$
$$\Gamma = -\alpha H u^3.$$

Substituting these expressions in equations (10), we obtain

$$\frac{dH}{dv} = -\alpha,$$

and thus

$$H = h - \alpha v, \qquad (11)$$

where h is a constant of integration. Furthermore, the first of equations (10) gives

$$\frac{d^2u}{dv^2} + u = k^2(1+m)H^{-2},$$

When the coefficient α is so small that terms of order α^2 are negligible, we obtain

$$\frac{d^2u}{dv^2} + u = k^2 h^{-2}(1 + 2\alpha h^{-1} v).$$

We have dropped the factor (1 + m) in writing the above equations, since this factorcan always be included in the coefficient k^2. The general integral of the last equation takes the form

$$u = k^2 h^{-2}[1 + 2\alpha h^{-1} v + e\cos(v - \omega)], \qquad (12)$$

where e and ω are arbitrary constants. We can compare the orbit given by this equation with the elliptic orbit

$$u = p_0^{-1}[1 + e_0\cos(v - \omega_0)], \qquad (13)$$

which describes the motion of the planet in the absence of the resistance of the medium. Evidently, we can replace the constant elements p_0, e_0 and π_0 in equation (13) by functions of v so that this equation becomes identical to equation (12). This can be achieved in the following way: Let the coordinates and their velocities take the values u_0, v_0, $\dot{u}_0$ and $\dot{v}_0$ at time $t = t_0$. We then evaluate the elements of the elliptic motion in terms of these values (vol. I, ch. IV). This would be the motion of the planet if the resistance of the medium was absent at the moment t_0. Such a motion is called osculatory (i.e. touching) relative to the motion under consideration. The elements p_0, e_0 and π_0 corresponding to this orbit are called osculating elements for moment t_0.

Let us derive expressions for the osculating elements as functions of v. We note that, in the moment t_0 the quantities

$$H = r^2\dot{v}, \quad u, \quad \frac{du}{dv} = \dot{u}\,\dot{v}^{-1}$$

should have the same values for both the real and osculating motions. For the osculatory (elliptic) motion we have

$$H = k\sqrt{p_0},$$

whereas, for the motion in a resisting medium, H will be given by equation (11). Hence $k^2 p_0 = (h - \alpha v_0)^2$,

$$k^2 h^{-2} = p_0^{-1}(1 - 2\alpha h^{-1} v_0). \qquad (14)$$

Substituting here the values of u and $\frac{du}{dt}$ at $v = v_0$, given by equations (12) and (13), we obtain

$$k^2 h^{-2}[1 - 2\alpha h^{-1} v_0 + \varepsilon\cos(v_0 - \omega)] = p_0^{-1}[1 + e_0\cos(v_0 - \pi_0)]$$
$$k^2 h^{-1}[2\alpha h^{-1} - \varepsilon\sin(v_0 - \omega)] = p_0^{-1}[-e_0\sin(v_0 - \pi_0)].$$

Taking equation (14) into account, and limiting ourselves to the first powers of α, we obtain

$$e_0 \cos(v_0 - \pi_0) = (1 - 2\alpha h^{-1} v_0)\,\varepsilon \cos(v_0 - \omega)$$
$$e_0 \sin(v_0 - \pi_0) = (1 - 2\alpha h^{-1} v_0)\,\varepsilon \sin(v_0 - \omega) - 2\alpha h^{-1}.$$

Combining these equations, we finally obtain

$$e_0 \cos(\pi_0 - \omega) = \varepsilon - 2\alpha h^{-1} \varepsilon v_0 - 2\alpha h^{-1} \sin(v_0 - \omega)$$
$$e_0 \sin(\pi_0 - \omega) = 2\alpha h^{-1} \varepsilon \cos(v_0 - \omega).$$

When $\alpha = 0$, these relations become

$$e_0 = \varepsilon, \quad \pi_0 = \omega,$$

Therefore, within accepted accuracy limits, the above relations may be replaced by

$$e_0 = \varepsilon - 2\alpha h^{-1} [\varepsilon v_0 + \sin(v_0 - \omega)]$$
$$\pi_0 = \omega + 2\alpha h^{-1} \varepsilon \cos(v_0 - \omega).$$

These formulae show that the peribelion longitude π_0 of the osculating orbit is a periodic function of v_0 and consequently of t_0. The element e_0 will vary not only periodically but also secularly, owing to the presence of a term proportional to v_0.

Increasing v_0 by 2π, the eccentricity e_0 is increased by

$$\Delta e = -4\pi\alpha h^{-1} \varepsilon = -4\pi\alpha h^{-1} e_0. \qquad (15)$$

In other words, in the presence of a resisting medium acting according to the above mentioned law, the eccentricity will decrease after each revolution of the planet by a quality equal to $4\pi\alpha h^{-1}$.

In the same way we can obtain from equation (14) stating that

$$p_0 = k^{-2} h^2 (1 - 2\alpha h^{-1} v_0).$$

that after each revolution of the planet the parameter p_o is changed by

$$\Delta p = -4\pi\alpha k^{-2} h.$$

since

$$p_0 = a_0 (1 - e_0^2),$$

then, assuming that these elements are infinitesimal, we obtain

$$\frac{\Delta p}{p_0} = \frac{\Delta a}{a_0} - \frac{2e_0 \Delta e}{1 - e_0^2}.$$

Therefore

$$\frac{\Delta a}{a_0} = -\frac{4\pi\alpha}{h} \frac{1 + e_0^2}{1 - e_0^2}.$$

Concluding, let us find the corresponding variation of the average daily motion defined by

$$n_0 = k a_0^{-\frac{3}{2}}.$$

We obtain

$$\frac{\Delta n}{n_0} = -\frac{3}{2} \frac{\Delta a}{a_0} = \frac{6\pi\alpha}{h} \frac{1 + e_0^2}{1 - e_0^2}. \tag{16}$$

Thus, if a planet or comet moves in a medium whose resistance is linearly proportional to the velocity and inversly proportional to the square of the distance from the sun, then the eccentricity of the osculating orbit decreases and the average daily motion increases. The magnitudes of these changes are given in the first approximation by equations (15) and (16).

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CHAPTER III

THE METHOD OF VARIATION OF ARBITRARY CONSTANTS

10. The osculating elements

Let us denote by x, y and z the heliocentric elliptic coordinates of a planet (or comet), P, having mass m. If the only force acting on this planet is the gravitational force of the sun (whose mass is set equal unity), then the equations of motion, derived in § 3, are given by

$$\begin{aligned} \ddot{x} + k^2(1+m)xr^{-3} &= 0 \\ \ddot{y} + k^2(1+m)yr^{-3} &= 0 \\ \ddot{z} + k^2(1+m)zr^{-3} &= 0. \end{aligned} \qquad (1)$$

In order to simplify, we shall replace the term k^2 (1 + m) by k^2. Due to the fact that the factor (1 + m) is always accompanied by k^2, it can always be included when necessary.

If a force, mF, having components mF_x, mFy and mF_z, acts on the planet in addition to the sun's gravitational force, equation (1) should then be replaced by the following equations of motion

$$\begin{aligned} \ddot{x} + k^2xr^{-3} &= F_x \\ \ddot{y} + k^2yr^{-3} &= F_y \\ \ddot{z} + k^2zr^{-3} &= F_z . \end{aligned} \qquad (2)$$

The motion determined by equations (1) is called the uperturbed or Kepler motion, whilst the motion described by equations (2) is called the perturbed motion. In the perturbation theory, one usually deals with motion along an approximately elleptic orbit. This is why the unperturbed motion is sometimes called the elleptic motion.

The complete solution of equation (1) is well known. In order to be precise we will limit ourselves to a motion along an ellipse. We then express the solution in the following way:

$$M = n(t - t_0) + M_0 \qquad (3)$$

$$n = k a^{-\frac{3}{2}} \qquad (4)$$

$$E - e \sin E = M \qquad (5)$$

$$r = a(1 - e\cos E) \qquad (6)$$

$$\operatorname{tg}\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2} \qquad (7)$$

$$u = v + \omega$$

$$\left.\begin{aligned} x &= r(\cos u\cos\Omega - \sin u\sin\Omega\cos i)\\ y &= r(\cos u\sin\Omega + \sin u\cos\Omega\cos i)\\ z &= r\sin u\sin i, \end{aligned}\right| \qquad (8)$$

These equations express the unknown coordinates x, y and z in functions of time and six arbitrary constants a, e, M_o, ω, Ω and i.

A similar integration of the equations of perturbed motions, eqs. (2), in terms of known functions is not possible. Therefore, one has to solve these equations in a different manner. One often makes use of the fact that the perturbing acceleration is in most cases considerably less than the acceleration caused by the gravitation of the sun. One can then study the perturbed motion using the method of successive approximations. The unperturbed motion is taken as the first approximation, then, by adding corrections ("perturbations" or "inequalities") to it, one gradually approaches the correct description of the real motion. The application of this method simplifies essentially the appropriate choice of thefunctions of time that determine the motion. It usually happens that it is more useful to use instead of the unknown functions x, y and z, other quantities that can determine the position of the planet. In particular, the osculating elements of the orbit can be used for this purpose.

Equations (3) - (8) giving

$$\left.\begin{aligned} x &= f_1(t, a, e, M_0, \omega, \Omega, i)\\ y &= f_2(t, a, e, M_0, \omega, \Omega, i)\\ z &= f_3(t, a, e, M_0, \omega, \Omega, i), \end{aligned}\right| \qquad (9)$$

represent an elliptical motion when the quantities a, e, ..., i are considered to be constant numbers. However, these equations can represent any arbitrary types of motion when the quantities a, e, ..., i are considered to be properly chosen functions of time. The functions a(t), e(t), ..., i(t) representing the perturbed motion in equation (9), are called the instantaneous elements. The totality of these elements determine the instantaneous orbit of the planet P. Once the instantaneous orbit is known, one can evaluate the coordinates of P for any subsequent moment using the formulae that describe the elliptical motion.

We have only the conditions, given by equations (9), to determine the six functions a(t), e(t), ..., i(t). Thus, we require that these functions satisfy another three supplementary conditions. It is required that not only the coordinates but also their derivatives x, y and z should be expressed in terms of the instantaneous elements by the expressions obtained for the elleptical motion. These conditions can easily be obtained from equations (3)-(8) in the form of

$$\dot{r} = \frac{ke \sin v}{\sqrt{p}} \tag{10}$$

$$\dot{v} = \frac{k\sqrt{p}}{r^2} \tag{11}$$

$$\left.\begin{aligned} \dot{x} &= \dot{r}r^{-1}x + r\dot{v}(-\sin u \cos\Omega - \cos u \sin\Omega \cos i) \\ \dot{y} &= \dot{r}r^{-1}y + r\dot{v}(-\sin u \sin\Omega + \cos u \cos\Omega \cos i) \\ \dot{z} &= \dot{r}r^{-1}z + r\dot{v}\cos u \sin i, \end{aligned}\right\} \tag{12}$$

where

$$u = v + \omega, \qquad p = a(1 - e^2).$$

Equations (9) and (12) define the quantities a(t), e(t), ... in functions of time. We call these functions the osculating elements and the corresponding orbit, which continuously changes its direction and form, the osculating orbit. The solution of equations (9) and (12) relative to the elements is given in Vol. 1 Ch. IV. One sees here that these equations have one and only one solution.

We see that we can apply the equations of the unperturbed motions to express the osculating elements for any moment t in terms of the values of x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$ in this moment. Therefore, the osculating elements can be interpreted as the elements of that unperturbed motion which would replace the perturbed motion if the perturbing acceleration vanished at this moment.

In the following, we recall the formulae which lead to the solution of equations (9) and (12) relative to the elements. From the following relations

$$\begin{aligned} k\sqrt{p}\,\sin i \sin \Omega &= y\dot{z} - z\dot{y} \\ k\sqrt{p}\,\sin i \cos \Omega &= x\dot{z} - z\dot{x} \\ k\sqrt{p}\cos i &= x\dot{y} - y\dot{x} \end{aligned} \tag{13}$$

we find the parameter p and the longitude of node Ω, together with the slope of the orbit i. The kinetic energy integral

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = k^2\left(\frac{2}{r} - \frac{1}{a}\right) \tag{14}$$

gives us the semi-major axis a and allows us to find the eccentricity e from the following relation

$$p = a(1 - e^2), \tag{15}$$

We obtain the true anomaly v from equation (10) and then find the perihelion distance from the node ω using the relations

$$\begin{aligned} r\sin(v + \omega) &= z \operatorname{cosec} i \\ r\cos(v + \omega) &= x\cos\Omega + y\sin\Omega, \end{aligned} \tag{16}$$

which can easily be obtained from equations (8). Finally, we find the average anomaly of the epoch M_o using equations (7), (5) and (3).

11. Differential equations for the determination of the osculating elements

In the previous section we have seen that, in order to study the motion of planet P, it is possible to use instead of coordinates x, y and z, the six elements a, e, M_0, ω, Ω and i. This change of variables is useful because the elements, which remain constant during the unperturbed motion, slowly vary during the perturbed motion, at least if the perturbing acceleration is small as compared to the acceleration produced by the sun. For this reason, the determination of elements a,e, ... using the method of successive approximations is more convenient than the determination of coordinates x, y and z.

Let us now derive the differential equations which determine elements a, e, ..., i. For this purpose, we substitute in equations (2), and rewrite them as

$$\left.\begin{aligned} \frac{dx}{dt} &= \dot{x}, \quad \frac{dy}{dt} = \dot{y}, \quad \frac{dz}{dt} = \dot{z} \\ \frac{d\dot{x}}{dt} &= -k^2 x r^{-3} + F_x \\ \frac{d\dot{y}}{dt} &= -k^2 y r^{-3} + F_y \\ \frac{d\dot{z}}{dt} &= -k^2 z r^{-3} + F_z \end{aligned}\right\} \qquad (17)$$

the expressions, given by Eqs. (9) and (12), that express x, y, z, $\dot{x}$, $\dot{y}$ and $\dot{z}$ in terms of the new unknowns a, e, M_0, ω, Ω and i. However, direct substitution will lead us to a series of complicated calculations and we therefore choose an indirect method which leads us more easily to our target.

Let us assume that the following relation holds

$$\Psi(a, e, \ldots, i, x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = 0, \tag{18}$$

This relation can be deduced from equations (3)-(8), (10), (11) and (12). Differentiating equation (18) with respect to time gives

$$\frac{\partial\Psi}{\partial a}\frac{da}{dt} + \ldots + \frac{\partial\Psi}{\partial i}\frac{di}{dt} + \frac{\partial\Psi}{\partial x}\frac{dx}{dt} + \ldots +$$
$$+ \frac{\partial\Psi}{\partial \dot{x}}\frac{d\dot{x}}{dt} + \ldots + \frac{\partial\Psi}{\partial t} = 0 \tag{19}$$

We now consider that an unperturbed motion results when the perturbing accleration F vanishes at moment t. We denote the corresponding coordinates and components of velocity by $\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}$ and $\dot{\zeta}$. The differential equation (1) describing this motion can be written, in anology to equations (7), as

$$\frac{d\xi}{dt} = \dot{\xi}, \quad \frac{d\eta}{dt} = \dot{\eta}, \quad \frac{d\zeta}{dt} = \dot{\zeta},$$
$$\frac{d\dot{\xi}}{dt} = -k^2\xi\rho^{-3}, \quad \frac{d\dot{\eta}}{dt} = -k^2\eta\rho^{-3}, \quad \frac{d\dot{\zeta}}{dt} = -k^2\zeta\rho^{-3},$$

where

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2.$$

For the moment t under consideration, we have

$$x = \xi, \quad y = \eta, \quad z = \zeta,$$
$$\dot{x} = \dot{\xi}, \quad \dot{y} = \dot{\eta}, \quad \dot{z} = \dot{\zeta},$$

and hence

$$\left.\begin{aligned} &\frac{dx}{dt} = \frac{d\xi}{dt}, \quad \frac{dy}{dt} = \frac{d\eta}{dt}, \quad \frac{dz}{dt} = \frac{d\zeta}{dt}, \\ &\frac{d\dot{x}}{dt} = \frac{d\dot{\xi}}{dt} + F_x, \quad \frac{d\dot{y}}{dt} = \frac{d\dot{\eta}}{dt} + F_y, \quad \frac{d\dot{z}}{dt} = \frac{d\dot{\zeta}}{dt} + F_z \end{aligned}\right\} \tag{20}$$

We now return back to equation (18), which evidently takes part in the unpeturbed motion. Differentiating this equation, we obtain for the case under consideration

$$\frac{\partial\Psi}{\partial x}\frac{dx}{dt}+\ldots+\frac{\partial\Psi}{\partial\dot{x}}\frac{d\dot{x}}{dt}+\ldots+\frac{\partial\Psi}{\partial t}=0.$$

Subtracting this equality, term by term from the equality given by Eq. (19), we finally obtain

$$\frac{\partial\Psi}{\partial a}\frac{da}{dt}+\ldots+\frac{\partial\Psi}{\partial i}\frac{di}{dt}+\frac{\partial\Psi}{\partial x}F_x+\frac{\partial\Psi}{\partial y}F_y+\frac{\partial\Psi}{\partial z}F_z=0. \qquad (21)$$

and we thus arrive to the following important conclusion:

Any relation, Eq. (18), between the elements, coordinates and components of velocity leads to a relation of the type of Eq. (21) between the derivatives of the elements and thecomponents of the perturbing acceleration.

The transition from equation (18) to equation (21) will, in short, be called the basic operation.

The longitudie of the node and the slope

Let us apply the basic operation to equations (13) and for abbreviation the following notation is introduced

$$\frac{1}{k\sqrt{p}}F_x=F'_x,\quad \frac{1}{k\sqrt{p}}F_y=F'_y,\quad \frac{1}{k\sqrt{p}}F_z=F'_z.$$

We then obtain

$$\begin{aligned}
&\frac{1}{2}p^{-1}\sin i\sin\Omega\frac{dp}{dt}+\sin i\cos\Omega\frac{d\Omega}{dt}+\cos i\sin\Omega\frac{di}{dt}=yF'_z-zF'_y\\
&\frac{1}{2}p^{-1}\sin i\cos\Omega\frac{dp}{dt}-\sin i\sin\Omega\frac{d\Omega}{dt}+\cos i\cos\Omega\frac{di}{dt}=xF'_z-zF'_x\\
&\qquad\frac{1}{2}p^{-1}\cos i\frac{dp}{dt}\qquad\qquad-\sin i\frac{di}{dt}=xF'_y-yF'_x.
\end{aligned}$$

Taking equations (8) into consideration, we find

$$\frac{dp}{dt} = 2pr\,[F'_x(-\sin u\cos\Omega - \cos u\sin\Omega\cos i) +$$
$$+ F'_y(-\sin u\sin\Omega + \cos u\cos\Omega\cos i) + F'_z\cos u\sin i]$$
$$\sin i\,\frac{d\Omega}{dt} = r\sin u\,[F'_x\sin\Omega\sin i - F'_y\cos\Omega\sin i + F'_z\cos i]$$
$$\frac{di}{dt} = r\cos u\,[F'_x\sin\Omega\sin i - F'_y\cos\Omega\sin i + F'_z\cos i].$$

In order to simplify these expressions, let us introduce the components of the perturbing acceleration along the radius vector, in the direction perpendicular to the radius vector in the plane of the osculating orbit and along the normal to the plane of the orbit. Denoting these components by S, T and W and assuming, as before,

$$\frac{1}{k\sqrt{p}}\,S = S', \quad \frac{1}{k\sqrt{p}}\,T = T', \quad \frac{1}{k\sqrt{p}}\,W = W',$$

we obtain

$$\left.\begin{aligned} S' &= F'_x(\cos u\cos\Omega - \sin u\sin\Omega\cos i) \\ &\quad + F'_y(\cos u\sin\Omega + \sin u\cos\Omega\cos i) + F'_z\sin u\sin i \\ T' &= F'_x(-\sin u\cos\Omega - \cos u\sin\Omega\cos i) + \\ &\quad + F'_y(-\sin u\sin\Omega + \cos u\cos\Omega\cos i) + F'_z\cos u\sin i \\ W' &= F'_x\sin\Omega\sin i - F'_y\cos\Omega\sin i + F'_z\cos i. \end{aligned}\right\} \qquad (22)$$

The coefficients that multiply F^1_x, F'_y and F'_z in the expression of S' are evidently equal to xr^{-1}, yr^{-1} and zr^{-1}.
The corresponding coefficients in the expression of T' are obtained from the previous ones by the replacement of u by u + 90° and, we therefore finally obtain

$$\frac{dp}{dt} = 2pr\,T' \qquad (23)$$

$$\sin i\,\frac{d\Omega}{dt} = r \sin u\,W' \qquad (24)$$

$$\frac{di}{dt} = r \cos u\,W'. \qquad (25)$$

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The semimajor axis and the eccentricity

Applying the basic operation to the kinetic energy integral, given by Eq. (14), we obtain

$$\cdots \frac{da}{dt} = 2\dot{x}F_x + 2\dot{y}F_y + 2\dot{z}F_z.$$

Taking equations (10), (11), (12) and (22) into account, we find that

$$\frac{da}{dt} = 2a^2e \sin v\,S' + 2a^2pr^{-1}\,T'. \qquad (26)$$

Substituting in euqation (15) we obtain

$$2ae\frac{de}{dt} = (1 - e^2)\frac{da}{dt} - \frac{dp}{dt}$$

Taking into account equation (6) as well as the following equation

$$pr^{-1} = 1 + e \cos v, \qquad (27)$$

which follows from the equation of the orbit, we obtain

$$\frac{de}{dt} = p \sin v\,S' + p\,(\cos v + \cos E)\,T'. \qquad (28)$$

The perihelion distance from the node

We now apply the basic operation to the second set of equations and we obtain

$$\cdots r \sin u \left[\left(\frac{dv}{dt}\right) + \frac{d\omega}{dt}\right] \quad (\quad x \sin\Omega \quad y \cos\Omega)\frac{d\Omega}{dt}.$$

Where (dv/dt) denotes the derivative corresponding to the dependence

of v on time that enters only through the osculating elements[1].
Substituting here expressions (8) for x and y, we obtain.

$$\frac{d\omega}{dt} = -\left(\frac{dv}{dt}\right) - \cos i\,\frac{d\Omega}{dt}$$

In order to find (dv/dt), we turn to equation (10).
Excluding the eccentricity e from this equation, we rewrite it as

$$\dot{r}\operatorname{ctg} v = \frac{k}{\sqrt{p}}\,e\cos v,$$

or, by using equation (27), as

$$\dot{r}\operatorname{ctg} v = \frac{k\sqrt{p}}{r} - \frac{k}{\sqrt{p}}.$$

We apply the basic operation to this equation and due to the fact the $\dot{r}$ is defined by the factor

$$r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z},$$

we obtain

$$S\operatorname{ctg} v - \frac{\dot{r}}{\sin^2 v}\left(\frac{dv}{dt}\right) = \frac{k}{2\sqrt{pr}}\left(1 + \frac{r}{p}\right)\frac{dp}{dt}.$$

Using equation (23), we obtain

$$e\left(\frac{dv}{dt}\right) = p\cos v\,S - (r+p)\sin v\,T.$$

Consequently

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(1) It is necessary to note that, in contrast to the radius vector r, the true anomaly v cannot be considered as a coordinate. In fact, v not only depends on x, y and z but also on x, y and z.

$$e\frac{d\omega}{dt} = -p\cos v\, S' + (r+p)\sin v\, T' - e\cos i\frac{d\Omega}{dt}. \tag{29}$$

<u>The mean anomaly of the epoch</u>

Let us now apply the basic operation to equations (5) and (6). We obtain

$$\left(\frac{dM}{dt}\right) = (1 - e\cos E)\left(\frac{dE}{dt}\right) - \sin E\frac{de}{dt}$$
$$\frac{r}{a}\frac{da}{dt} - a\cos E\frac{de}{dt} + ae\sin E\left(\frac{dE}{dt}\right) = 0.$$

Here, we denote by $(\frac{dM}{dt})$ and $(\frac{dE}{dt})$ the derivatives of the parts of M and E which depend on the variations of the osculating elements. Eliminating $(\frac{dE}{dt})$, we obtain

$$\frac{e}{\sqrt{1-e^2}}\left(\frac{dM}{dt}\right) = \operatorname{ctg} v\frac{de}{dt} - \frac{r}{a^2\sin v}\frac{da}{dt}.$$

Substituting here the values of the derivatives of the eccentricity and semimajor axis, obtained above, we find

$$\frac{e}{\sqrt{1-e^2}}\left(\frac{dM}{dt}\right) = (p\cos v - 2er)S' + \frac{p}{\sin v}(\cos^2 v + \cos v\cos E - 2)T'.$$

In order to simplify the coefficient T', we use the following equations

$$\begin{aligned} r\sin v &= a\sqrt{1-e^2}\sin E \\ r\cos v &= a(\cos E - e). \end{aligned} \tag{30}$$

Substituting equation (6) in the second set of these equations, we obtain

$$r\cos v\cos E = a\cos^2 E - ae\cos E = r - a\sin^2 E.$$

Elliminating sin E by means of equations (30), we obtain

$$\cos v\cos E = 1 - \frac{r}{p}\sin^2 v.$$

Using these relations, we finally obtain

$$\frac{e}{\sqrt{1-e^2}}\left(\frac{dM}{dt}\right) = -(p\cos v - 2er)\, S - (r+p)\sin v\, T.$$

We now apply our basic operation to equation (3). We find

$$\left(\frac{dM}{dt}\right) = \frac{dM_0}{dt} + (t-t_0)\frac{dn}{dt}. \qquad (31)$$

The complete differentiation of the same equation gives

$$\frac{dM}{dt} = \frac{dM_0}{dt} + (t-t_0)\frac{dn}{dt} + n.$$

Therefore

$$\frac{dM}{dt} = \left(\frac{dM}{dt}\right) + n.$$

Integrating this equation from t_o to t_1 we obtain

$$M(t) = M_0(t_0) + \int_{t_0}^{t}\left(\frac{dM}{dt}\right)dt + \int_{t_0}^{t} n\, dt,$$

or, expressed as

$$M(t) = M_0(t) + \int_{t_0}^{t} n\, dt, \qquad (32)$$

where

$$M_0(t) = M_0(t_0) + \int_{t_0}^{t}\left(\frac{dM}{dt}\right)dt. \qquad (33)$$

The osculating elements $M_o(t)$ and $n(t)$ are functions of time. We use he notation $M_o(t_o)$ to stress that the quantity $M_o(t)$ corresponds to the epoch of osculation t_o.

If we evaluate the position of the planet for an arbitrary moment t, we can find the corresponding mean anomaly $M(t)$ using the equation of elliptical motion

$$M(t) = M_0(t) + n(t)(t - t_0) \qquad (34)$$

It follows from equation (31) that the derivative of the osculating element M is given by

$$\frac{dM_0}{dt} = \left(\frac{dM}{dt}\right) - (t - t_0)\frac{dn}{dt},$$

where, on the basis of equation (4),

$$\frac{dn}{dt} = -\frac{3}{2}\frac{k}{a^2\sqrt{a}}\frac{da}{dt} = -\frac{3}{2}\frac{n}{a}\frac{da}{dt}$$

or

$$\frac{dn}{dt} = -3nae \sin v\, S' - 3napr^{-1}T'.$$

We see that thederivative of the element $M_0(t)$ includes terms relative to time. Hence, this element varies rapidly regardless to the degree of smallness of the S' and T' factors. This situation leads to a great deal of difficulties in the evaluation of perturbations. It obliges us to use equation (32) and not equation (34) in order to evaluate the mean anomaly. The function M_0 involved in this formula is defined on the basis of equation (33) by the following relation

$$\frac{dM_0}{dt} = \frac{\sqrt{1-e^2}}{e}(p\cos v - 2er)S' - \frac{\sqrt{1-e^2}}{e}(r+p)\sin v\, T'. \qquad (35)$$

Equations (32) and (35) are in practice commonly used. To simplify their final forms, we shall denote $\overline{M}_0$ simply by M_0 as far as this does not cause any confusion, and once and for all agree to use equation (32) to evaluate the mean anomaly.

12. Comparison between the different formulae

In most practical cases, the slopes of the orbits, i, are very small. In this case, some of the formulae derived in the previous two sections are preferably replaced by others.

Instead of the peribelion distance from the node, we introduce here the perihelion longitude π by the following equation.

$$\pi = \Omega + \omega.$$

Then, equation (29) gives

$$e\frac{d\pi}{dt} = 2e\sin^2\frac{i}{2}\frac{d\Omega}{dt} - p\cos v\, S + (r + p)\sin v\, T.$$

Using equation (24), we transform the first term on the right-hand side of the previous equation into

$$2r\sin u\frac{\sin^2\frac{i}{2}}{\sin i}W = r\sin u \operatorname{tg}\frac{i}{2}W.$$

This term can only decrease if the value of i decreases, whereas the corresponding term in formula (29) will still be large.

Furthermore, we introduce the mean longitude in the orbit by

$$\lambda = \pi + M = \Omega + \omega + M,$$

which will simply be called the mean longitude. We denote the mean longitude corresponding to the initial epoch t_o by λ_o, so that

$$\lambda_0 = \pi + M_0,$$
$$\lambda = \lambda_0 + n(t - t_0).$$

Taking into account equation (32), we obtain

$$\lambda = \varepsilon + \int_{t_0}^{t} n\,dt, \qquad (30)$$

where

$$\varepsilon = \pi + M_0(t).$$

Hence

$$\frac{d\varepsilon}{dt} = \frac{d\pi}{dt} + \frac{dM_0}{dt}.$$

The quantity ε will also be called the average longitude of the epoch, hoping that this will not lead to any misunderstanding.

We now combine together all the differential equations which determine the osculating elements:

$$
\left.
\begin{aligned}
\frac{da}{dt} &= 2a^2 e \sin v\, S' + 2a^2 p r^{-1} T' \\
\frac{de}{dt} &= p \sin v\, S' + p (\cos v + \cos E)\, T' \\
\sin i \frac{d\Omega}{dt} &= r \sin u\, W' \\
\frac{di}{dt} &= r \cos u\, W' \\
e \frac{d\pi}{dt} &= 2e \sin^2 \frac{i}{2} \frac{d\Omega}{dt} - p \cos v\, S' + (r + p) \sin v\, T' \\
\frac{d\varepsilon}{dt} &= 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt} - 2r \sqrt{1 - e^2}\, S' + \frac{e^2}{1 + \sqrt{1 - e^2}} [- p \cos v\, S' \\
&\quad + (r + p) \sin v\, T'],
\end{aligned}
\right\} \quad (37)
$$

where $S' = \frac{1}{k\sqrt{p}} S$, $T' = \frac{1}{k\sqrt{p}} T$, $W' = \frac{1}{k\sqrt{p}} W$,

and S , T and W denote the components of the perturbing acceleration. Since the average longitude is given by equation (36), the following equation should be added to the previous ones

$$
\frac{dn}{dt} = -3nae \sin v\, S' - 3nap r^{-1} T' \qquad (38)
$$

The integration of this system of differential equations gives the values of the osculating elements in moment t. The position of the planet in this moment is obtained by the usual formulae. Starting with equation (36) and

$$\left.\begin{aligned} E - e\sin E &= \lambda - \pi \\ r &= a(1 - e\cos E) \\ \mathrm{tg}\,\frac{v}{2} &= \sqrt{\frac{1+e}{1-e}}\,\mathrm{tg}\,\frac{E}{2} \\ u &= v + \pi - \Omega \end{aligned}\right| \qquad (39)$$

one finds r and u. Then, using equations (8), one obtains x, y and z.

13. The Lagrange equations

In the previous section we did not impose any limitations on the perturbing acceleration F. We now assume that the accleration is caused by a force for which a potential exists. In other words, we assume that there exists a function R such that

$$F_x = \frac{\partial R}{\partial x}, \qquad F_y = \frac{\partial R}{\partial y}, \qquad F_z = \frac{\partial R}{\partial z}$$

For example, in equation (1) of Chapter I, the perturbation to the motion of one of the planets, caused by the presence of the others, is expressed, by the perturbation function.

Let us transform the basic equations (37) so that they include the partial derivatives of the function R instead of the components of the perturbing acceleration S, T and W. For an arbitrary element a, the following relations hold

$$\begin{aligned} \frac{\partial R}{\partial a} &= \frac{\partial R}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial a} \\ &= F_x\frac{\partial x}{\partial a} + F_y\frac{\partial y}{\partial a} + F_z\frac{\partial z}{\partial a}, \end{aligned}$$

where equations (22) expressed as

$$
\begin{aligned}
F_x &= S(\cos u \cos \Omega - \sin u \cos \Omega \cos i) + \\
&\quad + T(-\sin u \cos \Omega - \cos u \sin \Omega \cos i) + W \sin \Omega \sin i \\
F_y &= S(\cos u \sin \Omega + \sin u \cos \Omega \cos i) + \\
&\quad + T(-\sin u \sin \Omega + \cos u \cos \Omega \cos i) - W \cos \Omega \sin i \\
F_z &= S \sin u \sin i + T \cos u \sin i + W \cos i.
\end{aligned}
$$

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can be used to express F_x, F_y and F_z in terms of S, T and W. The evaluation of the derivatives of the coordinates with respect to the elements, $\frac{\partial x}{\partial a}$, $\frac{\partial y}{\partial a}$, ..., $\frac{\partial x}{\partial \varepsilon}$, ..., is straightforward. We first find the derivatives of r and u, namely

$$
\left.
\begin{aligned}
\frac{\partial r}{\partial a} &= \frac{r}{a}, & \frac{\partial u}{\partial a} &= 0 \\
\frac{\partial r}{\partial e} &= -a \cos v, & \frac{\partial u}{\partial e} &= \frac{2 + e \cos v}{1 - e^2} \sin v = \frac{\sin v}{1 - e^2}\left(1 + \frac{r}{p}\right) \\
\frac{\partial r}{\partial \varepsilon} &= \frac{ae}{\sqrt{1 - e^2}} \sin v; & \frac{\partial u}{\partial \varepsilon} &= \frac{a^2 \sqrt{1 - e^2}}{r^2}.
\end{aligned}
\right| \quad (10)
$$

We then differentiate equations (8) with respect to the elements. In doing so, we should remember that each of the coordinates depends explicity on Ω, together with the relationship $u = v + \pi - \Omega$. Similarly, whilst differentiating with respect to π, we must take into account that u depends explicity on π and to v, since v is a function of

$$M = \int_{t_0}^{t} n\,dt + \varepsilon - \pi.$$

Therefore,

$$\frac{\partial}{\partial \pi} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}.$$

We finally obtain

$$\frac{\partial R}{\partial a} = nar\sqrt{1-e^2}\,S'$$

$$\frac{\partial R}{\partial e} = na^2\{-p\cos v\,S' + (r+p)\sin v\,T'\}$$

$$\frac{\partial R}{\partial i} = na^2\sqrt{1-e^2}\,r\sin u\,W'$$

$$\frac{\partial R}{\partial \Omega} = 2kr\sqrt{p}\sin^2\frac{i}{2}\,T' - na^2\sqrt{1-e^2}\sin i\,r\cos u\,W'$$

$$\frac{\partial R}{\partial \varepsilon} = na^2(e\sin v\,S' + pr^{-1}\,T')$$

$$\frac{\partial R}{\partial \pi} = -\frac{\partial R}{\partial \varepsilon} + kr\sqrt{p}\,T'.$$

It remains to substitute the values of S', T' and W' obtained by these equations into equations (37). In equations (37) the following combinations are present

$$r\sqrt{1-e^2}\,S' = \frac{1}{na}\frac{\partial R}{\partial a}$$

$$kr\sqrt{p}\,T' = \frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \varepsilon}$$

$$r\sin u\,W' = \frac{1}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}$$

$$r\cos u\,W' = -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \varepsilon}\right)$$

as well as the following combinations

$$-p\cos v\,S' + (r+p)\sin v\,T' = \frac{1}{na^2}\frac{\partial R}{\partial e}$$

$$e\sin v\,S' + pr^{-1}\,T' = \frac{1}{na^2}\frac{\partial R}{\partial \varepsilon}$$

Substituting these expressions into equations (37) and (38), we obtain after some manipulations

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \varepsilon} \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \pi} - \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{1}{na^2}\frac{\partial R}{\partial \varepsilon} \\
\frac{di}{dt} &= -\frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega} - \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \varepsilon}\right) \\
\frac{d\Omega}{dt} &= \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} \\
\frac{d\pi}{dt} &= \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e} \\
\frac{d\varepsilon}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{1}{na^2}\frac{\partial R}{\partial e} \\
\frac{dn}{dt} &= -\frac{3}{a^2}\frac{\partial R}{\partial \varepsilon}.
\end{aligned}\right\} \tag{41}$$

Combining the first and second equations, we obtain

$$\frac{dp}{dt} = -\frac{2\sqrt{1-e^2}}{na}\left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \varepsilon}\right), \tag{42}$$

which also follows from equations (23). Subsequently, we shall call equations (41), the Lagrange equations.

It is important to note that, in evaluating derivative $\frac{\partial R}{\partial a}$ involved at the end of equations (41), we ignored equation (4) and only considered the explicit dependence of R on a. This is the method by which equations (40) were derived, and based on these equations, the entire present deduction was developed.

In conclusion, we note that the Lagrange equations obtained here have the following properties:

(1) Time enters the Lagrange equations only through the derivatives of the perturbation function R.

(2) The elements of the orbit are divided into two groups, one consisting

of a, e and i and the other of Ω, π and ϵ. The differential equations, which determine the elements of one of these groups, include the partial derivatives of R with respect to only the elements of the other group.

(3) Let α and β be two elements belonging to different groups. If $\frac{d\alpha}{dt}$ contains $\frac{\partial R}{\partial \beta}$, then $\frac{d\beta}{dt}$ will contain $\frac{\partial R}{\partial \alpha}$, where the coefficients of $\frac{\partial R}{\partial \alpha}$ and $\frac{\partial R}{\partial \beta}$ will be equal in magnitude but of different signs.

14. Another derivation of the Lagrange Equations

The differential equations (41), or the more general equations (37), are some of the corner-stones of celestial mechanics. Therefore, it is interesting to investigate all questions concerning these equations. We have just obtained an elementary and relatively simple derivation for the Legendre equations.[1] Another interesting method for the deduction of these equations was suggested by Lagrange, to whom we owe the development of the method of variation of arbitrary constants. In the following, we give a brief outline of the derivation suggested by Lagrange. We shall not carry out all the calculations since we have already obtained the final equations.

Following Lagrange, and in keeping with his notations, we consider the following system of equations

(1) A derivation of the Legendre equations, having a geometrical character, may be found in: S.L. Kazakov, The method of variation of arbitrary constants, scientific Transactions of Moscow University (Sposob variacej proizvolnyh postojannyh, Ucenye Zapiski Moskovskogo Universiteta) 1905 and in: A.I. Krylov, Sur la variation des orbits elliptiques des planets, Proceeding of the Academy of Science (Collection of Transactions) 1905 vol. IV. Izvestija Akademii nauk (Sobranie trudov) . It is shown in the latter paper that one has the right to think that Newton has obtained the above equations namely on the basis of these arguments. Newton, however, published only some theorems which have no direct relation to equations (37).

$$\frac{dx}{dt} - \frac{\partial\Omega}{\partial x'} - X' = 0, \qquad \frac{dx'}{dt} + \frac{\partial\Omega}{\partial x} + X = 0$$
$$\frac{dy}{dt} - \frac{\partial\Omega}{\partial y'} - Y' = 0, \qquad \frac{dy'}{dt} + \frac{\partial\Omega}{\partial y} + Y = 0, \tag{43}$$

Let the number of these equations be 2h. This should equal the number of the conjugate variables x, x', y, y', ... The quantities Ω, X, X', Y, Y', ... are functions of t, x, x', y, y', ... We assume that we are able to integrate the following equations

$$\frac{dx}{dt} - \frac{\partial\Omega}{\partial x'} = 0; \qquad \frac{dx'}{dt} + \frac{\partial\Omega}{\partial x} = 0$$
$$\frac{dy}{dt} - \frac{\partial\Omega}{\partial y'} = 0, \qquad \frac{dy'}{dt} + \frac{\partial\Omega}{\partial y} = 0, \tag{44}$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

These equations are obtained from equations (43) by equating all the supplementary functions X, X', Y, Y', ... to zero. Let the general solution of equations (44) be given by

$$x = \varphi_1(t, a, b, \ldots, g), \qquad x' = \psi_1(t, a, b, \ldots, g) \tag{45}$$
$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

which involves 2h arbitrary constants a, b, ..., g.

The expressions given by Eqs. (45) satisfy equations (43) only if the quantities a, b, ..., g are treated as functions of t. Let us find the differential equations that these functions should satisfy. We substitute the following equations

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial a}\frac{da}{dt} + \frac{\partial x}{\partial b}\frac{db}{dt} + \cdots$$
$$\frac{dx'}{dt} = \frac{\partial x'}{\partial t} + \frac{\partial x'}{\partial a}\frac{da}{dt} + \frac{\partial x'}{\partial b}\frac{db}{dt} + \cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$

into equations (14), and taking equations (44) into consideration, we obtain

$$
\begin{aligned}
&\frac{\partial x}{\partial a}\frac{da}{dt} + \frac{\partial x}{\partial b}\frac{db}{dt} + \ldots - X' = 0\\
&\frac{\partial y}{\partial a}\frac{da}{dt} + \frac{\partial y}{\partial b}\frac{db}{dt} + \ldots - Y' = 0\\
&\ldots\ldots\ldots\ldots\\
&\frac{\partial x'}{\partial a}\frac{da}{dt} + \frac{\partial x'}{\partial b}\frac{db}{dt} + \ldots + X = 0\\
&\frac{\partial y'}{\partial a}\frac{da}{dt} + \frac{\partial y'}{\partial b}\frac{db}{dt} + \ldots + Y = 0.\\
&\ldots\ldots\ldots\ldots
\end{aligned}
\tag{46}
$$

These are the required differential equations. Lagrange suggested that these equations could be written in a much simpler way. He introduced the so-called Lagrange brackets $[a,a]$, $[a,b]$ ... in the following way

$$[a,b] = \frac{\partial x}{\partial a}\frac{\partial x'}{\partial b} - \frac{\partial x}{\partial b}\frac{\partial x'}{\partial a} + \frac{\partial y}{\partial a}\frac{\partial y'}{\partial b} - \frac{\partial y}{\partial b}\frac{\partial y'}{\partial a} + \ldots$$

It is easy to see that

$$[a,a] = [b,b] = \ldots = [g,g] = 0 \tag{47}$$

$$[a,b] + [b,a] = 0. \tag{48}$$

Assuming that

$$R_a = X\frac{\partial x}{\partial a} + Y\frac{\partial y}{\partial a} + \ldots + X'\frac{\partial x'}{\partial a} + Y'\frac{\partial y'}{\partial a} + \ldots$$

$$\ldots\ldots\ldots\ldots$$

we can replace the system (46) by

$$
\left.
\begin{aligned}
&[a,a]\frac{da}{dt} + [a,b]\frac{db}{dt} + \ldots + [a,g]\frac{dg}{dt} + R_a = 0\\
&[b,a]\frac{da}{dt} + [b,b]\frac{db}{dt} + \ldots + [b,g]\frac{dg}{dt} + R_b = 0\\
&\ldots\ldots\ldots\ldots\\
&[g,a]\frac{da}{dt} + [g,b]\frac{db}{dt} + \ldots + [g,g]\frac{dg}{dt} + R_g = 0
\end{aligned}
\right|
\tag{49}
$$

In order to obtain the first equation, we multiply equations (46) in sequence by $-\frac{\partial x'}{\partial a}, -\frac{\partial y'}{\partial a}, \dots, +\frac{\partial x}{\partial a}, +\frac{\partial y}{\partial a}, \dots$ and add. The other equations are obtained in a similar manner.

In the light of the relations given by equations (47) and (48), equations (49) are simpler than equations (46). In particular, the third property of the Lagrange brackets is responsible for the simplicity of equations (49). When expressions (45) are substituted in a Lagrange bracket, the independent variable t is eliminated according to this property. In other words

$$\frac{\partial}{\partial t}[a, b] = 0 \tag{50}$$

To prove this, let us differentiate the bracket $[a,b]$ term by term. We obtain

$$\frac{\partial}{\partial t}[a, b] = \frac{\partial^2 x}{\partial a \partial t}\frac{\partial x'}{\partial b} + \frac{\partial x}{\partial a}\frac{\partial^2 x'}{\partial b \partial t} - \frac{\partial^2 x}{\partial b \partial t}\frac{\partial x'}{\partial a} - \frac{\partial x}{\partial b}\frac{\partial^2 x'}{\partial a \partial t} + \dots$$
$$= \frac{\partial}{\partial a}\left(\frac{\partial x}{\partial t}\frac{\partial x'}{\partial b} - \frac{\partial x'}{\partial t}\frac{\partial x}{\partial b}\right) - \frac{\partial}{\partial b}\left(\frac{\partial x}{\partial t}\frac{\partial x'}{\partial a} - \frac{\partial x'}{\partial t}\frac{\partial x}{\partial a}\right) + \dots$$

Taking equation (44) into account, we find

$$\frac{\partial}{\partial t}[a, b] = \frac{\partial}{\partial a}\left(\frac{\partial \Omega}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial \Omega}{\partial x'}\frac{\partial x'}{\partial b} + \frac{\partial \Omega}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial \Omega}{\partial y'}\frac{\partial y'}{\partial b} + \dots\right) -$$
$$- \frac{\partial}{\partial b}\left(\frac{\partial \Omega}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial \Omega}{\partial x'}\frac{\partial x'}{\partial a} + \frac{\partial \Omega}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial \Omega}{\partial y'}\frac{\partial y'}{\partial a} + \dots\right).$$

Hence

$$\frac{\partial}{\partial t}[a, b] = \frac{\partial^2 \Omega}{\partial a \partial b} - \frac{\partial^2 \Omega}{\partial b \partial a} = 0.$$

Equation (40) suggests that one can evaluate the Lagrange brackets at the value t that makes calculations quite simple.

The present method will be particularly simple when we take constants a, b, ..., g to be the initial values x_0 , y_0 , ..., x_0' , y_0', ... corresponding to the value to the independent variable. We immediately evaluate the Lagrange brackets at $t = t_0$ due to the property just shown. We obtain

$$[x_0, x_0'] = \frac{\partial x_0}{\partial x_0}\frac{\partial x_0'}{\partial x_0'} - \frac{\partial x_0}{\partial x_0'}\frac{\partial x_0'}{\partial x_0} + \frac{\partial y_0}{\partial x_0}\frac{\partial y_0'}{\partial x_0'} - \frac{\partial y_0}{\partial x_0'}\frac{\partial y_0'}{\partial x_0} + \dots = 1$$

$$[x_0, y_0] = [x_0, y_0'] = \dots = 0$$

In the same way, we obtain

$$[y_0, y_0'] = 1, \quad [y_0, x_0] = [y_0, x_0'] = \dots = 0,$$

and so on. In this manner, equations (49) will have the following form

$$\frac{dx_0}{dt} = +R_{x_0'}, \quad \frac{dx_0'}{dt} = -R_{x_0}$$

$$\frac{dy_0}{dt} = +R_{y_0'}, \quad \frac{dy_0'}{dt} = -R_{y_0}$$

However, these equations are not used in celestial mechanics because equations (45) become very complicated when the constants of integration are chosen according to the above method.

Let us consider the determinant

$$F = \begin{vmatrix} [a, a], & [a, b], & \dots, & [a, g] \\ [b, a], & [b, b], & \dots, & [b, g] \\ \cdot & \cdot & \cdot & \cdot \\ [g, a], & [g, b], & \dots, & [g, g] \end{vmatrix}$$

We construct the complementary determinant F', whose elements are the algebraic complements to the corresponding elements of F, devided by the value of the determinant F. This determinant is given by

$$F' = \begin{vmatrix} (a, a), & (a, b), & \dots, & (a, g) \\ (b, a), & (b, b), & \dots, & (b, g) \\ \dots & \dots & \dots & \dots \\ (g, a), & (g, b), & \dots, & (g, g) \end{vmatrix}.$$

The elements of the determinant F' are called Poisson brackets. Evidently

$$FF' = 1.$$

We can express equations (49) in terms of Poison brackets as follows

$$\begin{aligned} \frac{da}{dt} + (a, a) R'_a + (a, b) R'_b + \dots + (a, g) R'_g &= 0 \\ \frac{db}{dt} + (b, a) R'_a + (b, b) R'_b + \dots + (b, g) R'_g &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots & \\ \frac{dg}{dt} + (g, a) R'_a + (g, b) R'_b + \dots + (g, g) R'_g &= 0. \end{aligned}$$

We now apply the method of variation of arbitrary constants to the perturbation of the motion of a planet that is given by equations (17). We rewrite these equations as follows

$$\left.\begin{aligned} \frac{dx}{dt} &= x', & \frac{dx'}{dt} &= -k^2\frac{x}{r^3} + \frac{\partial R}{\partial x} \\ \frac{dy}{dt} &= y', & \frac{dy'}{dt} &= -k^2\frac{y}{r^3} + \frac{\partial R}{\partial y} \\ \frac{dz}{dt} &= z', & \frac{dz'}{dt} &= -k^2\frac{z}{r^3} + \frac{\partial R}{\partial z} \end{aligned}\right\} \qquad (51)$$

Comparing these equations with equations (43), we find here that

$$\Omega = \frac{1}{2}(x'^2 + y'^2 + z'^2) - \frac{k^2}{r}$$

$$X = \frac{\partial \Omega}{\partial x}, \quad Y = \frac{\partial \Omega}{\partial y}, \quad Z = \frac{\partial \Omega}{\partial z},$$

$$X' = 0, \quad Y' = 0, \quad Z' = 0,$$

and thus

$$R_a = \frac{\partial R}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial R}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial a} = \frac{\partial R}{\partial a} \tag{52}$$

. . . .

when R = 0, we obtain from equations (51) a system corresponding to equations (44) characterizing the uperturbed motion. In solving this latter system, we choose as the integration constants, the usual elements of the elliptic motion a, e, i, Ω, ω and M_0. After long but relatively simple manipulations, we obtain the corresponding Lagrange brackets, which are

$$[\omega, \Omega] = 0, \quad [\omega, i] = 0, \quad [\omega, M_0] = 0$$

$$[\Omega, i] = -na^2\sqrt{1-e^2}\,\sin i, \quad [\Omega, M_0] = 0,$$

$$[\Omega, a] = \frac{1}{2}na\sqrt{1-e^2}\,\cos i, \quad [i, M_0] = 0,$$

$$[i, a] = 0, \quad [\omega, a] = \frac{1}{2}na\sqrt{1-e^2}, \quad [M_0, a] = -\frac{1}{2}na$$

$$[\omega, e] = \frac{-na^2e}{\sqrt{1-e^2}}, \quad [a, e] = 0, \quad [M_0, e] = 0$$

$$[\Omega, e] = \frac{-na^2e}{\sqrt{1-e^2}}\cos i, \quad [i, e] = 0.$$

The corresponding poisson brackets are immediately obtained by solving the system (49). Taking into consideratior equations (52), we finally obtain

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M_0},$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial M_0} - \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial \omega},$$

$$\sin i\,\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i},$$

$$\sin i\,\frac{di}{dt} = \frac{\cos i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} - \frac{1}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \Omega},$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e},$$

$$\frac{dM_0}{dt} = -\frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}.$$

In order to obtain the Lagrange equations in the form (41), it is sufficient to replace the variables ω and M_0 by π and ϵ using the relations

[illegible]

15. The perturbations of the elements

The lagrange equations, from which the osculating elements are obtained, are extremely complicated equations. They can only be solved using the method of successive approximation. In the following, we are going to investigate the form in which this solution is obtained.

Let us suppose that there exist only two planets whose masses are m and m'. We denote the elements of these planets respectively by a, e, ... and a', e',... On the basis of the results obtained in the previous section, we assume that the motion of these planets relative to the sun is defined by the following twelve equations

$$\left.\begin{aligned} \frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \varepsilon} \\ \frac{d\Omega}{dt} &= \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}, \\ & \dots\dots\dots \end{aligned}\right\} \qquad (54)$$

$$\left.\begin{aligned} \frac{da'}{dt} &= \frac{2}{n'a'}\frac{\partial R'}{\partial \varepsilon'} \\ \frac{d\Omega'}{dt} &= \frac{\operatorname{cosec} i'}{n'a'^2\sqrt{1-e'^2}}\frac{\partial R'}{\partial i'} \\ & \dots\dots\dots \end{aligned}\right\} \qquad (54')$$

In these equations (§ 3)

$$R = \kappa^2 m'\left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3}\right) \qquad (55)$$

$$R' = \kappa^2 m\left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r^3}\right), \qquad (55')$$

where x, y, z, r and x', y', z', r' denote the coordinates and radius vectors of the planets and Δ the distance between them. It is easy to see that R and R' depend only on the mutual location of the orbits, but not on their location relative to the ecliptic. In fact, if the angle between the radius vectors r and r' is denoted by H, then these functions will be given by

$$R = \kappa^2 m'\left(\frac{1}{\Delta} - \frac{r\cos H}{r'^2}\right), \qquad R' = \kappa^2 m\left(\frac{1}{\Delta} - \frac{r'\cos H}{r^2}\right).$$

They depend only on r, r' and H since

$$\Delta^2 = r^2 + r'^2 - 2rr'\cos H.$$

The method of successive approximations applies when the masses m and m' are small. This method yields the unknown functions in the form of a series of expansion in powers of masses m and m', such as

$$\left.\begin{array}{l} a = a_0 + \delta_1 a + \delta_2 a + \ldots \\ e = e_0 + \delta_1 e + \delta_2 e + \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (55)$$

$$\left.\begin{array}{l} a' = a'_0 + \delta_1 a' + \delta_2 a' + \ldots \\ e' = e'_0 + \delta_1 e' + \delta_2 e' + \ldots \\ \ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (55')$$

Here, by a_o , e_o , ..., a'_o , e'_o ... we denote the (constant) values of the osculating elements evaluated in the initial moment, by δ a, δ_1 e,... the functions of time that have m as a multiplying factor, and by δ_1a', δ_1 e',... the functions of time that have m' as a multiplier. In general, we denote by δ_na, δ_ne, ..., δ_na', δ_ne', ... terms of the n-th power in masses m and m'. The expressions δ_na, δ_ne,... δ_na', δ_ne', ... are to be called perturbations of the n^{th} order.

Putting in equations (54) and (54')

$$m = 0, \qquad m' = 0,$$

we obtain

$$a = a_0, \qquad e = e_0, \ldots, \qquad a' = a'_0, \ldots$$

These values will then be substituted in the right hand side of the same equations, i.e., equations (54) and (54'). Since R and R' are functions of t and a, e, ..., a', e', ..., then equations (54) and (54') will have, after this substitution, the following form

$$\frac{da}{dt} = m' f(t, a_0, e_0, \ldots, a'_0, \ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{da'}{dt} = m f_1(t, a_0, e_0, \ldots, a'_0, \ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Integrating these equations yields

$$a = a_0 + m'\int_{t_0}^{t} f(t,\ a_0,\ e_0,\ \dots)\,dt = a_0 + \delta_1 a, \tag{57}$$
$$\dots\dots\dots\dots\dots\dots\dots$$

and, hence, determines the first order perturbations.

To obtain the second-order perturbations, we substitute into the right-hand sides of equations (54) and (54') the values $a_0 + \delta a$, $e_0 + \delta_1 e$, ..., $a_0' + \delta_1 a'$,... which have been obtained for the elements a, e, ... , a', ... This substitution yields

$$\frac{da}{dt} = m' f(t,\ a_0,\ \dots) + m'\frac{\partial f}{\partial a_0}\delta_1 a + \dots + m'\frac{\partial f}{\partial a_0'}\delta_1 a' + \dots,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

After integration, we obtain

$$a = a_0 + \delta_1 a + \delta_2 a$$
$$\dots\dots\dots$$

These equations are accurate up to the second order in masses. Repeating the same procedure, we may obtain as many terms in the series-expansion of the elements as we desire.

Let us now investigate the analytic form of the expansions given by equations (56) and (56'). The coordinates of the ea the planets are periodic functions of the corresponding mean M or M'. Consequently, the perturbation functions R and R' are also periodic functions of M and M'. Hence, they can be expanded in a double Fourier series as follows

$$R = \Sigma N \cos(jM + j'M' + B)$$
$$R' = \Sigma N' \cos(jM + j'M' + B'),$$

where j and j' take all the integral values from $-\infty$ to $+\infty$

It is easy to see from equations (56) and (56') and from the expressions of the coordinates in the elliptic motion (§§ 77-82), that the functions R and R' and consequently the coefficients N and N' can be expanded in powers of the eccentrisities e and e' and the mutual' slope of the orbits J. (We have already pointed out that the functions R and R' depend only on the mutual slope of the orbits). Noting that

$$M = nt + \varepsilon - \pi, \qquad M' = n't + \varepsilon' - \pi',$$

we finally obtain both the functions R and R' in the form

$$\Sigma A e^{\alpha} e'^{\alpha'} J^{\beta} \cos D, \tag{58}$$

where

$$D = j(nt + \varepsilon) + j'(n't + \varepsilon') + C.$$

Substituting these expressions for R and R' into equations (54) and (54'), we obtain similar summations in the right-hand side of these equations.

Equation (57) shows that, in order to evaluate the second order perturbations, it is necessary to replace in the right-hand sides of equations (54) and (54') all the elements a, e, ... by their initial elements and integrate the resulting trigonometric series. This yields expressions for the first order perturbations in the form of turms having one of the following two types:

$$A_0 e_0^{\alpha} e_0'^{\alpha'} J_0^{\beta} \frac{\sin D_0}{j n_0 + j' n_0'},$$

j $jn_0 + j'n_0' \neq 0$, and

$$t A_0 e_0^{\alpha} e_0'^{\alpha'} J_0^{\beta} \cos C,$$

if $jn_0 + j'n_0 = 0$.

Evaluating the second-, third-, ... order terms in the above way, we obtain terms of the type

$$\frac{t^p A_0 e_0^{\alpha} e_0'^{\alpha'} J_0^{\beta} \cos(\nu t + C)}{(j_1 n_0 + j_1' n_0')^{k_1} (j_2 n_0 + j_2' n_0')^{k_2} \dots}, \tag{59}$$

where

$$\nu = j n_0 + j' n_0'.$$

If $\nu = 0$, Such a term is then called a periodic perturbation in the case of $p \geqslant 0$, we will have a secular perturbation if $\nu = 0$ and a mixed perturbation if $\nu = 0$. The sum

$$\alpha + \alpha' + \beta$$

is called the degree of perturbation. The higher this degree is, the smaller is the perturbation of the order under consideration.

Particular attention should be paid to the small terms in the denominator of equations (59). These terms are called the small subgroup. There are responsible for increasing the value of the perturbations. If q is the sum of the orders k_λ , k_μ , ... of all the small sub-groups of the term given by equation (59), then the larger q is the larger is the corresponding perturbation, as long as the other conditions are not altered.

Poincare called the difference n-p for the n-th order perturbation the rank and the difference $n - \frac{1}{2}p - \frac{1}{2}q$, the class. Once we know the order, rank and class ofa given perturbation, we have an estimate of the general character of this perturbation. For small intervals of time, most important will be the lowest order perturbations, and in particular the first order terms. On the other hand, for long intervals of time, the value of a perturbation is mainly determined by its class. For very long intervals of time, the contribution of the perturbation is best of all judged by its rank.

16. Long-periodic perturbations

Amongst the periodic perturbations, given by equation (59), particular attention should be paid to those, for which the coefficients ν involved in the arguments of the trigonometric functions are small. The periods of these perturbations, which are equal to $\frac{360^\circ}{\nu}$, can be considerably longer than the periods of inversion of the planets under consideration. These perturbations are called long-periodic perturbations.

Long-periodic perturbations play an important role in the theory of the motion of planets. It happens that the amplitudes of some of these perturbations are very large even when their degrees are large. Let us consider a perturbation of the first order. If the term

$$A_0 e_0^{a} e_0'^{a'} J_0^{b} \frac{\sin[(jn_0 + j'n_0')t + C]}{jn_0 + j'n_0'}$$

corresponds to a long-periodic perturbation, then the quantity $jn_o + j'n_o'$ appearing in the denominator is small and the amplitude is much larger than what is expected for a perturbation of such a degree.

The average longitude of the planet has a strong influence on the long-periodic perturbations. The average longitude is given by the following relation (§13)

$$\lambda = \varepsilon + \rho,$$

where

$$\rho = \int_{t_0}^{t} n\,dt,$$

$$\frac{dn}{dt} = -\frac{3}{a^2}\frac{\partial R}{\partial \varepsilon}.$$

To obtain ρ, let us integrate twice each term on the right-hand side of this equation. This yields

$$A_0 e_0^{a} e_0'^{a'} J_0^{b} \frac{\cos[(jn_0 + j'n_0')t + C]}{(jn_0 + j'n_0')^2}$$

Thus, long-periodic perturbations contribute to the average longitude by terms of the first-order in mass, i.e. of class zero, devided by the squares of small quantities.

The detailed theory of sereis-expansion of perturbation functions will be given in one of the following chapters. This theory shows that the following relation holds

$$\alpha + \alpha' + \beta = j + j' + \text{even integer}$$

Thus, the long-periodic term can have a considerable amplitude only when then numbers j and j' have small absolute values.

In order to find the values of j and j', for which the perturbation becomes long-periodic, it is most convenient to expand the ratio n_o/n_o' in a continued fraction. For example, for Jupiter and Saturn

$$n_0 = 299''.1283, \qquad n_0' = 120''.4547.$$

when the initial moment is chosen to be January 1.0, 1900. Accordingly,

$$\frac{n_0}{n_0'} = 2 + \cfrac{1}{2 + \cfrac{1}{14 + \ldots}}$$

The appropriate fraction may be the following

$$\frac{2}{1}, \ \frac{5}{2}, \ \frac{72}{29} \ldots$$

If we choose j = 1 and j' = -2, then

$$j n_0 + j' n_0' = 58''.6736,$$

which approximately equals 1/5 n_o or - ½ n_o' . Such a divider cannot be considered as small. On the contrary, when j = 2 and j' = -5, we obtain

$$j n_0 + j' n_0' \qquad 4''.0169,$$

which approximately equals $\frac{1}{74}\ n_o$ or $\frac{1}{30}\ n_o'$. The corresponding long-periodic inequality, whose period is approximately 900 years, has in the longitude of Saturn an amplitude of the order of 50'. Finally, if we consider the next fraction, we obtain

$$j n_0 + j' n_0' = 29\ n_0 - 72\ n_0' = 1''.9823.$$

The corresponding inequality, the degree of which is not less than /29-72/ = 43, is completely insensible.

A large inequality in the motion of Jupiter and Saturn, depending on the subgroup 2n-2n', was discovered empirically. Several unsuccessful attempts to interpret this inequality led Euler and Lagrange to assume the existance of an unknown type of gravitation in addition to the gravitation influenced by the Sun. The correct interpretation was given by Laplace who evaluated all the first order inequalities for the motion of Jupiter and Saturn up to the third degree.

Annotation:

In practical studies of the motion of planets, one is rarely met with more than one small subgroup. Actually, let the ratio of mean durnal motions be expanded in a continuous fraction, so that

$$\frac{n_0}{n_0'} = \alpha + \cfrac{1}{\alpha_1 + \cfrac{1}{\alpha_2 + \ldots}}$$

Let the first appropriate fraction leading to a small subgroup be

$$\frac{P_k}{Q_k} = \alpha + \cfrac{1}{\alpha_1 + \ldots + \cfrac{1}{\alpha_{k-1}}},$$

The next incomplete quotient α_k will be a large number. Thus, the next appropriate fraction

$$\frac{P_{k+1}}{Q_{k+1}} = \frac{P_k \alpha_k + P_{k-1}}{Q_k \alpha_k + Q_{k-1}}$$

will have a very large numerator and denominator. The perturbations that correspond to this fraction, as well as to all subsequent appropriate fractions, will be insensible.

17. Secular perturbations

It has been pointed out that the secular perturbations are obtained from the terms of the perturbation functions, in which the arguments of the trigonometric functions do not depend on time. The totality of these terms are called the secular parts of the perturbation function.

Any term in the perturbation function depends on and only, through the mean anomalies M and M'. Hence, terms in equation (58) in which $j = j' = 0$ do not depend on ϵ and ϵ'. If R_0 denotes the secular part of the perturbation function, then

$$\frac{\partial R_0}{\partial \epsilon} = 0.$$

Referring to the first ofequations (41), we note that the expression of $\frac{da}{dt}$ does not involve constant terms. In other words, a does not have a secular perturbation of the first order. The last of equations (41) indicates that the average durnal motion will also not have a secular perturbation of thefirst order. This result holds for the mutual perturbations of any arbitrary number of planets. It leads to the following fundamental theorem.

The semimajor axes of the orbits of planets and their average durnal motions do not have secular perturbations of the first order relative to masses.

Laplace (1773) proved this theorem for terms of degree not higher than the second. The general proof of this theorem was given by Lagrange (1776). In the year 1809, Poisson showed that there are no pure secular terms in the perturbations of the semimajor axes and among the second order terms[1]. In the year 1878, Spiru C. Haretu was able to find third-order secular terms.

The other elements e, i, Ω, ... have secular perturbations. For example, Leverrie found that for Jupiter

$$
\begin{aligned}
e &= 0.048\,335\,08 + 0.000\,166\,19\,T - 0.000\,000\,467\,T^2 - 0.000\,000\,0019\,T^3 \\
i &= 1^\circ 18' 31''.11 - 20''.506\,T + 0''.014\,T^2 \\
\Omega &= 99^\circ 26' 35''.59 + 3637''.908\,T + 1''.2680\,T^2 - 0''.03061\,T^3 \\
\pi &= 12^\circ 43' 14''.39 + 5793''.928\,T + 3''.5936\,T^2 - 0''.01732\,T^3 .
\end{aligned}
$$

where T denotes time in centuries (36525 days) counted starting from the mean value Paris midday time on January 1.o, 1900.

Secular perturbations have always been connected with the stability of the solar system. However, it is necessary to point out that, even if the convergence of the series (56) could not be proved for an arbitrary time t, the presence of secular terms in various-order perturbations would not be sufficient for concluding that the solar system would be unstable. In fact, the expansion of periodic functions of time in powers of the mass can involve an infinite number of secular terms. For example, let us consider the function sin (mat), where m is the mass of the perturbing planet and a is an arbitrary constant. Expanding this function in powers of m yields

$$\sin(mat) = mat - \frac{1}{6} m^3 a^3 t^3 + \cdots .$$

Thus, any method of integration of equations (54) and (54'), based on the expansion of the solution in powers of the perturbing masses, will lead to secular terms, even if the solution is expressed in terms of periodic functions of time.

---

(1) This result is known as the Poisson theorem.

18. Poisson's method

When the secular perturbations of the angular elements Ω, π and ϵ are large, it is sufficient to apply the method of integration suggested in § 15 because of the slow convergence of the successive approximations. This situation is met with in the theory of lunar motion. There, the secular perturbations produce variations in the perihelion and node longitudes. These variations are given by(1)

$$\begin{aligned} \pi &= 334^\circ 26' 16''.52 + 14\,648\,509''.18\,T - 37''.16\,T^2 - 0''.045\,T^3 \\ \Omega &= 259^\circ 7' 52''.79 - 6\,962\,911''.94\,T + 7''.18\,T^2 + 0''.0077\,T^3 \end{aligned}$$

where T denotes time in centrueis of the average Paris astronomic time, starting from January 1.0, 1900.

Poisson (1835) suggested a special method for the integration of these equations. This method is to include in the first approximation the contribution of the secular perturbations of the angular elements to the periodic inequalities. Denoting the average longitudes of the planets by

$$\lambda = \varepsilon + \rho, \qquad \lambda' = \varepsilon' + \rho',$$

where

$$\rho = \int_{t_0}^{t} n\,dt, \qquad \rho' = \int_{t_0}^{t} n'\,dt,$$

we rewrite equations (54) and (54') in the following general form

---

(1) The numerical coefficients in these equations are taken from the table of Radon (see 117). Heisen's (Gajzen) tables yield

$$\begin{aligned} \pi &= 334^\circ 26' 10''.[illegible] + 14\,648\,510''.08\,T - 36''.211\,T^2 - 0''.036600\,T^3 \\ \Omega &= 259^\circ 7' 10''.97 - 6\,962\,025''.21\,T + 8''.210\,T^2 + 0''.00[illegible]\,T^3 \end{aligned}$$

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$$
\begin{aligned}
\frac{d\Omega}{dt} &= m'\Theta(\varepsilon+\rho,\ \Omega,\ \pi,\ \ldots;\ \varepsilon'+\rho',\ \Omega',\ \ldots) \\
\frac{d\pi}{dt} &= m'\Phi(\varepsilon+\rho,\ \Omega,\ \pi,\ \ldots;\ \varepsilon'+\rho',\ \Omega',\ \ldots) \\
\frac{d\varepsilon}{dt} &= m'\Psi(\varepsilon+\rho,\ \Omega,\ \pi,\ \ldots;\ \varepsilon'+\rho',\ \Omega',\ \ldots) \\
\frac{da}{dt} &= m'F(\varepsilon+\rho,\ \Omega,\ \pi,\ \ldots;\ \varepsilon'+\rho',\ \Omega',\ \ldots) \\
&\ldots\ldots\ldots\ldots
\end{aligned}
$$

Let

$$\Omega = \theta + \alpha m't, \qquad \pi = \pi_1 + \beta m't, \qquad \varepsilon = \varepsilon_1 + \gamma m't,$$

where α, β and γ are arbitrary constants and θ, π and ϵ are the new unknown. Substituting these expressions in the previous ones, we obtain

$$
\left.
\begin{aligned}
\frac{d\theta}{dt} &= -m'\alpha + m'\Theta(\varepsilon_1+\rho+\gamma m't,\ \theta+\alpha m't,\ \ldots) \\
\frac{d\pi_1}{dt} &= -m'\beta + m'\Phi(\varepsilon_1+\rho+\gamma m't,\ \theta+\alpha m't,\ \ldots) \\
&\ldots\ldots\ldots\ldots
\end{aligned}
\right\} \qquad (60)
$$

As we have done in § 15, let us put

$$
\begin{aligned}
\theta &= \Omega_0 + \delta_1\theta + \delta_2\theta + \ldots \\
\pi_1 &= \pi_0 + \delta_1\pi + \delta_2\pi + \ldots \\
&\ldots\ldots\ldots
\end{aligned}
$$

where δ_n denotes terms proportional to the n-th power of masses m, m', ... We substitute these expressions in equations (60) and expand the functions Φ, ... in series in the following way

$$
\Theta = \Theta(\varepsilon_0 + \rho + \gamma m't,\ \Omega_0 + \alpha m't,\ \pi_0 + \beta m't,\ \ldots) + \frac{\partial\Theta}{\partial\varepsilon}(\delta_1\varepsilon + \delta_2\varepsilon + \ldots) + \frac{\partial\Theta}{\partial\Omega}(\delta_1\theta + \delta_2\theta + \ldots) + \ldots
$$

This means that we are going to take as the increments of the arguments only the periodic terms $\delta_1\epsilon, \delta_2\epsilon, \ldots \delta_1\Theta, \ldots$ and not the secular elements m't , m't and m't. We obtain in the first approximation

$$\left.\begin{array}{l} \frac{d\theta}{dt} = m'\alpha + m'\Theta(\epsilon_0 + \beta + \gamma m't, \pi_0 + \alpha m't, \ldots) \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{array}\right| \qquad (61)$$

Integrating these equations, and equating to zero the secular perturbations of θ, π and ϵ yields three equations for the determination of the quantities α , β and γ.

We note that the integration of equations (60) has been made in a way as simple as the method of successive approximation. The reason is that the elements ϵ , Ω and π appear in the expansion of the perturbation function only in the arguments of the trigonometric functions. Actually, since the perturbation function is a periodic function not only of λ and λ' , but also of Ω , Ω' , π and π' , then its expansion is given by

$$R = \Sigma N \cos(j\lambda + j'\lambda' + h\Omega + h'\Omega' + k\pi + k'\pi').$$

In the Poisson method, one is not very strict on the expansion in powers of the masses, since in the arguments of the periodic perturbations there will be terms like γ m't, β m't, ... having mass m'as a multiplier. In other words, part of the second-order terms will be taken into account in the first approximation.

CHAPTER IV

THE CANONICAL ELEMENTS

19. The canonical equations

Let us consider the motion of a system of n material points. We denote their masses by m_i and coordinates by x_i, y_i and z_i. We assume that the interaction between the particles is described by the force function U. The motion of the system will then be described by the following equations

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}, \quad m_i \ddot{y}_i = \frac{\partial U}{\partial y_i}, \quad m_i \ddot{z}_i = \frac{\partial U}{\partial z_i} \qquad (1)$$

$$(i = 0, 1, \ldots, n-1)$$

There are several methods to replace these 3n second-order differential equations by 6n first-order equations. The following method is of particular importance. Let us introduce the following notations

$$x_i' = m_i \dot{x}_i, \quad y_i' = m_i \dot{y}_i, \quad z_i' = m_i \dot{z}_i.$$

The kinetic energy of the system will then be

$$T = \frac{1}{2} \sum m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) =$$
$$= \frac{1}{2} \sum \frac{1}{m_i} (x_i'^2 + y_i'^2 + z_i'^2).$$

It is easy to see that equations (1) are equivalent to the following equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial x_i'}; \quad \frac{dx_i'}{dt} = -\frac{\partial H}{\partial x_i}$$
$$\frac{dy_i}{dt} = \frac{\partial H}{\partial y_i'}; \quad \frac{dy_i'}{dt} = -\frac{\partial H}{\partial y_i}$$
$$\frac{dz_i}{dt} = \frac{\partial H}{\partial z_i'}; \quad \frac{dz_i'}{dt} = -\frac{\partial H}{\partial z_i}$$

where

$$H = T - U,$$

is the total energy of the system, since the force function is equal to the negative of the potential energy. These equations are known as the canonical equations. The function H is called the Hamiltonian of the system.

Let us now consider the more general case, when the positions of the points of the system are defined by the S parameters $q_1, q_2, \ldots q_s$, which may be subject to a number of holonomic constraints. These parameters are called the generalized coordinates. The value of S defines the number of degrees of freedom of the system. In this case, the equations of motion (1) are transformed into the following Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad (i = 1, 2, \ldots, s) \qquad (2)$$

where

$$L = T + U$$

is the Lagrangian of the system.

We note that when we express the rectangular coordinates in terms of the generalized coordinates, the kinetic energy becomes

$$T = \frac{1}{2} \sum_i \sum_k A_{ik} \dot{q}_i \dot{q}_k + \sum_k A_k \dot{q}_k + A, \qquad (3)$$

where A_{ik}, A_k and A are functions of q_1, q_2, ... q_s and t. The summations over all of the indices are carried from 1 to S.

We can now show that equations (1) can be replaced by first-order equations having a canonical form. We introduce the subsidiary unknowns

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

which will be called the generalized momenta. Since U does not depend on the derivatives, then

$$p_i = \sum_k A_{ik} \dot q_k + A_i. \tag{4}$$

Equations (2) may then be replaced by

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}, \quad p_i = \frac{\partial L}{\partial \dot q_i}. \tag{5}$$

To eliminate $\dot q_i$, we use equations (4). We are able to solve these equations relative to $\dot q_k$ since the determinant formed by the coefficients A_{ik} cannot be equal to zero. Indeed, in the case we are interested in, the halomomic constraints do not involve time. Therefore, in equations (3) A_k=0 and A=0. If the determinants formed by the coefficients A_{ik} were equal to zero, then there would be nonvanishing values of $\dot q_k$, for which[1]

$$\sum_k A_{ik} \dot q_k = 0, \qquad (i = 1, \ldots, s)$$

and consequently T = 0. Evidently, this cannot take place.

Let us introduce the following quantity

$$H = \sum p_k \dot q_k - L,$$

which is a function of p_k , q_k and t, since $\dot q_i$ can be expressed in terms of p_i according to the above arguments.

---

(1) To simplify the formulae, we shall not indicate the limits of summation whenever all of the indices run through the same values 1, 2, ... s, as in the present case. It is also possible not to indicate the summation indices if we introduce the "rule of dummy indices": Summation is always carried out over the indices which are repeated in the summand at least twice. For example

$$\sum a_{ik} b_{kn} = a_{i1} b_{1n} + a_{i2} b_{2n} + \ldots + a_{is} b_{sn}.$$

Such indices are called dummy since they disappear after summation.

Varying p_k and q_k will lead on the one hand to

$$\delta H = \sum \frac{\partial H}{\partial p_k} \delta p_k + \sum \frac{\partial H}{\partial q_k} \delta q_k$$

and on the other hand to

$$\delta H = \delta \sum p_k \dot{q}_k - \delta L =$$
$$= \sum \dot{q}_k \delta p_k + \sum p_k \delta \dot{q}_k - \sum \frac{\partial L}{\partial q_k} \delta q_k - \sum \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k$$

Using the second of equations (5), we obtain

$$\delta H = \sum \dot{q}_k \delta p_k - \sum \frac{\partial L}{\partial q_k} \delta q_k .$$

Comparing the two expressions of δ H, we obtain

$$\dot{q} = \frac{\partial H}{\partial p_k}, \quad \frac{\partial L}{\partial q_k} = -\frac{\partial H}{\partial q_k} .$$

Taking into consideration the first of equations (5), we finally obtain

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} . \qquad (6)$$
$$(k = 1, 2, \ldots, s)$$

If the Lagrangian L, and consequently the function H, do not explicity depend on time, then equations (6) will have the following first integral

$$H = \text{const},$$

which is nothing else but the kinetic energy integral. In fact, this equation can be rewritten as

$$\sum \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{const},$$

or

$$\sum \dot{q} \frac{\partial T}{\partial \dot{q}} - L = \text{const}$$

In our case, T is a homogeneous function of the first order in q_k, and thus

$$\sum q_k \frac{\partial T}{\partial q_k} = 2T$$

Noting that L = T + U, we write the complete integral obtained above as

$$T - U = \text{const.}$$

This is nothing else but the law of conservation of energy.

The canonical equations have several remarkable properties. We are interested in the most elementary properties of these equations which can easily be deduced from the following theorem.

<u>Theorem I</u>

<u>If the general solution of equations (6) is given by the following equalities</u>

$$q_k = \varphi_k(t, \gamma_1, \gamma_2, \ldots, \gamma_{2s}), \quad p_k = \psi_k(t, \gamma_1, \gamma_2, \ldots, \gamma_{2s}) \tag{7}$$

where $\gamma_1, \gamma_2 \cdots \gamma_s$ are constants of integration, then equations (6) are equivalent to the following equations

$$\frac{\partial}{\partial t}\sum p_k \frac{\partial q_k}{\partial \gamma_i} - \frac{\partial}{\partial \gamma_i}\sum p_k \frac{\partial q_k}{\partial t} = -\frac{\partial H}{\partial \gamma_i} \tag{8}$$
$$(i = 1, 2, \ldots, 2s)$$

First, we show that equations (8) can be easily inferred from equations (6). Indeed, when equations (6) hold, the evident identity

$$\frac{\partial}{\partial t}\sum p_k \frac{\partial q_k}{\partial \gamma_i} - \frac{\partial}{\partial \gamma_i}\sum p_k \frac{\partial q_k}{\partial t} = \sum \frac{\partial p_k}{\partial t}\frac{\partial q_k}{\partial \gamma_i} - \sum \frac{\partial p_k}{\partial \gamma_i}\frac{\partial q_k}{\partial t} \tag{9}$$

leads to

$$\frac{\partial}{\partial t}\sum p_k \frac{\partial q_k}{\partial \gamma_i} - \frac{\partial}{\partial \gamma_i}\sum p_k \frac{\partial q_k}{\partial t} = -\sum \frac{\partial H}{\partial q_k}\frac{\partial q_k}{\partial \gamma_i} - \sum \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial \gamma_i}, \qquad (10)$$

which are equivalent to equations (8). Conversly, transforming equations (8) into the form (10) and using identity (9), we replace these equations by

$$\sum\left(\frac{\partial p_k}{\partial t} + \frac{\partial H}{\partial q_k}\right)\frac{\partial q_k}{\partial \gamma_i} - \sum\left(\frac{\partial q_k}{\partial t} - \frac{\partial H}{\partial p_k}\right)\frac{\partial p_k}{\partial \gamma_i} = 0.$$
$$(i = 1, 2, \ldots, 2s)$$

These equations can be regarded as a system of 25 linear equations in which expressions

$$\left(\frac{\partial p_k}{\partial t} + \frac{\partial H}{\partial q_k}\right), \quad -\left(\frac{\partial q_k}{\partial t} - \frac{\partial H}{\partial p_k}\right)$$

play the role of the unknown quantities. The determinant of this system does not equal zero, since equations (7) are solvable relative to $\gamma_1, \gamma_2 \cdots \gamma_{2s}$. Therefore, the expressions just written must vanish. Namely this is what equations (6) state.

Annotation

The canonical equations (6) will not be altered if we interchange the positions of p_k and q_k and at the same time replace t by -t. By means of such interchanges, we can obtain from each property of the canonical systems, a new property. For example doing such permutations in theorem I, we find that the equations are also equivalent to the following equations

$$\frac{\partial}{\partial t}\sum q_k \frac{\partial p_k}{\partial \gamma_i} - \frac{\partial}{\partial \gamma_i}\sum q_k \frac{\partial p_k}{\partial t} = \frac{\partial H}{\partial \gamma_i}.$$

20. The canonical transformations

Let us replace in the following canonical equations

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}$$

the variables q_k and p_k by new variables, Q_k and p_k, as defined by

$$\begin{aligned} Q_k &= \Phi_k(t, q_1, \ldots, q_s, p_1, \ldots, p_s) \\ P_k &= \Psi_k(t, q_1, \ldots, q_s, p_1, \ldots, p_s) \end{aligned} \tag{11}$$

The canonical system will then be transformed into a new system. The following theorem specifies the condition for the resulting system to also have a canonical form. The corresponding transformations will be called canonical trnasformations.

<u>Theorem II</u>:

If the relation between the new and old variables is such, that the expression

$$\Sigma p_k dq_k - \Sigma P_k dQ_k = dW \tag{12}$$

is a complete differential of some function W, then, after the transfromation, equations (6) may be represented in the following way

$$\frac{dQ_k}{dt} = \frac{\partial K}{\partial P_k}, \quad \frac{dP_k}{dt} = -\frac{\partial K}{\partial Q_k}, \tag{13}$$

where

$$K = H + \frac{\partial W}{\partial t},$$

Here, it is assumed that functions H and W have been expressed in terms of the new variables Q_k and P_k.

To prove this theorem, we note that equaticns (11) and (7) allow us to express the variables Q_k and P_k, and consequently the function W, in terms of $\gamma_1, \gamma_2, \ldots, \gamma_{2s}$ and t. In equation (12) we understand that dW is the complete differential of the function W only with respect to the variables Q_k and P_k, where we consider that p_k and q_k are expressed in terms of Q_k and P_k, although the function w may also depend on time t. Therefore, it follows from equation (12), that

$$
\left.
\begin{aligned}
&\sum p_k \frac{dq_k}{dt} - \sum P_k \frac{dQ_k}{dt} = \frac{dW'}{dt} - \frac{\partial W}{\partial t} \\
&\sum p_k \frac{\partial q_k}{\partial \gamma_i} - \sum P_k \frac{\partial Q_k}{\partial \gamma_i} = \frac{\partial W}{\partial \gamma_i},
\end{aligned}
\right\} \tag{14}
$$

where

$$
\frac{dW'}{dt} = \frac{\partial W}{\partial t} + \sum \left(\frac{\partial W}{\partial Q_k} \frac{dQ_k}{dt} + \frac{\partial W}{\partial P_k} \frac{dP_k}{dt} \right)
$$

$$
\frac{\partial W}{\partial \gamma_i} = \sum \left(\frac{\partial W}{\partial Q_k} \frac{\partial Q_k}{\partial \gamma_i} + \frac{\partial W}{\partial P_k} \frac{\partial P_k}{\partial \gamma_i} \right)
$$

Now, if W is expressed in terms of $\gamma_1, \gamma_2, \gamma_3, \ldots \gamma_{2s}$ and t, then

$$
\frac{d}{dt}\left(\frac{\partial W}{\partial \gamma_i}\right) = \frac{\partial}{\partial \gamma_i}\left(\frac{dW'}{dt}\right)
$$

Differentiating the first of equations (14) with respect to γ_i and the second with respect to t and subtracting them term by term, we obtain

$$
\frac{d}{dt} \sum p_k \frac{\partial q_k}{\partial \gamma_i} - \frac{\partial}{\partial \gamma_i} \sum p_k \frac{dq_k}{dt} - \frac{\partial}{\partial \gamma_i}\left(\frac{\partial W}{\partial t}\right) = \frac{d}{dt} \sum P_k \frac{\partial Q_k}{\partial \gamma_i} - \frac{\partial}{\partial \gamma_i} \sum P_k \frac{dQ_k}{dt}
$$

Applying theorem I to the left-hand side of this equation, and noting that

$$
\frac{\partial H}{\partial \gamma_i} = \sum \frac{\partial H}{\partial Q_k} \frac{\partial Q_k}{\partial \gamma_i} + \sum \frac{\partial H}{\partial P_k} \frac{\partial P_k}{\partial \gamma_i},
$$

$$
\frac{\partial}{\partial \gamma_i}\left(\frac{\partial W}{\partial t}\right) = \sum \frac{\partial^2 W}{\partial Q_k \partial t} \frac{\partial Q_k}{\partial \gamma_i} + \sum \frac{\partial^2 W}{\partial P_k \partial t} \frac{\partial P_k}{\partial \gamma_i},
$$

we obtain

$$\frac{d}{dt}\sum P_k \frac{\partial Q_k}{\partial q_i} - \frac{\partial}{\partial q_i}\sum P_k \frac{dQ_k}{dt} = -\frac{\partial K}{\partial q_i},$$

where

$$K = H + \frac{\partial W}{\partial t}.$$

On the basis of theorem I, it follows from these equations that

$$\frac{dQ_k}{dt} = \frac{\partial K}{\partial P_k}, \quad \frac{dP_k}{dt} = -\frac{\partial K}{\partial Q_k},$$

This is what we wanted to prove.

Annotation I

If the relation between q_k, p_k and Q_k, P_k are such, that

$$\Sigma q_k dp_k - \Sigma Q_k dP_k = dW',$$

then, after the transformation of equations (6), we obtain

$$\frac{dQ_k}{dt} = \frac{\partial K'}{\partial P_k}, \quad \frac{dP_k}{dt} = -\frac{\partial K'}{\partial Q_k},$$

where

$$K' = H - \frac{\partial W'}{\partial t}.$$

In order to prove this, it is sufficient to use the substitution indicated at the end of the previous section.

Annotation II

The conditions of theorem II are often expressed in a slightly different way. Let us add equation

$$\Sigma Q_k dP_k + \Sigma P_k dQ_k = \Sigma d(P_k Q_k)$$

term by term to equation (12). This yields

$$\Sigma p_k dq_k + \Sigma Q_k dP_k = dS,$$

where

$$S = W + \Sigma P_k Q_k.$$

or

$$p_k = \frac{\partial S}{\partial q_k}, \quad Q_k = \frac{\partial S}{\partial P_k}. \qquad (*)$$

Therefore, starting with the arbitrary function

$$S(t, q_1, \ldots, q_s, P_1, \ldots, P_s),$$

and using equations (*), we obtain a canonical transformation.

In conclusion, we give some examples for the applications of theorem II.

<u>Example I</u>

Let

$$q_1 = \sqrt{2Q_1}\cos P_1, \quad p_1 = \sqrt{2Q_1}\sin P_1,$$
$$q_i = Q_i, \quad p_i = P_i \quad (i = 2, 3, \ldots, s).$$

Since

$$q_1 dp_1 = 2Q_1 \cos^2 P_1 dP_1 + \cos P_1 \sin P_1 dQ_1,$$

then

$$q_1 dp_1 - Q_1 dP_1 = Q_1 \cos 2P_1 dP_1 + \frac{1}{2}\sin 2P_1 dQ_1 = d\left(\frac{1}{2} Q_1 \sin 2P_1\right).$$

Hence, the conditions of theorem II are fulfilled here. The canonical system obtained as a result of this change of variables may be written immediately.

<u>Example II</u>

At the beginning of the previous section, we have seen that the equations of motion in rectangular coordinates (1) may be easily represented in a canonical form. Theorem II allows us to show that the canonical form of the equations is not violated by the transition from the rectangular coordinates to any curvilinear coordinates. We have already obtained this result in the previous section by another method.

We leave it to the reader to prove this result using theorem II.

Example III

In practice, it oftenly happens that P_k are linear functions of p_k and Q_k are linear functions of q_k. In this case, it is sufficient for the transformation to be canonical, that

$$\Sigma P_k Q_k = \Sigma p_k q_k.$$

Indeed, dQ_k will be related to dq_k by the same relations that related Q_k to q_k. Consequently, it follows from the previous equality that

$$\Sigma P_k dQ_k - \Sigma p_k dq_k = 0.$$

21. Jacobi's method for solving canonical systems

Let us consider the canonical system

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \qquad (15)$$

Introducing the new variables Q_k and P_k, related to the old ones by

$$\Sigma p_k dq_k - \Sigma P_k dQ_k = dW, \qquad (16)$$

we obtain, on the basis of theorem II, that

$$\frac{dQ_k}{dt} = \frac{\partial K}{\partial P_k}, \quad \frac{dP_k}{dt} = -\frac{\partial K}{\partial Q_k}, \qquad (17)$$

where

$$K = \frac{\partial W}{\partial t} + H.$$

System (15) will be resolved once we find a function W for which K = 0. Actually, equations (17) yield

$$Q_k = \alpha_k, \quad P_k = \beta_k,$$

where α_k and β_k are constants of integration. On the other hand we obtain from condition (16) that

$$p_k = \frac{\partial W}{\partial q_k}, \quad P_k = -\frac{\partial W}{\partial Q_k},$$

where we suppose that W is expressed in terms of q_k and Q_k by means of equations (11). Replacing in W the quantities Q_k by α_k, we obtain the function $W(t, q_1, q_2, \ldots, q_s, \ldots, \alpha_1, \ldots, \alpha_s)$ that satisfies the following relations

$$\frac{\partial W}{\partial q_k} = p_k, \quad \frac{\partial W}{\partial \alpha_k} = \beta_k. \tag{18}$$

These relations permit us to express p_k and q_k in terms of t and the 2s arbitrary constants α_k and β_k and thus give the general solution of system (15).

We know the expression of the function H in terms of p_k and q_k. Let this expression be $H(t, q_1, q_2, \ldots, q_s, p_1, p_2, \ldots, p_s)$. We may write the equation K = 0 that defines W in the following way

$$\frac{\partial W}{\partial t} + H\left(t, q_1, \ldots, q_s, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_s}\right) = 0. \tag{19}$$

Thus, if we consider W as a function of the S +1 independent variables t and $q_1, q_2, \ldots q_s$, then this function will satisfy a first-order partial differential equation. Any solution of this euqation which will involve S + 1 unknown arbitrary constants, will be called a complete integral.

In the present case, the unknown function W enters equation (19) only by its derivatives. The solution of this equation will then involve S arbitrary constants, among which there is no additive constants. The complete integral is simply obtained by the introduction of the (S + 1)-th addetive constant and will have the form

$$W(t, q_1, q_2, \ldots, q_s, \alpha_1, \ldots, \alpha_s) \ldots \tag{20}$$

Thus, we are lead to the following result, which has been represented by the well-known Jacobi theorem.

Theorem III

In order to solve the canonical system (15), it is sufficient to find a complete integral of the type (20) for equation (19). The general solution of system (15) is obtained by finding q_k and p_k from the following equations

$$\frac{\partial W}{\partial \alpha_k} = \beta_k, \qquad \frac{\partial W}{\partial q_k} = p_k$$

where $\beta_1, \beta_2, \ldots, \beta_s$ are new arbitrary constants.

The Jacobi method consists in applying this theorem by integrating the canonical systems. The constants α_k and β_k that appear as a result of integrating the system by this method are called canonical constants or canonical elements..

It is easy to obtain the complete integral of equation (19) when the function H does not depend on t. Indeed, substituting in this equation

$$W = -\alpha t + W', \tag{21}$$

we obtain for the new unknown function W' the following equation

$$H\left(q_1, q_2, \ldots, q_s, \frac{\partial W'}{\partial q_1}, \ldots, \frac{\partial W'}{\partial q_s}\right) = \alpha,$$

where α is an arbitrary parameter. The solution of this equation involves s-1 arbitrary constants $\alpha_1, \alpha_2, \ldots, \alpha_{s-1}$ among which there are no additive constants. Once this solution is found, equation (21) can be used to obtain a solution for equation (19) involving the s constants $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_{s-1}$. The general solution of the canonical system (15) is then given by

$$\frac{\partial W}{\partial \alpha} = t + \beta, \quad \frac{\partial W}{\partial \alpha_k} = \beta_k, \quad \frac{\partial W}{\partial q_k} = p_k \qquad (22)$$
$$(k = 1, 2, \ldots, s-1).$$

where $\beta, \beta_1, \beta_2, \ldots, \beta_{s-1}$ are new constants.

It is interesting to note that the simplification of the integration of system (15), achieved this way, is a consequence of the existance of the first integral

$$H = \text{const.}$$

22. Application of the method of variation of arbitrary constants to the canonical elements

Let the following canonical system

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \qquad (15)$$

be solved by the Jacobi method. We have already seen that the solution

$$\begin{aligned} q_k &= \varphi_k(t, \alpha_1, \alpha_2, \ldots, \alpha_s, \beta_1, \beta_2, \ldots, \beta_s) \\ p_k &= \psi_k(t, \alpha_1, \alpha_2, \ldots, \alpha_s, \beta_1, \beta_2, \ldots, \beta_s) \end{aligned} \qquad (23)$$

is obtained from equation (18), i.e.

$$\frac{\partial W}{\partial q_k} = p_k, \quad \frac{\partial W}{\partial \alpha_k} = \beta_k, \qquad (24)$$

where W is the complete integral of the equation

$$\frac{\partial W}{\partial t} + H\left(t, q_1, \ldots, q_s, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_s}\right) = 0 \qquad (25)$$

Suppose that we have to solve a new canonical system

$$\frac{dq_k}{dt} = \frac{\partial (H + R)}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial (H + R)}{\partial q_k}, \qquad (26)$$

where R is a function of t, $q_1, \ldots q_s, p_1, \ldots p_s$. To apply the method of variation of arbitrary constants, we try to satisfy these equations again by expressions (23) considering the quantities α_i and β_i as functions of time and not as constants. For this purpose, we replace the variables q_k and p_k in equations (15) by the new variables α_k and β_k defined by equations (23). Using equations (24), we get

$$\sum p_k dq_k - \sum \alpha_k d\beta_k = \sum p_k dq_k + \sum \beta_k d\alpha_k - \sum d(\alpha_k \beta_k) =$$
$$= \sum \frac{\partial W}{\partial q_k} dq_k + \sum \frac{\partial W}{\partial \alpha_k} d\alpha_k - \sum d(\alpha_k \beta_k) = d\left(W - \sum \alpha_k \beta_k\right).$$

Applying theorem II, we write the transformed equations in the form

$$\frac{d\alpha_k}{dt} = \frac{\partial R}{\partial \beta_k}, \quad \frac{d\beta_k}{dt} = -\frac{\partial R}{\partial \alpha_k}, \tag{27}$$

since equation (25) in the present case yields

$$K = (H - R) + \frac{\partial W}{\partial t} = -R.$$

Hence, the application of the method of variation of arbitrary constants to the canonical elements immediately allows us to write the differential equations for these elements in a simple form.

23. Canonical elements of elliptic motion

We shall now apply the Jacobi method to the solution of the two-body problem. We denote by x, y and z the coordinates of the planet in the heliocentric coordinate system, and by m its mass. In order to write the equations of motion of the planet,

$$\ddot{x} = \frac{\partial U}{\partial x}, \quad \ddot{y} = \frac{\partial U}{\partial y}, \quad \ddot{z} = \frac{\partial U}{\partial z},$$

where

$$U = \frac{k^2(1+m)}{r},$$

in a canonical form, it is sufficient to assume that

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{k^2(1+m)}{r}.$$

Let us introduce the spherical coordinates

$$x = r\cos\varphi\cos\theta, \quad y = r\cos\varphi\sin\theta, \quad z = r\sin\varphi$$

and agree to write k^2 instead of $k^2(1+m)$. The Hamiltonian will then be written as

$$H = \frac{1}{2}(\dot r^2 + r^2\dot\varphi^2 + r^2\cos^2\varphi\,\dot\theta^2) - k^2 r^{-1}$$

In this case, the Hamilton-Jacobi equation reads

$$\frac{\partial W}{\partial t} + \frac{1}{2}\left\{\left(\frac{\partial W}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial W}{\partial \varphi}\right)^2 + r^{-2}\sec^2\varphi\left(\frac{\partial W}{\partial \theta}\right)^2\right\} - k^2 r^{-1} = 0.$$

This equation explicitly involves neither t nor θ. We substitute (see § 21).

$$W = -\alpha_1 t + \alpha_2\theta + W_1,$$

into this equation. We obtain

$$\left(\frac{\partial W_1}{\partial r}\right)^2 + r^{-2}\left(\frac{\partial W_1}{\partial \varphi}\right)^2 + r^{-2}\sec^2\varphi\,\alpha_2^2 - 2k^2 r^{-1} = 2\alpha_1.$$

It is sufficient for us to find a solution of this equation that involves one arbitrary constant. Therefore, we assume

$$\left(\frac{\partial W_1}{\partial \varphi}\right)^2 + \alpha_2^2\sec^2\varphi = \alpha_3^2,$$

which allows us to write the previous equation as

$$\left(\frac{\partial W_1}{\partial r}\right)^2 + \alpha_3^2 r^{-2} - 2k^2 r^{-1} = 2\alpha_1$$

so that the variables r and φ become separated. In other words we assume that

$$W_1 = W' + W'',$$

where W' is a function only of φ and W'' only of r. These functions can be found from the above-equations by means of quadratures. We thus obtain

$$W = -\alpha_1 t + \alpha_2 \theta + \int_0^{\varphi} \sqrt{\alpha_3^2 - \alpha_2^2 \sec^2 \varphi}\, d\varphi + \int_{r_0}^{r} \sqrt{2\alpha_1 + 2k^2 r^{-1} - \alpha_3^2 r^{-2}}\, dr.$$

We fix the lower limits of integration in order not to introduce unnecessary arbitrary constants. We take $\varphi = 0$ as a lower limit for the first integral and the smaller of the two roots of the expression inside the square root as a lower limit for the second integral.

According to theorem III, the general solution of the equations is given by

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \quad \frac{\partial W}{\partial \alpha_2} = \beta_2, \quad \frac{\partial W}{\partial \alpha_3} = \beta_3.$$

In the present case, this solution will have the form

$$\beta_1 = -t + \int_{r_0}^{r} (2\alpha_1 + 2k^2 r^{-1} - \alpha_3^2 r^{-2})^{-\frac{1}{2}}\, dr, \tag{28}$$

$$\beta_2 = \theta - \alpha_2 \int_0^{\varphi} \sec^2 \varphi\, (\alpha_3^2 - \alpha_2^2 \sec^2 \varphi)^{-\frac{1}{2}}\, d\varphi, \tag{29}$$

$$\beta_3 = \alpha_3 \int_0^{\varphi} (\alpha_3^2 - \alpha_2^2 \sec^2 \varphi)^{-\frac{1}{2}}\, d\varphi - \alpha_3 \int_{r_0}^{r} r^{-2} (2\alpha_1 + 2k^2 r^{-1} - \alpha_3^2 r^{-2})^{-\frac{1}{2}}\, dr. \tag{30}$$

These relations define the coordinates r, θ and φ as functions of t and the six arbitrary constants α_1, α_2, α_3, β_1, β_2 and β_3. The latter constants are the canonical elements of the elliptic motion.

Let us now find the relation between the canonical elements and the conventional ones. Equation (28) indicates that r should vary in the interval

$$r_0 \leq r \leq r_1,$$

where r_0 and r_1 are the roots of the equation

$$2\alpha_1 r^2 + 2k^2 r - \alpha_3^2 = 0.$$

In the elliptic motion, the limits of the radius vector are a(1-e) and a(1+e). Therefore,

$$2a = r_0 + r_1 = -\frac{k^2}{\alpha_1}, \quad a^2(1-e^2) = r_0 r_1 = -\frac{\alpha_3^2}{2\alpha_1}.$$

Hence,

$$\alpha_1 = -\frac{k^2}{2a}, \quad \alpha_3 = k\sqrt{a(1-e^2)} = k\sqrt{p}.$$

On the other hand, denoting by T the time at which the planet passes by the perihelion, we obtain from equation (28)

$$\beta_1 = -T,$$

because $r = r_0$ at the moment of passing by the perihelion.

It follows from equation (29) that

$$\alpha_3^2 - \alpha_2^2 \sec^2\varphi = 0,$$

or

$$\cos\varphi = \frac{\alpha_2}{k\sqrt{p}}.$$

Noting that the lower value of cos φ takes place when φ = ± i, where is is the slope of the orbit, we obtain

$$\alpha_2 = k\sqrt{p}\cos i.$$

When p = 0, the planet is at one of the nodes of its orbit. Therefore, we can consider, on the basis of equation (29), that

$$\beta_2 = \psi.$$

Now, we consider again equation (30). Instead of the latitude , we introduce the argument of the latitude u. Since

$$\sin\varphi = \sin i \sin u,$$

then, substituting the values obtained for α_2 and α_3, we obtain

$$\alpha_3 \int_0^{\varphi} (\alpha_3^2 - \alpha_2^2 \sec^2\varphi)^{-\frac{1}{2}} d\varphi = \int_0^{u} (1 - \cos^2 i \sec^2\varphi)^{-\frac{1}{2}} \sec\varphi \sin i \cos u \, du = u.$$

Hence

$$u - \beta_3 = \alpha_3 \int_{r_0}^{r} r^{-2}\left(2\alpha_1 + 2k^2 r^{-1} - \alpha_3^2 r^{-2}\right)^{-\frac{1}{2}} dr.$$

At the moment of passing by the perihelion, $u = \omega$ and $r = r_0$, so that

$$\beta_3 = \omega = \pi - \Omega.$$

Thus, we obtain the following system of canonical elements

$$\left.\begin{array}{ll} \alpha_1 = -\dfrac{k^2}{2a}; & \beta_1 = -T \\ \alpha_2 = k\sqrt{p}\cos i; & \beta_2 = \Omega \\ \alpha_3 = k\sqrt{p}; & \beta_3 = \pi - \Omega. \end{array}\right\} \qquad (31)$$

We shall evaluate the average longitude for the perturbed as well as the unperturbed motion by (§ 12)

$$\lambda = \varepsilon + \int_0^t n\,dt.$$

On the other hand, we have for the unperturbed motion

$$\lambda = \pi + n(t - T) = \varepsilon + nt,$$

Hence,

$$\beta_1 = -T = \frac{\varepsilon - \pi}{n}$$

or

$$\beta_1 = \frac{\varepsilon - \pi}{k}\, a^{\frac{3}{2}}. \qquad (32)$$

To conclude with, we express the elliptic elements in terms of the conventional ones. We obtain

$$\left.\begin{array}{ll} a = -\dfrac{k^2}{2\alpha_1}; & \Omega = \beta_2 \\ e^2 = 1 + \dfrac{2\alpha_1\alpha_3^2}{k^4}; & \pi = \beta_2 + \beta_3 \\ \cos i = \dfrac{\alpha_2}{\alpha_3}; & \varepsilon = \beta_2 + \beta_3 + \beta_1 k^{-2}(-2\alpha_1)^{\frac{3}{2}} \end{array}\right\} \qquad (33)$$

24. Application of the canonical elements to the derivation of the Lagrange Equations

The canonical elements have the advantage over the conventional ones in that the equations for their variation during the perturbed motion, are very simple (§ 22). These equations are

$$\frac{d\alpha_k}{dt} = \frac{\partial R}{\partial \beta_k}, \quad \frac{d\beta_k}{dt} = -\frac{\partial R}{\partial \alpha_k} \qquad (k = 1, 2, 3). \quad (34)$$

However, the conventional elements are used in the actual calculation of the positions of the planets. Therefore, it is useful to derive equations for the derivatives of the elliptic elements. For this purpose, we differentiate equations (33), and taking into account equations (34), we obtain

$$\frac{da}{dt} = +\frac{2a^2}{k^2}\frac{\partial R}{\partial \beta_1}$$

$$\frac{de}{dt} = \frac{a(1-e^2)}{k^2 e}\frac{\partial R}{\partial \beta_1} - \frac{na\sqrt{1-e^2}}{k^2 e}\frac{\partial R}{\partial \beta_2}$$

$$\frac{di}{dt} = \frac{\operatorname{cosec} i}{k\sqrt{a(1-e^2)}}\left(\cos i\,\frac{\partial R}{\partial \beta_3} - \frac{\partial R}{\partial \beta_2}\right)$$

$$\frac{d\Omega}{dt} = -\frac{\partial R}{\partial \alpha_2}$$

$$\frac{d\pi}{dt} = -\frac{\partial R}{\partial \alpha_2} - \frac{\partial R}{\partial \alpha_3}$$

$$\frac{d\varepsilon}{dt} = -\frac{\partial R}{\partial \alpha_2} - \frac{\partial R}{\partial \alpha_3} - n\frac{\partial R}{\partial \alpha_1} - \frac{3a}{k^2}(\varepsilon - \pi)\frac{\partial R}{\partial \beta_1}.$$

These equations enables us to express the derivatives of the perturbation function with respect to the canonical elements in terms of its derivative with respect to the elliptic elements. It is easy to see that

$$\frac{\partial R}{\partial\beta_1} = n\frac{\partial R}{\partial\epsilon}$$

$$\frac{\partial R}{\partial\beta_2} = \frac{\partial R}{\partial\Omega} + \frac{\partial R}{\partial\pi} + \frac{\partial R}{\partial\epsilon}$$

$$\frac{\partial R}{\partial\beta_3} = \frac{\partial R}{\partial\pi} + \frac{\partial R}{\partial\epsilon}$$

$$\frac{\partial R}{\partial\alpha_1} = \frac{2a^2}{k^2}\frac{\partial R}{\partial a} + \frac{a(1-e^2)}{k^2 e}\frac{\partial R}{\partial e} - \frac{3a}{k^2}(\epsilon-\pi)\frac{\partial R}{\partial\epsilon}$$

$$\frac{\partial R}{\partial\alpha_2} = \frac{-\operatorname{cosec} i}{k\sqrt{a(1-e^2)}}\frac{\partial R}{\partial i}$$

$$\frac{\partial R}{\partial\alpha_2} = -\frac{\sqrt{a(1-e^2)}}{kae}\frac{\partial R}{\partial e} + \frac{\operatorname{ctg} i}{k\sqrt{a(1-e^2)}}\frac{\partial R}{\partial i}.$$

Thus we obtain the following equations, which we have already obtained by another method in chapter III,

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial\epsilon}\\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial\pi} - \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{1}{na^2}\frac{\partial R}{\partial\epsilon}\\
\frac{di}{dt} &= -\frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial\Omega} - \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R}{\partial\pi} + \frac{\partial R}{\partial\epsilon}\right)\\
\frac{d\Omega}{dt} &= \frac{\operatorname{cosec} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i}\\
\frac{d\pi}{dt} &= \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial R}{\partial e}\\
\frac{d\epsilon}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} + \frac{\operatorname{tg}\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial i} + \frac{e\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\frac{1}{na^2}\frac{\partial R}{\partial e}.
\end{aligned}\right| \quad (35)$$

We remind the reader that in these formulae the quantity $ka^{-3/2}$ is denoted by n. We note that the last of these formulae can be understood in two ways. On the one hand, if the average longitude of the planet is evaluated by the formula

$$\lambda = \varepsilon + n(t - t_0) = \varepsilon + ka^{-3/2}(t - t_0), \tag{36}$$

then R will depend on "a" explicity and through λ . Hence

$$\frac{\partial R}{\partial a} = \left(\frac{\partial R}{\partial a}\right) + \frac{\partial R}{\partial \lambda}\frac{\partial \lambda}{\partial a} = \left(\frac{\partial R}{\partial a}\right) + \frac{\partial R}{\partial \varepsilon}\frac{\partial n}{\partial a}(t - t_0),$$

where the derivative on the left-hand sides corresponds to the total variation of "a", which $\left(\frac{\partial R}{\partial a}\right)$ is the derivative evaluated when is kept constant. On the other hand, if the average longitude is evaluated by

$$\lambda = \varepsilon + \int_{t_0}^{t} n\,dt, \tag{37}$$

where n denotes a function of time defined by

$$\frac{dn}{dt} = -\frac{3}{a^2}\frac{\partial R}{\partial \varepsilon}, \tag{38}$$

then, it is necessary to evaluate the derivatives of R with respect to "a" in the last of formulae (35) fixing the value of λ , i.e. to take instead of $\frac{\partial R}{\partial a}$. In this way we avoid obtaining a term, that increases with increasing time, on the right-hand side of the last of equations (35).

25. The canonical elements of Delaunay and Poincaré

The canonical elements, defined by equations (31) suffer from a shortcoming, closely related to what we have seen in the end of the previous section. The element α_1 enters the perturbation function explicity and through n. Thus, we will have on the right-hand side of the following equation

$$\frac{d\beta_1}{dt} = -\frac{\partial R}{\partial \alpha_1}$$

a term proportional to time. The canonical form of the equations will be lost if we try to bypass this shortcoming by the method suggested in the previous section, i.e. replacing the element ϵ, given by equation (36), by the element ϵ, given by equation (37).

Delaunay suggested the introduction of the elements

$$L = k\sqrt{a} = k^2(-2\alpha_1)^{-\frac{1}{2}}$$

$$l = n(t - T) = ka^{-\frac{3}{2}}(t + \beta_1) = k^{-2}(-2\alpha_1)^{\frac{3}{2}}(t + \beta_1).$$

instead of the elements α_1 and β_1. Since the difference

$$\beta_1 d\alpha_1 - l\,dL = -t\,d\alpha_1$$

is a complete differential of the function $W = -t\alpha_1$, then on the basis of theorem II (§ 20) we obtain again a canonical form after the transformation. Introducing the following notation

$$\alpha_2 = H, \qquad \alpha_3 = G, \qquad \beta_2 = h, \qquad \beta_3 = g,$$

we obtain

$$\left.\begin{aligned}
\frac{dL}{dt} &= \frac{\partial R'}{\partial l}, & \frac{dl}{dt} &= -\frac{\partial R'}{\partial L} \\
\frac{dG}{dt} &= \frac{\partial R'}{\partial g}, & \frac{dg}{dt} &= -\frac{\partial R'}{\partial G} \\
\frac{dH}{dt} &= \frac{\partial R'}{\partial h}, & \frac{dh}{dt} &= -\frac{\partial R'}{\partial H}
\end{aligned}\right| \tag{39}$$

where

$$R' = R - \alpha_1 = R + \frac{k^4}{2L^2}.$$

Delaunay elements are expressed in terms of the elliptic elements by

$$\left.\begin{array}{lll} L = k\sqrt{a}, & G = k\sqrt{a(1-e^2)}, & H = k\sqrt{a(1-e^2)}\cos i \\ l = n(t-T), & g = \pi - \Omega, & h = \Omega. \end{array}\right\} \quad (40)$$

Hence, in this system, one of the elements is the mean anomaly ℓ . The elements L, G and H have the dimension of areal velocity, while the other elements, ℓ , g and h, are angles.

It is possible to find other homogeneous canonical elements, which have some advantages over Delaunay elements. First of all, following Poincaré, we consider the following system of elements

$$\left.\begin{array}{ll} L = k\sqrt{a}; & \lambda = l + \pi = nt + \varepsilon; \\ \rho_1 = k\sqrt{a}\,(1 - \sqrt{1-e^2}); & \omega_1 = -\pi; \\ \rho_2 = k\sqrt{a(1-e^2)}\,(1 - \cos i); & \omega_2 = -\Omega, \end{array}\right\} \quad (41)$$

where λ is the mean longitude. This system has the advantage that at small eccentricities and slopes, the elements ρ_1 and ρ_2 are also small.

We prove that, in the transition to the elements (41), the differential equations (39) preserve the canonical form. We consider the expression

$$l\,dL + g\,dG + h\,dH - \lambda\,dL - \omega_1 d\rho_1 - \omega_2 d\rho_2,$$

which evidently equals

$$l\,dL + g\,dG + h\,dH - (l+g+h)\,dL + (g+h)(dL - dG) + h(dG - dH) = 0,$$

so that the conditions of theorem II are fulfilled.

In addition to system (41), Poincaré also introduced the following system

$$\left.\begin{array}{ll} L = k\sqrt{a}; & \lambda = l + \pi \\ \xi_1 = \sqrt{2\rho_1}\cos\omega_1; & \eta_1 = \sqrt{2\rho_1}\sin\omega_1, \\ \xi_2 = \sqrt{2\rho_2}\cos\omega_2; & \eta_2 = \sqrt{2\rho_2}\sin\omega_2, \end{array}\right| \qquad (42)$$

which is canonical in the light of the arguments given in § 20 (example I). The elements ξ_1 and η_1 are of order of magnitude of the eccentricity, while ξ_2 and η_2 are of the same order as the slope of the orbit.

The characteristic property of the canonical elements (40), (41) and (42) is the choice of the mean anomaly or the mean longitude as one of the variables. Levi-civita and Hill were able to find other canonical systems, in which one of the elements is the eccentrisity or the true anomaly. De Sitter and Ardoyer developed the general approach for obtaining such systems of elements(1).

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CHAPTER V

APPLICATION OF THE CANONICAL VARIABLES IN THE STUDY OF PERTURBATIONS

26. The canonical form of the equations of relative motion

The transformation of the equations of motion of a system, for which a force function exists, into a canonical form is carried out in § 19. In the present section, we consider the equations of relative motion. When the motion of points m_1 (x_1, y_1, z_1), ..., m_{n-1} $(x_{n-1}, y_{n-1}, z_{n-1})$ are related to point m_o (x_o, y_o, z_o), taken as the coordinate origin, the equations of motion are (§ 3)

$$\left.\begin{array}{l}\frac{d^2x_i}{dt^2} + k^2(m_0 + m_i)\frac{x_i}{r_i^3} = \frac{\partial R_i}{\partial x_i} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (i = 1, 2, \ldots, n-1),\end{array}\right| \tag{1}$$

where R_i denotes the perturbation function that corresponds to point m_i. To each point m_i there corresponds a force function

$$U_i = \frac{k^2(m_0 + m_i)}{r_i} + R_i.$$

Thus, equations (1) may be transformed into the following forms

$$\left.\begin{array}{ll}\frac{dx_i}{dt} = \frac{\partial H_i}{\partial x_i'}, & \frac{dx_i'}{dt} = -\frac{\partial H_i}{\partial x_i} \\ \frac{dy_i}{dt} = \frac{\partial H_i}{\partial y_i'}, & \frac{dy_i'}{dt} = -\frac{\partial H_i}{\partial y_i} \\ \frac{dz_i}{dt} = \frac{\partial H_i}{\partial z_i'}, & \frac{dz_i'}{dt} = -\frac{\partial H_i}{\partial z_i}\end{array}\right\}$$

where

$$H_i = T - U_i,$$

These equations are not canonical since the functions $H_1, H_2, \dots, H_{n-1}$ are different. Poincaré called these equations semicanonical.

It is useful to obtain the equations of relative motion in a canonical form. For this purpose, it is necessary to choose the relative coordinates in a different way. A convenient choice of relative coordinates is given in § 4. In these coordinates, the equations of motion are given by

$$\left.\begin{aligned} \mu_i \frac{d^2 x_i}{dt^2} &= \frac{\partial U}{\partial x_i}, \\ \mu_i \frac{d^2 y_i}{dt^2} &= \frac{\partial U}{\partial y_i}, \\ \mu_i \frac{d^2 z_i}{dt^2} &= \frac{\partial U}{\partial z_i}, \end{aligned}\right| \qquad (2)$$

where

$$\mu_i = \frac{m_i (m_0 + m_1 + \dots + m_{i-1})}{m_0 + m_1 + \dots + m_i},$$

$$U = \frac{1}{2} k^2 \sum_i \sum_j \frac{m_i m_j}{\Delta_{ij}}.$$

These equations have the same structure as the equations of the absolute motion in the presence of a force function. They can be transformed into a canonical form in the usual way (§ 19). Assuming that

$$T = \frac{1}{2} \sum \mu_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

$$H = T - U$$

$$x_i' = \mu_i \dot{x}_i, \qquad y_i' = \mu_i \dot{y}_i, \qquad z_i' = \mu_i \dot{z}_i,$$

we obtain

$$\left.\begin{aligned}
\frac{dx_i}{dt} &= \frac{\partial H}{\partial x'_i}, & \frac{dx'_i}{dt} &= -\frac{\partial H}{\partial x_i} \\
\frac{dy_i}{dt} &= \frac{\partial H}{\partial y'_i}, & \frac{dy'_i}{dt} &= -\frac{\partial H}{\partial y_i} \\
\frac{dz_i}{dt} &= \frac{\partial H}{\partial z'_i}, & \frac{dz'_i}{dt} &= -\frac{\partial H}{\partial z_i}
\end{aligned}\right\} \qquad (3)$$

If the masses m_1, m_2, ..., m_{n-1} are small compared to m_o, then their mutual gravitation may be neglected. We may then take instead of U the function

$$U_0 = k^2 \sum_1^{n-1} \frac{m_0 m_i}{r_i},$$

where

$$r_i^2 = x_i^2 + y_i^2 + z_i^2,$$

Consequently, the equations of motion are devided into n-1 separate systems. For example, equations (2) will have the form

$$\frac{d^2 x_1}{dt^2} = \text{[illegible]}$$

$$\cdots\cdots\cdots\cdots$$

Hence, in the first approximation, we only have to solve a two-body problem to obtain the coordinates x_i, y_i and z_i as functions of time and six constant orbital elements. We shall take as orbital elements the second set of canonical elements of Poincaré (§ 25). We denote the elements that correspond to point m_i by

$$L_i,\ \lambda_i,\ \xi_{i,1},\ \xi_{i,2},\ \eta_{i,1},\ \eta_{i,2}.$$

In section 25, it was convenient to avoid double indices. Here, we shall adopt the notation

$$\xi_{i,1} = \xi_{2i-1}, \qquad \xi_{i,2} = \xi_{2i},$$
$$\eta_{i,1} = \eta_{2i-1}, \qquad \eta_{i,2} = \eta_{2i}.$$

Thus, we obtain in the first approximation

$$\begin{aligned} x_i &= f(t,\ L_i,\ \lambda_i,\ \xi_{2i-1},\ \xi_{2i},\ \eta_{2i-1},\ \eta_{2i}) \\ y_i &= \varphi(t,\ L_i,\ \ldots \qquad \ldots\ldots\ldots) \\ z_i &= \psi(t,\ L_i,\ \qquad \ldots\ldots\ldots\ldots) \end{aligned}$$

as well as similar expressions for x_i' , y_i' and z_i' .

We apply the method of variation of arbitrary constants to obtain the general solution of equations (3). We replace the variables x_i y_i , z_i, x_i' , y_i' and z_i' by the variables L_j, λ_j, ξ_j and η_j using the equations just written. We choose as a perturbation function the quantity

$$R = U - U_0,$$

since, in this case,

$$H = T - U_0 - R.$$

We obtain the transformed equations in the following form

$$\left.\begin{aligned} \frac{dL_i}{dt} &= \frac{\partial R'}{\partial \lambda_i} & \frac{d\lambda_i}{dt} &= -\frac{\partial R'}{\partial L_i} \\ \frac{d\xi_j}{dt} &= \frac{\partial R'}{\partial \eta_j} & \frac{d\eta_j}{dt} &= -\frac{\partial R'}{\partial \xi_j} \end{aligned}\right\} \tag{4}$$

$$(i = 1,\ 2,\ \ldots,\ n-1; \qquad j = 1,\ 2,\ \ldots,\ 2n-2),$$

where (§ 25)

$$R' = R + \sum_{1}^{n-1} k^4 m_0^2 \left(\frac{m_0 + m_1 + \ldots + m_i}{m_0 + m_1 + \ldots + m_{i-1}}\right)^2 \frac{1}{2L_i^2},$$

since, in the present case, the quantity k^2 for point m_i should be replaced by
$$k^2 m_0 \frac{m_0 + m_1 + \ldots + m_i}{m_0 + m_1 + \ldots + m_{i-1}}.$$

It is useful to note that the replacement of R by R' is connected with the choice of the mean longitude λ_i as one of the variables.

We finally apply equations (4) to the motion of three bodies having masses m_0, m_1 and m_2. In this case,

$$R' = R + k^4 m_0^2 \left[\frac{(m_0 + m_1)^2}{2 m_0^2 L_1^2} + \frac{(m_0 + m_1 + m_2)^2}{2(m_0 + m_1)^2 L_2^2} \right],$$

where

$$R = U - U_0 = k^2 m_0 m_1 \left(\frac{1}{\Delta_{10}} - \frac{1}{r_1} \right) + k^2 m_0 m_2 \left(\frac{1}{\Delta_{20}} - \frac{1}{r_2} \right) + k^2 m_1 m_2 \frac{1}{\Delta_{12}}.$$

Equation (16) of Chapter I yields

$$\begin{aligned}
\Delta_{10}^2 &= x_1^2 + y_1^2 + z_1^2 = r_1^2 \\
\Delta_{20}^2 &= \left(x_2 + \frac{m_1 x_1}{m_0 + m_1} \right)^2 + \left(y_2 + \frac{m_1 y_1}{m_0 + m_1} \right)^2 + \left(z_2 + \frac{m_1 z_1}{m_0 + m_1} \right)^2 \\
\Delta_{12}^2 &= \left(x_2 - \frac{m_0 x_1}{m_0 + m_1} \right)^2 + \left(y_2 - \frac{m_0 y_1}{m_0 + m_1} \right)^2 + \left(z_2 - \frac{m_0 z_1}{m_0 + m_1} \right)^2.
\end{aligned}$$

In this way, we obtain

$$R = k^2 m_0 m_2 \left(\frac{1}{\Delta_{20}} - \frac{1}{r_2} \right) + k^2 m_1 m_2 \frac{1}{\Delta_{12}}. \tag{5}$$

27. The integrals of area

In § 4, it was claimed that the form of equations (2) was similar to the form of the equations of the absolute motion of n bodies, and for this reason, both sets of equation should have similar integrals of area. Thus, in the three-body problem (i = 1,2), equations (2) will have the following integrals, which correspond to integrals (6) of § 1,

$$\begin{aligned}
\mu_1 (y_1 \dot{z}_1 - z_1 \dot{y}_1) + \mu_2 (y_2 \dot{z}_2 - z_2 \dot{y}_2) &= C_1 \\
\mu_1 (z_1 \dot{x}_1 - x_1 \dot{z}_1) + \mu_2 (z_2 \dot{x}_2 - x_2 \dot{z}_2) &= C_2 \\
\mu_1 (x_1 \dot{y}_1 - y_1 \dot{x}_1) + \mu_2 (x_2 \dot{y}_2 - y_2 \dot{x}_2) &= C_3,
\end{aligned}$$

where

$$\mu_1 = \frac{m_1 m_0}{m_0 + m_1}, \qquad \mu_2 = \frac{m_1 (m_0 + m_1)}{m_0 + m_1 + m_2}.$$

We shall use the elliptic coordinates as an intermediate step to express these integrals in terms of the canonical elements. In the case of unperturbed motion, the areal velocities are expressed in terms of the elliptic elements by

$$\begin{aligned} y\dot{z} - z\dot{y} &= + G \sin i \sin \Omega \\ z\dot{x} - x\dot{z} &= - G \sin i \cos \Omega \\ x\dot{y} - y\dot{x} &= + G \cos i , \end{aligned}$$

where

$$G = k\sqrt{a(1-e^2)}.$$

These relations hold also for the perturbed motion because we are using osculating elements. The relations between the elliptic and canonical elements (§ 25) yield

$$G = L - \rho_1, \qquad G \cos i = L - \rho_1 - \rho_2,$$

and thus

$$G \sin i = \sqrt{2\rho_2 (L - \rho_1) - \rho_2^2}.$$

Consequently

$$\begin{aligned} G \sin i \sin \Omega &= G \sin i \frac{-\eta_2}{\sqrt{2\rho_2}} = -\eta_2 \sqrt{L - \rho_1 - \tfrac{1}{2}\rho_2} \\ G \sin i \cos \Omega &= G \sin i \frac{+\xi_2}{\sqrt{2\rho_2}} = +\xi_2 \sqrt{L - \rho_1 - \tfrac{1}{2}\rho_2}. \end{aligned}$$

We thus write the integrals of area in terms of the canonical elements in the following way

$$\begin{aligned} -\mu_1 \eta_{1,2} \sqrt{L_1 - \rho_{1,1} - \tfrac{1}{2}\rho_{1,2}} - \mu_2 \eta_{2,2} \sqrt{L_2 - \rho_{2,1} - \tfrac{1}{2}\rho_{2,2}} &= C_1 \\ +\mu_1 \xi_{1,2} \sqrt{L_1 - \rho_{1,1} - \tfrac{1}{2}\rho_{1,2}} + \mu_2 \xi_{2,2} \sqrt{L_2 - \rho_{2,1} - \tfrac{1}{2}\rho_{2,2}} &= C_2 \\ \mu_1 (L_1 - \rho_{1,1} - \rho_{1,2}) + \mu_2 (L_2 - \rho_{2,1} - \rho_{2,2}) \qquad &= C_3. \end{aligned}$$

If we take the invariable plane as the xy plane, then $C_1 = C_2 = 0$. In this case, the first two equalities yield

$$\frac{\eta_{1,2}}{\xi_{1,2}} = \frac{\eta_{2,2}}{\xi_{2,2}}$$

or (§ 25)

$$\operatorname{tg}\Omega_1 = \operatorname{tg}\Omega_2 .$$

In other words, the line of intersection of two relative osculating orbits is parallel to the invariable plane. This property permits us to introduce one node instead of two. This was the reason why Jacobi called this property the elimination of nodes. It was shown in section 2 that the elimination of nodes is a particular case of a more general property of dynamic systems.

28. Expression of rectangular coordinates in terms of canonical elements

Before integrating equations (4), we have to express the perturbation function R in terms of the canonical elements. The perturbation function is easily expressed in terms of the rectangular coordinates. Thus we start by expressing rectangular coordinates x_i, y_i and z_i in terms of the elements L_i, λ_i, ... of this point. We recall the formulae that connect the canonical elements of Poincaré with the elliptic elements (§ 25). Introducing the angle of eccentricity φ, we write these formulae in the following manner

$$\left.\begin{aligned}
&L = \sqrt{fMa}, && \lambda = \text{[illegible]},\\
&\rho_1 = 2L\sin^2\frac{\varphi}{2}, && \omega_1 = -\pi,\\
&\rho_2 = 2L\cos\varphi\sin^2\frac{i}{2}, && \omega_2 = -\Omega,\\
&\xi_1 = 2\sqrt{L}\sin\frac{\varphi}{2}\cos\pi, && \eta_1 = -2\sqrt{L}\sin\frac{\varphi}{2}\sin\pi,\\
&\xi_2 = 2\sqrt{L\cos\varphi}\sin\frac{i}{2}\cos\Omega; && \eta_2 = -2\sqrt{L\cos\varphi}\sin\frac{i}{2}\sin\Omega.
\end{aligned}\right\} \tag{1}$$

The well-known relations between the rectangular and alliptic coordinates lead to

$$x = r\cos u\cos\Omega - r\sin u\sin\Omega\cos i$$
$$y = r\cos u\sin\Omega + r\sin u\cos\Omega\cos i$$
$$z = r\sin u\sin i,$$

where

$$u = v + \pi - \Omega$$

is the argument of the latitude. These formulae can be represented in the following way

$$\left.\begin{aligned} x &= X\left[\cos^2\frac{i}{2}\cos\lambda + \sin^2\frac{i}{2}\cos(\lambda - 2\Omega)\right] - \\ &\quad - Y\left[\cos^2\frac{i}{2}\sin\lambda + \sin^2\frac{i}{2}\sin(\lambda - 2\Omega)\right] \\ y &= X\left[\cos^2\frac{i}{2}\sin\lambda - \sin^2\frac{i}{2}\sin(\lambda - 2\Omega)\right] + \\ &\quad + Y\left[\cos^2\frac{i}{2}\cos\lambda - \sin^2\frac{i}{2}\cos(\lambda - 2\Omega)\right] \\ z &= X\sin i\sin(\lambda - \Omega) + Y\sin i\cos(\lambda - \Omega), \end{aligned}\right\} \tag{7}$$

if

$$X = r\cos(v - M), \qquad Y = r\sin(v - M),$$

and $M = \lambda - \pi$ denotes the average anomaly.

The result which we are trying to obtain can be expressed in the form of the following theorem:

<u>Theorem</u>

<u>Each of the rectangular coordinates can be expanded in a series of the type</u>

$$\Sigma A \xi_1^{\alpha_1} \xi_2^{\alpha_2} \eta_1^{\beta_1} \eta_2^{\beta_2} \cos(k\lambda + H), \tag{8}$$

<u>where</u> $\alpha_1, \alpha_2, \beta_1, \beta_2$ <u>and k are positive integers or zeros, H is a constant and A depends on L.</u>

We first show that the coordinates can be expanded in series of positive integral powers of ξ_1, ξ_2, η_1 and η_2, such that the expansion coefficients depend on L and are periodic functions of λ. Indeed, the expressions of X and Y in equations (7) consist of $\sin\lambda$ and $\cos\lambda$ multiplied by the quantities

$$\cos^2\frac{i}{2}, \qquad \sin^2\frac{i}{2}\,{}^{\cos}_{\sin}\,2\Omega, \qquad \sin i\,{}^{\cos}_{\sin}\,\Omega. \tag{9}$$

However, on the basis of expressions (6'),

$$\xi_1^2+\eta_1^2 = 4L\sin^2\frac{\varphi}{2}, \qquad \xi_2^2+\eta_2^2 = 4L\cos\varphi\sin^2\frac{i}{2};$$

Consequently,

$$\sin^2\frac{i}{2} = \frac{\xi_2^2+\eta_2^2}{4L\left(1-\frac{\xi_1^2+\eta_1^2}{2L}\right)} = \frac{\xi_2^2+\eta_2^2}{4L-2(\xi_1^2+\eta_1^2)},$$

$$\cos^2\frac{i}{2} = \frac{4L-2(\xi_1^2+\eta_1^2)-\xi_2^2-\eta_2^2}{4L-2(\xi_1^2+\eta_1^2)}.$$

We see that $\sin^2\frac{i}{2}$ and $\cos^2\frac{i}{2}$ can be expanded in series of the required type. Moreover, we have

$$\sin i = 2\sqrt{\xi_2^2+\eta_2^2}\,\frac{\sqrt{4L-2(\xi_1^2+\eta_1^2)-\xi_2^2-\eta_2^2}}{4L-2(\xi_1^2+\eta_1^2)}.$$

On the other hand,

$$\operatorname{tg}\Omega = -\frac{\eta_2}{\xi_2}.$$

Hence

$$\sin\Omega = \frac{-\eta_2}{\sqrt{\xi_2^2+\eta_2^2}}, \qquad \cos\Omega = \frac{\xi_2}{\sqrt{\xi_2^2+\eta_2^2}}$$

$$\sin 2\Omega = \frac{-2\eta_2\xi_2}{\xi_2^2+\eta_2^2}, \quad \cos 2\Omega = \frac{\xi_2^2-\eta_2^2}{\xi_2^2+\eta_2^2}.$$

Comparing these expressions with the previous ones, we see that the five quantities (9) are expandable in series of positive integral powers of

ξ_1, ξ_2, η_1 and η_2. Let us now consider the quantities X and Y. Since

$$r\cos v = a(\cos E - e), \qquad r\sin v = a\sqrt{1-e^2}\,\sin E,$$

then

$$\frac{1}{a}X = -e\cos M + \frac{1+\sqrt{1-e^2}}{2}\cos(E-M) + \frac{1-\sqrt{1-e^2}}{2e^2}\,e^2\cos(E+M)$$
$$\frac{1}{a}Y = +e\sin M + \frac{1+\sqrt{1-e^2}}{2}\sin(E-M) - \frac{1-\sqrt{1-e^2}}{2e^2}\,e^2\sin(E+M).$$

Each of the quantities

$$\frac{1+\sqrt{1-e^2}}{2}, \qquad \frac{1-\sqrt{1-e^2}}{2e^2},$$
$$\genfrac{}{}{0pt}{}{\cos}{\sin}(E-M), \qquad \genfrac{}{}{0pt}{}{\cos}{\sin}(E+M)$$

can be expanded in positive integral powers of e sin M and e cos M. This is evident as far as the expansion of the first two quantities in powers of e' is concerned. On the other hand, the Kepler equation

$$M = E - e\sin E$$

leads to

$$E - M - e\sin M\cos(E-M) - e\cos M\sin(E-M) = 0. \qquad (10)$$

Assuming that

$$w = E - M, \qquad z_1 = e\sin M, \qquad z_2 = e\cos M,$$

this equation may be rewritten as

$$f(w, z_1, z_2) = 0,$$

where the left-hand side is a holomorphic function of w, z, and z_2 at the point $w = z_1 = z_2 = 0$. According to a well-known theorem on implicit functions, if

$$\frac{\partial f}{\partial w} \neq 0 \text{ для } w = z_1 = z_2 = 0,$$

then thisequation has one and only one solution $w = \varphi(z_1, z_2)$ being holomorphic in the vecinity of point $z_1 = 0$, $z_2 = 0$. This property is satisfied in the present case since

$$\left(\frac{\partial f}{\partial w}\right)_0 = 1.$$

Therefore, we obtain from equation (10) the quantity w = E - M in the form of a series expansion in positive integral powers of z_1 and z_2. We conclude that sin (E-M) and cos (E-M) are also expandable in similar series.

Finally, the equalities

$$\begin{gathered} e^2 \cos (E+M) = e^2 \cos 2M \cos (E-M) - e^2 \sin 2M \sin (E-M) \\ e^2 \sin (E+M) = e^2 \cos 2M \sin (E-M) + e^2 \sin 2M \cos (E-M) \\ e^2 \cos 2M = (e \cos M)^2 - (e \sin M)^2 \\ e^2 \sin 2M = 2(e \sin M)(e \cos M) \end{gathered}$$

indicate that the expression

$$e^2 {\cos \atop \sin} (E+M)$$

has the required property. In this manner, the possibility of expansion of X and Y in positive integral powers of e sin M and e cos M is proved.

Since M = $\lambda - \pi$, then

$$\begin{gathered} e \sin M = e \cos \pi \sin \lambda - e \sin \pi \cos \lambda \\ e \cos M = e \cos \pi \cos \lambda + e \sin \pi \sin \lambda \end{gathered}$$

Thus, X and Y can be expanded in positive integral powers of e cos π and e sin π, and the expansion coefficients will be functions of time.

The theorem will be completely proved when we prove the possibility of expanding e cos π and e sin π in positive integral powers of ξ_1 and η_1. First of all, if follows from equations (6') that

$$\xi_1^2 + \eta_1^2 = 4L \sin^2 \frac{\gamma}{2} = 2L(1 - \cos \gamma) = 2L\left(1 - \sqrt{1 - e^2}\right)$$

From this equation, it follows that

$$e = \sqrt{\frac{1}{L}(\xi_1^2 + \eta_1^2) - \frac{1}{4L^2}(\xi_1^2 + \eta_1^2)^2}$$

Consequently,

$$e = \frac{\sqrt{\xi_1^2 + \eta_1^2}}{\sqrt{L}} \left[1 - \frac{1}{4L}(\xi_1^2 + \eta_1^2)\right]^{\frac{1}{2}}.$$

On the other hand, the same formulae (6') yields

$$\frac{e\cos\pi}{\xi_1} = \frac{e\sin\pi}{-\eta_1} = \frac{e}{\sqrt{\xi_1^2+\eta_1^2}}.$$

Hence

$$e\cos\pi = \frac{\xi_1}{\sqrt{L}}\left[1-\frac{1}{4L}(\xi_1^2+\eta_1^2)\right]^{\frac{1}{2}}$$

$$e\sin\pi = \frac{-\eta_1}{\sqrt{L}}\left[1-\frac{1}{4L}(\xi_1^2+\eta_1^2)\right]^{\frac{1}{2}}.$$

We have proved that each of the coordinates x, y and z can be represented by a series of the type

$$\Sigma B\xi_1^{\alpha_1}\xi_2^{\alpha_2}\eta_1^{\beta_1}\eta_2^{\beta_2}\Psi(\lambda),$$

where B is a function of L and $\Psi(\lambda)$ a periodic function of λ, the period of which equals 2π. The function $\Psi(\lambda)$ can be expanded in a Fourier series

$$\Psi(\lambda) = \Sigma(C_k\cos k\lambda + D_k\sin k\lambda).$$

Since

$$\sin k\lambda = \cos\left(k\lambda+\frac{\pi}{2}\right),$$

then, this proves our theorem.

The series (8) obtained above converge for small values of ξ_1, ξ_2, η_1 and η_2. A more exact determination of the region of convergence of these series will not be given here.

29. Expression of the perturbation function in terms of the canonical variables

The perturbation function R, which we are going to study now, is equal to the difference U - U_0 (see section 26). Consequently

$$R = \frac{1}{2}k^2\sum_{0}^{n-1}\sum_{0}^{n-1}\frac{m_k m_l}{\Delta_{kl}} - k^2\sum_{1}^{n-1}\frac{m_0 m_i}{r_i} \qquad (11)$$
$$(k \neq l),$$

where

$$r_i^2 = x_i^2 + y_i^2 + z_i^2$$
$$\Delta_{ki}^2 = (x_k - x_i)^2 + (y_k - y_i)^2 + (z_k - z_i)^2$$

If neither the quantity r_i nor Δ_{ij} equals zero at any time, when

$$\xi_j = 0, \qquad \eta_j = 0, \qquad (j = 1, 2, \ldots, n-1),$$

then both of them, and consequently the perturbation function will be holomorphic functions in the vicinity of point $\xi_j = 0$, $\eta_j = 0$. Hence, the function R can be expanded in positive powers of ξ_j and η_j in the vecinity of this point. The expansion coefficients willobe finite and continuous for all values of t. They will be periodic functions of λ_i having a period of 2π. Hence, these coefficients may be expanded in multiple Fourier series of the type.

$$\sum \{C \cos(\sum k_i \lambda_i) + D \sin(\sum k_i \lambda_i)\},$$

where the summation is carried over the indices k_i run over all the positive as well as the negative integral values. This result can be stated in the following way:

Theorem I

If points m_0, m_1, ..., m_{n-1} move in such a way, that their mutual separations Δ_{kl} and radius vectors r_i are never equal to zero, then the corresponding perturbation functions R can be exnanded in a series of the type

$$R = \sum A \mathfrak{M} \cos(\sum k_i \lambda_i + H),$$

where H is a constant, the coefficients A depend only on L_i and m is a product of positive powers of ξ_i and η_i, i.e. $m = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \eta_1^{\beta_1} \eta_2^{\beta_2} \ldots$ The summation is over the indices α_1, α_2, ..., β_1, β_2 ... and also over k_i which all run over all possible positive and negative integral values. The series (12) converges for sufficiently small values of ξ_i and η_i.

We now investigate how the function R can be expressed in terms of the variables ρ_j and ω_j, related to ξ_j and η_j by

$$\xi_j = \sqrt{2\rho_j}\cos\omega_j, \qquad \eta_j = \sqrt{2\rho_j}\sin\omega_j. \tag{13}$$

We prove the following theorem:

Theorem II

Under the same conditions, as those which hold in theorem I, the function R can be expanded in the series

$$R = \sum A\rho_1^{q_1}\rho_2^{q_2}\ldots\cos(\sum k_i l_i + \sum p_j \omega_j + H), \tag{14}$$

which converges for sufficiently small values of ρ_i. Each of the quantities $2q_i$ takes the values 0, 1, 2, ..., while the indices k_i and p_i run over all the integral values from $-\infty$ to $+\infty$. For each term of this expansion, the following condition holds

$$2q_j \geq |p_j|; \qquad 2q_j \equiv p_j \pmod 2). \tag{15}$$

where the coefficients A depend only on L_i while the quantities H are constants.

In order to prove this theorem, we transform each term of the series (12) by introducing the new variables ρ and ω instead of the variables ξ and η by means of equations (13). We start by those terms for which the product m = 1. Evidently, these terms will not be changed, their form will remain to be

$$A\cos(\sum k_i l_i + H). \tag{16}$$

Accordingly, they satisfy the conditions of the theorem.

We now consider the following expansion:

$$\Phi = A\rho_1^{q_1}\rho_2^{q_2}\ldots\cos(\sum k_i l_i + \sum p_j \omega_j + h).$$

where q and p satisfy the conditions required by the theorem. We show that the multiplication of these terms by ξ_h and η_h produces a sum of terms having the same type. For this purpose, we consider expression

$$\xi_h = \sqrt{2 z_h}\cos\omega_h, \text{ and } \eta_h = \sqrt{2 z_h}\sin(\omega_h)$$

It is better to consider the general expression

$$\sqrt{2 z_h}\cos(\omega_h + B'),$$

which is evidently equal to

$$\frac{1}{\sqrt{2}} A z_1^{q_1} z_2^{q_2} \cdots \sqrt{z_h}\,[\cos(\Sigma k_i \ell_i + \Sigma p_j \omega_j + \omega_h + B + B') + \cos(\Sigma k_i \ell_i + \Sigma p_j \omega_j - \omega_h + B - B')].$$

Let us consider the particular case in which h = 3. After the transformation, the coefficients $2q_1$, $2q_2$, $2q_4$, ... and $2p_1$, p_2 , p_4 , ... will not be altered and therefore satisfy condition (15). The coefficient $2q_3$ will be replaced by $2q_3 + 1$ while the coefficient p_3 will be replaced by $p_3 - 1$. Since, by condition,

$$2q_3 \equiv p_3 \pmod 2, \qquad 2q_3 \geq p_3,$$

then

$$2q_3 + 1 \equiv p_3 + 1 \pmod 2$$
$$2q_3 + 1 \geq p_3 + 1, \qquad 2q_3 + 1 \geq p_3 - 1$$

In this way, starting with a term of type (16), and progressing successively to other terms of expansion (12) by means of multiplications by ξ and η , we will only obtain terms satisfying condition (15).

30. <u>Poincaré's theorem on the rank</u>

In the previous sections, we have studied the forms of the expansions of the perturbation function R. Now, we consider again the integration of equations (4), which may be rewriten in terms of the new coordinates as follows

$$\left.\begin{array}{ll}
\dfrac{dL_i}{dt} = \dfrac{\partial R'}{\partial \lambda_i}, & \dfrac{d\lambda_i}{dt} = -\dfrac{\partial R'}{\partial L_i}, \\[2ex]
\dfrac{d\xi_j}{dt} = \dfrac{\partial R'}{\partial \eta_j}, & \dfrac{d\eta_j}{dt} = -\dfrac{\partial R'}{\partial \xi_j} \\[2ex]
(i = 1, 2, \ldots, n-1; & j = 1, 2, \ldots, 2n-2),
\end{array}\right\} \qquad (17)$$

where

$$R' = R_0 + R,$$

and

$$R_0 = k^4 m_0^3 \left[\frac{(m_0 + m_1)^2}{2 m_0^3 L_1^2} + \frac{(m_0 + m_1 + m_2)^2}{2 (m_0 + m_1)^2 L_2^2} + \ldots \right],$$

while the function R is defined by equation (5). On the basis of theorem I of section 29, the perturbation function can be expanded in a series of the type

$$R = \sum A \mathfrak{M} \cos (\sum k_i \lambda_i + H),$$

where m is a product of positive powers of ξ_i and η_i.

We remind the reader that the coefficients A depend only on elements L_i and can have as a multiplying factor only one of the masses m_i. This can easily be seen from equations (11). Accordingly, in the first approximation in which all $m_i = 0$, we may write

$$L_i = L_i^0, \quad \xi_j = \xi_j^0, \quad \eta_j = \eta_j^0, \quad \lambda_i = n_i t + \lambda_i^0,$$

where the upper index zero denotes constant values,

$$n_i = \frac{\partial R_0}{\partial L_i} = k^4 M_i^3 (L_i^0)^{-3},$$

and

$$M_1 = m_0 m_1, \quad M_2 = m_0 \frac{m_0 + m_1 + m_2}{m_0 + m_1}, \ldots$$

We substitute the values (18) into the right-hand side of equations (17). We obtain expressions of the following type:

$$\sum B \cos (\nu t + H),$$

where $\nu = \sum k_i n_i$.

In the second approximation, we obtain

$$\left.\begin{array}{ll} L_i = L_i^0 + \delta_1 L_i, & \lambda_i = n_i t + \lambda_i^0 + \delta_1 \lambda_i \\ \xi_i = \xi_i^0 + \delta_1 \xi_i, & \eta_i = \eta_i^0 + \delta_1 \eta_i, \end{array}\right\} \quad (20)$$

in which we denote by $\delta_1 L_i, \ldots, \delta_1 \eta_i$ the sums of the type

$$B_0 t + \sum_\nu B \sin(\nu t + h').$$

The secular elements $B_o t$ are obtained as a result of the integration of the three terms of series (19), in which $\nu = 0$.

It is easy to see that $\delta_1 L_i$ does not involve any secular elements. This result is equivalent to the Laplace-Legendre theorem, given in section 17, and which states that the semimajor axes of the orbits are invariable.

Substituting expressions (20) into the right-hand side of equations (17), we obtain the third approximation, and so on. In this way, we obtain, after an arbitrary number of approximations,

$$\left.\begin{array}{ll} L_i = L_i^0 + \delta L_i, & \lambda_i = n_i t + \lambda_i^0 + \delta\lambda_i \\ \xi_i = \xi_i^0 + \delta \xi_i, & \eta_i = \eta_i^0 + \delta \eta_i, \end{array}\right\} \quad (21)$$

where each of the quantities $\delta L_i, \ldots, \delta \eta_i$ is represented by a series having the form

$$\sum A t^p \mathfrak{M} \cos(\nu t + h'),$$

in which m is a product of nonegative powers of ξ_i^0 and η_i^0, while the coefficient A depends only on L_i^0 and has a multiplying factor of $m_1^{m'} m_2^{m''} \ldots$ where m', m'', ... are integers satisfying the relation

$$m' \geq 0, \quad m'' \geq 0, \ldots, \quad m' + m'' + \ldots > 0.$$

We remind the reader that the sum m' + m'' + ... is called the order of the corresponding series while the expression m' + m'' + ... - p is its rank. Poincaré proved the following theorem:

<u>Theorem</u>

If the mean motion of n planets is such that the following relation holds

$\nu = \sum k_i n_i =$

where k_i are integers, then

(1) The rank of each term of the expansions of δL_i, $\delta \lambda_i$, $\delta \xi_j$ and $\delta \eta_i$ is more than or equal to zero.

(2) The rank of each mixed term equals at least unity, and

(3) The expansion of δL_i does not involve zero-rank terms.

We have just seen that this theorem is valid for first-order terms. Indeed, the quantitis $\delta_1 L_i$, $\delta_1 \lambda_i$, $\delta_1 \xi_j$ and $\delta_1 \eta_j$ do not involve mixed terms. They can involve secular terms only of the type At, i.e. having a rank of more than or equal to zero. Finally, the expression $\delta_1 L_i$ involves no secular terms. We shall now prove that once the theorem is valid for all terms having order $\leqslant$ m, it will also be valid for the (m + 1)-order terms. We divide our proof into three parts. First, we deduce the expressions required for the calculation of the (m + 1) - order term. We substitute equations (21), in which $\delta L_i, \ldots, \delta \eta_i$ are understood as the aggregate ofterms having orders $\leqslant$ m, into the right-hand side of equations (17). Beforehand, we write these equations in the following way

$$\frac{dL_i}{dt} = \frac{\partial R}{\partial \lambda_i}, \qquad \frac{d\lambda_i}{dt} = -\frac{\partial R}{\partial L_i} - \frac{\partial R_0}{\partial L_i}$$

$$\frac{d\xi_j}{dt} = \frac{\partial R}{\partial \eta_j}, \qquad \frac{d\eta_j}{dt} = -\frac{\partial R}{\partial \xi_j}.$$

Integrating the three equations that do not involve R_o, we obtain

$$\delta L_i = \int_0^t \frac{\partial R}{\partial \lambda_i} dt, \qquad \delta \xi_j = \int_0^t \frac{\partial R}{\partial \eta_j} dt, \qquad \delta \eta_j = -\int_0^t \frac{\partial R}{\partial \xi_j} dt. \qquad (22)$$

The quantity R is of the first order relative to the masses m_1 , m_2, ...

The substitution of expressions (21) into the right-hand side of these equations, the error of which is of the (m + 1) order, will thus yield right-hand sides having an error of the (m + 2) - order. This allows us to evaluate all of the (m + 1)-order terms in $\delta L_i, \delta \xi_j$ and $\delta \eta_j$

In order to evaluate the (m + 1) - order terms in the expressions of the mean longitudes λ_i, we first consider the expansion of $\partial R/\partial L_i$.

Since

$$
\begin{aligned}
R_t &= R_0(L_1^0 + \delta L_1,\ L_2^0 + \delta L_2,\ \ldots) = \\
&= R_0(L_1^0,\ L_2^0,\ \ldots) + \sum_i \frac{\partial R_0}{\partial L_i^0}\,\delta L_i + \frac{1}{2}\sum_{ik} C_{ik}\,\delta L_i\,\delta L_k + \Phi,
\end{aligned}
$$

where

$$
C_{ik} = \frac{\partial^2 R_0}{\partial L_i^0\,\partial L_k^0},
$$

and by Φ is denoted the aggregate of the third- and higher-order terms relative to δL_i, then

$$
\frac{\partial R}{\partial L_i} = \frac{\partial R_0}{\partial L_i^0} + \sum_k C_{ik}\,\delta L_k + \frac{\partial \Phi}{\partial L_i}.
$$

Noting that

$$
-\frac{\partial R_0}{\partial L_i^0} = n_i, \qquad \delta L_k = \int_0^t \frac{\partial R}{\partial \epsilon_k}\,dt,
$$

we obtain, after integrating,

$$
\delta\lambda_i = -\sum_k C_{ik}\int_0^t dt\int_0^t \frac{\partial R}{\partial \epsilon_k}\,dt - \int_0^t \frac{\partial \Phi}{\partial L_i}\,dt - \int_0^t \frac{\partial R}{\partial L_i}\,dt \tag{23}
$$

We want to be sure whether the substitution of expression (21) into the right-hand side of this equation is done within an error of the (m + 2)-order relative to the masses. This is evidently correct as far as the first and the last terms are concerned. The reason is that the perturbation function R is a first-order quantity. Only the second term is required to be considered. First of all, we note that the partial derivative $\partial\Phi/\partial L_i$ is a sum of terms having at least the second order relative to the perturbations δL_k. Each of these perturbations is of the first order relative to the masses. We replace the quantities δL_k by their approximate values $\delta' L_k$ which involve errors of the order m+1. Identities of the type

$$AB - A'B' = A(B-B') + B'(A-A'),$$
$$ABC - A'B'C' = AB(C-C') + AC(B-B') + B'C'(A-A'),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

show that this replacement produces the partial derivative $\partial\Phi/\partial L$ to within an error of the (m + 2)-order. Thence, if we replace the perturbation involved in the right-hand side of equations (22) and (23) by their approximate values, which are correct to within m-order terms, we obtain all of the (m + 1)-order terms in the left-hand side.

We now prove the theorem as far as it concerns quantities δL_i, $\delta \xi_j$ and $\delta \eta_j$. We first note that the multiplication of two terms of positive ranks yields a sum of positive-rank terms. Similarly, the multiplication of terms of negative rank yields terms of negative rank. Hence, when we substitute expressions (21), which consist of terms of negative rank, into the right-hand side of equations (22), we obtain a sum of terms of non-negative rank in the expression of the integr . Moreover, since each term of the perturbation function R is multiplied by m_1 or m_2 , or ..., the ranks of all of these terms will be greater than or equal to unity.

In integrating the secular terms, their ranks are decreased by a unity as the following formula indicates

$$\int_0^t A\,t^p dt = \frac{A\,t^{p+1}}{p+1},$$

Hence, expressions (22) consist of terms whose rank is not less than zero. The above-mentioned reduction of the rank takes place only for purely secular terms. Hence, equations (22) cannot involve mixed terms having zero-ranks.

We still have to prove that the expr ssion

$$\delta L_i = \int_0^t \frac{\partial R}{\partial \lambda_i}\, dt$$

involves no terms of the zero rank. We substitute expression (21) into that of the partial derivative $\partial R/\partial\lambda_i$. Expanding this partial derivative in a series, we obtain

$$\frac{\partial R}{\partial \lambda_i} = \sum D_0 \mathfrak{R}, \tag{24}$$

where D_0 denotes those partical derivatives of $\partial R/\partial\lambda_i$ with respect to the elements, in which the values of the elements are replaced by the following initial values:

$$L_i^0,\quad n_i t + \lambda_i^0,\quad \xi_j^0,\quad \eta_j^0;$$

and R denotes a product of non-negative powers of δL_i, $\delta\lambda_i$, $\delta\xi_j$ and $\delta\eta_j$ On account of theorem I in section 29, each of the partial derivatives $\partial R/\partial\lambda_i$ may be expanded in a series of the type

$$\sum A \mathfrak{M} \cos\left(\sum k_i \lambda_i + H\right),$$

where all of the indices k_i may evidently be assumed not to equal zero. Replacing the elements by their above-mentioned initial values, we obtain

$$D_0 = \sum A \mathfrak{M}_0 \cos(\nu t + H'),$$

where

$$\nu = \sum k_i n_i$$

The quantity ν cannot be equal to zero since neither of the indices k_i equal zero. The partial derivatives D_0 will thus consist entirely of periodic terms. The rank of each of these terms is $\geqslant 1$, because the function R is a first-order quantity relative to the masses.

Now, considering product R, it is easy to see that each zero-rank term of this expression can only be a result of the multiplication of zero-order terms relative to δL_i, $\delta\lambda_i$, $\delta\xi_j$ and $\delta\eta_j$. We are assuming here that these latter quantities can only have secular terms of zero rank. Hence, each zero-rank term in R must be secular. When we multiply quantity D_0 by product R, we will obtain only terms the ranks of which are more than or equal to unity. The first-rank terms will be either periodic or mixed secular. In both cases, the integration cannot reduce the rank as can be seen from the following well-known formula:

$$\int^{t} At^{p}\cos(\nu t+H)\,dt=\frac{A}{\nu}t^{p}\sin(\nu t+H)+p\frac{A}{\nu^{2}}t^{p-1}\cos(\nu t+H)-$$

The theorem is already proved as far as it concerns δL_i, $\delta \xi_{j}$ and $\delta \eta_j$. It remains for us to consider the expression, given by equation (23). This expression consists of three terms. Everything that is applicable to equation (22) holds true for the last of these three terms. This term can thus lead to terms having neither negative nor a zero rank. We now consider the second term

$$\int_0^t \frac{\partial\Phi}{\partial L}\,dt$$

We have shown that the derivative $\partial \Phi/\partial L_i$ consists of terms at least of the second order relative to δ L. Since the quantity δ L is equal to a sum of terms all of which have rank $\geqslant$ 1, then the rank of each term of the partial derivative $\delta \Phi/\partial L_i$; will be $\geqslant$ 2. The integration reduces the rank of each term by a unity, yet the ranks will still be $\geqslant$ 1 as the theorem implies.

It now remains to consider the first term

$$\sum c_i \int_0^t dt \int_0^t \frac{\partial \Phi}{\partial \lambda}\,dt$$

It follows from the above arguments that there are no zero-order terms in the expression of the partial derivative $\partial R/\partial \lambda_k$ First order terms are either periodic or mixed. In both cases the rank does not change after the double integration. The remaining terms will lead either to periodic or to mixed terms having ranks $\geqslant 2$, or to secular terms having zero ranks.

Hence, among the (m-1) order terms of the expression of the quantity there will be no terms having negative ranks and no mixed terms of zero rank.

The theorem is thus completely proven.

31. <u>Poisson's theorem</u>

Poincaré's theorem, proved in the previous section, is a generalization of the well-known Laplace-Lagrange theorem on the absence of secular perturbations in semimajor axes. Poisson's theorem, mentioned in section 17, gives a generalization of the Laplace-Lagrange theorem in another direction.

The semimajor axis a_i of an orbit is related (section 25) to the element L_i by

$$a_i = k^{-2} M_i^{-1} L_i^2,$$

where M_i is a factor, which depends on masses and is slightly different from a unity when the mass of the sun is chosen as the mass unit. Denoting by $\delta_m a_i$ and $\delta_m L_i$ the m-order perturbation of the elements a_i and L_i and by a_i^0 and L_i^0 the osculating elements for the moment t = 0, we obtain

$$a_i^0 + \delta_1 a_i + \delta_2 a_i + \ldots = k^{-2} M_i^{-1} (L_i^0 + \delta_1 L_i + \delta_2 L_i + \ldots)^2,$$

where

[illegible]

We have already proved that $\delta_1 L_i$ consists only of periodic terms. Accordingly, secular terms will be present in $\delta_2 a_i$ only if they are present in $\delta_2 L_i$. Hence, Poisson's theorem may be reformulated as follows:

<u>Poisson's theorem:</u>

If the mean motions n_i of a planet are such that the relation

$$\nu = \sum k_i n_i = 0$$

does not hold for any integral value of k_i, then, no secular terms can be present among the terms of δL_i that have a second order relative to the masses.

In order to prove this theorem, we consider equations (22), which yield

$$\frac{d}{dt}(\delta L_i) = -\frac{\partial R}{\partial \lambda_i}.$$

Putting

$$L_k = L_k^0 + \delta_1 L_k, \qquad \lambda_k = n_k t + \lambda_k^0 + \delta_1 \lambda_k, \qquad \xi_j = \xi_j^0 + \delta_1 \xi_j, \qquad \eta_j = \eta_j^0 + \delta_1 \eta_j,$$

in the right-hand side of this equation, we obtain

$$\frac{\partial R}{\partial \lambda_i} = \left(\frac{\partial R}{\partial \lambda_i}\right)_0 + \sum \left(\frac{\partial^2 R}{\partial \lambda_i \, \partial L_k}\right)_0 \delta_1 L_k + \sum \left(\frac{\partial^2 R}{\partial \lambda_i \, \partial \lambda_k}\right)_0 \delta_1 \lambda_k + \sum \left(\frac{\partial^2 R}{\partial \lambda_i \, \partial \xi_j}\right)_0 \delta_1 \xi_j + \sum \left(\frac{\partial^2 R}{\partial \lambda_i \, \partial \eta_j}\right)_0 \delta_1 \eta_j$$

According to equations (22), the first order perturbations $\delta_1 L_k$, $\delta_1 \xi_j$ and $\delta_1 \eta_j$ are equal to

$$\delta_1 L_k = -\int_0^t \left(\frac{\partial R}{\partial \lambda_k}\right)_0 dt, \qquad \delta_1 \xi_j = \int_0^t \left(\frac{\partial R}{\partial \eta_j}\right)_0 dt, \qquad \delta_1 \eta_j = -\int_0^t \left(\frac{\partial R}{\partial \xi_j}\right)_0 dt.$$

We divide $\delta_1 \lambda_k$ according to equation (23) into two parts, such that

$$\delta_1 \lambda_k = \delta_1' \lambda_k + \delta_1'' \lambda_k,$$

where

$$\delta_1' \lambda_k = -\int_0^t \left(\frac{\partial R}{\partial L_k}\right)_0 dt, \qquad \delta_1'' \lambda_k = -\sum c_k \int_0^t dt \int_0^t \left(\frac{\partial R}{\partial \lambda_k}\right)_0 dt$$

The last term in equation (23) produces perturbations of orders not lower than the second and therefore, can be neglected in the present discussions. In this way,

$$
\begin{aligned}
\frac{d}{dt}(\delta_2 L_i) = & \sum \left\{ \left(\frac{\partial^2 R}{\partial \lambda_i \partial L_k} \right)_0 \delta_1 L_k + \left(\frac{\partial^2 R}{\partial \lambda_i \partial \rho_k} \right)_0 \delta_1 \rho_k \right\} + \\
& + \sum \left\{ \left(\frac{\partial^2 R}{\partial \lambda_i \partial \xi_j} \right)_0 \delta_1 \xi_j + \left(\frac{\partial^2 R}{\partial \lambda_i \partial \eta_j} \right)_0 \delta_1 \eta_j \right\} + \\
& + \sum \left(\frac{\partial^2 R}{\partial \lambda_i \partial \lambda_k} \right)_0 \delta_1 \lambda_k .
\end{aligned} \tag{25}
$$

It was shown in the preceding section that each of the second derivatives involved in the previous expression can be expanded in a series of the type

$$\Sigma A \mathfrak{M} \cos(\nu t + H),$$

where $\nu = \sum k_i n_i$ cannot be equal to zero otherwise the corresponding term vanishes after differentiating with respect to λ_i. Secular terms can appear in the expression of the quantity $\delta_2 L_i$ only when constant terms are present in the expansions (25). It is easy to see that the first of the sums involved in equation (25) cannot produce a term. Indeed, each term

$$\mathfrak{T} = A \mathfrak{M} \cos(\Sigma k \lambda_i + H)$$

of series (12) in which R is expanded, gives the following part of this sum

$$
\begin{aligned}
& \left(\frac{\partial^2 \mathfrak{T}}{\partial \lambda_i \partial L_k} \right) \int^t \left(\frac{\partial \mathfrak{T}}{\partial \lambda_k} \right) dt - \left(\frac{\partial^2 \mathfrak{T}}{\partial \lambda_i \partial \lambda_k} \right)_0 \int^t \left(\frac{\partial \mathfrak{T}}{\partial L_k} \right)_0 dt = \\
& = k_i \frac{\partial A_k}{\partial L_k} \mathfrak{M}_k \sin(\Sigma k \lambda_i + H) \frac{k_k A_k \mathfrak{M}_k}{\nu} [\cos(\Sigma k_i \lambda_i + H) - \\
& - \cos(\Sigma k_i \lambda_i^0 + H)] - k_i k_k A_k \mathfrak{M} \cos(\Sigma k_i \lambda_i + H) \frac{1}{\nu} \frac{\partial A}{\partial L_k} \mathfrak{M}_k [\sin(\Sigma k_i \lambda_i + H) - \\
& - \sin(\Sigma k_i \lambda_i^0 + H)].
\end{aligned}
$$

These expressions equal a sum of periodic terms, none of which are constant. Similarly, we can show that no term in the series expansion of R can give a constant term in the second sum of equation (25). The third term can contribute only with periodic and secular terms since

$$\left(\frac{\partial^2 \Omega}{\partial \lambda_i \partial \lambda_j}\right)_0 \int dt \int \left(\frac{\partial \Omega}{\partial \lambda_j}\right) dt$$

$$= A_0' \mathfrak{M}_0 \frac{k_i k_j}{\nu^2} \cos(\sum k \lambda + H) \{-\sin(\sum k' \lambda + H') + \sin(\sum k \lambda + H) + \ldots \cos(\sum k \lambda + H)\}$$

and

$$\cos(\sum k\lambda + H)\sin(\sum k\lambda + H) = \frac{1}{2}\sin 2(\sum k\lambda + H)$$

$$\cos(\sum k\lambda + H)\cos(\sum k'\lambda + H') = \frac{1}{2}\cos(\sum k\lambda - \sum k'\lambda) + \frac{1}{2}\cos(\sum k\lambda + \sum k'\lambda + 2H)$$

This completely proves Poisson's theorem, we note that the third sum in equation (25) yields a mixed term of the second rank in the expression of the second rank in $\delta_2 L_i$.

Poincaré generalized Poisson's theorem by proving that the expression of δL cannot involve secular terms not only of the zero rank (section 29) but also of the first rank(1).

32. Poincaré's theorem on the class

In section 30, we began to study the expressions of the perturbations δL_i, $\delta \lambda_i$, $\delta \xi_j$ and $\delta \eta_j$ obtained as a result of applying the method of successive approximations to equation (17). Each of these quantities is obtained in the form of a series of terms having the following structure:

$$\frac{A_0 \mathfrak{M}}{\nu_1 \nu_2 \ldots} t^p \cos(\sum k_i \lambda_i + H),$$

where

$$\nu = \sum k_i n_i \qquad (i = 1, 2, \ldots)$$

are factors introduced by the integration. If the coefficients A_0 are of the m-order relative to the perturbing masses, then (section 15).

---

(1) M. Poincare, Lecon de Mechanique Celeste, t. 1, Paris 1905, 294.

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$$m - \frac{1}{2} p - \frac{1}{2} q,$$

is called the class of the term under considerations relative to the corresponding divisor ν_i.

Theorem

If the mean motions n_i is such that, for the arbitrary integer, k_i the following relation holds

$$\Sigma k_i n_i \neq 0,$$

then, the class of each term in the expansion of δL_i, $\delta \xi_j$ and $\delta \eta_j$ relative to any divisor is not less than ½. In the expansion of $\delta \eta_i$, the class of each term is not less than zero.

In order to prove this theorem, we first note that the theorem is valid for all the first order perturations. In this case, m = 1 and p + q $\lesssim$ 1 for the expansions of $\delta_1 L_i$, $\delta \xi_i$ and $\delta_1 \eta_1$, and m = 10, p = 0 and q $\lesssim$ 2 for the expansion of $\delta_1 \lambda_i$.

Let us assume that the theorem is valid for all of the perturbations that have orders less than or equal to m. We then show that the theorem will also be valid for all the (m + 1) - order perturbations. We make use of equations (22) and (23) to calculate the (m + 1) - order perturbations in terms of the m-order ones. We first of all obtain

$$\delta L_i = \int_0^t \frac{\partial R}{\partial \lambda_i} dt, \quad \delta \xi_i = \int_0^t \frac{\partial R}{\partial \eta_i} dt, \quad \delta \eta_i = -\int_0^t \frac{\partial R}{\partial \xi_i} dt. \qquad (26)$$

Each of the partial derivative of the function R; involved in this expression, can be expanded in a series of the type:

$$\Sigma D_0 R. \qquad (27)$$

in which we denote by D_0 the second, third, ... derivatives in which the elements are replaced by their initial values:

$$L_i^0, \quad n_i t + \lambda_i^0, \quad \xi_i^0, \quad \eta_i^0.$$

It has allready been pointed out in section 30 that the quantity D_o be expanded in a series consisting entirely of periodic terms. For these terms, m = 1, p = 0 and q = 0 since they have not been obtained as a result of an integration. In other words, the class of each term of the expansion of D_o equals to unity. The quantity R is a product of positive integral powers of δL_i, $\delta\lambda_i$, $\delta\xi_j$ and $\delta\eta_j$ evaluated up to terms having masses of m-order inclusively. Since the product of each two terms can yield only terms having classes less than the original two, then the expansion of each of the derivatives

$$\frac{\partial R}{\partial \lambda_i}, \quad \frac{\partial R}{\partial \eta_j}, \quad \frac{\partial R}{\partial \xi_j}$$

will involve only terms having classes ≥ 1.

In the integration, the class of a term relative to the mass does not change. However, the value of one of the coefficients p or q may be increased by one unit. Therefore, the classes of the terms of expansion (26) will be $\geq \frac{1}{2}$. Hence, the theorem holds for forms of the m + 1 order.

Now, we investigate the class of the (m + 1)-order terms in the expansion of the perturbation

$$\delta\lambda_i = -\sum C_{ik} \int_0^t dt \int_0^t \frac{\partial R}{\partial L_k} dt - \int_0^t \frac{\partial R}{\partial L_i} dt - \int_0^t \frac{\partial \Phi}{\partial L_i} dt \qquad (28)$$

The integrands of the first two terms in the right-hand side may be expanded in series of the type just considered. Noting that the double integration increases the sum p + q by two units, we conclude that these two terms can only lead to terms having classes ≥ 0. The partial derivative $\partial\Phi/\partial L_i$, involved in the third term of equation (27), consists of terms at least of the second order in δL_i. The product of two or more quantities δL_i consists only of terms having classes ≥ 1, since the class of each term in δL_i is $\geq \frac{1}{2}$ as shown above. The integration with increase the sum p + q by one unit, yet the terms of the expansion will still have a class $\geqslant \frac{1}{2}$.

The theorem is thus proven. We note that the double integration yields the least-class term in $\delta\lambda_i$.

33. <u>The least-class perturbations</u>

Let us consider the structure of those terms, the class of which relative to some given divisor

$$\nu_0 = \sum k_i^0 n_i$$

is the least. We show that all of the terms of class ½ in the expression of δL, $\delta\xi$ and $\delta\eta$ as well as terms of class zero in the expansion of $\delta\lambda$ are of the following form

$$A t^p \cos(\beta\nu_0 t + H), \tag{29}$$

where β is an integer. We assume that this holds for perturbations evaluated up to the m-order inclusively, and then show that the same form can represent the least-class terms of the (m + 1) - order. We refer to equations (26) and see under which conditions can terms of class ½ be obtained in the right-hand side. First of all, it is necessary that the term under consideration

$$A t^p \cos(\nu t + H), \tag{30}$$

involved in the expression of the corresponding derivative $\partial R/\partial\lambda_i$, $\partial R/\partial\eta_j$ or $\partial R/\partial\xi_j$, has a class equal to unity. Indeed, each of these derivatives may be expanded in a series of the type (27), where the factor D_0 consists of terms of class unity as we have already seen. Hence, the class of the term (30) cannot be less than unity.

Furthermore, it is necessary that the integration of the term (30) decreases its class by ½. This can only take place in two cases; 1) if v = 0, then the integration increases by one unit the exponent p, and if $\nu = \beta\nu_0$, where β is an integer, then the power q of the devisor ν_0

increases by one unit after integration.

Hence, all terms of class ½ in the expressions of the perturbations δL_i, $\delta \xi_j$ and $\delta \eta_j$ must have the form (29). We now show that in order to obtain all the terms of class ½ in the expressions of δL_i, $\delta \eta_j$ and $\delta \xi_j$, it is sufficient to consider those terms of the perturbation function R, the arguments of which are multiplies of

$$\theta = \Sigma k_i^0 \lambda_i.$$

We again consider expression (27), in which we have denoted by D_o those partial derivatives of the perturbation function R, in which the elements L_i, λ_i, ξ_j and η_j are replaced by their initial values L_i^o, $\lambda_i^o + n_i t$, ξ_j^o and η_j^o. In other words, the quantity D_o consists of terms of the following type:

$$B_0 \cos (\nu t + H_0), \tag{31}$$

where

$$\nu = \Sigma k_i n_i.$$

Here, H_o is a constant while B_o is a function of L_i^o, ξ_j^o and η_j^o. This term is obtained by the term

$$B \cos(\Sigma k_i \lambda_i + H) \tag{32}$$

involved in the expansion of the function R and, subsequently, substituting the above-mentioned initial values.

The factor R in the expansion (27) is a product of non-negative powers of δL_i, $\delta\lambda_i$, $\delta \xi_j$ and $\delta \eta_j$ evaluated inclusively up to the m-order terms relative to the masses. In the frame of our assumptions, the terms of factor R that have the least class (zero) will assume form (29) since they are obtained as a product of the least class terms in the expressions of δL_i, $\delta \lambda_i$, ...

Hence, the least-class terms in the partial derivatives $\partial R/\partial \xi_i$, $\partial R/\partial \eta_i$ and $\partial R/\partial \lambda_i$ are obtained as the product of expressions

(29) and (30). Their arguments will thus have the form

$$(\beta\nu_0 + \nu)\, t + \text{const.}$$

We have already seen that in order to obtain terms having class ½ after integration, each such argument should be of the form

$$\gamma\nu_0 t + \text{const.}$$

Consequently,

$$\nu = \pm(\gamma - \beta)\nu_0,$$

or

$$\nu = \sigma\nu_0,$$

where σ is an integer. Therefore

$$k_i = \sigma k_i^0.$$

Accordingly, terms (32) of the function R which lead to least-class perturbations. will have arguments of the form

$$\Sigma k_i \lambda_i + H = \sigma\theta + H,$$

i.e. all multiplied by θ. This is what has been required to prove.

It remains for us to investigate the structure of the least-order terms of the expansion of $\delta\lambda_i$. The easiest manner by which these terms are obtained is also required to be shown. We start by considering formula (28). In the previous section, we have seen that the least-class (zero) terms in the expression of $\delta\lambda_i$ may be only obtained from the first term of formula (28). In order to obtain these terms, we take

$$\delta\lambda_i = -\sum C_{ik} \int_0^t dt \int_0^t \frac{\partial R}{\partial L_k}\, dt,$$

or, considering equation (26),

$$\delta\lambda_i = -\Sigma C_{ik} \int_0^t \delta L_k\, dt. \tag{33}$$

We again use expression (27) for considering the partial derivative $\partial R/\partial\lambda_k$. Similar arguments show that the least possible class for terms of this type is unity. Hence, in order to obtain terms of class zero in the expression of $\delta\lambda_i$, it is necessary that the double integration increases the sum p + q by two units. This is only possible if the arguments of these terms have the form $\beta \nu_0 t + H$

Finally, we combine the differential equations which allow us to obtain the least-class terms of δ L, $\delta\xi$, $\delta\eta$ and $\delta\lambda$. We start by $\delta\lambda$. In order to evaluate the zero-class terms in $\delta\lambda$, we use formula (33). From this formula, it follows that

$$\frac{d[illegible]}{dt} - \sum C [illegible] L_i = \sum [illegible] L - L_i$$

Noting that $\lambda_i = n_i t + \lambda_i^o + \delta\lambda_i$, we obtain

$$\frac{d[illegible]}{dt} = [illegible] - \sum C_i \; L_i - L_i$$

Assuming that (cf. section 30)

$$\Phi_0 = C_0 - \sum n \; (L - L [illegible] - \sum\sum C, \; L \; [illegible] \; L_i - L_i$$

where C_o is a constant, we obtain

$$\frac{d[illegible]}{dt} = - \frac{[illegible]}{[illegible]}$$

This equation yields only zero-class terms if the L - L_o involved in the expression of the function Φ_o is understood as the aggregate of terms of class ½.

We have already seen that in order to obtain terms of class ½ in the expressions of δL_i, $\delta \xi_j$ and $\delta \eta_j$, it is sufficient to keep only terms of the function R, the arguments of which are of the type $\sigma\theta$. Let us denote the aggregate of such terms by Ψ. Using equations (26), we obtain the following equations

[illegible]

which define the terms of class ½. The right-hand side of these equations can be further simplified. We have seen that, in order to obtain terms of class ½, it is necessary to keep only the zero-rank terms of R in the expressions (27) of the derivatives $\frac{\partial R}{\partial j}$, ... However the zero-class terms of R can be obtained only by replacing the quantity $\delta\lambda_i$ by its zero-class terms, and putting

[illegible]

that is to say

[illegible]

Since such substitution is already done in D_o, then denoting by

[illegible]

the results of this substitution into

[illegible]

we obtain

$$\frac{dL_i}{dt} = \frac{\partial\Phi}{\partial\lambda_i}, \quad \frac{d\lambda_i}{dt} = -\left(\frac{\partial\Phi}{\partial L_i}\right)_0, \quad \frac{d[illegible]}{dt} = [illegible]\left(\frac{\partial\Phi}{\partial[illegible]}\right).$$

Noting that

$$\frac{\partial\Phi_0}{\partial\lambda_i} = 0, \quad \frac{\partial\Phi_0}{\partial L_i} = 0,$$

we formulate our conclusions in the form of the following theorem

<u>Theorem</u>:

In order to obtain the least-class perturbations, it is necessary to integrate the following equations:

$$\frac{dL_i}{dt} = \frac{\partial(\Phi_0 + \Psi_0)}{\partial \lambda_i}, \qquad \frac{d\lambda_i}{dt} = -\frac{\partial(\Phi_0 + \Psi_0)}{\partial L_i}. \tag{36}$$

$$\frac{d\xi_j}{dt} = \left(\frac{\partial \Psi}{\partial \eta_j}\right)_0, \qquad \frac{d\eta_j}{dt} = -\left(\frac{\partial \Psi}{\partial \xi_j}\right)_0. \tag{37}$$

34. <u>The Delaunay-Hill method for calculating long-periodic perturbations</u>

We assume that the main motion n_i of the system of material points under consideration is such that the quantity

$$\nu_0 = \sum k_i^0 n_i$$

is small. In this case, the perturbation of least class relative to the divisor v_0 will be of particular interest, since the amplitudes of these perturbations will be particularly large. The theorem, given in the previous section, enables us to determine these perturbations independently for the others. Putting again

$$\theta = \sum k_i^0 \lambda_i,$$

and denoting by Ψ the aggregate of the terms of the expansion of the cuntion R, the arguments of which are multiples of θ, we can conclude that Ψ depends only on θ, L_i, ξ_j and η_j. Consequently

$$\Psi_0, \quad \left(\frac{\partial \Psi}{\partial \xi_j}\right)_0, \quad \left(\frac{\partial \Psi}{\partial \eta_j}\right)_0$$

are functions only of θ. This situation enables us to obtain the solution of the system (36) and (37), which defines the least class term, by means of quadratures. The first of equations (36) yields

$$\frac{dL_i}{dt} = \frac{\partial \Psi_0}{\partial \lambda_i} = \frac{\partial \Psi_0}{\partial \theta} k_i^0.$$

Introducing the auxiliary function U by means of the relation

$$\frac{dU}{dt} = \frac{\partial \Psi_0}{\partial \theta},$$

we obtain from the previous equations

$$\frac{dL_i}{dt} \quad k_i' \frac{dU}{dt} = 0.$$

Let U = 0 for t = 0. Then, integrating the previous equation from t = 0 to t = t yields

$$L_i = k_i U + L_i^0 \qquad ([illegible])$$

Substituting these values of L_i into equation (34), we obtain

$$\Phi_0 = C - 2BU - AU^2,$$

where A, B and C are constants. On the other hand, equations (36) have the evident integral

$$\Phi_0 + \Psi_0 = \text{const},$$

defining the dependence of U on θ in a closed form. This integral may be represented in the following way

$$C - \Psi_0 = 2BU + AU^2$$

from which it follows by expressing U in terms of θ that

$$AU + B = \sqrt{B^2 + AC - A\Psi_0} \qquad ([illegible])$$

Since

$$\frac{dU}{dt} = \sum k_i \frac{dL_i}{dt} \quad \sum k_i \frac{\partial \Phi}{\partial L_i} = -\sum \frac{dL_i}{dU} \frac{\partial \Phi}{\partial L_i} = -\frac{\partial \Phi_0}{\partial U},$$

or,

$$\frac{dU}{dt} = 2B + 2AU = 2\sqrt{A\Psi_0 + B^2 + AC}$$

then we obtain the following relation between θ and t:

$$t = \int \frac{d\theta}{2\sqrt{A\Psi_0 + B^2 + AC}} \qquad ([illegible])$$

We now consider the second of equations (36) which yields

$$\frac{d\theta}{dt} = \frac{\partial \Phi}{\partial L} \quad [illegible] \sum c_i (L_i - L_i^0) =$$
$$= [illegible] \sum k_i c_i \qquad ([illegible])$$

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Replacing the quantity U by its value given by equation (39) and integrating, we obtain element λ_i as a function of θ. Similarly, integrating equations (37) leads to expressions for the coordinates ξ_j and η_j in terms of functions of θ. Equation (40) allows us to express the coordinates λ_i, ξ_j and η_j in terms of time t. This gives the complete solution of the three-body problem under consideration.

In this way, in order to obtain the perturbations of least class relative to the argument θ, it is necessary to pick all of the terms, the arguments of which are multiples of θ, out of the perturbation function R. Replacing thequantities L_i, ξ_i and η_i in the function ψ, obtained in this way, by their initial values we obtain the function ψ_o.

Equations (38) and (39) allow us to express the quantity L_i in terms of the argument θ. Integrating in the same way equations (41) and (37), we obtain λ_i, ξ_j and η_j as functions of θ.

Finally, equation (41) defines the dependence of the argument θ on time t.

Delaunay was the first to note that it is possible to obtain all periodic perturbation by integrating those equations of motion, in which the perturbation function is replaced by some of its separate terms. We applied this method to construct the most complete analytical theory of lunar motion(1).

Tisserand considerably simplified the Delaunay method by relating

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(1) C. Delaunay, Theorie du mouvement de la Lune, Memoires de l'Academie des Science de Paris, 28 (1860), 29 (1867).

it to the general theory of canonical transformation[1]. On the other hand, Hill significantly generalized this method by showing how the terms of the perturbation function, the arguments of which were multiples of a given argument, could be taken into account[2].

The method obtained in this way is known as the Delaunay - Hill Method. Finally, the work of Poincaré[3] helped in clarifying the main mathematical points of this method.

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(1) F. Tisserand, Traite de Mathematique Celeste, 3, Ch XI, 1894.

(2) G.W. Hill, On the Extension of Delaunay's Method in the Lunar Theory to the General Problem of Planetary Motion, Transations of the American Mathem. Soc. 1, 1900, 205-242 = The collected Mathem. Works, 4, 1907.

(3) H. Poincare, Le methodes nouvelles de la Mecanique Celeste, 2, Ch. XIX, Paris 1893;
H. Poincare, Lecon de Mechanique Celeste, 1, Ch. XIII, Paris 1907.

CHAPTER VI

SOME PARTICULAR CASES OF THE THREE BODY PROBLEM

35. Introduction

In 1772, Lagrange was awarded a prize by the Paris Academy for his well-known memoir "Essai sur le problème des trois corps (Oeuvres, 6, 229-324)". Lagrange pointed out in the preface of this work, that this included a method for the solution of the three-body problem, which was very different from all previous contributions. This method was shown by Lagrange to consist in reducing the determination of the relative coordinates of the three bodies, which requires the integration of a twelve-order system, to the determination of the sides of the triangle formed by the three bodies. This requires the integration of a 7-order system consisting of two second-order equations and one third-order equation. These equations involve two arbitrary constants introduced by the kinetic energy integral and the integral of areas. Accordingly, the mutual distances of the three bodies will depend on nine arbitrary constants. When the mutual distances are known, the determination of the relative coordinates which introduces another three arbitrary constants is quite simple.

Eliminating time from the above-mentioned 7-order system, we finally reduce the solution of the problem to the integration of a 6-order system.. The reduction, performed by Lagrange is essentially identical to that indicated in section 2. However, the special form in which Lagrange obtained the equations of motion enabled him to formulate and solve the problem of finding all the three body types of motion when their mutical distances always keep constant ratios. These types of motion are called Lagrangean. We shall see that a Lagrangean motion will necessarily be coplanar.

If we have to proceed along the same path as Lagrange, when studying the lagrangean motion, we then have to initially deduce the differential equations that define the mutual distances of the three bodies. However, as Lagrange himself pointed out (loc. cit., page 431), the particular case under consideration can be resolved in a much simpler way if we, beforehand, assume that the three bodies are moving in an invariable plane. Indeed, restricting the problem by this subsidiary condition, Laplace was able to derive a very simple deduction for the eqations of the lagrangian motion[1].

Lagrange assumed that the solution of the general problem, i.e. without restricting it to coplaner motions alone, is indispensibly connected to several difficulties. Andoyer and Caratheodory[2] proved that this was not true. They developed a simple method for obtaining the general solution of the problem suggested by Lagrange. We shall give the details of this method in the next sections. This is very interesting since it can very easily be extended to the n-body problem.

By this method, it is easy to show that the lagrangean motion of n-bodies will also take place only in an invariable plane, except in some almost trivial cases, when the motion proceeds along straight lines

(1) Laplace, Mechanique céleste, Seconde partie, Livee X, Ch. VI (Oeuvres, 4). Laplace's method is explained in: Charlier, Die Mechanik des Himmels, 2, 89-102, 1907; A simple geometrical method for obtaining the results of Laplace is given by C.D. Černyj in the paper: Geometrische Losung zweier spezieller Falle des problems der drei Körper, Astr. Nachr. 171, 1906, 129-136.

(2) H. Andoyér, Sur l'équilibre relatif de n corps, Bulletin Astr., 23, 50-59, 1906.
C. Caratheodory, Uber die strenge Losungen des Dreikorproblems, Sitzungs berichte der math. naturwiss. Abteilung der [illegible]yerschen Akademie der Wiss. zu Munchen, 1933, 257-276.

passing through a common centre of gravity.

36. Equations of Lagrangean Motion. The Case of Moncollinear Motion

We shall construct the equation of relative motion of three bodies, starting with the assumption that the ratios of the mutual distances of these bodies remain constant. We denote by m_1, m_2 and m_3 the masses of points P_1 , P_2 and P_3 and by x_i , y_i and z_i the coordinates of point P_i. We take the origin of the coordinate at the centre of gravity O, and the xy plane as the plane of the tri-angle $P_1P_2P_3$.

The distance from the centre of gravity to the vertices P_1, P_2 and P_3 are proportional to the dimensions of the triangle. Hence we can use in the case of a lagrangian motion, a rotating coordinate system and put

$$x_i = a_i\rho, \quad y_i = b_i\rho, \quad z_i = 0 \qquad (1)$$
$$(i = 1,\ 2,\ 3),$$

where $\rho = \varphi(t)$ is a properly defined function of time, and a_i and b_i are constants. We denote by p , q and r the components of the angular velocity of the system along the axes x, y and z. The components of the velocity of a point, whose coordinates are x, y and z, are well-known and equal to

$$\dot{x} - yr + zq, \quad \dot{y} - zp + xr, \quad \dot{z} - xq + yp.$$

Consequently, the components of acceleration are given by

$$\frac{d}{dt}(\dot{x} - yr + zq) - r(\dot{y} - zp + xr) + q(\dot{z} - xq + yp)$$
$$\frac{d}{dt}(\dot{y} - zp + xr) - p(\dot{z} - xq + yp) + r(\dot{x} - yr + zq)$$
$$\frac{d}{dt}(\dot{z} - xq + yp) - q(\dot{x} - yr + zq) + p(\dot{y} - zp + xr).$$

Substituting the values (1) of the coordinates under consideration into these expressions, and noting that the components of acceleration of point P_i caused by the attraction of the other two points, are equal to

$$A_i\rho^{-2}, \qquad B_i\rho^{-2}, \qquad 0,$$

where A_i and B_i are constant factors depending on a_1, a_2, ..., b_3, m_1, m_2, m_3 and the constant of gravitation, we obtain the following equation of motion of the point

$$a_i[\ddot\rho - (q^2+r^2)\rho] - b_i[2r\dot\rho + (\dot r - pq)\rho] = A_i\rho^{-2} \tag{2}$$

$$a_i[2r\dot\rho + (\dot r + pq)\rho] - b_i[\ddot\rho - (p^2+r^2)\rho] = B_i\rho^{-2} \tag{3}$$

$$a_i[2q\dot\rho + (\dot q - pr)\rho] + b_i[2p\dot\rho + (\dot p + qr)\rho] = 0. \tag{4}$$

In the case of a collinear motion, in which the three points P_1, P_2 and P_3 are always on one straight line, or in other words, when

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3},$$

will not be considered in this section. Hence, equations (2), (3) and (4) lead to the following relations

$$\ddot\rho - (q^2+r^2)\rho = A'\rho^{-2}; \qquad 2r\dot\rho + (\dot r - pq)\rho = B'\rho^{-2} \tag{2'}$$

$$\ddot\rho - (p^2+r^2)\rho = A''\rho^{-2}; \qquad 2r\dot\rho + (\dot r + pq)\rho = B''\rho^{-2} \tag{3'}$$

$$2q\dot\rho + (\dot q - pr)\rho = 0; \qquad 2p\dot\rho + (\dot p + qr)\rho = 0, \tag{4'}$$

where A', B', A" and B' are new constants though they can be expressed in terms A_i , B_i , a_i and b_i. The term-by-term substration of equation (2') and (3') yields

$$p^2 - q^2 = A\rho^{-3}, \qquad pq = B\rho^{-3},$$

from which it follows that

$$p = \alpha\rho^{-\frac{3}{2}}, \qquad q = \beta\rho^{-\frac{3}{2}}, \tag{5}$$

where α and β are constants. Substituting these values into equation

(4), we obtain

$$\beta\dot{\rho} - 2\alpha r\rho = 0, \qquad \alpha\dot{\rho} + 2\beta r\rho = 0, \tag{6}$$

from which it follows that

$$(\alpha^2 + \beta^2)\dot{\rho} = 0. \tag{7}$$

It is now easy to show that the motion proceeds in the invariable plane. For this purpose, we show that we can obtain $p = q = 0$ by means of an appropriate choice of the axes Ox and Oy. If $\alpha^2 + \beta^2 = 0$, then $\alpha = \beta = 0$, and hence the relation (5) yields

$$p = q = 0,$$

which proves that the plane $P_1P_2P_3$ is invariable.

We now investigate whether the sum $\alpha^2 + \beta^2$ cannot be equal to zero. If this can take place, we may then conclude from equation (7) that $\dot{\rho} = 0$. Since $\rho \neq 0$, then the relations (6) yield $r = 0$. Once $\dot{\rho} = 0$ and $r = 0$, it then follows from equations (4') that

$$p = \text{const}, \qquad q = \text{const}.$$

We add the vectors p and q which are directed along the x- and y-axes, and take the direction of the resulting vector as a new x-axis. In this case, we get $q = 0$ in the new coordinate system and, hence, equations (2) yield $A_1 = 0$. In other words, the projections of all of the forces on the axis Ox vanish, so that all of the forces will be parallel to the axis Oy. This is impossible since we agreed to only consider the case in which the three bodies are not located along one straight line.

We take the plane Oxy of the fixed coordinate system as the invariable plane in which the motion takes place and draw the perpendicular axis Oz from the centre of gravity of the system. The motion of the points P_i in the case under consideration will then consist of the rotation of triangle $P_1P_2P_3$ as a whole, around axis Oz with an angular velocity r

and with the motion of each point P_i along the ray OP_i. Hence, the areal velocity of point P_i will differ only by a constant factor from the quantity $r^2 \varphi$, so that the integral of area for the axis O_z will give

$$r\rho^2 = \text{const},$$

from which it follows that

$$2r\dot{\rho} + \dot{r}\rho = 0.$$

In this way, equations (2) and (3) become

$$a_i[\ddot{\rho} - r^2\rho] = A_i\rho^{-2}, \qquad b_i[\ddot{\rho} - r^2\rho] = B_i\rho^{-2},$$

which yield

$$\frac{A_1}{a_1} = \frac{A_2}{a_2} = \frac{A_3}{a_3} = \frac{B_1}{b_1} = \frac{B_2}{b_2} = \frac{B_3}{b_3}. \tag{8}$$

These equations show that the resultant of all the forces that act on each of the points P_i passes through the centre of gravity of the system.

It is now already not difficult to determine the form of the triangle $P_1P_2P_3$ (figure 1). Denoting, as previously, the sides of this triangle by

$$P_1P_2 = \Delta_{12}, \quad P_1P_3 = \Delta_{13}, \quad P_2P_3 = \Delta_{23},$$

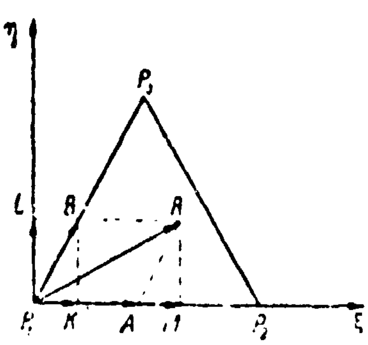

Fig. 1

We obtain the following expressions for the three accelerations that points P_2 and P_3 produce on point P_1.

$$\vec{P_1A} = k^2 m_2 \Delta_{12}^{-2},$$
$$\vec{P_1B} = k^2 m_3 \Delta_{13}^{-2}.$$

We determine the angle α, which is formed between the geometrical sum P_1R of these axcelerations and the straight line P_1P_2. For this perpose, we project the accleration P_1B on the axes $P_1 \xi$ and $P_2 \eta$ into the components

$$\overrightarrow{P_1K} = \overrightarrow{P_1B}\cos\varphi_1, \qquad \overrightarrow{P_1L} = \overrightarrow{P_1B}\sin\varphi_1,$$

where φ_1 is the angle $P_2P_1P_3$. Then

$$\operatorname{tg}\alpha = \frac{RM}{P_1M} = \frac{P_1L}{P_1K + P_1A} = \frac{m_3\Delta_{13}^{-2}\sin\varphi_1}{m_2\Delta_{12}^{-2} + m_3\Delta_{13}^{-2}\cos\varphi_1}.$$

On the other hand, denoting by ξ_0 and η_0 the coordinates of the centre of gravity of the system, we obtain

$$\xi_0 = \frac{m_2\Delta_{12} + m_3\Delta_{13}\cos\varphi_1}{m_1 + m_2 + m_3}, \qquad \eta_0 = \frac{m_3\Delta_{13}\sin\varphi_1}{m_1 + m_2 + m_3}.$$

On account of equations (8), the straight line P_1R passes through point O. We therefore evidently obtain the following equation

$$\frac{m_3\Delta_{13}^{-2}\sin\varphi_1}{m_2\Delta_{12}^{-2} + m_3\Delta_{13}^{-2}\cos\varphi_1} = \frac{m_3\Delta_{13}\sin\varphi_1}{m_2\Delta_{12} + m_3\Delta_{13}\cos\varphi_1}.$$

Taking into account that $m_2 \neq 0$, $m_3 \neq 0$ and $\sin\varphi_1 \neq 0$, we obtain from the previous equation

$$\Delta_{12}^3 = \Delta_{13}^3$$

or

$$\Delta_{12} = \Delta_{13}.$$

If $m_1 \neq 0$, we can then prove the equality of the other two sides of the triangle in an exactly similar way. We will then finally obtain

$$\Delta_{12} = \Delta_{13} = \Delta_{23}.$$

The case in which two of the masses m_1, m_2 and m_3 are infinitesimal will not be considered here, since it is trivial.

It remains for us to only consider the case in which one of the

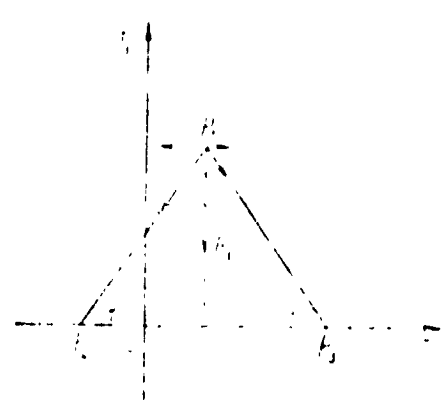

Fig 2.

masses, say m_1, is infinitesimal. In this case, the previous approach will be valid so far as it concerns the vertices P_2 and P_3. We can only conclude that the sides Δ_{12} and Δ_{13} are equal. It is however easy to see that the triangle $P_1P_2P_3$ will be equilateral.

Thus let $m_1 = 0$. We draw the coordinate axes as shown in figure 2, taking into account that the centre of gravity in this case is located on the straight line P_2P_3. The coordinates of points P_1, P_2 and P_3 will be equal to

$$(a_1\rho,\ b_1\rho),\quad (a_2\rho,\ 0)\ и\ (a_3\rho,\ 0).$$

Since the origin of the coordinates is at the centre of gravity, then

$$-\frac{a_2}{a_3} = \frac{m_3}{m_2}.$$

On the other hand, since the sides P_1P_2 and P_1P_3 are equal, then

$$2a_1 = a_2 + a_3.$$

Hence denoting by φ each of the angles at the base of the triangle, we easily find

$$b_1 = (a_1 - a_2)\,\mathrm{tg}\,\varphi = -a_2\,\frac{m_2 + m_3}{2m_3}\,\mathrm{tg}\,\varphi.$$

It folows from equation (8) that

$$\frac{A_2}{a_2} = \frac{B_1}{b_1}.$$

In the pre sent case,

$$A_2 = \frac{1}{2}k^2 m_3 \Delta_{23}^{-2}, \qquad B_1 = -k^2(m_2 + m_3)\Delta_{12}^{-2}\sin\varphi,$$

as it can be easily seen from figure 2. Hence

$$\frac{1}{2}\Delta_{12}^{-3}\cos\varphi = \Delta_{23}^{-3}.$$

Since

$$\Delta_{23} = 2\Delta_{12}\cos\varphi,$$

we finally then obtain

$$\cos\varphi = \frac{1}{2}, \qquad \varphi = 60^{\circ}.$$

Hence, when three material points move under the action of their mutual attraction in such a way, that the distances between them keep constant ratios, then if these points are not on a straight line, they will always form an equilateral triangle. The plane of this triangle will keep an invariable position in space.

Let us assume that the initial positions of two of the three points, say P_1 and P_2, are fixed, similarly as the plane in which the motion is taking place. In this case, in order to obtain a motion of the type under consideration, we have to place the third point at the vertex of one of the two equilateral triangles that can be formed at both sides of P_1P_2, i.e. at one of pints L_4 and L_5 of figure 3, in which the points P_1 and P_2 are denoted by m and m' . These points are called the triangular points of libration(1).

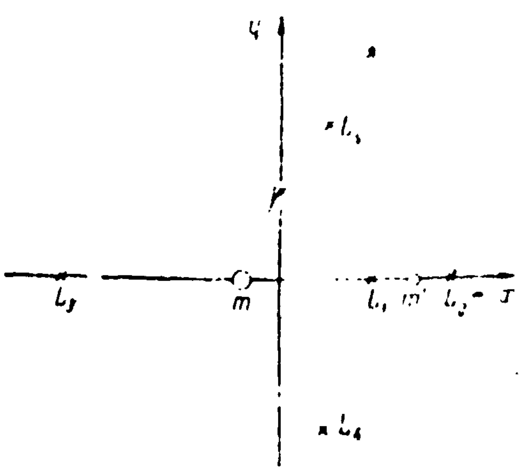

Fig. 3.

In conclusion, we show that, in in the case under consideration, the motion of each of the points P_i relative to the common centre of inertia 0 proceeds in such a way, as if each of these points was attracted by a mass equal to the masses of the two other points,

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(1) These points are also called the equilateral points of libration. Guilder called these points, as well as the points that we shall Consider later, by the centres of libration.

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located at O. In other words, the motion proceeds according to the generalized laws of Kepler.

We now turn to equations (23) of section 6 which define the motion relative to the centre of gravity. Denoting by Δ the common value of the presently equal distances Δ_i, and taking into account that

$$m_1x_1 + m_2x_2 + m_3x_3 = 0,$$

we obtain the following equations of motion for the point P_i:

$$\left.\begin{aligned} \ddot{x}_i + k^2(m_1 + m_2 + m_3)x_i\Delta^{-3} &= 0 \\ \ddot{y}_i + k^2(m_1 + m_2 + m_3)y_i\Delta^{-3} &= 0 \\ \ddot{z}_i + k^2(m_1 + m_2 + m_3)z_i\Delta^{-3} &= 0. \end{aligned}\right\}$$

where

$$\Delta^2 = c_i(x_i^2 + y_i^2 + z_i^2),$$

in which c_i is constant. These equations are identical in form to the equations of motion of the two-body problem. This proves the validity of our assumption.

37. The case of collinear lagrangian motion

We now consider the case in which the three bodies P_1, P_2 and P_3 are always on one straight line. Taking this line as the x-axis, we obtain the following expression for the coordinates of point P_i

$$x_i = a_i\rho(t), \qquad y_i = 0, \qquad z_i = 0,$$

so that the problem is reduced to the determination of the function $\rho(t)$ and the constants a_1, a_2 and a_3.

Since all of the forces are along the Ox, axis we can choose a coordinate system which does not rotate around this axis, i.e. we can consider $\underline{p} = 0$. Taking into account that in the present case $b_1 = 0$ and $B_1 = 0$, we obtain, from equations (3) and (4)

$$2\mathbf{r}\dot{\rho} + \dot{\mathbf{r}}\rho = 0, \qquad 2\mathbf{q}\dot{\rho} + \dot{\mathbf{q}}\rho = 0. \tag{9}$$

We assume that $\underline{q} \neq 0$. Multiplying these equations by $\underline{q}$ and $\underline{r}$ and substracting them term-by-term from one another, we obtain

$$\mathbf{r}\dot{\mathbf{q}} - \mathbf{q}\dot{\mathbf{r}} = 0.$$

Consequently,

$$\mathbf{r} = A\mathbf{q},$$

where A is a constant. Hence, taking the direction of the geometrical sum of the vectors $\underline{r}$ and $\underline{q}$ as the new Oz axis, we will obtain $\underline{q} = 0$. Thus, we can always consider that the Oy and Oz axes are chosen in such a way that $\underline{q} = 0$.

Consequently, the motion of the straight line $P_1P_2P_3$ in space will consist of a rotation of this straight line around the Oz axis with an angular velocity equal to $\underline{r}$. Integrating the first of relation (9), we obtain the integral of area

$$\mathbf{r}\rho^2 = \text{const.} \tag{10}$$

In the present case, equations (2) read

$$\ddot{\rho} - \mathbf{r}^2\rho = -\frac{A_i}{a_i}\rho^{-2}, \tag{10'}$$

and hence yield

$$\frac{A_1}{a_1} = \frac{A_2}{a_2} = \frac{A_3}{a_3}.$$

Assuming that

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and putting

$$z = \frac{a_3 - a_2}{a_2 - a_1}$$

we easily obtain

$$\frac{m_2 + m_3(1+z)^{-2}}{a} = \frac{-m_1 + m_3 z^{-2}}{a} = \frac{-m_1(1+z)^{-2} - m_2 z^{-2}}{a}$$

since

$$\left.\begin{aligned} A_1 &= -k^2 m_2 (a_1 - a_2)^{-2} - k^2 m_3 (a_1 - a_3)^{-2} \\ A_2 &= -k^2 m_1 (a_2 - a_1)^{-2} - k^2 m_3 (a_2 - a_3)^{-2} \\ A_3 &= -k^2 m_1 (a_3 - a_1)^{-2} - k^2 m_2 (a_3 - a_2)^{-2} \end{aligned}\right| \qquad (11)$$

Since the origin of the coordinates is at the centre of gravity of the system, then

$$m_1 a_1 + m_2 a_2 + m_3 a_3 = 0.$$

If follows from equations (11) that a_1 , a_2 and a_3 are proportional to each other. Taking this into account, we obtain the following equation for the determination of z:

$$\left.\begin{aligned} (m_1 + m_2) z^5 + (3m_1 + 2m_2) z^4 + (3m_1 + m_2) z^3 - \\ - (m_2 + 3m_3) z^2 - (2m_2 + 3m_3) z - (m_2 + m_3) = 0. \end{aligned}\right\} \qquad (13)$$

This equation has at least one positive root since the left-hand side has different signs at z = 0 and z = + ∞ . On the other hand, according to a theorem by Descartes, equation (13) can have no more than one positive root since its coefficients change sign only once. Hence, whatever the values of the masses are, we only obtain one positive value for z. Equations (11) enable us to obtain the ratios $a_1 : a_2 : a_3$ which correspond to this value of z.

The three different masses can be located on a straight lines by three different manners. This leads to three collinear lagrangean motions.

In astronomical applications, we denote the masses by m, m' and m' and assume that the mass m is very large (the mass of the sum), the mass m' is small (the mass of a planet) and the mass m" is very small (the mass of a planetcaid, comet, meteorite, etc.). Putting into euqation (13) $m_1 = m$, $m_2 = m'$ and $m_3 = m''$, we obtain an equation, the positive root of which is very small. Keeping the most important terms in this equation, we obtain

$$(3m + m')z^3 - (m' + m'') = 0,$$

from which it follows that

$$z = \left(\frac{m' + m''}{3m + m'}\right)^{\frac{1}{3}}.$$

In this manner, denoting the distance between the planet and the sum by r, and the distance between the astroid and planet by r', we obtain

$$r' = r\left(\frac{m' + m''}{3m + m'}\right)^{\frac{1}{3}}. \tag{14}$$

In the case, when $m_1 = m$, $m_2 = m'$ and $m_3 = m'$, i.e. when the planetoid is between the planet and the sum, we obtain

$$r' = r\left(\frac{m - m'}{3m - m}\right)^{\frac{1}{3}} \tag{15}$$

Finally, if the planet and planetoid are at different sides of the sum, s that $m_1 = m'$, $m_2 = m$ and $m_3 = m'$, then equation (13) reads

$$f(z) = 0,$$

where

$$f(z) = m(z^5 + 2z^4 + z^3 - z^2 - 2z - 1) + m'(z^5 + 3z^4 + 3z^3) + m''(3z^2 + 3z + 1).$$

Evidently, the positive root of this equation slightly differs from unity. Hence we may approximate the value of this root by

$$z = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{7m' - m''}{12m + 26m' + 3m''}.$$

Consequently

$$r'' = r\left(1 - \frac{7m' - m''}{12m + 26m' + 3m''}\right), \tag{16}$$

where r" is the distance from the planetoid to the sun.

Each of the three positions in which the third mass could be located on the straight line joining the other two masses m and m', will be called a collinear point of libration. These positions are denoted by L_1, L_2 and L_3 in figure 3.

When the magnitudes of the masses satisfy the above condition, the positions of the collinear points of librations will be defined by formulae (14), (15) and (16). If the mass m" is negligibly small in comparison with the two others, then the position of the five points of libration will be given in the first approximation by

$$\begin{aligned}
&\text{for } L_1 \quad && r'' = r - r\left(\frac{m'}{3m}\right)^{\frac{1}{3}} \\
&\quad\text{"} \ L_2 && r'' = r + r\left(\frac{m'}{3m + m'}\right)^{\frac{1}{3}} \\
&\quad\text{"} \ L_3 && r'' = r - r\,\frac{7m'}{12m + 26m'} \\
&\quad\text{"} \ L_4 \text{ and } L_5 && r'' = r' = r.
\end{aligned} \tag{17}$$

The following table gives the positions of the first three libration points for the different planets of the solar system. The table gives the values of the distances r" of the points of libration from the sun, expressed in fractions of the radius vectors of the planets

	L_1	L_2	L_3
Mercury	0.9966	1.0034	1 - 0.000.000.07
Venus	0.9907	1.0093	1 - 0.000.001.43
Earth	0.9899	1.0101	1 - 0.000.001.78
Mars	0.9952	1.[illegible]8	1 - 0.000.000.19
Jupiter	0.9332	1.0698	1 - 0.000.557
Saturn	0.9550	1.0164	1 - 0.000.167
Uranus	0.9758	1.0216	1 - 0.000.026
Neptune	0.9743	1.0261	1 - 0.000.030

It is interesting to note that all of the planets have sattelites at distances much smaller than the distances to the libration points L_1 and L_2. For example, the distance from the earth to the moon is approximately four times smaller than the distance to any of these points.

After obtaining the value of z, and from equations (11) the values of the ratios $a_1 : a_2 : a_3$, we can start to study the motion of point p_1. For this purpose, we can apply equations (10) and (10'). In order to obtain the function $\wp$ (t), we give one of the constants a_i an arbitrary nonvanishing value. We put, for example, $a_1 = 0$. Then, the above mentioned equations yield.

$$r\rho^2 = C, \quad \rho^3\ddot{\rho} - r^2\rho^4 = A_1,$$

where C is an arbitrary constant, while A_1 is defined by formulae (12). We denote by u the angle between the straight line $P_1P_2P_3$ and an arbitrarily given direction in the plane xOy. Since r is equal to du/dt, then the equations of motion will have the final form

$$\rho^2 \frac{du}{dt} = C, \quad \rho^2 \frac{d^2\rho}{dt^2} - \left(\frac{du}{dt}\right)^2 \rho^4 = A_1.$$

from which it follows that

$$\frac{d^2\rho}{dt^2} = C^2 \rho^{-3} + A_1 \rho^{-2},$$

Multiplying by 2 $\frac{d\rho}{dt}$ and integrating, we obtain

$$\left(\frac{d\rho}{dt}\right)^2 = h - 2A_1\rho^{-1} - C^2\rho^{-2},$$

where h is a new constant.

In order to obtain an equation for the orbit, we eliminate dt by means of the integral of area. We obtain

$$\frac{C^2}{\rho^4}\left(\frac{d\rho}{du}\right)^2 = h + \frac{A_1^2}{C^2} - \left(\frac{C}{\rho} + \frac{A_1}{C}\right)^2$$

or

$$\left(\frac{ds}{du}\right)^2 = h + \frac{A_1^2}{C^2} - s^2,$$

where

$$s = \frac{C}{\rho} + \frac{A_1}{C}.$$

The integration of the latter term yields

$$\rho = \frac{p}{1 + e\cos(u - \omega)},$$

where

$$e = \sqrt{1 + hC^2A_1^{-2}}, \quad p = -A_1^{-1}C^2,$$

and ω is a new arbitrary constant.

Thus, the motion of any of the points P_i around the common centre of gravity proceeds by a conic section which satisfies the law of areas. In other words, this motion proceeds according to the laws of Kepler.

CHAPTER VII

THE RESTRICTED PROBLEM OF THREE BODIES

38. The Equations of Motion, The Jacobi Integral

The so-called restricted problem of three bodies is one of the particular cases of the three body problem, which has been relatively well studied, and which is of particular interest in astronomy. It consists of the following: It is required to investigate the motion of a body, P, having an infinitesimal mass, in the field of gravitation of two bodies, S and J, having finite masses and moving in circular orbits about their own centre of gravity.

Such a problem is met with in the study of the motion of a planetoid or a comet under the action of the gravitation of the sun and Jupiter, when Jupiter's orbit is approximated by a circle. The lunar motion can also be considered as a particular case of the limited problem in the first approximation. In this case, one has to neglect not only the eccentricity of the earth and the gravitation of the other planets, but also the mass of the moon. That is, one has to neglect the force of gravitation with which the moon acts on the earth and sun.

Let us denote by m and m' the masses of bodies S and J and assume that $m \gg m'$. We choose the common centre of mass 0 as the origin of coordinates, the plane in which bodies S and J move as the xy plane and the sraight-line SOJ as the x-axis. In this coordinate system, we denote the coordinates of points S and J by (-a, 0, 0) and $(a_2, 0, 0)$ where $a_1 > 0$ and $a_2 > 0$. Further, we denote by n the constant angular velocity with which the straight line SOJ rotates around point 0. According Kepler's third law

$$k^2(m_1 + m_2) = n^2(a_1 + a_2)^3, \tag{1}$$

where $a_1 + a_2$ is the semimajor axis of the orbit along which one of the bodies S and J moves under the action of the mutual gravitation.

We choose the positive direction of the axis Oy so that n is always positive. Let x, y and z be the coordinates of point P. Since the coordinate system rotates with angular velocity n around the z axis, then the components of the absolute velocity of this point are

$$\dot{x} - ny, \quad \dot{y} + nx, \quad \dot{z}.$$

If we denote by m_0 the mass of point P, the kinetic energy of this point is given by

$$T = \frac{1}{2} m_0 \left[(\dot{x} - ny)^2 + (\dot{y} + nx)^2 + \dot{z}^2 \right].$$

Applying the Lagrange equations (§ 19), we obtain

$$\begin{aligned} \ddot{x} - 2n\dot{y} - n^2 x &= \frac{\partial U}{\partial x}, \\ \ddot{y} + 2n\dot{x} - n^2 y &= \frac{\partial U}{\partial y}, \\ \ddot{z} &= \frac{\partial U}{\partial z}, \end{aligned}$$

where U is the force function acting on point P divided by m_0. In the present case, point P moves under the action of the gravitation of points S and J. Therefore

$$U = \frac{k^2 m_1}{r_1} + \frac{k^2 m_2}{r_2}.$$

Assuming that

$$\Omega = \frac{1}{2} n^2 (x^2 + y^2) + k^2 \left(\frac{m_1}{r_1} + \frac{m_2}{r_2} \right), \qquad (2)$$

the equations of motion in the restricted problem of three bodies will be given by

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial\Omega}{\partial x} \\ \ddot{y} + 2n\dot{x} &= \frac{\partial\Omega}{\partial y} \\ \ddot{z} &= \frac{\partial\Omega}{\partial z} \end{aligned}\right\} \qquad (3)$$

Multiplying these equations by x, y and z, adding and integrating, we obtain

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2\Omega - C, \qquad (4)$$

where C is an arbitrary constant. This relation is known as the Jacobi integral. The constant C will be called the Jacobi constant.

The Jacobi integral enables us to draw many important conclusions on the character of motion of point P. This will be now investigated.

39. <u>The Surface of Zero-Velocity</u>

Let us denote by v the velocity of point P relative to the moving coordinate system. We then write the Jacobi integral, given by equation (4), as follows

$$v^2 = 2\Omega - C.$$

Using this relation, we are able to determine the relative velocity v in each point in the rotating space, for all motions characterized by a given value of the Jacobi constant C. Inversely, if the constant C and velocity v are given, then this relation defines the locus of points of the rotating space, in which body P can exist.

We consider the totality of motion of point P, for which the constant C has a given value. Evidently, these motion are possible in the space region, in which $2\Omega - C \geqslant 0$, otherwise the velocity v of body P is imaginary. The surface

$$2\Omega - C = 0 \qquad (5)$$

defines the boundary between the regions in space, in which the motions corresponding to a given value of C are possible and the regions in which these motions are impossible. This surface is called the surface of zero-velocity, since v = 0 at each of its points.

In the following, we study the form of the surface of zero-velocity for different values of C. We choose the units of length and time such that

$$SJ = a_1 + a_2 = 1, \quad k = 1$$

Using equation (1), we obtain

$$n^2 = m_1 + m_2.$$

Taking into account expression (2), we write equation (5) in the following way

$$m_1\left(x^2 + y^2 + \frac{2}{r_1}\right) + m_2\left(x^2 + y^2 + \frac{2}{r_2}\right) = C, \qquad (6)$$

where

$$r_1 = \sqrt{(x + a_1)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x - a_2)^2 + y^2 + z^2}$$

The surface, represented by equation (6), evidently lies inside the cylinder

$$(m_1 + m_2)(x^2 + y^2) = C$$

and asymptotically approaches the cylinder when z increases to infinity.

The equation of the curve resulting from the intersection of surface (6) with the plane xOy, is obtained by substituting Z = 0 in equation (6). This substitution yields

$$(m_1 + m_2)(x^2 + y^2) + \frac{2m_1}{\sqrt{(x + a_1)^2 + y^2}} + \frac{2m_2}{\sqrt{(x - a_2)^2 + y^2}} = C \qquad (7)$$

Evidently, this curve is symmetric relative to the x-axis.

Let us assume that C is a large number. In this case, equation (7) is satisfied by points of one of the following types:

1- Points, for which the quantity $x^2 + y^2$ is large. For such points, the second and third terms of equation (7) are small, so that this equation reads

$$x^2 + y^2 = \frac{C - \epsilon_0}{m_1 + m_2}, \qquad (8)$$

where ϵ_0 is a small positive quantity.

2- Points for which the radius vector

$$r_1 = \sqrt{(x - a_1)^2 + y^2}$$

is small. For these points, the first and third terms of equation (7) are small, so that this equation reads

$$(x - a_1)^2 + y^2 = \left(\frac{2m_1}{C - \epsilon_1}\right)^2, \qquad (9)$$

where ϵ_1 is a small positive quantity

3- Points for which the distance to J, i.e.

$$r_2 = \sqrt{(x - a_2)^2 + y^2},$$

is sufficiently small. For these points, equation (7) may be written as

$$(x - a_2)^2 + y^2 = \left(\frac{2m_2}{C - \epsilon_2}\right)^2. \qquad (10)$$

Hence, for large values of C, curve (7) consists of three separate closed parts, each having a form slightly differing from a circle. The larger is mass m_1 as compared to m_2, the greater are the dimensions of curve (9) as compared to those of curve (10).

As C decreases the dimensions of curves (9) and (10) increase, and their forms become more and more stretched along the axis Ox. At some

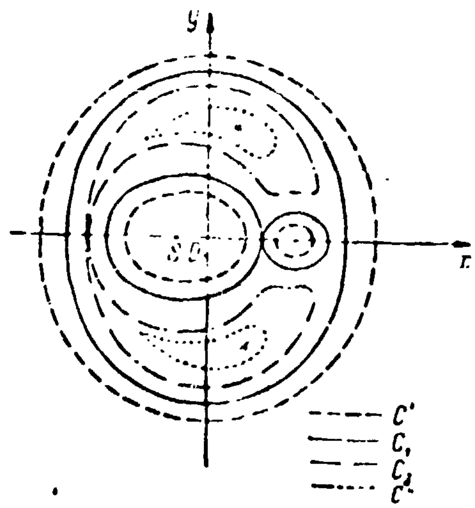

Рис. 4.

Figure 4

value $C = C_1$, these curves touch each other. At smaller values of C we have two separate ovals and one curve enclosing points S and J (figure 4). On the other hand, when C is decreased, the dimensions of curve (8) decrease. At some values $C = C_2$ and $C = C_3$, this curve touches the two internal curves, mentioned just above. Subsequently, these curves are amalgamated.

At large values of C, the domain of the plane xOy, in which the motion of body P is forbidden, consists of points external with respect to curve (9) and (10) and internal with respect to curve (8). At smaller values of C, this region consists only of points lying inside curves C" (figure 4) which decrease when C decreases and turn into points at some value $C = C_4$ and then completely disappear. Thus, at sufficient values of C, body P will have the possibility of moving over all plane xOy.

Figure 4 gives a schematic representation for the curves of these curves for decreasing values of C, namely $C' > C_1 > C_2 > C''$.

Using similar arguments, we may find a representation for the form of the curve that results by the intersection of surface (6) with the plane xOz. The equation of this curve is obtained by the substitution y = 0 in equation (6). This substitution yields

$$(m_1+m_2)x^2+\frac{2m_1}{r_1}+\frac{2m_2}{r_2}=C, \qquad (11)$$

where

$$r_1=\sqrt{(x-a_1)^2+z^2}, \qquad r_2=\sqrt{(x-a_2)^2+z^2}$$

When the value of C is very large, this equation can be simplified by three different ways; either by making x^2 very large, or by making any of the quantities r_1 and r_2 very small. Accordingly, curve (11) consists of three separate parts, respectively defined by the following equations

$$(m_1+m_2)x^2=C-\varepsilon,$$
$$r_1=\frac{2m_1}{C-\varepsilon'},$$
$$r_2=\frac{2m_2}{C-\varepsilon''}.$$

This case is represented by curves C' in figure 5. On decreasing C, we pass again by the critical values C_1 , C_2 and C_3 where different parts of curve (11) get into contact. Finally, when the value of the

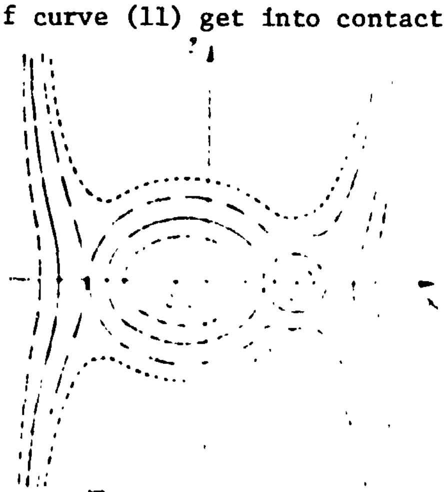

Fig. 5

Figure 5

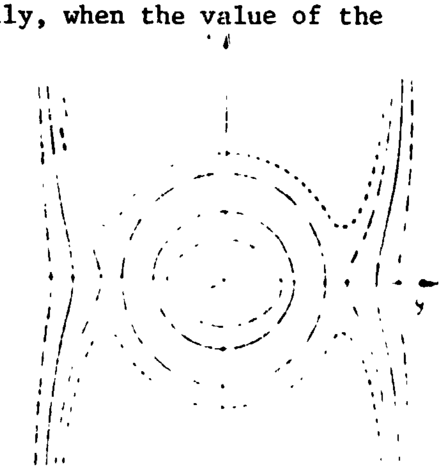

Fig. 6

Figure 6

Jacobi constant becomes sufficiently small, e.g. C = C", curve (11) does not cut the Ox axis.

The intersection of surface (6) with the plane yOz is defined by

$$(m_1 + m_2)y^2 + \frac{2m_1}{r_1} + \frac{2m_2}{r_2} = C, \qquad (12)$$

where

[illegible]

The corresponding curves are represented in figure 6 for the different values of C. The curves are obtained on the assumption that mass m_1 is considerably larger than mass m_2.
The comparison between the three cross sections of surface (6), represented by figures 4, 5 and 6, enable us to have a clear picture on the shape of this surface for the different values of the Jacobi constant C.

After this qualitative study of the surface (6), we turn into the study of specific points on this surface.

40. Specific Points on the Surfaces of Zero-Velocity

A specific point on the surface

$$F(x, y, z) = 0$$

is defined by the following equations

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0,$$

They can be solved in combination with the equation of the surface. For a surface of zero-velocity, defined by an equation of the type

$$2\Omega - C = 0, \qquad (13)$$

where

$$2\Omega = (m_1 + m_2)(x^2 + y^2) + \frac{2m_1}{r_1} + \frac{2m_2}{r_2}$$

$$r_1 = \sqrt{(x + a_1)^2 + y^2 + z^2}, \qquad r_2 = \sqrt{(x - a_2)^2 + y^2 + z^2}$$

the specific points are given by

$$\left.\begin{aligned}
&\frac{\partial\Omega}{\partial x}=(m_1+m_2)\,x-\frac{m_1(x-a_1)}{r_1^3}-\frac{m_2(x-a_2)}{r_2^3}=0\\
&\frac{\partial\Omega}{\partial y}=(m_1+m_2)\,y-\frac{m_1 y}{r_1^3}-\frac{m_2 y}{r_2^3}=0\\
&\frac{\partial\Omega}{\partial z}=-\frac{m_1 z}{r_1^3}-\frac{m_2 z}{r_2^3}=0.
\end{aligned}\right\}\qquad(14)$$

Solving equations (14), we find the coordinates of the specific points. Subsequently, we use equation (13) to find the corresponding values of the Jacobi constant C.

It is easy to find the mechanical meaning of the specific points. Comparing equations (14) with the equations of motion of body P, equations (3), we find that at each of the specific points not only

$$\dot{x}=\dot{y}=\dot{z}=0,$$

but also

$$\ddot{x}=\ddot{y}=\ddot{z}=0.$$

Thus, once body P arrives at a specific point and its corresponding value of C, its velocity and acceleration vanish. The body then remains eternally at this point, hence, the specific points are the positions of relative equilibrium of point P. In these points the body can remain at rest relative to the moving coordinate system. When body P is at a specific point, the ratio of the distances between the three bodies S, J and P remains unchanged. We thus conclude that the specific points are nothing else but the libration points, which we have studied in the previous chapter.

Let us now find the coordinates of the libration points and evaluate the corresponding values of the Jacobi constant. The last of equations (14) yields z = 0 so that the libration points lay in the plane xOy.

They may be identified with the specific points of curve (7). We shall make use of this situation in the practical evaluation of the coordinates of the libration points. We first observe that when z = 0.

$$m_1 r_1^2 + m_2 r_2^2 = (m_1 + m_2)(x^2 + y^2) + m_1 a_1^2 + m_2 a_2^2.$$

This is because the origin of coordinates is taken in the centre of mass, and hence

$$-m_1 a_1 + m_2 a_2 = 0.$$

Equation (7) can then be written as

$$m_1 (r_1^2 + 2r_1^{-1}) + m_2 (r_2^2 + 2r_2^{-1}) = C', \tag{15}$$

where

$$C' = C + m_1 a_1^2 + m_2 a_2^2 = C + \frac{m_1 m_2}{m_1 + m_2}, \tag{16}$$

since, evidently,

$$a_1 = \frac{m_2}{m_1 + m_2}, \qquad a_2 = \frac{m_1}{m_1 + m_2}.$$

Writing the equation of curve (15) in the form

$$f(x, y) = 0,$$

we obtain the following equations for the specific points of curve (7)

$$\frac{\partial f}{\partial x} = 0, \qquad \frac{\partial f}{\partial y} = 0.$$

In the present case, these points may be represented by

$$\left.\begin{aligned} \frac{\partial f}{\partial r_1}\frac{\partial r_1}{\partial x} + \frac{\partial f}{\partial r_2}\frac{\partial r_2}{\partial x} = 0 \\ \frac{\partial f}{\partial r_1}\frac{\partial r_1}{\partial y} + \frac{\partial f}{\partial r_2}\frac{\partial r_2}{\partial y} = 0 \end{aligned}\right\} \tag{17}$$

They may be satisfied in two ways; either to put

$$\frac{\partial f}{\partial r_1} = 0, \qquad \frac{\partial f}{\partial r_2} = 0,$$

which yields

[illegible]

or to put

$$\frac{\partial(r_1, r_2)}{\partial(x, y)} = \frac{1}{r_1 r_2} \ldots a_2 = 0 \qquad [illegible]$$

from which it follows that y = 0. In the first case, we obtain two libration points, L_4 and L_5 (figure 3), which form equilateral triangles with points S and J. Thus, points L_4 and L_5 are isolated points of curve (15). In other words, they are double points of complex tangents. The corresponding value $C = C_4$ are easily obtained from equations (15) and (16) as

$$C_4 \; 3(m_1 \; m_2) \; \frac{[illegible]}{m_1 \cdot m_2} \qquad [illegible]$$

In the second case, in which the double points are subject to condition (18) and lay on the axis Ox, one of the following conditions holds

1) $r_1 + r_2 = 1$,
2) $r_1 = 1 + r_2$,
3) r_1 [illegible] r_2 [illegible] 1

depending on the situation of the double point relative to S and J. We shall successively consider each of these cases.

<u>The first case</u>

Let $r_1 + r_2 = 1$. Then

$$r_1 = -x + a_1, \quad r_2 \; [illegible], \qquad \frac{dr_1}{dx} = -1, \quad \frac{dr_2}{dx} = 1,$$

so that the first of equations (14) yields

$$\frac{m_2}{m_1} \; [illegible] \qquad [illegible]$$

or

$$(m_1 + m_2)r_2^5 - (3m_1 + 2m_2)r_2^4 + (3m_1 + m_2)r_2^3 - m_2 r_2^2 + 2m_2 r_2 - m_2 = 0. \quad (20)$$

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If the ratio m_1/m_2 is small, we obtain the required positive root in the form of a series-expansion. Actually, expanding the right-hand side of equation (20) in a power series, we obtain

$$\frac{m_2}{m_1} = 3r_2^3\left(1 - r_2 + \frac{1}{3}r_2^2 + \ldots\right),$$

Taking the cubic root of each side, and introducing the notation

$$\nu = \sqrt[3]{\frac{m_2}{3m_1}},$$

we obtain

$$\nu = r_2\left(1 - \frac{1}{3}r_2 - \frac{1}{9}r_2^2 + \ldots\right),$$

Hence,

$$r_2 = \nu + \frac{1}{3}\nu^2 + \frac{1}{9}\nu^3 + \ldots \quad (21)$$

From equation (15), we obtain the corresponding values of the Jacobi constant;

$$\begin{aligned} C_1 &= m_1(3 + 9\nu^2 - 2\nu^3 + \ldots) + \frac{m_1 m_2}{m_1 + m_2} \\ &= m_1(3 + 9\nu^2 - \nu^3 + \ldots). \end{aligned} \quad (22)$$

If we want to obtain a more accurate value than that which the series (21) yields, we shall find it easier to numerically solve equation (20).

<u>The second case</u>

Let $r_1 = 1 + r_2$. Then, $r_1 = x + a_1$, $r_2 = x - a_2$ and

$$\frac{\partial r_1}{\partial x} = 1, \qquad \frac{\partial r_2}{\partial x} = 1.$$

the first of equations (17) then yields

$$\frac{m_2}{m_1} = \frac{r_1 - r_1^{-2}}{r_2^{-2} - r_2} = \frac{r_2^2(3 + 3r_2 + r_2^2)}{(1 - r_2^3)(1 + r_2)^2} \quad (23)$$

or

$$(m_1+m_2)r_2^5+(3m_1+2m_2)r_2^4+(3m_1+m_2)r_2^3-m_2r_2^2-2m_2r_2-m_2=0. \qquad (23')$$

This equation is identical to equation (14) of § 37 when $m_3 = 0$ is substituted in the latter equation. Hence, it has one and only one positive root. Let us expand this root into a power series. Expanding the right-hand side of equation (23) in powers of r_2, we obtain

$$\nu^3 = r_2^3\left(1 - r_2 + \frac{4}{3}r_2^2 - \dots\right).$$

Raising both sides to the power 1/3 e obtain

$$\nu = r_2\left(1 - \frac{1}{3}r_2 + \frac{1}{3}r_2^2 + \dots\right),$$

Solving this equation, we obtain

$$r_2 = \nu + \frac{1}{3}\nu^2 - \frac{1}{9}\nu^3 + \dots \qquad (24)$$

The corresponding value of the Jacobi constant is

$$C_2 = m_1(3 + 9\nu^2 - 5\nu^3 + \dots) - \frac{m_1 m_2}{m_1 + m_2}. \qquad (25)$$

<u>The third case</u>

Let $r_1 = r_2 = 1$. Then, $r_1 = -x - a_1$, $r_2 = -x + a_2$ and

$$\frac{dr_1}{dx} = -1, \quad \frac{dr_2}{dx} = -1.$$

In analogy with the previous case, we obtain

$$\frac{m_2}{m_1} = -\frac{r_1 - r_1^{-2}}{r_2 - r_2^{-2}}. \qquad (26)$$

Since $r_2 > 1$, then $r_1 < 1$. We assume that

$$r_1 = 1 - \alpha, \quad r_2 = 2 - \alpha.$$

We obtain the following equation for the quantity α

$$3\nu^3 = \frac{(2-\alpha)^2(3\alpha - 3\alpha^2 + \dots)}{(1-\alpha)^2(\dots 12\alpha + 6\alpha^2 - \dots)}.$$

or

$$3\nu^3 = \alpha\,\frac{12 - 24\alpha + 19\alpha^2 - 7\alpha^3 + \alpha^4}{7 - 26\alpha + 37\alpha^2 - 25\alpha^3 + 8\alpha^4 - \alpha^5}$$

Then,

$$\alpha = \nu^3\left(\frac{7}{4} - 3\alpha + \ldots\right).$$

Solving this equation by the method of successive approximations, we obtain

$$\alpha = \frac{7}{4}\nu^3 - \frac{21}{4}\nu^6 + \ldots \qquad (27)$$

The Jacobi constant will then be given by

$$C_3 = m_1(3 + 3\alpha^2 + \ldots) + m_2\left(5 - \frac{7}{2}\alpha + \frac{5}{4}\alpha^2 + \ldots\right) - \frac{m_1 m_2}{m_1 + m_2}$$

In other words,

$$C_3 = m_1\left(3 + 12\nu^3 - \frac{3}{16}\nu^6 + \ldots\right) \qquad (28)$$

If we assume that the sum of the masses m_1 and m_2 equals unity and introduce the following notation

$$m_1 = 1 - \mu, \qquad m_2 = \mu,$$

then all the quantities under consideration will be functions of the variable μ only. It is sufficient for this variable to vary from 0 to ½ in order to cover all of the possible cases[(1)].

---

(1) The coordinates of the libration points as well as the corresponding values of the Jacobi constants, $C_1(\mu)$, $C_2(\mu)$, ... are studied in the following paper
M.Martin, On the libration points of the restricted problem of three bodies, American journal of Mathematics, 53, 1931, 167-177.
Corrections and addenda to this paper are given in A.A. Markov, Progress in Astronomical Sciences (Uspehi Astronomiceskih nauk) 3, 1933, 75-77.
Tables of the abscissae of points L_1, L_2 and L_3 are given in the following paper
J. Rosenthal, Table for the libration points of the restricted problem of three bodies, Astr. Nachr. 244, 1931, 1969.

In figure 7, the curves that correspond to the critical values of the Jacobi constant are shown for the case

$$m_1 = 10, \qquad m_2 = 1.$$

These critical values, as well as the bipolar coordinates of the libration points are given in the following table

L_1:	$r_1 =$ 0.7175,	$r_2 =$ 0.28[illegible]5,	C_1 [illegible]
L_2:	1.3170,	0.3170,	C_2 [illegible]
L_3:	0.91[illegible]9,	1.91[illegible]9,	C_3 [illegible]
L_4 и L_5:	1.000,	1.0[illegible],	C_4 32.091

For the case

$$m_1 = 1, \qquad m_2 = 1$$

The corresponding values are

L_1	$r_1 =$ 0.5[illegible]	$r_2 =$ 0.[illegible]	C_1 8.000
L_2	1.[illegible]	0.[illegible]	C_2 [illegible]912
L_3	0.[illegible]	1.[illegible]	C_3 [illegible]912
L_4 and L_5	1.0[illegible]	1.000	C_4 [illegible]

Curves obtained for these values of the Jacobi constants are shown in Figure 8.

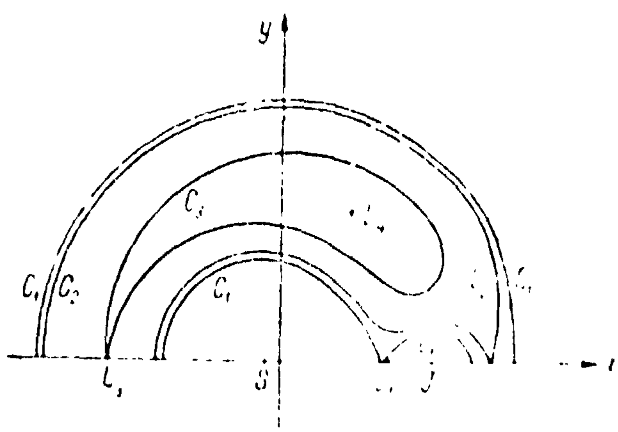

Рис. 7.

Figure 7

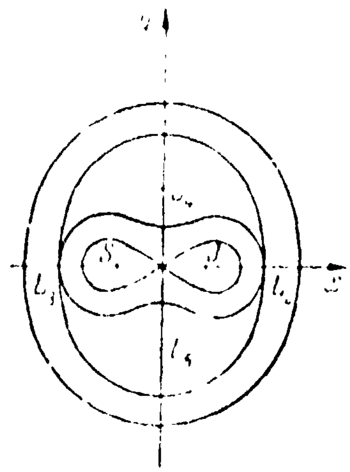

Рис. 8.

Figure 8

41. Periodic solutions of the restricted problem of three bodies

In the previous chapter, we studied the Lagrange motions in which bodies simultaneously move in elliptic orbits. These motions are examples of periodic orbits in the three-body problem. In these motions, the coordinates of all the three bodies are expressed by periodic functions of time having equal periods.

Hill gave another example of periodic orbits. He developed a method for the independent determination of some inequalities in the motion of the moon, caused by the gravitation of the sun (Chapter XVIII). Later, Poincaré suggested a method for finding and studying the general classes of periodic solutions of the three-body problem. Periodic solutions are thus the first targets attained in the three body problem that have never been solved analytically. With the start of the periodic solutions, the study of other interesting types of solutions, such as the asymptotic solution, became possible.

The study of periodic solutions is just in its initial stage. Even in the simple case ofthe restricted problem, only a few groups of periodic orbits have been more or less perfectly studied.

The most well-studied orbits are plane periodic orbits, passing close to the libration points L_1 , L_2, ... L_5. These are the orbits that inclose planet J but not S, and the orbits that enclose the sun only at such a distance, that the ratios of the period of revolutions along them to the period of revolution of planet J are simple, such as 1:3, 2.3 and so on.

The orbits of the first type are useful in the investigation of the motion of the so-called "Trojans". These are the small planets that move nearly along the Jupiter trajectory. The elements of the orbits of the known "Trojans" are shown in the next table. The elements are given relative to the ecliptic and equinox 1925.0. The average longitude of epoch ϵ is given for the moment 1925 January 10 T.U. The elements of Jupiter in that table are the average elements for the above mentioned moment. The elements of the small planets are the osculating elements of different epochs, grouped around 1935.

Name	a	n	φ	i	Ω	π	ϵ
Jupiter	5.2026	299″.128	2.775	1.307	[illegible]	[illegible]	277.142
588 Achilles	5.2267	296.940	8.541	10.309	315.778	82.790	335.322
617 Patroclus	[illegible]	[illegible]	[illegible]	[illegible]	13.581	316.976	221.077
624 Hector	5.1173	[illegible]	[illegible]	18.261	[illegible]	161.521	[illegible]
659 Nestor	[illegible]	[illegible]	6.270	4.521	350.188	322.382	[illegible]
884 Priamus	[illegible]	295.193	6.832	8.882	300.729	270.525	[illegible]
911 Agamemnon	[illegible]	305.125	3.711	21.930	[illegible]	55.383	[illegible]
1143 Odysseus	5.1600	[illegible]	[illegible]	[illegible]	220.311	93.999	310.097
1172 Aneas	[illegible]	[illegible]	5.765	16.675	246.117	291.186	[illegible]
1173 Anchises	5.1047	[illegible]	7.853	6.971	[illegible]	313.889	[illegible]
1208 Troilus	[illegible]	[illegible]	1.061	[illegible]	[illegible]	[illegible]	[illegible]

The last column of this table indicates that the planets 588, 621, 659, 911 and 1143 are close the libration point L_4, while the others are around L_5.

The theory of the second-type orbits, which enclose Jupiter J at a short distance, are closely related to the theory of satellites.

The orbits of the third type make it possible to construct a theory for the motion of small planets, the average motion of which is commeasurable with the motion of Jupiter. It is often more useful to use these periodic orbits as a first approximation to the orbits of planets, rather than to use the Kepler ellipses.

Poincaré divided the periodic elements of the restricted problem of three bodies into three grades. He related the orbits of bodies S and J that lay in the xOy plane to the first and second grade, and those

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of the other planets to the third one. The difference between periodic elements of the first and second grade is the following. If the mass of body J tends to zero, the periodic orbits will tend to Kepler ellipses. The orbits of the first grade will be those, for which the eccentricities of the limiting ellipses are zero. In other words, the orbits of the first grade are slightly different from circles when the value of μ is small. The second-grade orbits are near to elliptic orbits.

We shall not consider in detail the properties of periodic orbits. We shall only consider infinitesimal orbits around libration points in the following section(1).

Alongisde the analytical methods of finding periodic solutions, the method of numerical integration of differential equations is applied by the initiative of Darwin and Tile. The numerical integration of the equations of motion has an advantage over the corresponding analytical methods. The former method is simpler than the latter when one considers a given concrete case. One is then able to obtain the numerical solution using the most elementary methods of calculation. However, this solution is only useful for the interval of time at which the calculation were made. This is the most serious drawback of numerical solutions. Periodic solutions are evidently free from this deficiency. It is sufficient to obtain an analytical solution for one period in order to

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(1) Apart from the classical work:
H. Poincaré, Le methodes nouvelles de la Mechanique Celeste, t. I, II, III, Paris 1892-1899,
the theory of periodic orbits is given in:
F.R. Moulten, Periodic Orbits, Washington, 1920.
A detailed bibliography is given in the article:
E.T. Whittaker, Prinziplen der Storungstheorie und allgemeine Theorie der Bahnkurven in dynamischen Problemen, Encyklopädie der Math. Wissenschaften, Bd. VI, 2 (1921) 512-556.

obtain a complete picture of the motion that corresponds to the given initial conditions.

At present, analytical methods are applied for the study of only periodic orbits in the case when the mass of J is considerably smaller than the mass of S. On the other hand, the numerical methods are easily used for the arbitrary ratios of masses. Darwin used this method for the case when the mass of J equal to one-tenth that of S and he was able to find a number of periodic elements. Elis Strömgren, as well as Tile, studied the case when the masses of J and S were equal. Such studies were started by Burrau in the year 1900. Since 1913, research on this was continued by Elis Strömgren, a scientist of the Copenhagen observatory. These investigations gave the possibility not only to find a large number of periodic elements, but also to follow the transition of some classes of these orbits into others and to observe the disappearing process of some classes of periodic orbits when the initial conditions are changed. These observations naturally led to a considerable simplification of the analytical solutions corresponding to these processes[1].

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(1) The results of Darwin are given in his classical work: G. Darwin, Periodic Orbits, Acta Math., 21, 1897, 99-216. Some additions are given in Math. Ann., 51, 1899. The results obtained in the Copenhagen observatory are given in a series of memoirs: Publikationer og mindre Middelelser tra Kobenhavns Observatorium. The conclusion are given in Elis Stromgren's Paper "Connaissance actuelle des orbites dans le probleme des trois corps", which is published in No. 100 (1936) of this Journal. This paper contains the full bibliography of the work of the Copenhagen School and is also published in: Bull. astr., 2-e serie, 9, 1936.

42. Motion near collinear libration points

Let (a,b,c) be an arbitrary point of the uniformly rotating space Sxyz. We investigate whether it is possible that among the motions, defined by

$$\ddot{x} - 2n'\dot{y} = \frac{\partial\Omega}{\partial x}$$

$$\ddot{y} + 2n'\dot{x} = \frac{\partial\Omega}{\partial y}$$

$$\ddot{z} = \frac{\partial\Omega}{\partial z}$$

where

$$\Omega = \frac{1}{2}(m_1 + m_2)(x^2 + y^2) + \frac{m_1}{r_1} + \frac{m_2}{r_2}$$

there exists such a motion, that body P is always as close as possible to point (a,b,c).

We can consider that the function Ω is holomorphic in the vicinity of point (a,b,c), except in the case when this point coincides with one of bodies S and J. Accordingly

$$x = a + \xi, \quad y = b + \eta, \quad z = c + \zeta$$

we expand the right-hand side of the equations of motion in powers of the small quantities ξ, η and ζ. Keeping only the first powers of these quantities, we obtain

$$\begin{aligned}
\ddot{\xi} - 2n'\dot{\eta} &= \frac{\partial\Omega}{\partial a} + \xi\frac{\partial^2\Omega}{\partial a^2} + \eta\frac{\partial^2\Omega}{\partial a\,\partial b} + \zeta\frac{\partial^2\Omega}{\partial a\,\partial c} \\
\ddot{\eta} + 2n'\dot{\xi} &= \frac{\partial\Omega}{\partial b} + \xi\frac{\partial^2\Omega}{\partial a\,\partial b} + \eta\frac{\partial^2\Omega}{\partial b^2} + \zeta\frac{\partial^2\Omega}{\partial b\,\partial c} \\
\ddot{\zeta} &= \frac{\partial\Omega}{\partial c} + \xi\frac{\partial^2\Omega}{\partial a\,\partial c} + \eta\frac{\partial^2\Omega}{\partial b\,\partial c} + \zeta\frac{\partial^2\Omega}{\partial c^2}
\end{aligned} \tag{30}$$

When ξ, η and ζ are sufficiently small, the motion takes place in the vicinity of the points, the coordinates of which are given by

$$\frac{\partial\Omega}{\partial a} = \frac{\partial\Omega}{\partial b} = \frac{\partial\Omega}{\partial c} = 0,$$

i.e., in the vicinities of the libration points, since these equations are identical with equations (14).

In this section, we consider the case of motions, proceeding infinitely close to collinear libration points. Hence, we put

$$a = x_k, \quad b = 0, \quad c = 0,$$

where x_k denotes the abscissa of the libration point L_K (K = 1,2,3). In order to find the second derivatives of function Ω, involved in equations (30), we differentiate expression (29) and replace the variables x, y and z by the above mentioned coordinates of the libration point L_K. We then obtain

$$\begin{aligned}
\frac{\partial^2\Omega}{\partial a^2} &= n^2 + \frac{2m_1}{r_1^3} + \frac{2m_2}{r_2^3}, & \frac{\partial^2\Omega}{\partial a\,\partial b} &= 0,\\
\frac{\partial^2\Omega}{\partial b^2} &= n^2 - \frac{m_1}{r_1^3} - \frac{m_2}{r_2^3}, & \frac{\partial^2\Omega}{\partial a\,\partial c} &= 0,\\
\frac{\partial^2\Omega}{\partial c^2} &= -\frac{m_1}{r_1^3} - \frac{m_2}{r_2^3}, & \frac{\partial^2\Omega}{\partial b\,\partial c} &= 0,
\end{aligned}$$

where

$$r_1 = x_k + a_1, \quad r_2 = x_k - a_2.$$

Introducing the following notations

$$A_k = \frac{m_1}{r_1^3} + \frac{m_2}{r_2^3},$$

we write equation (30) as follows

$$\left.\begin{aligned}
\ddot{\xi} - 2n\dot{\eta} &= (n^2 + 2A_k)\xi\\
\ddot{\eta} + 2n\dot{\xi} &= (n^2 - A_k)\eta\\
\ddot{\zeta} &= -A_k\zeta
\end{aligned}\right\} \qquad (31)$$

The last of these equations is independent of the others. If immediately gives

$$\zeta = C_1 \sin\sqrt{A_z}\,t + C_2 \cos\sqrt{A_z}\,t, \tag{32}$$

where C_1 and C_2 are constants of integration. We search for the solutions of the first two equations in the form

$$\xi = Ge^{\lambda t}, \quad \eta = He^{\lambda t}.$$

Substituting these expressions in equations (31) yields the following relations between the parameters G, H and

$$\left.\begin{aligned} [\lambda^2 - (n^2 + 2A_x)]\,G - 2n\lambda H &= 0 \\ 2n\lambda G + [\lambda^2 - (n^2 - A_x)]\,H &= 0. \end{aligned}\right| \tag{33}$$

we denote by λ_1, λ_2, λ_3 and λ_4 the roots of the characteristic equation

$$[\lambda^2 - (n^2 + 2A_x)][\lambda^2 - (n^2 - A_x)] + 4n^2\lambda^2 = 0$$

or

$$\lambda^4 + (2n^2 - A_x)\lambda^2 + (n^2 - A_x)(n^2 + 2A_x) = 0 \tag{34}$$

and by q_1, q_2, q_3 and q_4 the corresponding values of the ratio H:G, defined by equations (33). The following equations

$$\left.\begin{aligned} \xi &= G_1 e^{\lambda_1 t} + G_2 e^{\lambda_2 t} + G_3 e^{\lambda_3 t} + G_4 e^{\lambda_4 t} \\ \eta &= G_1 q_1 e^{\lambda_1 t} + G_2 q_2 e^{\lambda_2 t} + G_3 q_3 e^{\lambda_3 t} + G_4 q_4 e^{\lambda_4 t}, \end{aligned}\right| \tag{35}$$

where G_1, G_2, G_3 and G_4 are arbitrary constants, together with equation (34) define the general solution of system (31).

Evidently, the nature of the body P, that has an infinitesiamal mass, depends on the type of roots of equation (34). One easily sees that this equation has two real and two imaginary roots for the libration points L_1 , L_2 and L_3. Indeed, we shall show that the quantity

$$n - A_k = m_1 + m_2 - \frac{m_1}{r_1^3} - \frac{m_2}{r_2^3} \qquad (36)$$

is negative for k = 1, 2, 3. Consequently, there will be two real roots of opposite signs for equation (34), being considered as a second-order equation with respect to .

For point L_1 , expression (36) is negative because, in this case,

$$r_1 < 1, \quad r_2 < 1$$

For points L_2 and L_3, equations (23) and (26) give

[illegible]

Eliminating m from equation (36), we obtain

[illegible]

Since, for point L_2,

r_1 [illegible] 1, r_2 [illegible] 1,

and, for point L_3,

r_1 [illegible] 1, r_2 [illegible] 1.

expression (36) is evidently negative. Hence, two roots of the characteristic equation are purely imaginary complex-conjugate quantities. The other two roots are real and have opposite signs. Accordingly, the libration points L_1 , L_2 and L_3 are positions of unstable relative equilibrium. In other words, when body P is displaced from any of these points by an arbitrary small distance with an arbitrary small velocity, it may leave for ever the vicinity of this libration point.

Let us denote the real roots of equation (34) by λ_3 and λ_4. If we choose the initial conditions so that $G_3 = G_4 = 0$, we obtain a motion, in which body P will remain forever in the vicinity of the corresponding libration point. This is because the coordinates ξ, η and ζ relative to this point will be bounded for all values of t.

The existance of orbits, arbitrarily close to the libration points L_1, L_2 and L_3 is a necessary but not sufficient condition for the existance of periodic orbits in the vicinity of these points. If we limit ourselves to the accuracy, that the approximate integration of equation (31) achieves, we easily obtain periodic orbits by the appropriate choice of the initial values of coordinates, ξ_o, η_o and ζ_o, and components of velocity, $\dot{\xi}_o$, $\dot{\eta}_o$ and $\dot{\zeta}_o$.

We first set ζ_o and $\dot{\zeta}_o$ equal to zero. In equation (32) we will have $C_1 = C_2 = 0$, i.e. the motion of body P is planer. We then choose $\dot{\xi}_o$ and $\dot{\eta}_o$ using conditions $G_3 = 0$ and $G_4 = 0$. Formulae (35) will subsequently yield

[illegible]

where $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$ and β is a real number. On the basis of equation (33),

[illegible]

where

[illegible]

Expressing the exponential functions in terms of trigonometric functions, we obtain

[illegible]

where

[illegible]

We solve these equations for cos β t and sin β t, then square the resulting expressions and add. We obtain the following equation for the trajectory

[illegible]

or

[illegible] [illegible]

That is, the motion proceeds by an ellipse, the axes of which coincide with the coordinate axes. Denoting the semiaxes in directions $L_k \xi$ and $L_k \eta$ respectively, by a and b, we obtain

$$\frac{b}{a} = q.$$

It is easy to show that $q > 0$ for all three libration points. Hence, the eccentricity of the elliptic orbits obtained is equal to

$$e = \frac{\sqrt{q^2 - 1}}{q}$$

Consequently, the form of these orbits does not depend on the initial position (ξ_0, η_0) of point P, which affects only the dimensions of the orbits.

The values of q and e that correspond to three values of the ratio of masses of S and J are given in the following table. The first two ratios are relative to the work of stromgren and Darwin, discussed in 41, the third value takes place in the earth-sun system.

		L_1		L_2		L_3	
$m_2 : m_1$	μ	q	e	q	e		e
1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

We have thus obtained three systems of infinitesimal periodic orbits, each of which depends on two parameters. Naturally, the existance of these orbits is in sufficient to prove the existance of finite periodic orbits near the libration points L_1 , L_2 and L_3. However, it is possible to show that such periodic orbits actually exist(1).

In conclusion, we mention some words on the work by Hulden (Goulden) and Moulton, in which they applied the theory developed here to explain the antiaurora effect. They assumed that the libration point L_2 for the earth may be taken as the centre of an accumulation of meteors, occupying the interplanet space. Indeed, meteors for which $G_3 = G_4 = 0$ always remain in the vicinity of this point. Those, for which G_3 and G_4 are small, remain for a long time near this libration point. The light of the sun is reflected by the cluster of such meteors, we thus observe the reflected light as an antiaurora.

The distance of point L_2 from the earth is equal to 0.0101 astronomic units (§ 37), i.e. about 1.490.000 kms. If the above mentioned assumption is correct, the antiaurora will have a parallax of the order of 15'. Unfortunately, the antiaurora is of such a diffused effect, that there is no hope to check the validity of this assumption by observing its parallax.

43. Motion Near Triangular Libration Points

We will now consider the motion of body P near the libration points L_2 and L_2. We adopt that

$$m_1 = 1 - \mu, \quad m_2 = \mu,$$

where $n^2 = m_1 + m_2 = 1$, and assume that

$$0 < \mu < \frac{1}{2}.$$

The coordinates of bodies S and J are then equal to

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(1) F.R. Moulton, Periodic Orbits, Ch. V; and references cited therein.

$$-a_1 = -\mu, \quad a_2 = 1-\mu,$$

Hence, the coordinates of point L_4 are

$$a = \frac{1}{2}(1-2\mu), \quad b = \frac{\sqrt{3}}{2}, \quad c = 0.$$

The coordinates of point L_5 are obtained from the coordinates of point L_4 by changing the sign of $\sqrt{3}$. We can thus study the motion near points L_4 and then obtain the corresponding result for point L_5 by changing this sign.

Differentiating the function Ω, and replacing x, y and z by the coordinates of point L_4, obtained above, we obtain the equations of motion (30) in the form

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \frac{3}{4}\xi + \frac{3\sqrt{3}}{4}(1-2\mu)\eta \\ \ddot{\eta} + 2\dot{\xi} &= \frac{3\sqrt{3}}{4}(1-2\mu)\xi + \frac{9}{4}\eta \\ \ddot{\zeta} &= -\zeta. \end{aligned} \tag{39}$$

The general integral of the last of these equations is

$$\zeta = C_1 \cos t + C_2 \sin t, \tag{40}$$

where C_1 and C_2 are arbitrary constants. This solution shows that the projection of point P on the axis Oz performs periodic vibrations about the projection of the libration point on the same axis. The period of this vibration equals 2π, i.e. coincides with the period of rotation of the finite masses S and J around their centre of gravity. We should always remember that this result is only valid for linear vibrations about the position of relative equilibrium, i.e. for such a motion, for which we neglect in equations (30) terms involving second and higher powers of ξ, η and ζ.

We now solve the first two equations of system (39). We can write these equations as follows

$$\ddot{\xi} - 2\dot{\eta} = P\xi + S\eta, \quad \ddot{\eta} + 2\dot{\xi} = S\xi + T\eta, \tag{41}$$

if we consider that

$$P = \frac{3}{4}, \quad S = \frac{3\sqrt{3}}{4}(1-2\mu), \quad T = \frac{9}{4}.$$

Substituting

$$\xi = Ge^{\lambda t}, \quad \eta = He^{\lambda t},$$

into equation (41), we obtain the following equations for the unknown constants

$$\left.\begin{aligned} (\lambda^2 - P)\,G - (2\lambda + S)\,H &= 0 \\ (2\lambda - S)\,G + (\lambda^2 - T)\,H &= 0, \end{aligned}\right\} \tag{42}$$

These equations yield

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0. \tag{43}$$

Denoting by λ_1, λ_2, λ_3 and λ_4 the roots of the latter equation, we obtain the solution of system (41) in the form

$$\left.\begin{aligned} \xi &= G_1e^{\lambda_1 t} + G_2e^{\lambda_2 t} + G_3e^{\lambda_3 t} + G_4e^{\lambda_4 t} \\ \eta &= H_1e^{\lambda_1 t} + H_2e^{\lambda_2 t} + H_3e^{\lambda_3 t} + H_4e^{\lambda_4 t}, \end{aligned}\right\} \tag{44}$$

where G_1, G_2, G_3 and G_4 may be considered as arbitrary constants. The corresponding quantities H_1, H_2, H_3 and H_4 are obtained from equations (42).

The nature of the motion, represented by formulae (44), essentially depends on the type of roots of equation (43). These roots are given by the following equations

$$\left.\begin{aligned} \lambda_1 = -\lambda_2 &= \sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{M}} \\ \lambda_3 = -\lambda_4 &= \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{M}}, \end{aligned}\right\} \tag{45}$$

where

$$M = 1 - 27\mu(1 - \mu).$$

Increasing the value of μ from 0 to ½, we find that M is positive in the beginning, then vanishes at $\mu = \mu_0$, where

$$\mu_0 = 0.038\,520\,896\ldots \approx \frac{1}{26},$$

and, subsequently, M. remains negative up to $\mu = ½$. Hence, we conclude that the values of μ are in the interval

$$0 < \mu < \mu_0.$$

Equation (43) has different purely imaginary roots. Consequently, the general solution (44) may be written as

$$\begin{aligned} \xi &= G' \cos s_1 t + G'' \sin s_1 t + G''' \cos s_2 t + G'''' \sin s_2 t \\ \eta &= H' \cos s_1 t + H'' \sin s_1 t + H''' \cos s_2 t + H'''' \sin s_2 t, \end{aligned}$$

where

[illegible]

and G', G", G'" and G"" are new arbitrary constants. The coefficients H', H", ... are expressed in terms of G', G", ... by relations similar to those connecting G_1 , G_2, ... with H_1 , H_2, ...

It is thus clear that, in the case under consideration, the libration points are positions of stable relative equilibrium of body P, whose mass is infinitesimal. Actually, in order that the absolute values of ξ, and η remain less than an any given small quantity, it is necessary that the values of ξ , η , $\dot{\xi}$ and $\dot{\eta}$ are infinitesimal.

Whatever the initial conditions are the projection of the motion of body P on the plane $\xi\eta$ may be considered as a superposition of two elliptic motions, defined by and

$$\xi = G' \cos \beta t + G'' \sin \beta t, \quad \eta = H' \cos \beta t + H'' \sin \beta t \tag{46}$$

$$\xi = G''' \cos \gamma t + G'''' \sin \gamma t, \quad \eta = H''' \cos \gamma t + H'''' \sin \gamma t. \tag{47}$$

If we choose the initial conditions in such a way, that the constants C_1 and C_2 involved in formula (40) vanish, and that either $G''' = G'''' = 0$ or $G' - G'' = 0$, we obtain periodic orbits having period equal to $2\pi/\beta$ in the first case and the $2\pi/\gamma$ in the second one. Each of these periodic orbits depends on two arbitrary constants.

We have considered the case when $0 < \mu < \mu_0$. If $\mu > \mu_0$ then $M < 0$ and all of the roots λ, ... are complex numbers having nonvanishing real parts. Hence, the general solution (44) becomes

$$\xi = e^{\sigma t}(G' \cos \beta t + G'' \sin \beta t) + e^{\tau t}(G''' \cos \gamma t + G'''' \sin \gamma t)$$
$$\eta = e^{\sigma t}(H' \cos \beta t + H'' \sin \beta t) + e^{\tau t}(H''' \cos \gamma t + H'''' \sin \gamma t),$$

where σ, τ, β and γ are nonvanishing real numbers. In this case, L_4 and L_5 are positions of unstable relative equilibrium of body P.

The intermediate case of $\mu = \mu_0$ will be considered in the following section.

44. Application of Normal Coordinates

We have expanded the motion that proceeds infinitely close to the libration points L_4 and L_5 into the elleptic motions, described by equations (46) and (46). These elliptic motions proceed along orbits, the axes of which are inclined to the coordinate axes. In order' to simplify the study of these orbits, we transform the equations of motion (41) into a form, similar to that of the first two equations of (41). Equation (41) may be rewritten as

$$\ddot{\xi} - 2\dot{\eta} = \frac{1}{2}\frac{\partial F}{\partial \xi}, \quad \ddot{\eta} + 2\dot{\xi} = \frac{1}{2}\frac{\partial F}{\partial \eta},$$

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where

$$F = R\xi^2 + 2S\xi\eta + T\eta^2.$$

These equations have the following integral

[illegible]

Thus, the curves of zero velocity, given by the following equation

$$R\xi^2 + 2S\xi\eta + T\eta^2 = C, \qquad (48)$$

are conic sections, the centres of which are located at the libration point L_4.

In order to transform equation (48) into the canonical form

$$A\xi_1^2 + B\eta_1^2 = C,$$

it is necessary to rotate the coordinate axes by any angle θ, defined by

$$\operatorname{tg} 2\theta = \frac{2S}{R - T} = \sqrt{3}(1 - 2\mu).$$

The new coordinates will be expressed in terms of the old ones by

$$\xi_1 = \xi\cos\theta + \eta\sin\theta, \quad \eta_1 = -\xi\sin\theta + \eta\cos\theta,$$

where the coefficients A and B will be the roots of the secular equation, given by

$$\begin{vmatrix} R - \omega & S \\ S & T - \omega \end{vmatrix} = 0$$

or, in an unfolded form,

$$\omega^2 - 3\omega + \frac{27}{4}\mu(1 - \mu) = 0.$$

Consequently,

$$A = \frac{3}{2} - \frac{3}{2}\sqrt{1 - 3\mu(1 - \mu)}, \quad B = \frac{3}{2} + \frac{3}{2}\sqrt{1 - 3\mu(1 - \mu)}.$$

After the above transformations, equations (41) will become

$$\left.\begin{aligned} \ddot{\xi}_1 - 2\dot{\eta}_1 &= \frac{1}{2}\frac{\partial F}{\partial \xi_1} = A\xi_1 \\ \ddot{\eta}_1 + 2\dot{\xi}_1 &= \frac{1}{2}\frac{\partial F}{\partial \eta_1} = B\eta_1 . \end{aligned}\right| \qquad (49)$$

Adopting again that

$$\xi_1 = Ee^{\lambda t}, \quad \eta_1 = Fe^{\lambda t},$$

we obtain

$$\begin{aligned} (\lambda^2 - A)E - 2\lambda F &= 0 \\ 2\lambda E + (\lambda^2 - B)F &= 0. \end{aligned}$$

We thus obtain for λ, the previous values given by equations (45), while the corresponding ratios F:G will be different.

For $\lambda_1 = \beta i$ and $\lambda_2 = -\beta i$, we find

$$F = pEi \text{ and } F = -pEi,$$

where

$$p = \frac{\beta^2 + A}{2\beta} = \frac{2\beta}{\beta^2 + B}.$$

Similarly, for $\lambda_3 = \gamma i$ and $\lambda_4 = -\gamma i$, we obtain

$$F = qEi \text{ and } F = -qEi,$$

where

$$q = \frac{\gamma^2 + A}{2\gamma} = \frac{2\gamma}{\gamma^2 + B}.$$

Thence, equations (46) and (47) are replaced by

$$\xi_1 = E' \cos \beta t + E'' \sin \beta t, \quad \eta_1 = E'' p \cos \beta t - E' p \sin \beta t \qquad (50)$$

and

$$\xi_1 = E''' \cos \gamma t + E'''' \sin \gamma t, \quad \eta_1 = E''''q \cos \gamma t - E'''q \sin \gamma t.$$

The first pair of formulae represent a motion along the ellipse

$$p^2 \xi^2 + \eta^2 = p^2 (E'^2 + E''^2);$$

while the equation of the trajectory of the motion, represented by the second pair of formulae, is given by

$$q^2 \xi_1^2 + \eta_1^2 = q^2 (E'''^2 + E''''^2).$$

It is clear that the ratios of the semiaxes of these ellipses are p and q respectively. Their eccentiricities do not depend on E', E", ..., i.e. on the initial conditions.

The following table includes the values that characterize the motion near the libration points L_4 and L_5 .

μ	θ	A	B	β	γ
0.000	30° 0′ 0″	0.0000	3.0000	1.00000	0.00000
0.001	29 54 1	0.0090	2.9910	0.98607	0.10630
0.008	29 47 57	0.0180	2.9820	0.97119	0.23831
0.012	29 41 19	0.0269	2.9731	0.95513	0.29618
0.016	29 35 36	0.0359	2.9641	0.93761	0.34769
0.020	29 29 19	0.0448	2.9552	0.91819	0.39611
0.024	29 22 57	0.0537	2.9463	0.89618	0.44370
0.028	29 16 0	0.0625	2.9375	0.87033	0.49247
0.032	29 9 59	0.0714	2.9286	0.83801	0.51565
0.036	29 3 22	0.0802	2.9198	0.79088	0.61197

In figure 9, the elliptic orbits for the case μ = 0.01 are shown in a strongly magnified form. The table shows that the smaller μ is, the more complete is the coincidence of the semimajor axes of the ellipses under consideration with the tangent of the ellipse, along which point J rotates around S.

C-3

Let us now consider the case, when

$$\mu = \mu_0 = 0.038\ 5\ .$$

and, consequently,

$$\lambda_1 = \lambda_3 = \frac{i}{\sqrt{2}}, \qquad \lambda_2 = \lambda_4 = -\frac{i}{\sqrt{2}},$$
$$\theta = -28°59'10'', \qquad A = 0.085\ 786\ 44, \qquad B = 2.914\ 213\ 56.$$

Apart from the motion along the ellipse represented by equations (50) or (51), which will now be considered identical, we shall have a particular solution of the type

[illegible]

involving the arbitrary constants K and t_o. Accordingly, the motion becomes unstable when μ becomes equal to μ_o.

45. Tisseran's criterion

In conclusion of this chapter, we will consider one of the applications of the Jacobi integral, which has been suggested by Tisseran. It is well known that the elements of a conet's orbit may be strongly violated when it passes near a planet. Hence, it is often difficult to identify two comets only by their elements. Moreover, the appearance and even the brightness of a comet strongly vary certainly, one can evaluate the perturbation of one of the two comets under considerations from the time it appears until the time the other comet appears. However, this calculation is quite cumbersome. It is only worthwhile doing this calculation if the chances for the successful identification of a comet is good.

The orbital elements of a comet changes strongly enough to violate the orbit only if the comet approaches very closely a planet. For a planet as heavy and as far from the sun as Jupiter, a comet passing at a distance of 0.3 will be affected by the Jupiter's gravitation more than by the gravitation of the sun (§ 72). The changes in the orbit, that will occur in the short time in which the comet is in contact with Jupiter, are stronger than the perturbations induced by other planets. As a first approximation, the perturbation induced by other planets are neglected so that the case under consideration may be regarded as a restricted three body problem. In addition, the eccentricity of Jupiter is small and since the interaction of Jupiter with the comet lasts only for a short time, Jupiter's orbit would only slightly deviate from its circuit. Keeping the notation of § 38, we see that the coordinates x, y and z of the comet satisfy equation (4), i.e.

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = n^2(x^2 + y^2) + 2k^2\left(\frac{m_1}{r_1} + \frac{m_2}{r_2}\right) - C \qquad (52)$$

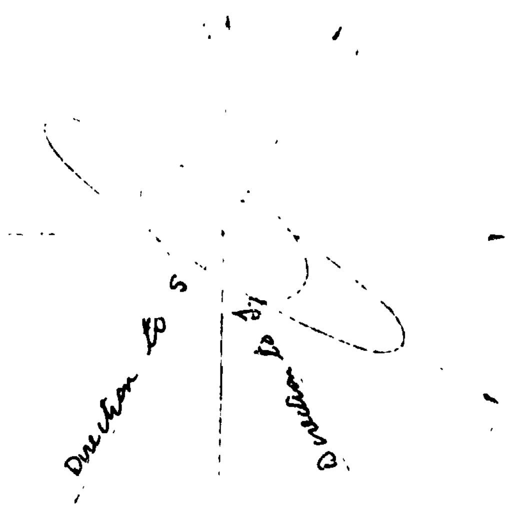

Fig. 9.

This relation leads to the following necessary condition for the identity of two comets: Two comets will appear to us identical if and only if they have the same value of the Jacobi constant C.

In order to make use of this condition, it is necessary to evaluate for each comet the relative coordinates x, y and z and components of velocity x, y and z. Then, equation (52) gives the corresponding value of C for each comet.

In order to simplify the application of this criterion, we make a transition into the fixed heliscentric coordinat e system $S\xi\eta\zeta$ in which the $S\zeta$ axis is parallel to the 0z axis. Evaluating the time starting from the moment when the axes Sx and 0ξ coincide, we obtain

$$\begin{aligned} &[illegible]\ \xi [illegible] + \eta\ [illegible] \\ &y = -\xi \sin nt\ [illegible]\ \eta\ [illegible] \\ &[illegible] \end{aligned}$$

From this it follows that

$$\begin{aligned} &[illegible]\ (\xi \cos [illegible]\ \eta\ [illegible])\ [illegible] \\ &[illegible]\ n(\xi\ [illegible]\ \eta)\ [illegible] \end{aligned}$$

In the new coordinate system, equation (52) becomes

$$\begin{aligned} &[illegible] + 2n(\xi\dot\eta - \eta\dot\xi) = 2k^2\left(\frac{m_1}{r}\ [illegible] \right) - \\ &- 2n'a_1(\xi \cos nt + \eta \sin nt) + n^2a_1^2\ [illegible]\ C \end{aligned} \qquad (53)$$

This formula can be used to calculate C when the comet is so far from the perturbing planet, that the comet moves almost entirely under the influence of the gravitation of the sun. In this situation, formula (53) can be considerably simplified. Let us denote by a, e, i, ... the elements of the comet in its motion around the sun. Taking the mass of the sun as unity, we put $m_1 = 1$. Then, the integral of area and the integral of the kinetic energy of the two-body problem enables us to write the following equations

$$\xi\dot{\eta} - \eta\dot{\xi} = k\sqrt{p}\cos i$$
$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = k^2\left(\frac{2}{r} - \frac{1}{a}\right),$$

where $r = r_1$ is the radius vector of the comet. Hence, equation (53) may be replaced by

$$a^{-1} + 2n'k^{-1}\sqrt{p}\cos i = C_0 + \delta,$$

where the average durnal motion of Jupiter n is replaced by n' in order to be distinguished from the elements of the comet. Moreover, the following notations have been used

$$C_0 = C k^{-2}$$
$$\delta = 2n'^2k^{-2}a_1(\xi\cos n't + \eta\sin n't) - n'^2k^{-2}a_1^2 - 2m_2r_2^{-1}.$$

In the cases that are usually met with in practice, the coordinates ξ and η are not large. Since a_1 and m_2 are small quantities (of the order of 0.001), and

$$n'k^{-1} = 0.084\,015,$$

then δ may be dropped.

We see that, after the comet has passed out of Jupiter's sphere of action, the expression

$$a^{-1} + 2n'k^{-1}\sqrt{p}\cos i$$

conserves its value. This equation represents Tisseran's criterion that defines the necessary (but not sufficient) condition for the identification of two comets.

Relation (54) becomes more accurate if on the one hand we take into consideration the corrective term δ, and on the other hand replace the average durnal motion of Jupiter by the angular heliocentric velocity.

$$[illegible]$$

Neglecting the square of the eccentricity of Jupiter, we obtain

$$[illegible]$$

Hence, the more exact relation will be given by

$$a\ [illegible]\ \cos\ [illegible]\ C_0. \qquad (55)$$

The radius-vector of Jupiter is best of all evaluated at the moment when the comet approaches Jupiter.

As an illustration of the present theory, we shall consider the approach of the Wolf comet to Jupiter in 1922 and which has been studied by M. Kanlenski[1].

The osculating elements of this comet before entering in Jupiter's sphere of action and after leaving it are

	1922 June 8.0	1922 December [illegible].0	
M_0	191 8' 5?.?"	23? 11 2?.?	
[illegible]	31 11 45.9	[illegible] 9.7	
n	528".1530	43?".12?9	
[illegible]	205 38 ?.3	9?.?' 22.?2	1925.0
[illegible]	18 ? 38.5	[illegible] 21 9.1	
[illegible]	[illegible]	[illegible] 1 0.0	
a	[illegible]	[illegible]	
[illegible]	[illegible]	[illegible]	
[illegible]	[illegible]	[illegible]	

The variations in the elements are strong because the minimal distance between the comet and Jupiter reached the value $\Delta = 0.1247$ on

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(1) The numbers given here are taken from the work:
M. Kamlenski, Recherches sur le mouvement de la comete periodique de Wolf, Bulletin de l'Academie' Polonaise des Science et des Lettres, Serie, A, 1925.

September 27, 1922.

Adopting that $\lg a' = 0.74624$ and $\lg r' = 0.73604$ and neglecting in equation (55) the small quantity δ , we obtain for the two comets the following values

$$C_o = 0.19221 \text{ and } 0.1919[illegible]$$

We thus see how small the change in the quantity C_o is, even during so large a variations of the elements.

PART TWO

THE NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS AND ITS APPLICATION TO THE STUDY OF THE MOTION OF STARS

CHAPTER VIII

THE NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

46. Introduction

All the problems of celestial mechanics are reduced to the solution of some differential equations. That is why celestial mechanics is always inseparably linked with the development of the methods for the solution of differential equations.

The integration of differential equations in a closed form is only possible in the most simple cases, such as thetwo-body problem. In general, the solution cannot be obtained in terms of the well-known functions. One then has to try other methods for the solution of differential equations. Amongst these methods, the two most general and effective methods are (1) the method of integration by series of expansions and (2) the method of numerical integration. In this chapter, we study in detail the numerical integration of differential equations.

The first successful application of the numerical method was given by Clero (1813 - 1765) in a study of the perturbation of Halley's comet. His method was later developed by Dalamber, Euler and in particular by Laplace. The final stage of this method was achieved by Gauss(1)

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(1) C.E. Gauss, Exposition d'une nouvelle methode de calculer les perturbations planetaires (Nachlass), Werke, 7, 1900, 439-472 Gauss' formulae were published for the first time by Encke (J.E.Encke, Uber mechanische Quadratur, .erliner Astr. Jahoclaich fur 1837, Berlin 1835, and published again in Gesammelte mathematische und astronomische Abhandlungen, Berlin 1888, 21 60). The application of the so-called "mechanics" of quadratures is not only out of date, but also may cause a lot of misunderstanding.

who suggested the so-called method of quadratures.

The method of quadratures was developed for its use in the solution of the particular problem of evaluating the perturbations of comets and small planets. This explains why the method of quadratures was not always accepted as a general method. It was considered as a particular way of evaluating perturbations, i.e. calculating small corrections to an already known approximate solution.

In 1908, the eighth satellite of Jupiter was discovered. The motion of this satellite could not be interpretted by Kepler's law. It was then necessary to investigate the general character of the corresponding dynamical problem. Cowell suggested that "mechanical quadratures" should be rejected. He proposed a new method for the general integration of the differential equations involved. This method was the origin of the method of quadratures. A great deal of attention was paid to this method, especially after it had succeeded in predicting the return of Halley's comet in 1910[1]. When the work on the motion of Halley's comet was over, Cowell made an important conclusion on the basis of his wide experience on the numerical integration of differential equations[2]. This conclusion was that the Cowell's method can be significantly improved. Although this conclusion was theoretically evident, it remained unnoticed for a long time. When Cowell's method

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(1) P.H. Cowell and A.D. Crommelin, The Orbit of Jupiter's Eighth Satellite. Monthly Notes, 68, 577-581.
P.H. Cowell and A.D. Crommelin, Essay on the return of Halley's Comet, Pablikation der Astr. Gesellochaft, 23, 1910.

(2) Appendix to the Volume of Greenwich Observations for the 1909, 81.

was improved, it became identical in form with the method of quadratures that Gauss had suggested.

In the past decade, the method of numerical integration of differential equations has been widely applied in fields other than celestical mechanics. In this connection, Adams, Stormer, Rugge and others, suggested some other methods. These methods are not as perfect as the method of quadratures. Hence, they were not widely used in celestial mechanics(1). However, we shall not only consider here the method of quadratures, but also the other methods. This will give us an idea on the advantages of the method of quadratures.

47. Evaluation of derivatives in terms of differences

We shall consider the values of each function that correspond to the values of the independent variable t that form an arithmetic progression, i.e.

$$\ldots\ t_0 - 2w,\ t_0 - w,\ t_0,\ t_0 + w,\ t_0 + 2w,\ \ldots$$

The values of a given function, say f(t), will be denoted by

$$f_k = f(t_k),$$

where

$$t_k = t_0 + kw.$$

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(1) These methods, which are of interest to engineers, are given in: A.N. Kryllov, Lectures on approximate calculations (Lekcii o priblizennyh vycislenijah) 3d. edition, 1935.

The following scheme shows the system of

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	Sums of 2nd Order	Sums of 1st Order	Values of the function	Differences of 1st Order	Differences of 2nd Order	Differences of 3rd Order	
-2	f_{-2}^{-2}		f_{-2}		. . .		
		$f_{-3/2}^{-1}$		$f_{-3/2}^{1}$			
-1	f_{-1}^{-2}		f_{-1}		f_{-1}^{2}		
		$f_{-1/2}^{-1}$		$f_{-1/2}^{1}$		$f_{-1/2}^{3}$	
0	f_0^{-2}		f_0		f_0^2		. . .
		$f_{1/2}^{-1}$		$f_{1/2}^{1}$		$f_{1/2}^{3}$	
1	f_1^{-2}		f_1		f_1^2	.	. . .
		$f_{3/2}^{-1}$		$f_{3/2}^{1}$		. .	
2	f_2^{-2}		f_2		. . .		. . .
		. . .		. . .		. . .	
. . .	. . .		.		. . .		

notations which we shall subsequently use for the differences. Accordingly

$$f^1_{k+\frac{1}{2}} = f_{k+1} - f_k$$
$$f^2_k = f^1_{k+\frac{1}{2}} - f^1_{k-\frac{1}{2}}$$

for arbitrary integral values of k. One of the values of the column of the sums of the 1st order may be chosen arbitrarily. The other values may be obtained using the following relation

$$f^{-1}_{k+\frac{1}{2}} = f^{-1}_{k-\frac{1}{2}} + f_k. \tag{1}$$

Similarly, assuming that one of the values of the second-order sums, say f_0^{-2} is arbitrary we evaluate the other numbers in this column using the following equation

$$f^{-2}_{k+1} = f^{-2}_k + f^{-1}_{k+\frac{1}{2}}. \tag{2}$$

and making k = 0, 1, ... and k = -1, -2, 3, ...

The scheme of differences indicated above is often completed by the semisums of two neighbouring quantities of the same column. For these semisums, the following notation is used

$$\left.\begin{array}{ll} f_{k+\frac{1}{2}} = \frac{1}{2}(f_k + f_{k+1}), & f_k^1 = \frac{1}{2}(f^1_{k-\frac{1}{2}} + f^1_{k+\frac{1}{2}}), \quad \ldots \\ f_k^{-1} = \frac{1}{2}(f^{-1}_{k-\frac{1}{2}} + f^{-1}_{k+\frac{1}{2}}), & f^{-2}_{k+\frac{1}{2}} = \frac{1}{2}(f_k^{-2} + f^{-2}_{k+1}), \quad \ldots \end{array}\right| \qquad (3)$$

Let us now try to express derivatives in terms of differences. We here use the stirling formula

$$f(t_k + zw) = f_k + z f_k^1 + \frac{z^2}{2!} f_k^2 + \frac{z(z^2-1^2)}{3!} f_k^3 + \frac{z^2(z^2-1^2)}{4!} f_k^4 + \ldots$$

Differentiating with respect to z and putting z = 0, we obtain

$$\left.\begin{array}{l} w\left(\frac{df}{dt}\right)_k = f_k^1 - \frac{1}{6} f_k^3 + \frac{1}{30} f_k^5 - \frac{1}{140} f_k^7 + \ldots \\ w^2\left(\frac{d^2f}{dt^2}\right)_k = f_k^2 - \frac{1}{12} f_k^4 + \frac{1}{90} f_k^6 - \frac{1}{560} f_k^8 + \ldots \\ w^3\left(\frac{d^3f}{dt^3}\right)_k = f_k^3 - \frac{1}{4} f_k^5 + \frac{7}{120} f_k^7 - \ldots \\ w^4\left(\frac{d^4f}{dt^4}\right)_k = f_k^4 - \frac{1}{6} f_k^6 + \frac{7}{240} f_k^8 - \ldots \\ w^5\left(\frac{d^5f}{dt^5}\right)_k = f_k^5 - \frac{1}{3} f_k^7 + \ldots \\ w^6\left(\frac{d^6f}{dt^6}\right)_k = f_k^6 - \frac{1}{4} f_k^8 + \ldots \end{array}\right| \qquad (4)$$

Similarly, Bessel's formula

$$f(t_k + zw) = f_k + z f^1_{k+\frac{1}{2}} + \frac{z(z-1)}{2!} f^2_{k+\frac{1}{2}} + \frac{z(z-1)\left(z-\frac{1}{2}\right)}{3!} f^3_{k+\frac{1}{2}} +$$
$$+ \frac{(z+1)z(z-1)(z-2)}{4!} f^4_{k+\frac{1}{2}} + \frac{(z+1)z(z-1)(z-2)\left(z-\frac{1}{2}\right)}{5!} f^5_{k+\frac{1}{2}} + \ldots$$

enables us to express the derivatives of the function $f(t)$ in terms of the differences given in the line $n = k + \frac{1}{2}$. We obtain

$$\left.\begin{aligned}
w\left(\frac{df}{dt}\right)_k &= f^1_{k+\frac{1}{2}} - \frac{1}{2} f^2_{k+\frac{1}{2}} - \frac{1}{12} f^3_{k+\frac{1}{2}} + \frac{1}{12} f^4_{k+\frac{1}{2}} - \frac{1}{120} f^5_{k+\frac{1}{2}} \dots \\
w^2\left(\frac{d^2f}{dt^2}\right)_k &= f^2_{k+\frac{1}{2}} - \frac{1}{2} f^3_{k+\frac{1}{2}} - \frac{1}{12} f^4_{k+\frac{1}{2}} + \frac{1}{21} f^5_{k+\frac{1}{2}} + \dots \\
w^3\left(\frac{d^3f}{dt^3}\right)_k &= f^3_{k+\frac{1}{2}} - \frac{1}{2} f^4_{k+\frac{1}{2}} + \dots \\
w^4\left(\frac{d^4f}{dt^4}\right)_k &= f^4_{k+\frac{1}{2}} - \frac{1}{2} f^5_{k+\frac{1}{2}} \dots
\end{aligned}\right\} \quad (5)$$

The following formulae are used in the method of the numerical integration of equations, suggested by Adams and Stormer,

$$\left.\begin{aligned}
w\left(\frac{df}{dt}\right)_k &= f^1_{k-\frac{1}{2}} + \frac{1}{2} f^2_{k-1} + \frac{1}{3} f^3_{k-\frac{3}{2}} + \frac{1}{4} f^4_{k-2} + \frac{1}{5} f^5_{k-\frac{5}{2}} \dots \\
w^2\left(\frac{d^2f}{dt^2}\right)_k &= f^2_{k-1} + f^3_{k-\frac{3}{2}} + \frac{11}{12} f^4_{k-2} + \frac{5}{6} f^5_{k-\frac{5}{2}} + \dots \\
w^3\left(\frac{d^3f}{dt^3}\right)_k &= f^3_{k-\frac{3}{2}} + \frac{3}{2} f^4_{k-2} + \frac{7}{4} f^5_{k-\frac{5}{2}} + \dots \\
w^4\left(\frac{d^4f}{dt^4}\right) &= f^4_{k-2} + \frac{11}{5} f^5_{k-\frac{5}{2}} + \dots
\end{aligned}\right\} \quad (6)$$

They express the derivatives of $f(t)$ in terms of the differences, located in the ascending diagonal. These relations are obtained by the similar interpolation of Newton's formula:

[illegible]

Annotation

There are several other ways to denote the differences. The quantity f^i_n, where i is an integer and n a half integer (even or odd), will be denoted by $f^i(n)$, or $f_i(n)$ or by $f^i\,(t_0 + nw)$, or finally by Δ^i_n.

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48. Integration of First-Order Equations. The Method of Differences

Let us consider the following differential equation

$$\frac{dx}{dt} = F(x, t) \qquad (7)$$

We want to calculate a table for the values of the function X(t), that satisfies this equation as well as the initial condition

$$x(t_o) = x_o$$

where t_o and x_o are given numbers. Let us assume that $x(t_o + kw) = x_k$, and evaluate x_1, x_2, x_3, ... Our problem is to find the way to follow in order to calculate the differences

$$\Delta_{k+\frac{1}{2}} = x_{k+1} - x_k, \qquad (k = 0, 1, 2, \ldots)$$

Since

$$x_{k+1} = x(t_k + w) = x_k + \frac{w}{1!}\left(\frac{dx}{dt}\right)_k + \frac{w^2}{2!}\left(\frac{d^2x}{dt^2}\right)_k + \ldots$$

then,

$$\Delta_{k+\frac{1}{2}} = \frac{w}{1!}\left(\frac{dx}{dt}\right)_k + \frac{w^2}{2!}\left(\frac{d^2x}{dt^2}\right)_k + \ldots$$

Adopting that

$$wF(x, t) = f(t), \qquad wF(x_k, t_k) = f_k$$

and taking equation (7) into consideration, we obtain

$$\Delta_{k+\frac{1}{2}} = f_k + \frac{w}{2}\left(\frac{df}{dt}\right)_k + \frac{w^2}{6}\left(\frac{d^2f}{dt^2}\right)_k + \frac{w^3}{24}\left(\frac{d^3f}{dt^3}\right)_k + \ldots \qquad (8)$$

This formula essentially solves our problem. However, this formula cannot be easily applied(1), since we have to calculate the derivatives

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(1) The method of integration of equation (7), that is based on the use of formula (8) is called Euler's method. It is only applied in the case when one can keep in equation (8) only two or three terms, i.e. when the interval w is very small, or when one does not require an accurate solution.

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$$\left(\frac{df}{dt}\right)_k, \quad \left(\frac{d^2f}{dt^2}\right)_k, \ldots$$

We now express the derivatives involved here in terms of differences. Using equation (6), we obtain

[illegible]

This formula represents Adams' method. Equations (5) leads to the following formula

$$\ldots - \frac{1}{12} \ldots \frac{11}{720} \ldots$$

which may be written as

$$\ldots - \frac{1}{12} f_k \ldots \frac{11}{720} f_k \ldots \frac{191}{60480} f \ldots$$
$$\frac{2197}{3628800} f \ldots \qquad (9)$$

since

$$f \ldots \frac{1}{2} (f_k \ldots f_{k-1}) = \frac{1}{2} (f_k + f_k \ldots f \ldots) \ldots f_k \ldots \frac{1}{2} f_k$$

This formula leads to the method of integration that may be called Cowell's method, since a similar method of integrating second-order equations has been suggested by Cowell.

Once x_1 , x_2, ... , x_k are calculated equation (9) immediately gives x_{k+1}. On the other hand, Cowell's formula, given by equation (10), expresses the unknown difference $x_{k+1} - x_k$ in terms of the differences f^2 , f^1, ... which depend on f_{k+1} , f_{k+2}, ... and consequently on x_{k+1} , x_{k+2}, ... The difference f^2_{k+2}, , $f^2_{k+\frac{1}{2}}$, ... are found, in the first approximation, by extrapolation. After the evaluation of the corresponding values, x_{k+1} , x_{k+2}, ..., these differences are calculated in the usual way. If it is necessary, the quantities x_{k+1}, x_{k+2}, ... are evaluated once more. Owing to the rapid decrease of the coefficients in equation (10), the above procedure converges so rapidly, that the second approximation might be unnecessary, provided that the interval w is not large.

In the cases, which we are met with in celestial mechanics, namely in evaluating the perturbations of the elements of orbits, the right-hand side of equation (7) slowly varies with the variation of the variable X. When the variable x slowly varies, the application of equation (10) becomes particularly simple because the values of the function f and all its differences may be evaluated in advance for several intervals using the approximate values of x.

Equations (9) and (10) can be used only when some of the initial values of the unknown function, x_1, x_2, ..., are given so that the evaluation of the differences f^1, f^2, ..., involved in these formulae is possible. The values x_1, x_2, ... (and also x_{-1}, x_{-2}, ...) are usually evaluated by the expansion of the integral in a series, i.e.

$$x(t) = x_0 + x_0'(t - t_0) + \frac{1}{2!} x_0''(t - t_0)^2 + \ldots$$

The coefficients of expansion can be found by the multiple differentiation of equation (7) and subsequently the substitution $t = t_0$. Sometimes, the initial values x_1, x_{-1}, x_2, x_{-2}, ... are found by a successive approximation (§ 57). An example of this approach will be given in § 55. It is also possible to find some of the initial values, say x_1, x_2, x_3..., using Euler's method.

49. The method of quadratures for the first-order equations:

The limitations of the method of differences, considered in the previous section, is the accumulation of errors when x_1, x_2, ... are evaluated in terms of their differences Δ. Indeed, in order to evaluate x_n, we use the following equations

$$x_1 - x_0 = \Delta_{\frac{1}{2}}, \quad x_2 - x_1 = \Delta_{\frac{3}{2}}, \quad \ldots, \quad x_n - x_{n-1} = \Delta_{n-\frac{1}{2}}$$

The term by term addition of these equations gives

$$x_n - x_0 = \sum_{0}^{n-1} \Delta_{k+\frac{1}{2}}. \qquad (11)$$

If the error in evaluating each of the differences Δ is within the limits from $-\epsilon$ to $+\epsilon$, then the maximal possible error in x_n is $\pm n\epsilon$. It is well known that the probability that the error in x_n achieves this maximal value is very small. However, we still have to be very careful on the progressive decrease in accuracy of the evaluation of x_n at large values of n.

In the following, we show how the above-mentioned accumulation of errors may be reduced. As an example, we will consider eatuion (10), which may be written as

$$\Delta_{k+\frac{1}{2}} = f_{k+\frac{1}{2}} + \text{Red},$$

where Red stands for the correction that must be added to a given value f in order to obtain the corresponding difference Δ. For simplificyt, we assume that the values f of the function are exact. Even in this case, the correction Red will have a finite error occurring as a result of the rounding off as well as the dropping of terms in equation (10). Hence, the accumulation of errors in evaluating x_n using equation (10) occurs due to two reasons. First, we use a limited number of laws in evaluating the function f, and second, we make errors in evaluating Red when we round the numbers off and drop the small terms. The summation of errors of the first type is not important because the accuracy of the calculation of f can always be put under control. The summation of the errors of Red is more harmful, since these errors cannot be easily controled.

It can be avoided by an appropriate change in method.

Let us substitute expression (10) into equation (11). This yields

$$x_n = x_0 + \sum_{0}^{n-1} f_{k+\frac{1}{2}} - \frac{1}{12}\sum_{0}^{n-1} f^{2}_{k+\frac{1}{2}} + \frac{11}{720}\sum_{0}^{n-1} f^{4}_{k+\frac{1}{2}} - \dots \qquad (12)$$

From equations (1) and (3), we obtain

$$f_k = \frac{1}{2}\left(f_{k+\frac{1}{2}} + f_{k-\frac{1}{2}}\right) - \frac{1}{2}\left(f \dots f_k \dots f\right)$$

However, it follows from equation (3) that

$$f_k^{1} = \frac{1}{2}\left(f^{1}_{k+\frac{1}{2}} + f^{1}_{k-\frac{1}{2}}\right), \quad f \dots \frac{1}{2}\left(f \dots f_k \dots\right).$$

Therefore,

$$f_{k+\frac{1}{2}} = f^{-1}_{k+1} - f^{-1}_k.$$

Similarly

$$f^{2}_{k+\frac{1}{2}} = f^{1}_{k+1} - f^{1}_k, \quad f^{1}_{k+\frac{1}{2}} = f_{k+1} - f_k, \dots$$

Using these equations, we can easily be convinced that

$$\sum_{0}^{n-1} f_{k+\frac{1}{2}} = f_n^{-1} - f_0^{-1}, \quad \sum_{0}^{n-1} f^{2}_{k+\frac{1}{2}} = f^{1}_n - f^{1}_0, \dots$$

Consequently, equation (12) can be written as follows

$$x_n = f_n^{-1} - \frac{1}{12} f_n^{1} + \frac{11}{720} f_n^{3} - \frac{191}{60\,480} f_n^{5} + \dots$$
$$+ x_0 - f_0^{-1} + \frac{1}{12} f_0^{1} - \frac{11}{720} f_0^{3} + \frac{191}{60\,480} f_0^{5} - \dots$$

Since one term in the column of the first sums may be arbitrarily chosen, then the quantity f_0^{-1} is usually defined from the following condition

$$x_0 = f_0^{-1} - \frac{1}{12} f_0^{1} + \frac{11}{720} f_0^{3} - \dots \qquad (13)$$

Then

$$x_n = f_n^{-1} - \frac{1}{12} f_n^1 + \frac{11}{720} f_n^3 - \dots \qquad (14)$$

The calculation using this formula is free from the above-mentioned limitation of the accumulation of errors. The error in x_n depends on the rounding off and neglection of the terms in the correction.

$$\text{Red} = -\frac{1}{12} f_n^1 + \frac{11}{720} f_n^3 - \dots$$

When the interval w is properly chosen, this error does not affect the value $f_n = w\,F(x_n, t_n)$ and hence does not affect the subsequent values of x.

The method of integration of equation (7), based on the application of a formula of the type given by equation (14), is called the method of quadratures since, if the right hand side of equation (7) does not involve X, this formula is reduced to the formula of quadratures (Sec. 56).

The method of differences, suggested by Adams and based on equation (9), corresponds to the method of quadratures in which the following formula is applied

$$x_n = f^{-1}_{n-\frac{1}{2}} + \frac{1}{2} f_{n-1} + \frac{5}{12} f^1_{n-\frac{3}{2}} + \frac{3}{8} f^2_{n-2} + \dots \qquad (15)$$

where the initial values of the column of sums are defined by

$$f^{-1}_{-\frac{1}{2}} = x_0 - \frac{1}{2} f_{-1} - \frac{5}{12} f^1_{-\frac{3}{2}} - \frac{3}{8} f^2_{-2} - \dots$$

To distinguish between this method and the previous method suggested by Gauss, we shall call it Adams' method of quadratures.

50. <u>A second form for the method of quadratures of first-order equations</u>

We shall consider the following problem. Let the integral of the equation

$$\frac{dx}{dt} = F(x, t)$$

be given by

$$x\left(t_0 - \frac{w}{2}\right);$$

It is required to calculate a table for the values of this integral that correspond to the following values of the argument

$$t_k = t_0 + kw,$$

where k is an arbitrary integer.

We first show how we can calculate the unknown function X(t) for the following values of the argument

$$t = t_0 + \left(k + \frac{1}{2}\right) w = t_k + \frac{1}{2} w$$

Let us define the difference

$$\delta_k = x\left[t_0 + \left(k + \frac{1}{2}\right) w\right] - x\left[t_0 + \left(k - \frac{1}{2}\right) w\right].$$

Expanding this quantity in a Taylor series, we obtain

$$\delta_k = x\left(t_k + \frac{w}{2}\right) - x\left(t_k - \frac{w}{2}\right) = w\left(\frac{dx}{dt}\right)_k + \frac{w^3}{24}\left(\frac{d^3x}{dt^3}\right)_k + \frac{w^5}{1920}\left(\frac{d^5x}{dt^5}\right)_k + \dots$$

Using equation (7) and applying formulae (4), we obtain

$$\delta_k = f_k + \frac{1}{24} f_k'' - \frac{17}{5760} f_k^{IV} + \frac{367}{967680} f_k^{VI} - \dots \qquad (10')$$

This formula is more convenient than formula (10). It involves only differences, while formula (10) involves semisums of differences.

The initial values of the unknown function, e.g.

$$x\left(t_0 - \frac{w}{2}\right), \quad x\left(t_0 + \frac{w}{2}\right), \quad x\left(t_0 + \frac{3w}{2}\right), \dots$$

are obtained either by expanding x(t) in a series, or by means of the method of successive approximations.

Formula (14) leads to a method of integration similar to the method used by Cowell. The version of the method of quadratures that corresponds to this formula is obtained by summing equation (16) from k = 0 to k = n-1. This summation yields

If $f^{-1}_{-\frac{1}{2}}$ is defined by

then, we obtain the following simple formula

(17)

The method based on the application of formulae (16) and (17) cannot be widely used because these formulae give the values x $(t_k \pm \frac{w}{2})$, whereas we have to know the values $x(t_k)$ in order to evaluate the right-hand sides of these formulae. Therefore, when these formulae are applied we can find the values $x(t_k)$ by integrating in average. Hence, the application of formulae (16) and (17) is useful only when the differences of $x(t_k)$ may be neglected.

In order to avoid this difficulty, we deduce from equation (17) a formula that yields $X_n = X(t_n)$. For this purpose, we use the well-known formula on the integration in average

which is obtained from Bessel's formula, given in § 47, by putting $z = \frac{1}{2}$. Adopting in this formula that and noting that

$$\varphi_n [illegible] = \frac{1}{2} f\left(t_n + \frac{w}{2}\right) + \frac{1}{2} f\left(t_n [illegible] \frac{}{2}\right)$$

$$f_n + \frac{1}{24} f_n'' [illegible] \frac{17}{5760} f [illegible] \dots$$

and, consequently,

$$\varphi^2_{n+\frac{1}{2}} = f_n'' + \frac{1}{24} f_n [illegible] \frac{17}{5760} f + \dots$$

we obtain

[illegible]

Hence, the evaluation of x_n will be carried out using a formula indentical with that given by equation (14). However, the initial term of the column of sums will be given in the present case by

[illegible]

Annotation I

The methods of integration of equation (7), considered above, can be applied without change to systems of equations of the type.

$$\frac{dx}{dt} = F(x, [illegible], t), \quad \frac{dy}{dt} = \Phi(x, y, [illegible], t), \quad \frac{dz}{dt} = [illegible]$$

In this case the integration will be carried out in parallel on three separate sheets.

Annotation II

In the application of the above-mentioned formulae, it is useful to use the following approximate equations

11	720	1	65	[illegible]
[illegible]	[illegible]	1	17	[illegible]
[illegible]	[illegible]	1	1 15[illegible]	[illegible]
17	[illegible]	1	[illegible]	[illegible]
[illegible]	[illegible]	1	26[illegible]	[illegible]

51. An example of integrating first-order eqauations

Let us calculate a table for the values of the integral of the following equation

$$\frac{dx}{dt} = \frac{1}{2} xt,$$

which satisfies the initial condition $t_o = 0$, $X_o = 1$.

Choosing the interval w = o.1, we obtain

$$f(t) = 0{,}05\, xt. \qquad (*)$$

In order to determine the first values of the integral, we differentiate the given equation and put t = 0. We obtain

$$x_0^{I} = 0, \quad x_0^{II} = \frac{1}{2}, \quad x_0^{III} = 0, \quad x_0^{IV} = \frac{3}{4}, \ldots .$$

Consequently,

$$x = 1 + \frac{1}{4} t^2 + \frac{1}{32} t^4 + \ldots$$

This series-expansion enables us to find the values of x for t = ± 0.1, ± 0.2. Furthermore the calculation will be carried by the following scheme, in which the values of x obtained by the series-expansion as well as the corresponding values f, f', ... are printed in bold type. The semisums of the values f, f', ... are typed in the spaces between the corresponding lines.

Table A

t							
[illegible]		[illegible]					
			[illegible]				
[illegible]		[illegible]		[illegible]			
			[illegible]		[illegible]		
[illegible]		[illegible]	[illegible]	[illegible]	[illegible]		
		[illegible]	[illegible]	[illegible]	[illegible]		
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]			
		[illegible]	[illegible]				
0.2	[illegible]	[illegible]	[illegible]		[illegible]		[illegible]
		[illegible]		[illegible]			
0.3	[illegible]		[illegible]		[illegible]		
		[illegible]		[illegible]			
0.4	[illegible]		[illegible]		[illegible]	[illegible]	[illegible]

In table A, the initial values are given, and the first approximation is obtained for $X_3 = X\ (0.3)$ and $X_4 = (0.4)$.

For the line t = 0, we evaluate

$$[illegible]$$

and obtain Red = - 417 (expressed in units of the 6th digit, which also applies to all quantities f, f^1, ...). We substitute this value in the corresponding column, and find the principal term in the column of sums

$$[illegible]$$

In order to find the next terms in the column of sums, we construct the semisums in the column of f and use the following relations [illegible] . . . .

We then extrapolate the quantities required for evaluating x_3 and x_4. Assuming that the forth difference is zero, we obtain

[illegible] 74.

Then, by successive addition, we obtain

[illegible] 111, [illegible] 18,

and so on, until we obtain f_3^{-1} and f_1^{-1}. Finally, we evaluate Red using equation (**), and obtain the values of x(t) which are subsequently tabled. At this stage, the first approximation is complete.

In order to obtain the second approximation, we evaluate f_3 and f_1 using equation (*). We then repeat the calculation of x_3 and x_4 using the new and more accurate values for the differences.

The final results are given in table B. In practice, the first approximation is written in pencil while the second is inked in. In Table B, the quantities obtained by extrapolation and those not corrected by the second approximation are printed in italics

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		[illegible]
[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		[illegible]	[illegible]
		[illegible]		[illegible]		[illegible]		
[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		[illegible]	[illegible]
		[illegible]		111		[illegible]		
[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		[illegible]	[illegible]
		[illegible]		[illegible]		[illegible]		
[illegible]	[illegible]	[illegible]	[illegible]		[illegible]	[illegible]	[illegible]	[illegible]
		0018075		[illegible]		[illegible]		
[illegible]	[illegible]	[illegible]	[illegible]		[illegible]		[illegible]	[illegible]
		[illegible]		[illegible]				
[illegible]	*[illegible]*		[illegible]				[illegible]	*[illegible]*
		[illegible]						
[illegible]	*[illegible]*						[illegible]	*[illegible]*

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It is useful to modify the calc[illegible]on method after evaluating values [illegible] Red. Instead of obtaining the diff[illegible]nces required for the evaluation of Red by extrapolation it is better [illegible] extrapolate immediately the R[illegible]d quantity itself. Such an extra[illegible] is shown in table C, where the ex[illegible]polated values are printed [illegible]ics.

[illegible]

[illegible]	[illegible]	[illegible]
[illegible]		
	[illegible]	
1.0		[illegible]
	[illegible]	
1[illegible]		[illegible]
	1[illegible]	
11[illegible]		[illegible]
	[illegible]	
[illegible]		[illegible]
	[illegible]	
[illegible]		[illegible]
	[illegible]	
[illegible]		

The values of x_5 and x_6, given in table B, have been found by the above mentioned extrapolation. The first value is exact and the second value is incorrect by only one unit., However, the values f_5 and f_6 evaluated using these values for x_5 and x_6 are final and do not require re-evaluation.

In conclusion, we shall demonstrate how the same problem could be solved using Adam's method (of differences). In this method, the calculation is carried out using equation (9), which may be rewritten as follows

$$\mathrm{Red} = \frac{1}{2} f'_{n-\frac{1}{2}} + \frac{5}{12} f_n + \frac{3}{8} f'_{n-\frac{3}{2}} + \frac{251}{720} f_n + \frac{1}{3} f_n \ldots$$

$$\Delta_{n+\frac{1}{2}} = f_n + \mathrm{Red}$$

$$x_{n+1} = x_n + \Delta_{n+\frac{1}{2}}.$$

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The results of the calculation are given in table D. The exact values, from which we start the numerical integration, are printed in bold type. The function f and its differences as well as the Red correction are calculated in six digits.

Table D

t	x	Δ	f	f^1	f^2	f^3	f^4	f^5	Red
−0.2	**1.010 05**		**−0.010 100**						
				+5 087					
−0.1	**1.002 59**		**−0.005 013**		**−74**				
				+5 013		**+74**			
0.0	**1.000 00**		**0.000 000**		**0**		**0**		
				+5 013		**+74**		**+6**	
0.1	**1.002 50**		**0.005 013**		**+74**		**+6**		
				+5 087		**+80**		**−6**	
0.2	**1.010 05**		**0.010 100**		**+154**		**0**		+2 602
		+12 70		+5 241		+80		+7	
0.3	1.022 75		0.015 341		+231		+7		+2 719
		+18 06		+5 475		+87		+2	
0.4	1.040 81		0.020 816		+321		+9		+2 863
		+23 68		+5 796		+96		+1	
0.5	1.064 49		0.026 612		+417		+10		+3 069
		+28 68		+6 213		+106			
0.6	1.094 17		0.032 825		+513				+3 320
		+36 15		+6 736					
0.7	1.130 32		0.039 561						+3 629
		+43 19							
0.8	1.173 51								

All the calculated values of x are quite accurate and the difference in accuracy between Adams' method and the method of quadratures will be considerable. only if the numerical integration is continued significantly further (or if larger intervals, w, are used) especially when function f is evaluated with a single spare index, as it has been done in the present example.

52. Integration of Second-Order Equations. Evaluation of an Integral Assigned by Two Values

We shall now consider methods for the integration of equations of type

$$\frac{d^2x}{dt^2} = F(x, t).$$

Each integral of a second-order equation can be assigned either by its values in two points, e.g. by the values

$$x_{-1} = x(t_0 - w), \quad x_0 = x(t_0),$$

or by the values

$$x_0 = x(t_0), \quad x_0' = \left(\frac{dx}{dt}\right)_{t=t_0},$$

that this integral and its derivative take at the initial point t_o.

In this section we shall only consider the first case. We assume that the initial values x_{-1} and x_o are given. We then have to evaluate the subsequent values x_1, x_2, ... of the unknown function.

Alongisde the differences

$$\Delta_{k-\frac{1}{2}} = x_k - x_{k-1}, \quad \Delta_{k+\frac{1}{2}} = x_{k+1} - x_k$$

we introduce into consideration the second difference

$$\Delta_k^2 = \Delta_{k+\frac{1}{2}} - \Delta_{k-\frac{1}{2}} = x_{k+1} - 2x_k + x_{k-1}.$$

Expanding $x_{k+1} = x\ (t_k + w)$ and $x_{k-1} = x(t_k - w)$ in powers of w, we obtain

$$\Delta_k^2 = \frac{2}{2!} w^2 \left(\frac{d^2x}{dt^2}\right)_k + \frac{2}{4!} w^4 \left(\frac{d^4x}{dt^4}\right)_k + \ldots$$

Assuming that

$$w^2 F(x,t) = f(t), \quad w^2 F(x_n, t_n) = f_n,$$

we obtain

$$\Delta_k^2 = f_k + \frac{1}{12} w^2 \left(\frac{d^2 f}{dt^2}\right)_k + \frac{1}{360} w^4 \left(\frac{d^4 f}{dt^4}\right)_k + \dots$$

Using equations (4), we express the derivatives in terms of differences. We obtain

$$\Delta_k^2 = f_k + \frac{1}{12} f_k^2 - \frac{1}{240} f_k^4 + \frac{31}{60480} f_k^6 - \frac{289}{3628800} f_k^8 + \dots, \qquad (19)$$

which represents Cowell's method. Applying equations (6) yields the principal formula of Stormer's method (similar to Adam's method, which reads

$$\Delta^2 = f_{k-1} + \frac{1}{12} f^2_{k-2} + \frac{1}{12} f^3_{k-3} + \frac{19}{240} f^4_{k-2} + \frac{3}{40} f^5_{k-3} + \frac{863}{12096} f^6_{k-3} + \dots \qquad (20)$$

The comparison between the last two formulae immediately indicates the superiority of Cowell's method. If the values of f_k are evaluated with an error equal to $\pm\epsilon$, then the first, second, third, ... order differences will have errors within the limits $\pm 2\epsilon$, $\pm 4\epsilon$, $\pm 8\epsilon$, ... In equation (20), all the coefficients are of the 1/12 order and hence the errors in the higher-order differences will significantly modify the value of Δ_k^2. Equation (19) is free from this defect because of the rapid decrease of its coefficients.

Calculation using Cowell's method is done in the following way. The values of x_{-1} and x_0 are given, and hence we can evaluate

$$\Delta_{-\frac{1}{2}} = x_0 - x_{-1};$$

Neglecting for the moment all the unknown terms in equation (19), we find the approximate value

$$\Delta_0^2 = f_0.$$

This yields

$$\Delta_{\frac{1}{2}} = \Delta_{-\frac{1}{2}} + \Delta_0^2$$
$$x_1 = x_0 + \Delta_{\frac{1}{2}}.$$

Having obtained the values f_{-1}, f_o and f_1, we determine f_o^2 and subsequently use the more accurate value

$$\Delta_0^2 = f_0 + \frac{1}{12} f_0^2,$$

to evaluate x_1 once again. We similarly evaluate x_{-2}. We then find f_o^4 and use the more accurate expression

$$\Delta_0^2 = f_0 + \frac{1}{12} f_0^2 - \frac{1}{240} f_0^4$$

and so on until we obtain the final value of this quantity and consequently the value of x_1. Thus obtaining a few values x_1, x_2, ... we evaluate Δ_k^2 by extrapolating the values of the unknown differences. If the interval w is not large, the extrapolation is done so well that it is never necessary to improve the accuracy of the resulting values of Δ_k^2.

Instead of finding x_{k+1} in terms of Δ_k^2 by using the double summation

$$\Delta_{k+\frac{1}{2}} = \Delta_{k-\frac{1}{2}} + \Delta_k^2, \quad x_{k+1} = x_k + \Delta_{k+\frac{1}{2}}, \tag{21}$$

it is possible to apply the following formula

[illegible]

which is more easily done using a calculating machine. However, avoiding the writing of differences does not save much time and it only prevents the possibility of checking and controling the calculations.

The double summation that explicitly appears in formulae (21) and which is implicitly involved in formula (22) leads to a greater accumulation of errors than the single summation obtained by applying Cowell's method to first order equations. It is thus clear that the replacement of Cowell's method by the corresponding method of quadratures is very essential in the integration of second-order equations, especially when the calculations are prolonged.

Summing equation (21) from k = 0 to k = n - 1 yields

$$\Delta_{n-\frac{1}{2}} = \Delta_{-\frac{1}{2}} + \sum_{0}^{n-1} \Delta_k^2, \qquad x_n = x_0 + \sum_{0}^{n-1} \Delta_{k+\frac{1}{2}} \tag{23}$$

Formula (19) gives

$$\sum_{0}^{n-1} \Delta_k^2 = \sum_{0}^{n-1} f_k + \frac{1}{12}\sum_{0}^{n-1} f_k^2 - \frac{1}{240}\sum_{0}^{n-1} f_k^4 + \dots$$

Since, according to our system of notation,

$$\begin{aligned} f_0^m &= f_{\frac{1}{2}}^{m-1} - f_{-\frac{1}{2}}^{m-1} \\ f_1^m &= f_{\frac{3}{2}}^{m-1} - f_{\frac{1}{2}}^{m-1} \\ &\dots\dots\dots \\ f_{n-1}^m &= f_{n-\frac{1}{2}}^{m-1} - f_{n-\frac{3}{2}}^{m-1}. \end{aligned}$$

The addition of these equations give

$$\sum_{0}^{n-1} f_k^m = f_{n-\frac{1}{2}}^{m-1} - f_{-\frac{1}{2}}^{m-1}.$$

Taking this into consideration, we write the first of equations (23) in the following manner

$$\begin{aligned} \Delta_{n-\frac{1}{2}} &= f_{n-\frac{1}{2}}^{-1} + \frac{1}{12} f_{n-\frac{1}{2}}^{1} - \frac{1}{240} f_{n-\frac{1}{2}}^{3} + \dots \\ &+ \Delta_{-\frac{1}{2}} - f_{-\frac{1}{2}}^{-1} - \frac{1}{12} f_{-\frac{1}{2}}^{1} + \frac{1}{240} f_{-\frac{1}{2}}^{3} - \dots \end{aligned}$$

We then choose the arbitrary initial term in the column of first sums in such a manner that the second line of this equation vanishes. By assuming that

$$f^{-1}_{-\frac{1}{2}} = \Delta_{-\frac{1}{2}} - \frac{1}{12} f^{1}_{-\frac{1}{2}} + \frac{1}{240} f^{3}_{-\frac{1}{2}} - \frac{31}{60\,480} f^{5}_{-\frac{1}{2}} + \dots \qquad (24)$$

We finally obtain

$$\Delta_{n-\frac{1}{2}} = f^{-1}_{n-\frac{1}{2}} + \frac{1}{12} f^{1}_{n-\frac{1}{2}} - \frac{1}{240} f^{3}_{n-\frac{1}{2}} + \dots$$

We replace n by k + 1 and sum from k = 0 to k = n-1. The second of equations (23) may then be written as follows

$$x_n = x_0 + \sum_{0}^{n-1} f^{-1}_{k+\frac{1}{2}} + \frac{1}{12} \sum_{0}^{n-1} f^{1}_{k+\frac{1}{2}} - \frac{1}{240} \sum_{0}^{n-1} f^{3}_{k+\frac{1}{2}} + \dots$$

However,

$$\sum_{0}^{n-1} f^{m}_{k+\frac{1}{2}} = f^{m-1}_{n} - f^{m-1}_{0} .$$

Consequently,

$$x_n = f^{-2}_n + \frac{1}{12} f_n - \frac{1}{240} f^{2}_n + \frac{31}{60\,480} f^{4}_n - \dots$$
$$+ x_0 - f^{-2}_0 - \frac{1}{12} f_0 + \frac{1}{240} f^{2}_0 - \frac{31}{60\,480} f^{4}_0 + \dots$$

We assume that the initial term in the column of the second sums is given by

$$f^{-2}_0 = x_0 - \frac{1}{12} f_0 + \frac{1}{240} f^{2}_0 - \frac{31}{60\,480} f^{4}_0 + \dots, \qquad (25)$$

Then, the latter formula will be given by

$$x_n = f^{-2}_n + \frac{1}{12} f_n - \frac{1}{240} f^{2}_n + \frac{31}{60\,480} f^{4}_n - \frac{289}{3\,628\,800} f^{6}_n + \dots \qquad (26)$$

In applying these formulae, it is worthwhile to take into consideration that

$$\frac{31}{60\,480} = \frac{1}{1951} + 0.0000000$$

$$\frac{289}{3\,628\,800} = \frac{1}{12\,557} + 0.0000000.$$

Formula (26), supplemented with equations (24) and (25), represents the method of quadratures. This method is applied in the same way as Cowell's method. First, starting from the given valuex x_o and $\Delta_{-\frac{1}{2}} = x_o - x_{-1}$, the adjacent values x_1 , x_{-2} , x_2, ... are found using the method of successive approximations. Then, the differences f_n^2 , f_n^4, ... required for the application ofequation (26) are found by extrapolations and corrected if necessary in the second approximation.

In Cowell's method, the calculation is carried out using formulae (21) and (22), which may be written as

$$\Delta_i = f_i + \mathrm{Red}_i$$

$$\Delta_{n+\frac{1}{2}} = x_0 - x_{-1} + \sum_0^n \Delta_k$$

$$x_n = x_0 + \sum_0^{n-1} \Delta_{k+\frac{1}{2}}.$$

The Red correction isthus subject to a double summation. However, in the method of quadratures, the calculation is done using a formula of the type

$$x_n = f_n^{-2} + \mathrm{Red}.$$

Thus, the errors made when evaluating Red are not increased by the summation.

The method of quadratures that corresponds to Stormer's method consists in the application of the following formula

$$x_n = f_n^{-2} + \frac{1}{12} f_{n-1} + \frac{1}{12} f^1_{n-\frac{1}{2}} + \frac{19}{240} f^2_{n-1} + \frac{3}{40} f^3_{n-\frac{1}{2}} + \dots$$

In order to obtain the initial terms of the columns of sums, we put in this formula n = 0 and n = -1. We then obtain

$$f_0^{-2} = x_0 - \frac{1}{12} f_{-1} - \frac{1}{12} f^1_{-\frac{1}{2}} - \frac{19}{240} f^2_{-2} - \ldots,$$
$$f_{-1}^{-2} = x_{-1} - \frac{1}{12} f_{-2} - \frac{1}{12} f^1_{-\frac{5}{2}} - \frac{19}{240} f^2_{-1} - \ldots,$$

We obtain for the column of the first sums

$$f^{-1}_{-\frac{1}{2}} = f_0^{-2} - f^{-2}_{-1}.$$

Störmer's formula is applied in the following form

$$x_n = f_n^{-2} + \frac{1}{12}\left(f_{n-1} + f^1_{n-\frac{3}{2}} + f^2_{n-2}\right) - \frac{1}{240} f^2_{n-2} + \ldots$$

The tabular interval w is chosen so small that the third difference may be neglected.

Annotation

All the methods of numerical integration of the equation

$$\frac{d^2x}{dt^2} = F(x, t)$$

are applied without alteration to systems of the type

$$\frac{d^2x}{dt^2} = F(x, y, z, t), \quad \frac{d^2y}{dt^2} = G(x, y, z, t), \quad \frac{d^2z}{dt^2} = H(x, y, z, t).$$

Naturally, the integration of equations coupled in a system is carried in paralle.

53. A Second Case in the Evaluation of the Integral of a Second-Order Equation Assigned by Two Values

The method of quadratures forthe second-order as well as for the first-order equations can be represented in two ways depending on whether we want to calculate the values $x_n = x(t_o + nw)$ or the values of the functions in the middle of the intervals, i.e. $x\,(t_o + (n + \frac{1}{2})w)$. The first case was considered in the previous section. Here, we shall derive the necessary formulae for the second case. We assume that

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$$\delta_k = x\left(t_k + \frac{w}{2}\right) - x\left(t_k - \frac{w}{2}\right)$$

$$\delta^2_{k+\frac{1}{2}} = \delta_{k+1} - \delta_k .$$

Taylor's formula gives

$$\delta^2_{k+\frac{1}{2}} = x\left(t_k + \frac{3w}{2}\right) - 2x\left(t_k + \frac{w}{2}\right) + x\left(t_k - \frac{w}{2}\right) =$$
$$= w^2\left(\frac{d^2x}{dt^2}\right)_k + \frac{1}{2}w^3\left(\frac{d^3x}{dt^3}\right)_k + \frac{5}{24}w^4\left(\frac{d^4x}{dt^4}\right)_k + \frac{1}{16}w^5\left(\frac{d^5x}{dt^5}\right)_k + \dots$$

Consequently

$$\delta^2_{k+\frac{1}{2}} = f_k + \frac{1}{2}w\left(\frac{df}{dt}\right)_k + \frac{5}{24}w^2\left(\frac{d^2f}{dt^2}\right)_k + \frac{1}{16}w^3\left(\frac{d^3f}{dt^3}\right)_k + \dots ,$$

or, using equation (5),

$$\delta^2_{k+\frac{1}{2}} = f_k + \frac{1}{2}f^1_{k+\frac{1}{2}} - \frac{1}{24}f^2_{k+\frac{1}{2}} + \frac{17}{1920}f^4_{k+\frac{1}{2}} - \dots ,$$

or, finally, using the following formula

$$f_{k+\frac{1}{2}} = f_k + \frac{1}{2}f^1_{k+\frac{1}{2}} ,$$

which has already been applied in Sec. 48,

$$\delta^2_{k+\frac{1}{2}} = f_{k+\frac{1}{2}} - \frac{1}{24}f^2_{k+\frac{1}{2}} + \frac{17}{1920}f^4_{k+\frac{1}{2}} - \frac{367}{193536}f^6_{k+\frac{1}{2}} + \dots$$

This formula gives a method of integration similar to Cowell's method

Summing this formula from k = 0 to k = n-;, we obtain

$$\delta_n - \delta_0 = f_n^{-1} - \frac{1}{24}f_n^1 + \frac{17}{1920}f_n^3 - \dots$$
$$- f_0^{-1} + \frac{1}{24}f_0^1 - \frac{17}{1920}f_0^3 + \dots$$

Assuming that

$$f_0^{-1} = \delta_0 + \frac{1}{24}f_0^1 - \frac{17}{1920}f_0^3 + \dots , \qquad (27)$$

Then

$$\delta_n = f_n^{-1} - \frac{1}{24} f_n^{1} + \frac{17}{1920} f_n^{3} - \dots$$

Replacing in this equation n by k, and summing from k = 0 to k = n-1, we obtain

$$x\left(t_n - \frac{w}{2}\right) - x\left(t_0 - \frac{w}{2}\right) = f^{-2}_{n-\frac{1}{2}} - \frac{1}{24} f_{n-\frac{1}{2}} + \frac{17}{1920} f^{2}_{n-\frac{1}{2}} - \dots$$
$$- f^{-2}_{-\frac{1}{2}} + \frac{1}{24} f_{-\frac{1}{2}} - \frac{17}{1920} f^{2}_{-\frac{1}{2}} + \dots$$

Assuming that

$$f^{-2}_{-\frac{1}{2}} = x\left(t_0 - \frac{w}{2}\right) - \frac{1}{24} f_{-\frac{1}{2}} + \frac{17}{1920} f^{2}_{-\frac{1}{2}} + \dots \tag{28}$$

we finally obtain

$$x\left(t_n - \frac{w}{2}\right) = f^{-2}_{n-\frac{1}{2}} - \frac{1}{24} f_{n-\frac{1}{2}} + \frac{17}{1920} f^{2}_{n-\frac{1}{2}} - \dots \tag{29}$$

When the initial values $x\left(t_o - \frac{w}{2}\right)$ and $x\left(t_o + \frac{w}{2}\right)$ that define the integral under consideration of equation (18) are given, then formula (29) together with equations (27) and (28) enable us to successively obtain the values of x(t) for $t = t_n - \frac{w}{2}$, where n = 2, 3, 4, ...

It is advisable to have formula (28) written in a slightly different form. Since

$$f^{-1}_{\frac{1}{2}} = \frac{1}{2}\left(f_0^{-1} + f_1^{-1}\right) = f_0^{-1} + \frac{1}{2} f^{0}_{\frac{1}{2}}$$
$$f_{\frac{1}{2}} = f_0 + \frac{1}{2} f^{1}_{\frac{1}{2}}, \dots$$

then this formula may be replaced by

$$f_0^{-2} = x\left(t_0 - \frac{w}{2}\right) + \frac{1}{2} f^{-1}_{\frac{1}{2}} + \frac{1}{24}\left(f_0 - \frac{1}{2} f^{1}_{\frac{1}{2}}\right) -$$
$$- \frac{17}{1920}\left(f_0^{2} - \frac{1}{2} f^{3}_{\frac{1}{2}}\right) + \dots \tag{28bis}$$

Similarly, we replace formula (27) by

$$f^{1}_{\frac{1}{2}} = z_0 + \frac{1}{2} f_0 + \frac{1}{24} f_0^{1} - \frac{17}{1920} f_0^{1} + \dots \qquad (27\text{bis})$$

In evaluating the right-hand side of formula (29), one has to know the values $x(t_n)$. The application of this formula is thus complicated by the requirement of finding the values of $x(t_n)$ by means of interpolation into the middle of the intervals. We can obtain a formula that immediately gives the values $x(t_n)$. We apply the formula of the interpolation in the average, given in Sec. 50, to the function $x(t_n - \frac{w}{2})$, defined by formula (29). Assuming again that

$$\varphi(\tau_n) = x\left(t_n - \frac{w}{2}\right),$$

we obtain

$$\varphi_{n+\frac{1}{2}} = f_n - \frac{1}{24} f_n + \frac{17}{1920} f_n^{2} - \dots$$
$$\varphi^{1}_{n+\frac{1}{2}} = f_n - \frac{1}{24} f_n^{2} + \frac{17}{1920} f_n^{1} - \dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$

Therefore

$$[illegible]$$

where the initial values of the columns of sums are defined by formula (27) and (28) or (27 bis) and (28 bis).

54. Evaluation of the Integral Assigned by the Initial point and initial velocity

We have been considering the evaluation of the integral of the euqation

$$\frac{d^2x}{dt^2} = f(x, t),$$

that is assigned by the two values of its independent variable, e.g. giving values of $x(t)$ for $t = t_o$ and $t = t_o - w$ and for $t = t_o - \frac{w}{2}$

and $t = t_o + \frac{w}{2}$. We now consider another problem of particular interest to celestial mechanics which is the evaluation of the integral defined by the values x and $\frac{dx}{dt}$ in some point. We shall also consider here two cases:

The first case

Let the unknown integral x(t) be assigned by the initial values

$$x(t_0) = x_0, \quad \left(\frac{dx}{dt}\right)_{t_0} = x'_0. \qquad (30)$$

We shall now see how the adjacent values x_1, x_2, ... can be obtained Considering equations (26), (24) and (25) which born the basis of the method of quadratures. We manipulate theconstants $f^{-1}_{-\frac{1}{2}}$ and f^{-2}_{o} in such a way that formula (26) gives a function that satisfies the initial conditions (30). It is clear that the quantity f^{-2}_{o} should be left in the same form as given by equation (25); the first condition of which is given by equations (30) would be satisfied. Now, denoting by $\Delta^1, \Delta^2, \Delta^3, \ldots$ the successive differences of the function x(t), we obtain on the basis of equation (4) the following relation

$$w\left(\frac{dx}{dt}\right)_n = \Delta^1_n - \frac{1}{6}\Delta^3_n + \frac{1}{30}\Delta^5_n - \frac{1}{140}\Delta^7_n + \ldots$$

Formula (26) yields

$$\Delta^m = f^{m-2}_n + \frac{1}{12}f^m_n - \frac{1}{240}f^{m+2}_n + \ldots$$

Consequently

$$w\left(\frac{dx}{dt}\right)_n = f^{-1}_n - \frac{1}{12}f^1_n + \frac{11}{720}f^3_n - \frac{191}{60480}f^5_n + \ldots$$

Noting that

$$f^{-1}_n = \frac{1}{2}\left(f^{-1}_{n+\frac{1}{2}} + f^{-1}_{n-\frac{1}{2}}\right) = f^{-1}_{n+\frac{1}{2}} - \frac{1}{2}f_n$$

and puting n = 0, we finally obtain

$$f^{-1}_{\frac{1}{2}} = wx'_0 - \frac{1}{2} f_0 + \frac{1}{12} f^{\mathrm{I}}_0 - \frac{11}{720} f^{\mathrm{III}}_0 + \frac{191}{60\,480} f^{\mathrm{V}}_0 - \dots \qquad (31)$$

It is therefore sufficient to calculate the initial values of the sum columns using formulae (31) and (25) to enable formula (25) give the values of the integral that satisfy the initial conditions (30).

<u>The second case</u>

Consider now the case, when the integral is assigned by the following initial conditions

$$x\left(t_0 - \frac{w}{2}\right) = X_0, \qquad x'\left(t_0 - \frac{w}{2}\right) = X'_0, \qquad (32)$$

It is required to calculate $x_n = x(t_o + nw)$. Let us start by applying the previous approach that has been applied to formula (29). For a change, we will use another method which will as rapidly lead us to our aim. Replacing equation (18) by the system

$$\frac{dx'}{dt} = F(x, t), \qquad \frac{dx}{dt} = x' \qquad (33)$$

We integrate the first of this equation by using formula (17). Since, in our present notations,

$$w F(x, t) = \frac{1}{w} f(t),$$

we obtain

$$wx'\left(t_n + \frac{w}{2}\right) = f^{-1}_{n+\frac{1}{2}} + \frac{1}{24} f^{\mathrm{I}}_{n+\frac{1}{2}} - \frac{17}{5\,760} f^{\mathrm{III}}_{n+\frac{1}{2}} + \dots, \qquad (34)$$

where the first sums f^{-1} are defined by condition (16), namely

$$f^{-1}_{-\frac{1}{2}} = wX'_0 - \frac{1}{24} f^{\mathrm{I}}_{-\frac{1}{2}} + \frac{17}{5\,760} f^{\mathrm{III}}_{-\frac{1}{2}} - \dots \qquad (35)$$

We now consider the second of equations (33). Adopting that

$$wx'(t) = h(t),$$

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then, using equation (17), we obtain

$$x\left(t_n + \frac{w}{2}\right) = h^{-1}_{n+\frac{1}{2}} + \frac{1}{24} h^{1}_{n+\frac{1}{2}} - \frac{17}{5\,760} h^{3}_{n+\frac{1}{2}} + \dots \tag{36}$$

In order to evaluate the right-hand-side expression, we have to know the quantity

$$h_n = h(t_0 + nw) = wx'(t_0 + nw),$$

because formula (34) yields the values of $x'(t)$ for points $t_n + \frac{w}{2}$, which lie in the middle of our intervals.

Hence, we recall the formula of integration in the average, (Sec. 50),

$$\varphi\left(\tau_n + \frac{w}{2}\right) = \varphi_{n+\frac{1}{2}} - \frac{1}{8}\varphi^{2}_{n+\frac{1}{2}} + \frac{3}{128}\varphi^{4}_{n+\frac{1}{2}} - \dots$$

Substituting in this formula

$$\tau_n = t_n - \frac{w}{2}$$

$$\varphi(\tau_n) = \varphi_n = h\left(t_n - \frac{w}{2}\right)$$

and taking into account that on the basis of equation (34)

$$\varphi_{n+\frac{1}{2}} = \frac{1}{2}(\varphi_n + \varphi_{n+1}) = \frac{w}{2}\left[x'\left(t_n - \frac{w}{2}\right) + x'\left(t_n + \frac{w}{2}\right)\right] =$$
$$= f_n^{1} + \frac{1}{24} f_n^{1} - \frac{17}{5\,760} f_n^{3} + \dots$$

and, consequently,

$$\varphi^{2}_{n+\frac{1}{2}} = f^{1}_n + \frac{1}{24} f^{3}_n - \frac{17}{5\,760} f^{5}_n + \dots,$$

we finally obtain

$$h_n = \varphi\left(\tau_n + \frac{w}{2}\right) = f^{-1}_n - \frac{1}{12} f^{1}_n + \frac{11}{720} f^{3}_n - \frac{191}{60\,480} f^{5}_n + \dots$$

Substituting these values of h_n into formula (36) yields

$$x\left(t_n + \frac{w}{2}\right) = f^{-2}_{n+\frac{1}{2}} - \frac{1}{24} f_{n+\frac{1}{2}} + \frac{17}{1\,920} f^{2}_{n+\frac{1}{2}} - \frac{367}{193\,536} f^{4}_n - \dots \tag{37}$$

Putting n = -1, we obtain the following formula

$$f^{-2}_{-\frac{1}{2}} = X_0 + \frac{1}{24} f_{-\frac{1}{2}} - \frac{17}{1920} f^{2}_{-\frac{1}{2}} + \dots \qquad (38)$$

which determines the initial term of the column for the second sums.

Formulae (37), (38) and (35) give the solution to the problem under consideration. Formula (38) may be replaced by another formula, which gives an expression for f_0^{-2}. This will be more convenient. Since,

$$f^{-2}_{-\frac{1}{2}} = \frac{1}{2}(f_0^{-2} + f_{-1}^{-2}) - f_0^{-2} - \frac{1}{2} \dots$$
$$f_{-\frac{1}{2}} = f_0 - \frac{1}{2} f^1_{-\frac{1}{2}} \dots$$

then equation (38) may be transformed into

$$f_0^{-2} = X_0 + \frac{1}{2} f^{-1}_{-\frac{1}{2}} + \frac{1}{24} f_0 - \frac{1}{48} f^1_{-\frac{1}{2}} - \frac{17}{1920} f_0^2 + \frac{17}{3840} f^3_{-\frac{1}{2}} + \dots$$

Adding this equation to equation (35) multiplying each term by ½, we obtain

$$f_0^{-2} = X_0 + \frac{n'}{2} X [illegible] \frac{1}{24} f [illegible] f^1_{-\frac{1}{2}} [illegible] \frac{17}{5760}(3f_0^2 - 2f [illegible]) + \dots$$

or, finally,

$$f_0^{-2} = X_0 + \frac{n'}{2} X_0 + \frac{1}{24} f [illegible] \frac{17}{5760}(2f [illegible] f_0^2) [illegible]$$
$$[illegible] \frac{367}{967\,680}(3f^4 [illegible] 2f_0^4) - \dots$$

There only remains to replace formula (37) by another formula which gives the values of $x(t_n)$. However, this has been done in the previous section.

Thus, in both cases, the values of the integral are evaluated using the following formula

$$x_n = f_n^{-2} + \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \frac{31}{60480} f_n^4 - \ldots, \qquad (37 bis)$$

However, in the first case, the initial values of the column of sums are calculated by using formulae (31) and (25), while in the second case, they are obtained from formulae (35) and (39)

55. An example of integrating second-order equations

Let us consider the equation

$$\frac{d^2x}{dt^2} = \frac{1}{2}\left(1 + \frac{1}{2} t^2\right) x. \qquad (*)$$

We want to calculate a table for the values of the integral, determined by the condition

$$x(-0.05) = x(+0.05) = 1.0006252.$$

This can be solved using either the formulae given in Sec. 52, or those of Sec. 53. In the first case we obtain a table for the arguments t = - 0.05, 0.15, ... In the second case, the arguments of the table will have the values t = 0.0, 0.1, 0.2 ... .

We choose the second way and accordingly we put

$$t_0 - \frac{w}{2} = -0.05, \qquad t_0 + \frac{w}{2} = +0.05$$
$$t_0 = 0.0, \qquad w = 0.1$$
$$f = 0.005\left(1 + \frac{1}{2} t^2\right) x.$$

We shall apply the formulae of Sec. 53 in our calculations, namely

$$x_n = f_n^{-2} + \mathrm{Red},$$
$$\mathrm{Red} = \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \frac{1}{1951} f_n^4 - \ldots$$
$$f_{\frac{1}{2}}^{-1} = x\left(t_0 + \frac{w}{2}\right) - x\left(t_0 - \frac{w}{2}\right) + \frac{1}{2} f_0 + \frac{1}{24} f_0^1 - \frac{1}{113} f_0^3 + \ldots$$
$$f_0^{-2} = x\left(t_0 - \frac{w}{2}\right) + \frac{1}{2} f_{-\frac{1}{2}}^{-1} + \frac{1}{24}\left(f_0 - \frac{1}{2} f^1_{-\frac{1}{2}}\right) - \frac{1}{113}\left(f_0^2 - \frac{1}{2} f^3_{-\frac{1}{2}}\right) + \ldots$$

We obtain the initial values x_{-2} , x_{-1} , x_o , x_1 , x_2 by successive approximations and as a first approximation we take

$$x_{-1} = x_0 = x_1 = 1.000\,16$$

and evaluate the corresponding values of f. These are given in table A. Using the above formulae, we obtain

Table A

t	f^{-2}	f^{-1}	f	f^{1}	f^{2}	Red	x
-0.1	1.002 09		0.005 03			+12	1.002 51
		-0.002 50		-3			
0.0	0.999 59		0.005 00		+6	+12	1.000 01
		+0.002 50		+3			
0.1	1.002 09		0.005 03			+12	1.002 51

$$f_{\frac{1}{2}}^{-1} = 0.002\,50, \qquad f_0^{-2} = 0.999\,59$$

and by successive additions we fill the column of sums. In order to obtain new more accurate values for x, we have to calculate the red correction. With these values, we obtain the second approximation given in table B. In this table, the values x_{-2} , x_2 and x_3.

Table B

t	f^{-2}	f^{-1}	f	f^{1}	f^{2}	Red	x
0.2	1.009 622		0.005 152		+76	+427	1.010 05
		-0.007 538		-114			
-0.1	1.002 084		0.005 038		+76	+418	1.002 50
		-0.002 500		-38			
0.0	0.999 584		0.005 000		+76	+416	1.000 00
		+0.002 500		+38			
0.1	1.002 084		0.005 038		+76	+418	1.002 50
		+0.007 538		+114			
0.2	1.009 622		0.005 152		+76	+427	1.010 05
		+0.012 690		+190			
0.3	1.022 312		0.005 342		+76	+445	1.022 76
		+0.018 032					
0.4	1.040 344					+468	1.040 81

are obtained by extrapolating the values of f_{-2}, f_2 and f_3 (indicated as all other extrapolated values, in italics). Finally, x_4 is calculated by extrapolating the values of Red (cf. Sec. 51).

Table C

t	f	f^1	f^2	f^3	f^4	f^5	[illegible]	[illegible]	[illegible]	x
0.0	[illegible]		[illegible]				+416			[illegible]
		0.002 [illegible]		+[illegible]				2		
0.1	1.002 081		0.00[illegible]		75		118		7	1.002 5[illegible]
		0.007 738		113		5		9		
0.2	1.009 622		0.005 151		+80		+427		7	1.01[illegible]
		0.012 89[illegible]		+193		[illegible]		16		
0.3	1.022 311		0.00[illegible] 211		+83		+455		7	1.022 76
		0.018 033		+276		*4*		23		
0.4	1.040 244		0.005 620		*+87*		468		7	1.040 81
		0.023 653		*+363*		*4*		30		
0.5	1.063 897		*0.005 983*		*+91*		+498		8	1.06[illegible] 49
		0.029 636		*+454*				38		
0.6	*1.093 633*		*0.006 437*				+536		8	1.09[illegible] 17
		0.036 073						*40*		
0.7	*1.129 706*						+582			1.130 2[illegible]

In the third approximation, the extrapolation is carried further by three intervals. The values of x_5 and x_6 are exactly obtained, but there is an error of two units in the fifth decimal place in the value obtained for f_{x7}.

If we wish to avoid successive approximations in evaluating x_5, x_6, x_7, ..., we can use Störmer's formula

$$x_n = f_n^{-2} + \mathrm{Red}$$
$$\mathrm{Red} = \frac{1}{12}(f_{n-1} + f^1_{n-\frac{1}{2}} + f^2_{n-1}) - \frac{1}{240} f^2_{n-2} + \cdots, \qquad (**)$$

The application of this formula in the case when the third difference is negligible, is as simple as calculating by means of the conventional formula

$$\mathrm{Red} = \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \cdots \qquad (*)$$

The advantages of the two methods are made use of from time to time in order to correc^t the values obtained by formula (**) using the more exact formula (***), If for example we tabulate the values of the integral of equation (*) that satisfies conditions

$$x(-0.05) = 1.0006252, \quad x'(-0.05) = -0.0250156$$

for t = 0.0, 0.1, 0.2, ..., we evaluate the initial terms of the columns of sums using formulae (35) and (39), which may be written as

$$f_{-\frac{1}{2}} = hx'\left(t_0 - \frac{h}{2}\right) - \frac{1}{24} f'_1 + \frac{1}{3\,9[illegible]} f'_1 - \frac{1}{[illegible]} f^3_1 + \dots$$

$$f_{-1}^{-2} = x\left(t_0 - \frac{h}{2}\right) + \frac{h}{2} x'\left(t_0 - \frac{h}{2}\right) + \frac{1}{24} f_1 - \frac{1}{3\,[illegible]} (2f^2_1 + f'_0) + \frac{1}{2[illegible]} (3f'_1 + 2f'_0) - \dots$$

Finally, we note that the function

$$x(t) = e^{\frac{1}{4}t^2},$$

calculated in this section, is identical to the function calculated in Sec. 51. The comparison between these two examples suggests that the numerical integration of the second-order equation is not more difficult, even easier, th^n the numerical integration of first-order equations.

56. <u>The formulae of quadratures</u>

Applying the formula derived in Secs. 49 and 50, to the integration of the following equation

$$\frac{dx}{dt} = F(t). \tag{40}$$

The solution of this equation that satisfies the initial condition $t = t_0$, $x = 0$ is given by the integral

$$x = \int_{t_0}^{t} F(t)\,dt.$$

On the other hand, the particular value of this function at $t = t_n$ is given by formula (14). Hence,

$$\int^{t_n} F(t)\,dt = f_n^{-1} - \frac{1}{12} f_n^{1} + \frac{11}{720} f_n^{3} - \dots \tag{41}$$

where

$$f(t) = w\,F(t),$$

The first term of the column of sums is defined by

$$f_0^{-1} = \frac{1}{12} f_0^{1} - \frac{11}{720} f_0^{3} + \dots \tag{42}$$

Similarly, applying formulae (17) and (17 bis) to the calculation of the integral of equation (40) that satisfies the initial condition $t = t_o - \frac{w}{2}$, $x = 0$, we obtain

$$\int_{t_0 - \frac{w}{2}}^{t_n + \frac{w}{2}} F(t)\,dt = f_{n+\frac{1}{2}}^{-1} + \frac{1}{24} f_{n+\frac{1}{2}}^{1} - \frac{17}{5760} f_{n+\frac{1}{2}}^{3} + \dots \tag{43}$$

$$\int_{t_0}^{t_n} F(t)\,dt = f_n^{-1} - \frac{1}{12} f_n^{1} + \frac{11}{720} f_n^{3} - \dots \tag{44}$$

where, in this case, the initial term of the column of sums is given by

$$f_{-\frac{1}{2}}^{-1} = -\frac{1}{24} f_{-\frac{1}{2}}^{1} + \frac{17}{5760} f_{-\frac{1}{2}}^{3} - \dots \tag{45}$$

We obtain another formulae of quadratures if in equations (43) and (44), we make the initial term of the column of sums subject to condition (42). We then put n = 0 in equation (44) and substract it from equation (43). In this way, we obtain

$$\int_{t_0}^{t_n + \frac{w}{2}} F(t)\,dt = f_{n+\frac{1}{2}}^{-1} + \frac{1}{24} f_{n+\frac{1}{2}}^{1} - \frac{17}{5760} f_{n+\frac{1}{2}}^{3} + \dots \tag{46}$$

In order to obtain the formula of quadratures that solves a double integral, we consider the second-order equation

$$\frac{d^2 x}{dt^2} = F(t). \tag{47}$$

The solution of this equation, which satisfies the conditions $x(t_o) = 0$ and $x'(t_o) = 0$, is given by

$$x = \int_{t_0}^{t} dt \int_{t_0}^{t} F(t)\, dt.$$

On the other nand, this solution is given by formula (26). Therefore, we find that

$$\int_{t_0}^{t_n} dt \int_{t_0}^{t} F(t)\, dt = f_n^{-2} + \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \dots \tag{48}$$

where the initial terms of the column of sums is given by

$$\left.\begin{aligned} f_{-\frac12}^{-1} &= -\frac{1}{2} f_0 + \frac{1}{12} f_0^1 - \frac{11}{720} f_0^3 + \frac{191}{60480} f_0^5 - \dots \\ f_0^{-2} &= -\frac{1}{12} f_0 + \frac{1}{240} f_0^2 - \frac{31}{60480} f_0^4 + \dots \end{aligned}\right\} \tag{49}$$

We now consider the solution of equation (47) that satisfies the conditions $x\,(t_o - \frac{w}{2}) = 0$ and $x'\,(t_o - \frac{w}{2}) = 0$. Using formulae (37) and (37 bis), we obtain

$$\int_{t_0-\frac{w}{2}}^{t_n+\frac{w}{2}} dt \int_{t_0-\frac{w}{2}}^{t} F(t)\, dt = f_{n+\frac12}^{-2} - \frac{1}{24} f_{n+\frac12} + \frac{17}{1920} f_{n+\frac12}^2 - \dots \tag{50}$$

$$\int_{t_0}^{t_n} dt \int_{t_0-\frac{w}{2}}^{t} F(t)\, dt = f_n^{-2} + \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \dots \tag{51}$$

where we now obtain

$$\left.\begin{aligned} f_{\frac12}^{-1} &= -\frac{1}{24} f_{\frac12}^{1} + \frac{17}{5760} f_{\frac12}^{3} - \frac{367}{967680} f_{\frac12}^{5} + \dots \\ f_{0}^{-2} &= \frac{1}{24} f_{\frac12} - \frac{17}{5760} (2f_{\frac12}^{2} + f_0^2) + \frac{367}{967680} (3f_{\frac12}^{4} + 2f_0^4) - \dots \end{aligned}\right\} \tag{52}$$

Applying a similar approach, to the one used in obtaining formula (46) from equations (50) and (51), we obtain

$$\int_{t_0}^{t_n+\frac{w}{2}} dt \int_{t_0}^{t} F(t)\, dt = f_{n+\frac12}^{-2} - \frac{1}{24} f_{n+\frac12} + \frac{17}{1920} f_{n+\frac12}^2 - \dots, \tag{53}$$

where the columns of sums are determined by condition (49).

All the formulae, which have been obtained, may be unified in the following manner

$$\left.\begin{aligned}
&\frac{1}{w}\int_A^{t_n} f(t)\,dt = f_n^{-1} - \frac{1}{12}f_n^{1} + \frac{11}{720}f_n^{3} - \frac{191}{60\,480}f_n^{5} + \dots \\
&\frac{1}{w}\int_A^{t_n+\frac{w}{2}} f(t)\,dt = f_{n+\frac{1}{2}}^{-1} + \frac{1}{24}f_{n+\frac{1}{2}}^{1} - \frac{17}{5760}f_{n+\frac{1}{2}}^{3} + \frac{367}{967\,680}f_{n+\frac{1}{2}}^{5} - \dots
\end{aligned}\right\}\ (\mathrm{I})$$

$$\left.\begin{aligned}
&\frac{1}{w^2}\int_A^{t_n} dt\int_A^{t} f(t)\,dt = f_n^{-2} + \frac{1}{12}f_n - \frac{1}{240}f_n^{2} + \frac{31}{60\,480}f_n^{4} - \dots \\
&\frac{1}{w^2}\int_A^{t_n+\frac{w}{2}} dt\int_A^{t} f(t)\,dt = f_{n+\frac{1}{2}}^{-2} - \frac{1}{24}f_{n+\frac{1}{2}} + \\
&\qquad + \frac{17}{1920}f_{n+\frac{1}{2}}^{2} - \frac{367}{193\,536}f_{n+\frac{1}{2}}^{4} + \dots
\end{aligned}\right\}\ (\mathrm{II})$$

where, if $A = t_0$, the initial terms of the column of sums are evaluated by equations (42) or (49), and if $A = t_0$ they are evaluated by formulae (45) or (52). For other values of A these initial terms are evaluated by the general formulae, that have been obtained in Secs. 49, 50 and 51.

It is interesting to note that equation (42) is equivalent to the first of equations (49).

Annotation

The following effect, which is met with in any interpolation formulae, and in particular, in numerical integrations, is easily shown on the simple equations (40) and (47): For any interval w there exist some functions F(t), for which formulae (I) or (II) give results that differ by an arbitrarily large amount from the actual results, even if all the terms on the right-hand side of these equations, that affect the result, will be taken into consideration. Indeed, let us consider the following function

$$\varphi(t) = A \sin^2 \frac{2\pi(t - t_0)}{w}.$$

All the values of this function that correspond to $t = t_o + nw$ are zero. Formulae (41) and (42) give for the integral

$$\int_{t_0}^{t_n} \varphi(t)\,dt$$

values, which are also equal zero. However, the integral is evidently not equal zero. It can be as large as possible when the initial value A is properly chosen. Hence, the integrals of functions F(t) and F(t) + φ (t) will differ by an arbitrarily large amount while the numerical integration will give the same value for the two functions. This example illustrates that in numerical integration as well as in any other processes involving interpolations, it is not sufficient to have a table for the values of the function, but it is also necessary to know its analytical character.

57. Basis of the Successive Approximation Method

Let us take a first-order equation, e.g.

$$\frac{dx}{dt} = F(x, t)$$

and investigate the calculation procedure of its integral, which satisfies the initial condition t_o, x_o. The unknown integral evidently satisfies the following integral equation

$$x = x_o + \int_{t_0}^{t} F(x, t)\,dt \qquad (54)$$

Inversely, the function that satisfies equation (54) also satisfies the differential equation given above as well as the required initial condition. Hence, obtaining the required particular solution of the above-mentioned differential equation is equivalent to solving the integral equation (54).

Let us take an arbitrary function ξ_0 (t) that satisfies the condition $\xi_0(t_o) = x_o$, and then evaluate a new function ξ_1 (t) using the following formula

$$\xi_1(t) = x_0 + \int_{t_0}^{t} F(\xi_0(t), t)\,dt.$$

The new function also satisfies the condition $\xi_1(t_o) = x_o$. Continuing this procedure, we obtain a set of functions

$$\xi_0(t),\ \xi_1(t),\ \xi_2(t),\ \ldots$$

connected together by the following relation

$$\xi_{n+1}(t) = x_0 + \int_{t_0}^{t} F(\xi_n(t), t)\,dt. \qquad (55)$$

We shall prove that

$$\lim_{n\to\infty} \xi_n(t) = x(t).$$

For this purpose let us substract equation (55) from euqation (53), term by term, and consider the difference

$$x - \xi_{n+1} = \int_{t_0}^{t} \{F(x,t) - F(\xi_n, t)\}\,dt$$

The theorem of finite increments gives

$$|F(x,t) - F(\xi_n,t)| < M\,|x - \xi_n|,$$

where M denotes the upper limits of the partial derivatives $F'_x(x_1 t)$ in the required variation region x and t.

Consequently,

$$|x - \xi_{n+1}| < M\,|t - t_0|\,|x - \xi_n|.$$

We subordinate the interval $t - t_o$ to the condition

$$M|t - t_0| < q, \qquad (56)$$

where q is a proper fraction such that $0 < q < 1$. Then

$$|x - \xi_{n+1}| < q\,|x - \xi_n|.$$

Applying this inequality for n = 0, 1, 2, ..., n-1, n, and multiplying the resulting inequalities term by term, we obtain

$$|x - \xi_{n+1}| < q^{n+1}|x - \xi_0|.$$

It is thus clear that when condition (56) holds, function $\xi_n(t)$ tends to the unknown function x(t) when n tends to infinity. Therefore, if we take an arbitrary function $\xi_o(t)$ and apply formula (55) a sufficiently large number of times, we obtain x(t) with an arbitrary accuracy. The required accuracy will be obtained more rapidly, with the smaller numbers for q, i.e. the smaller the interval $t - t_o$ is and with the better choice for the initial approximation $\xi_o(t)$.

In the example considered in Sec. 51, we could take $\xi_o(t)$ to be the constant value x_o. After two successive approximations, we would be able to find the correct values of x(t) for values of t near the initial value t_o. When a few values for x(t) are obtained, it would be better not to use the previous trail function but to constrctu by extrapolation a talk of values for x(t). It is then possible to construct a new function, using this table, which would be closer to the unknown function x (t). We then substitute this function into formula (55), and proceed in the same way as we have done before in applying the method of quadratures.

The application of successive approximations to the method of quadratures for second-order equations can be justified in a similar way.

58. Different methods for the reduction of the number of successive approximations

The methods of numerical integration of differential equations can be divided into two groups. The methods of Adams and Sotrmer and the corresponding methods of quadratures belong to the first group, while Cowell's method and the conventional method of quadratures belong to the second group.

The methods of the first group make use of the differences located in the ascending diagonal. Each value of the unknown function is thus evaluated using only its preceding values. These methods may be called extrapolational. On thecontrary, the methods of the second group made use of the differences lying on the horizontal line. Therefore, the preceeding and the following values equivalently, participate in the calculation of each of the values of the unknown function. That is why these methods may be called interpolational.

The interpolational methods lead to more accurate results, however, in the calculations of this method, one has to make use of several successive approximations. If the number of the required successive approximations is not greater than two, this is then not considered as a weakness, because successive approximations help in avoiding errors of calculations. However calculations with a large number of successive approximations are not valid. In the following, we shall consider the method for reducing the number of approximations.

The first method

The simplest and most convenient method to reduce the number of approximation consists of decreasing the interval w. This interval may be taken of such dimensions that the extrapolation of the values of the unknown function becomes sufficiently accurate in order to obtain the final values of the function f. In this case, the second approximation will only change thedifferences and sums of the first approximation and will be accepted as a final result.

The authoritive astronomer Comri mentioned only this method in his precept for using numerical integrations in problems of celestial mechanisms[(1)] and particularly warned against using large intervals during interpolations. He wrote: "The computer should be warned against attempting to use too large an interval, the result of which is that checking by difference, which is essential in these methods, becomes ineffective. A safe guide is that the sixth difference should not exceed two figures". Comri suggested the use of the following rule: "the interval should be such, that the sixth difference should not be more than two significant figures.

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(1) Planetary Co-ordinates for the years 1800-1940 referred to the Equinox of 1950.0. London, 1933.

<u>The second method:</u>

If we express in the conventional formula for quadratures,

$$x_n = f_n^{-2} + \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \frac{31}{60\,480} f_n^4 - \ldots \qquad (57)$$

the equantities f_n , f_n^2 , ... (the presence of which makes the successive approximation indespensible) in terms of quantities located in the ascending diagonal, we obtain the quadratic form of Stormer's method namely

$$x_n = f_n^{-2} + \frac{1}{12} f_{n-1} + \frac{1}{12} f^1_{n-\frac{3}{2}} + \frac{19}{240} f^2_{n-2} + \ldots \qquad (58)$$

Formula (58) does not require successive approximations. It yields however somewhat less accurate results than formula (57) (cf.Sec. 52).

In order to unify the conclusions obtained from these two formula, we pause midway in transition from equation (57) to equation (58). We have

$$\begin{aligned} f_n &= f_{n-1} + f^1_{n-\frac{1}{2}} \\ f_n &= f_{n-1} + f^1_{n-\frac{3}{2}} + f^2_{n-1} \\ f_n &= f_{n-1} + f^1_{n-\frac{3}{2}} + f^2_{n-2} + f^3_{n-\frac{3}{2}} \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

Stopping, for example on the second of these expressions, we obtain

$$x_n = \bar{x}_n + \sigma_n,$$

where

$$\bar{x}_n = f_n^{-2} + \frac{1}{12}(f_{n-1} + f^1_{n-\frac{3}{2}} + f^2_{n-2}) - \frac{1}{240} f_n^2 + \frac{31}{60\,480} f_n^4 - \ldots$$

$$\sigma_n = \frac{1}{12} f^3_{n-\frac{3}{2}}.$$

We immediately obtain the final result for x_n, since the errors in extrapolating the values f_n^2, f_n^1, ... will not effect the value of x_n as an result of the smaliness of their coefficients. Only the correction σ_n will be alterted in the subsequent approximations. However, this correction is very small and its evaluation is not difficult.

The method of Tietjen

Tietjen(1) observed that the need for successive approximations results from the presence of the term 1/12 f_n in the right-side of equation (58). All the other terms can be obtained by extrapolation and the resulting values are practically quite accurate. Therefore, taking into account that

$$f_n = w^2 F(x_n, t_n),$$

Tietjen replaced formula (57) by

$$x_n - \frac{w^2}{12} F(x_n, t_n) = S_n, \tag{59}$$

where

$$S_n = f_n^{-2} - \frac{1}{240} f_n^{2} + \frac{31}{60480} f_n^{4} \quad . \; . \; . \tag{60}$$

The final value of S_n can be immediately obtained. The solution of equation (59) yields the unknown value x_n and in this way, only the differences f_n^2, f_n^1, ... will be altered during the successive approximations. This method will be of practical interest only if the solution of equation (59) for x_n is sufficiently simple.

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(1) F. Tietjen, Specielle Storungen in Bezug auf Polar coordinaten, Berl. Astr. Jahrbuch fur 1877, M.F. Subbotin, On the Numerical Integration of Differential Equations (O cislennom integrirovanij differencial'nyh uravnenij) Proceedings of the Academy of Science of USSR (Izvestija Academii Nauk SSSR), 1933, No. 7, pp. 895-902.

The method of Numerov

The same procedure which was followed by Tietjen in developing the method of quadratures was later on applied by Numerov[1] in order to develop Cowell's method. Let us consider the principal formula in Cowell's method namely

$$\Delta_k = f_k + \frac{1}{12} f_k^2 - \frac{1}{240} f_k^4 + \frac{31}{60480} f_k^6 - \dots \qquad (61)$$

which can be used together with the following equation

$$x_{k+1} = 2x_k - x_{k-1} + \Delta_k$$

to obtain successively the values x_1 , x_2 , ... Let us introduce the new variable x by adopting that

$$x = x - \frac{1}{12} f$$

When $t = t_k$, we shall have

$$x_k = x_k - \frac{1}{12} f_k \qquad (62)$$

"Denoting by Δ , Δ^2, ... the differences of x, we obtain as a consequence of equation (62) the following relation

$$\Delta_k^2 = \Delta_k^2 - \frac{1}{12} f_k^2$$

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(1) B. Numerov, Methode nouvelle de la determination des orbites et le calcul des ephemerides en tenant compte des perturbations, Transactions of the Principal Russian Astrophysical Observatory (Trudy Gl. Rossijskoj Astrofiz. Observatorri) vol. II, 1923.

Thus, we obtain for the new variable

$$\Delta_k^2 = f_k - \frac{1}{240} f_k^{iv} + \frac{31}{60\,480} f_k^{vi} - \dots$$

and, as for any function,

$$x_{k+1} = 2x_k - x_{k-1} + \Delta_k.$$

The two equations are quite equivalent to equation (60).

In order to calculate $f_k = w^2 F(x_k, t_k)$, it is necessary to know x_k. Therefore, it is necessary to express the special coordinates x_k obtained in this method in terms of the conventional coordinates x_k. It is necessary for this purpose to solve equation (62), which may be written as

$$x_k = x_k - \frac{w^2}{12} F(x_k, t_k) \qquad (63)$$

and which corresponds to equation (59) in the method of Tietjen. Let us assume that the interval w is chosen in such a manner that the terms

$$\frac{1}{240} f_k^{iv} + \frac{31}{60\,480} f_k^{vi} \dots = \frac{1}{240} w^6 \left(\frac{d^6 x}{dt^6}\right) + \dots \qquad (64)$$

may be neglected. We then obtain a very simple formula for the subsequent evaluation of the "special coordinate" x, namely

$$x_{k+1} = 2x_k - x_{k-1} + f_k \qquad (65)$$

This formula is equivalent to equation (60), which may be written as

$$\Delta_k^2 = f_k$$

The so-called method of Numerov or method of extrapolation[1] consists in the application of this formula.

In the method of Numerov, one does not need to construct the differences of the quantities f_k. If these differences could be constructed, it would be easier to calculate using formula (61) than to make a

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(1) This latter name is not convenient, for the word extrapolation has a generally accepted wider meaning.

transition into the "special coordinates", and the accuracy would not then be easily controled. The guarantee against errors may be obtained only by the use of various special control calculations.

This modification of Cowell's method may be applied when the following two conditions hold: 1) It is possible to guarantee beforehand the insensitivity of terms (64) during the whole process of integration. (In Cowell's method and in the method of quadratures it can always be seen which of the terms can be neglected and which cannot). 2) The solution of equation (63) for x_k is sufficiently simple. The drawbacks of Cowell's method become more severe in this case (Sec. 52).

If we proceed by dropping terms in formula (61) and finally choosing a new variable

$$z = x - \frac{1}{12} f + \frac{1}{240} f'' - \frac{31}{60480} f^{IV} + \ldots ,$$

we then arrive to the following perfectly accurate equations

$$\Delta^2 z_k = f_k,$$
$$z_{k+1} - 2z_k + z_{k-1} = f_k.$$

In this case, Cowells method is improved to the maximum and we obtain, as we easily see from equation (57),

$$z_k = f_k ,$$

i.e., in other words, the method of quadratures.

59. Laplace's Method of Quadratures and Related Methods of the Numerical Integration of Equations

The methods of numerical integrations, considered in the previous sections, can be modified in many different ways. Let us consider the methods given in Sec. 49 for the integration of the following equation

$$\frac{dx}{dt} = f(x, t). \qquad (66)$$

These methods are based on the calculation of the difference

$$\Delta_{k+\frac{1}{2}} = x_{k+1} - x_k = x(t_{k+1}) - x(t_k)$$

by means of the successive differences of the function

$$f(t) = wF(x, t)$$

Let us represent this difference in the following way

$$\Delta_{k+\frac{1}{2}} = x(t_{k+1}) - x(t_{k+1} - w) = w\left(\frac{dx}{dt}\right)_{k+1} - \frac{w^2}{2}\left(\frac{d^2x}{dt^2}\right)_{k+1} + \dots$$

or, using again equation (6),

$$\Delta_{k+\frac{1}{2}} = f_{k+1} - \frac{1}{2}f^1_{k+\frac{1}{2}} - \frac{1}{12}f^2_k - \frac{1}{24}f^3_{k+\frac{1}{2}} - \frac{19}{720}f^4_k - \frac{3}{160}f^5_{k+\frac{1}{2}} - \frac{863}{60\,480}f^6_k - \dots \qquad (67)$$

This formula leads to a method of integration, which is similar to Cowell's method in thesense that it avoids the extrapolation in the evaluation of f_{k+1}, ... Summing equation (67) from k = 0 to k = n-1 yields

$$x_n - x_0 = f^{-1}_{n+\frac{1}{2}} - f^{-1}_{\frac{1}{2}} - \frac{1}{2}(f_n - f_0) - \frac{1}{12}(f^1_{n-\frac{1}{2}} - f^1_{-\frac{1}{2}}) - \frac{1}{24}(f^2_n - f^2_0) - \frac{19}{720}(f^3_{n-\frac{1}{2}} - f^3_{-\frac{1}{2}}) - \dots$$

On applying this formula to the case, when the right-hand side of equation (6) does not depend on x, we obtain

$$x_n - x_0 = \int_{t_0}^{t_n} F(t)\,dt = \frac{1}{w}\int_{t_0}^{t_n} f(t)\,dt$$

Consequently

$$\frac{1}{w}\int_{t_0}^{t_n} f(t)\,dt = \frac{1}{2}f_0 + f_1 + f_2 + \dots + f_{n-1} + \frac{1}{2}f_n - \frac{1}{12}(f^1_{n-\frac{1}{2}} - f^1_{\frac{1}{2}}) - \frac{1}{24}(f^2_n - f^2_0) - \dots$$

This formula is in many cases not convenient because it expresses the integral not only in terms of the values of the functions, f_0, f_1, ..., f_{n-1}, f_n, but also in terms of the values f_{-1}, f_{-2}, ..., which correspond to values of the argument lying outside the limits of integration. This inconvenience can be avoided if we use the following relations

$$\begin{aligned}
f^1_{-\frac{1}{2}} &= f^1_{\frac{1}{2}} - f^2_1 + f^3_{\frac{3}{2}} - f^4_2 + \cdots \\
f^2_{-1} &= f^2_1 - 2f^3_{\frac{3}{2}} + 3f^4_2 - 4f^5_{\frac{5}{2}} + \cdots \\
f^3_{-\frac{3}{2}} &= f^3_{\frac{3}{2}} - 3f^4_2 + 6f^5_{\frac{5}{2}} - 10f^6_3 + \cdots \\
f^4_{-2} &= f^4_2 - 4f^5_{\frac{5}{2}} + 10f^6_3 - 20f^7_{\frac{7}{2}} + \cdots \\
\cdots & \qquad\qquad \cdots \qquad\qquad \cdots
\end{aligned}$$

Finally, we obtain the following formula of quadratures

$$\begin{aligned}
\frac{1}{w}\int_a^{a+nw} f(t)\,dt &= \frac{1}{2}f_0 + f_1 + \cdots + f_{n-1} + \frac{1}{2}f_n \\
&- \frac{1}{12}\left(f^1_{n-\frac{1}{2}} - f^1_{\frac{1}{2}}\right) - \frac{1}{24}\left(f^2_{n-1} - f^2_1\right) \\
&- \frac{19}{720}\left(f^3_{n-\frac{3}{2}} - f^3_{\frac{3}{2}}\right) - \frac{3}{160}\left(f^4_{n-2} - f^4_2\right) - \cdots
\end{aligned}$$

which has been obtained by Laplace. This formula had been widely used till the forties of the 19th century in the calculation of perturbations. Later on, it was replaced by Gauss method discussed in Sec. 56.

60. The coefficients of the Formulae of Numerical Integration

In conclusion, we point out another way of calculating the coefficients of the formulae which are applied in the numerical integration of equations. When the form of a formula of numerical integration is established, we can apply this formula to a properly chosen specific case and obtain its coefficients. For example, if we consider the following equation

$$\frac{dx}{dt} = e^{t}$$

and apply formula (10) to its solution $x = e^{t}$. Choosing $t_c = 0$, the differences of the function $\varphi = e^{t}$ will be given by

$$\varphi^1_{k+\frac{1}{2}} = e^{(k+1)w} - e^{kw} = e^{kw}(e^{w} - 1)$$

$$\varphi^2_k = [e^{kw} - e^{(k-1)w}](e^{w} - 1) = e^{(k-1)w}(e^{w} - 1)^2$$

$$\varphi^3_{k+\frac{1}{2}} = e^{(k-1)w}(e^{w} - 1)^3$$

$$\cdots\cdots\cdots\cdots$$

or, assuming that

$$u = e^{\frac{w}{2}} - e^{-\frac{w}{2}},$$

we will obtain the following expressions

$$\varphi^1_{k+\frac{1}{2}} = e^{(k+\frac{1}{2})w}u$$

$$\varphi^2_k = e^{kw}u^2$$

$$\varphi^3_{k+\frac{1}{2}} = e^{(k+\frac{1}{2})w}u^3$$

$$\cdots\cdots\cdots\cdots$$

Consequently, we will have for the function $f = we^{t}$

$$f_{k+\frac{1}{2}} = \frac{w}{2}(e^{(k+1)w} + e^{kw}) = we^{(k+\frac{1}{2})w}\,\mathrm{ch}\,\frac{w}{2}$$

$$f^2_{k+\frac{1}{2}} = wu^2e^{(k+\frac{1}{2})w}\,\mathrm{ch}\,\frac{w}{2}$$

$$f^4_{k+\frac{1}{2}} = wu^4e^{(k+\frac{1}{2})w}\,\mathrm{ch}\,\frac{w}{2},$$

$$\cdots\cdots\cdots\cdots$$

where

$$\mathrm{ch}\,\frac{w}{2} = \frac{1}{2}(e^{\frac{w}{2}} + e^{-\frac{w}{2}}) = \sqrt{1 + \frac{1}{4}u^2}$$

Substituting these expressions into formula (10), we finally obtain the following identity

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$$\frac{u}{w\,\mathrm{ch}\frac{w}{2}} = 1 - \frac{1}{12}u^2 + \frac{11}{720}u^4 - \frac{191}{60480}u^6 + \dots$$

Since

$$w = 2\ln\left(\frac{1}{2}u + \sqrt{1 + \frac{1}{4}u^2}\right),$$

then this identity may be written in the following way

$$\frac{u}{2\sqrt{1+\frac{1}{4}u^2}\,\ln\left(\frac{1}{2}u + \sqrt{1+\frac{1}{4}u^2}\right)} = 1 - \frac{1}{12}u^2 + \frac{11}{720}u^4 - \dots$$

Thus, in order to obtain the coefficients of formula (10), it is sufficient to expand the function on the left-hand side in power series. Similarly, the coefficients of formulae (11), (19) and (26) can be found using the following identities

$$\frac{u}{w} = 1 + \frac{1}{24}u^2 - \frac{17}{5760}u^4 + \frac{367}{967680}u^6 - \dots,$$

$$\frac{u^2}{w^2} = 1 + \frac{1}{12}u^2 - \frac{1}{240}u^4 + \frac{31}{60480}u^6 - \dots,$$

$$\frac{u^2}{w^2\,\mathrm{ch}\frac{w}{2}} = 1 - \frac{1}{24}u^2 + \frac{17}{1920}u^4 - \frac{367}{193536}u^6 + \dots,$$

the proof of which is relatively simple.

CHAPTER IX

APPLICATION OF THE NUMERICAL INTEGRATION TO THE STUDY OF THE UNPERTURBED MOTION

61. Introduction

At the beginning of the previous chapter, we pointed out that the numerical integration of differential equations has been considered as a method for the calculation of perturbations. It has never been applied to the calculation of unperturbed coordinates. The only exception is the approach suggested by Krueger for the evaluation of a true anomaly (Sec. 62). Only when numerical integration was applied to the study of perturbed coordinates (and not perturbations) by the initiative of Cowell, the possibility of applying this method to the calculations of unperturbed coordinates was frequently considered.

In this chapter, we shall consider the calculations, by numerical integration, of the orbital coordinates r and v (or ξ and η), and the equivalent heliocentric coordinates x, y and z used in the calculations of the ephemeride.

In the calculation of a more or less long ephemeride, the method considered in this chapter is usually preferred than the calculation of the coordinates of a star by the conventional methods, studied in the first volume.

62. The calculation of coordinates defining the position of a luminary in an orbit

It is easier to apply the following equations to the simultaneous calculation of a true anomaly v and radius vector r

$$\frac{dv}{dt} = \frac{k\sqrt{p}}{r^2} \qquad (1)$$

$$r = \frac{p}{1 + e\cos v} \qquad (2)$$

In order to obtain a table for the values of r and v for the moments t_0, $t_0 + w$, $t_0 + 2w$, ..., it is sufficient to calculate these quantities for the moment t_0 by conventional methods (Volume I, Chapter III).

It may sometimes happen that the integration of the following equation

$$\frac{d^2 v}{dt^2} = -\frac{2k^2 e \sin v}{r^3},$$

which is easily obtained from equations (1) and (2), rather than equation (1), is more useful.

Krueger[1] suggested the following particularly convenient order of calculation. First of all, the coordinates v and r are found by Kepler's formulae for the first three moments. Then, a table of the approximate values of r is constructed by extrapolation. This gives the possibility of calculating the right-hand side of equation (1), which after integration yields v. With these values of v, the values of r are reevaluated. Repeating the integration, we obtain more accurate values for v (cf Sec. 57).

If we have to only calculate the values of the radius vector, we can apply Legendre's formula (Sec. 5).

$$\frac{d^2 r^2}{dt^2} = 2k^2\left(\frac{1}{r} - \frac{1}{a}\right). \qquad (3)$$

Integrating this equation, we obtain a table for the values of r. We then evaluate v from equation (1) by means of a simple quadrature. Of course, we can evaluate v using equation (2) if we have already obtained r. However, the calculation of the true anomaly by integrating equation (1) leads to more accurate results and requires less amount of

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(1) A. Krueger, Die Wiederkehr des Olbers'sehen Cometen 1887, Astr. Nachr. 117, 1887.

work. Hence, it is generally recommended to follow this procedure.

In order to start integrating equation (3), it is possible to calculate the values of r for two moments, for example $t_0 - w$ and t_0, or $t_0 - \frac{w}{2}$ and $t_0 + \frac{w}{2}$. It is also possible to calculate r only for one moment, but it is then necessary to calculate for this moment the derivative

$$\frac{dr}{dt} = \frac{k e \sin v}{\sqrt{p}} = \frac{k \operatorname{tg} \varphi \sin v}{\sqrt{a}}. \qquad (4)$$

The method of calculation of r and v for a series of different moments, which we have suggested above, is particularly useful when the calculation of these values by conventional formulae, is complicated, and in particular, for those orbits whose eccentricities slightly differ from unity.

In order to calculate the rectangular orbital coordinates ξ and η it is possible to use the following differential equations

$$\frac{d^2\xi}{dt^2} = -k^2 \xi r^{-3}, \qquad \frac{d^2\eta}{dt^2} = -k^2 \eta r^{-3},$$
$$r = \sqrt{\xi^2 + \eta^2}.$$

It is, however, simpler to express the coordinates ξ and η in terms of the quantity σ (Vol. I, Sec. 23), defined by

$$\frac{d\sigma}{dt} = \frac{k}{\sqrt{2}} q^{-3/2} \frac{\sqrt{1 + \alpha\sigma^2}}{1 + \sigma^2}, \qquad (5)$$

where

$$q = a(1-e), \qquad \alpha = \frac{1}{2}(1-e),$$

and subject to the condition that $\sigma = 0$ in the moment, when the luminary passes by the perihelion. The coordinates ξ and η are evaluated by the formulae (loc. cit.)'

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$$\ldots q(1 - \ldots), \qquad \ldots = 2q \ldots \sqrt{1-e}\,\sqrt{1-\ldots}\,.$$

In conclusion, we point out that the numerical integration of equation (1) or equation (3) may be applied for correcting or checking the calculated values of v or r. For example, taking as a first approximation the value of v_1 given in table XV (Vol. I) with an accuracy of 0.00005, and integrating equation (1), we obtain a value of v with a larger number of decimal places (Sec. 57).

63. An example of calculating the orbital coordinates using the numerical integration

Let the orbit of a comet be defined by the elements

$$e = 0.96[illegible]\,1567, \qquad \lg q = 9.765\,[illegible]500.$$

It is required to calculate r and v for the moments $t_0 + kw$, where

$$t_0 = 65[illegible].51100, \qquad w = 2^d,$$

and when time is evaluated starting from the moment the comet passes by the perihelion.

This problem can be solved by several methods

Krueger's method:

First of all, we use the conventional formulae (Vol. I, Sec. 23) to calculate the followin values of v and r

t	v	r
$t_0 - w$	100.00000	1.378 76
t_0	101.08251	1.409 32
$t_0 + w$	102.12697	1.4[illegible]9 77.

In equation (1), the function

$$f = \frac{wk\sqrt{p}}{r}$$

can be represented by

$$f = -2.111\,215\, r^{-2} - [0.324\,5321]\, r^{-4},$$

since the Gaussian constant, expressed in degrees,equals

$$k = 0^\circ.985\,6075, \qquad \lg k = 9.993\,7010 - 10;$$

In the present case, w = 2 and

$$1\,\bar{p} \quad 1\,q(1 + e) \quad [0.029\,7981].$$

The calculation of the function f for the indicated values of r and for the two extrapolated values (printed in italics) are shown in the following table

TABLE A

t	r	differences	r^{-3}	r^{-2}	f
t_{-2}	[illegible]		[illegible]	[illegible]	1.16170
		[illegible]			
t_{-1}	[illegible]	11	[illegible]	[illegible]	1.11659
		[illegible]			
t_0	[illegible]	11	[illegible]	[illegible]	[illegible]
		[illegible]			
t_1	[illegible]	−11	[illegible]	0.182107	[illegible]
		[illegible]			
t_2	[illegible]		0.080221	[illegible]	[illegible]

Since we integrate equation (1) using the formula

$$s_1 \quad f_n \quad \frac{1}{12} f_n' \quad \frac{11}{720} f_n \quad \ldots$$

we immediately write in table B the arithmetic averages of the values of f, obtained in table A.

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TABLE B

t	${}^{I}f$	f	f^1	f^2	f^3	Red	v
t_{-2}	98 859 77		− 5294		− 27	+ 111	98 861 18
		1 136 14		357			
t_{-1}	99 995 91		− 4937		− 27	+ 111	100 000 02
		1.086 77		+ 330			
t_0	101 082 68		4607		− 27	+ 383	101.086 51
		1 040 70		+ 303			
t_1	102 123 38		− 4304		− 25	358	102.126 96
		0 997 66		+ 278			
t_2	103 121 01		− 1026			+ 335	103 124 39
		0 957 40					
t_3	104 078 44					+ 374	104.081 58

TABLE C

t	$1 + e\cos v$	r	differences	r^1	r^2	f
t_0		1 109 3[illegible]	11			
			+ 3015			
t_1		1 139 77	− 12			1 018 46
			+ 3033			
t_2	0 780 281	1 170 10	− 13	0 680 2[illegible]	0 162 707	0 976 87
			3020			
t_3		1.500 30		0 666 533	0 111 266	0 937 91

The initial value of the column of sums is defined by the condition

$$v_0 = {}^{I}f_0 + \frac{1}{12} f_0^1 - \frac{11}{720} f_0^3 - \dots \quad f_0 + \text{Red},$$

which yields in the present case

$${}^{I}f_0 = 101.082\,68.$$

The details of the calculation are given in tables B and C in the following way.

1. Using the extrapolated value Red = + 335, we obtain $v_2 = 103^{\circ}.12439$.

2. Using this value for v_2, we calculate r_2 by means of formula (2), noting that in the present case p = 1.147088.
3. The obtained value r_2 allows us to find the final value f_2 = 0.97687. This shows that the value f_3 = 0.99766 used in the integration may not be changed.
4. We find r_3 by extrapolation and then evaluate f_3.
5. We write in table B the corresponding value $f_{5/2}$ = 0.95740.

Taking the extrapolated correction Red = + 314, we calculate v_3.

In conclusion, we note that the accepted accuracy in calculating r does not provide us with a fifth decimal place number for the true anomaly. For this purpose, it is necessary to find a value of r consisting of seven significant figures. However, such an accuracy is never expected to be necessary.

The Second method

First of all, we integrate equation (5) using the following initial values

$$t_0 = 65.54100, \qquad \sigma_0 = 1.2103123.$$

Since σ varies very slowly, the interval w can be increased and taken as $w = 4^d$. We can either calculate ξ and η, or find r and v by using the following formulae

$$r = q(1 + e\sigma^2), \qquad \mathrm{tg}\,\frac{v}{2} = \sigma\sqrt{\frac{1-e}{1-e\sigma^2}}.$$

We can also evaluate r by using the first of these formulae and then find v by integrating equation (1). This requires an evaluation of a quadrature (Sec. 56), since the right-hand side of this equation will be already known for all the moments under consideration.

The third method

Using the initial values

$$t_0 = \frac{v}{2} = 63\,511\,00, \quad r = 1.378\,762, \quad \frac{dr}{dt} = 0.015\,305\,62$$

we integrate equation (3) taking $w = 4^d$. The true anomaly may then be found similarly to the previous method.

64. Calculation of the ephemeride by means of the numerical integration of the equations of motion:

The rectangular coordinates of the luminary, x, y and z_1 required for the calculation of the ephemeride may be obtained by the numerical integration of the following equations

$$\frac{d^2x}{dt^2} + \frac{k^2x}{r^3} = 0, \quad \frac{d^2y}{dt^2} + \frac{k^2y}{r^3} = 0, \quad \frac{d^2z}{dt^2} + \frac{k^2z}{r^3} = 0. \qquad (4)$$

We start integrating either by calculating the coordinates

$$(x_{-1}, y_{-1}, z_{-1}), \quad (x_0, y_0, z_0)$$

for moments: $t_o - w$ and t_o, or by calculating the quantities

$$(x_0, y_0, z_0), \quad (x_0', y_0', z_0'),$$

defining the position and velocity of the luminary in the moment t_o. We can apply for this purpose the conventional Kepler equations. For example, if the motion proceeds by an allipse of moderate eccentricity, we may then apply the following formulae (Vol. 1):

$$\left.\begin{aligned} x &= aP_x(\cos E - e) + bQ_x \sin E \\ y &= aP_y(\cos E - e) + bQ_y \sin E \\ z &= aP_z(\cos E - e) + bQ_z \sin E \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} x' &= \frac{k}{r\sqrt{a}}(-aP_x \sin E + bQ_x \cos E) \\ y' &= \frac{k}{r\sqrt{a}}(-aP_y \sin E + bQ_y \cos E) \\ z' &= \frac{k}{r\sqrt{a}}(-aP_z \sin E + bQ_z \cos E) \end{aligned}\right\} \quad (7')$$

where the eccentric anomaly E is defined by the condition

$$E - e \sin E = M.$$

The following equations may be applied to control the calculation

$$r = \sqrt{x^2 + y^2 + z^2} = a(1 - e \cos E),$$
$$ke\sqrt{a} \sin E = xx' + yy' + zz'.$$

The unperturbed ephemeride is usually calculated for a short time interval. The initial moment t_o is chosen in the middle of this interval. Under these conditions, the method of quadratures has no advantages ove Cowell's method since the number of integrals involved in the integration is not large. Both methods can equally be recommended.

For the integration of equations (6), we have to calculate the functions

$$f = -w^2k^2r^{-3}x, \quad -w^2k^2r^{-3}y, \quad -w^2k^2r^{-3}z.$$

It is easier to calculate this using a table which gives the values of $w^2k^2r^{-3}$ that correspond to different values of the argument $r^2 = x^2 + y^2 + z^2$. Such a table is given in volume I. A more detailed table has been given by Comrie (Comrie, Planetary Co-ordinates for the years 1800-1940, London 1933). On using this table, one should take into consideration that

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for [illegible]	2[illegible]	[illegible]	11 [illegible]
	14[illegible]		17 [illegible]
	[illegible] 8[illegible]		189 [illegible]

The evaluation of the heliocentric ephemeride of a planet by means of this method has already been given in volume I.

We note that it is more useful to calculate by means of formulae (7) several position of the planet, rather than to calculate one or two positions that are necessarily sufficient for solving system (6). This will simplify the integration and will also render possible a good control.

65. Other Methods for Calculating the Ephemeride by Numerical Integration

Let us assume the coordinates, x_o , y_o and z_o, and velocity components x'_o , y'_o and z'_o , of a luminary are known for some moment t_o. It was shown in volume I that the coordinates x, y and z of this planet at any particular time could be expressed in terms of the above mentioned initial values by the following equations

$$
\begin{aligned}
x &= Fx_0 + Gx'_0\\
y &= Fy_0 + Gy'_0\\
z &= Fz_0 + Gz'_0
\end{aligned}
\qquad (8)
$$

The functions F and G are given by the following series-expansions

$$
\begin{aligned}
F &= 1 - \frac{1}{2}\theta^2 r_0^{-3} + \frac{1}{2}\theta^3 r_0^{-5} r'_0 - \dots\\
G &= \theta - \frac{1}{6}\theta^3 r_0^{-3} + \frac{1}{4}\theta^4 r_0^{-5} r'_0 - \dots
\end{aligned}
\qquad (9)
$$

where $\theta = k(t - t_o)$, and r_o and r'_o are given by

$$r_0^2 = x_0^2 + y_0^2 + z_0^2, \qquad r_0 r'_0 = x_0 x'_0 + y_0 y'_0 + z_0 z'_0.$$

The dashes here denote, the derivatives with respect to $\theta = k\,(t - t_o)$, so that

$$x' = \frac{1}{k}\frac{dx}{dt}, \qquad y' = \frac{1}{k}\frac{dy}{dt}, \qquad z' = \frac{1}{k}\frac{dz}{dt}, \qquad r' = \frac{1}{k}\frac{dr}{dt}.$$

The evaluation of the heliocentric coordinates x, y and z required to obtain the ephemeride can be made by means of formulae (8). The functions F and G can be easily calculated by means of the series (9) for the near moments t. For further points, it is more convenient to apply the numerical integration than to use the final expressions of these functions, which has been given in volume I. By differentiating equations (8) twice, and noting that

$$\frac{d^2x}{dt^2} = -k^2 x r^{-3}, \qquad \frac{d^2y}{dt^2} = -k^2 y r^{-3}, \qquad \frac{d^2z}{dt^2} = -k^2 z r^{-3},$$

we obtain

$$\frac{d^2F}{dt^2} = -k^2 r^{-3} F, \qquad \frac{d^2G}{dt^2} = -k^2 r^{-3} G, \tag{10}$$

where

$$r^2 = r_0^2 F^2 + r_0'^2 G^2 + 2 r_0 r_0' FG. \tag{11}$$

The system of equations (10), in which r is defined by equation (11), is easily integrated by means of the methods described in the previous chapter.

Equations (9) show that

$$F_0 = 1, \quad \left(\frac{dF}{dt}\right)_0 = 0, \quad G_0 = 0, \quad \left(\frac{dG}{dt}\right)_0 = k;$$

It is thus convenient to define the integral curve by the initial position and the initial velocity.

There is another method for calculating the rectangular heliocentric coordinates. It consists in taking formulae

$$x = n_1 x_1 + n_2 x_2, \qquad y = n_1 y_1 + n_2 y_2, \qquad z = n_1 z_1 + n_2 z_2, \qquad (12)$$

which express the condition that the three positions of a luminary are located in one plane which passes through the centre of sun, and considering that in these formulae (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of the luminary in the moments t_1 and t_2, and (x, y, z) the unknown coordinates corresponding to moment t. In this case, the quantities n_1 and n_2 are considered as functions of time that satisfy the following equations(1):

$$\frac{d^2 n_1}{dt^2} = -k^2 n_1 r^{-3}, \qquad \frac{d^2 n_2}{dt^2} = -k^2 n_2 r^{-3}, \qquad (13)$$

where

$$r^2 = r_1^2 n_1^2 + r_2^2 n_2^2 + 2 r_1 r_2 n_1 n_2 \ . \qquad (14)$$

The initial values of the functions $n_1(t)$ and $n_2(t)$ are suitably taken as

$$n_1(t_1) = 1, \qquad n_1(t_2) = 0,$$
$$n_2(t_1) = 0, \qquad n_2(t_2) = 1.$$

The integration of systems (10) or (13) is simpler than the direct integration of the equations of motions which have been considered in the previous section. Each of these systems involves only two equations and not three. However, this simplification is considerably mullified

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(1) Banachiewioz indicated the existance of these equations (T. Banachiewicz, Sur quelques points fundamentaux de la theorie des orbites, Acta astronomica, Ser., a, 3, 1933).

by the difficulty of equations (11) and (14) and by the indispensible returns to equations (8) or (12) in order to obtain the unknown coordinates.

The moments t_1 and t_2 are suitably chosen not far from the middle of the ephemeride.

CHAPTER X

CALCULATION OF THE PERTURBATIONS IN THE ELEMENTS

66. General Considerations:

The differential equations that determine the osculating elements in the general case of an arbitrary perturbing acceleration have already been given in Sec. 12. For the timebeing, we shall abandon the orbits with eccentricities near unity. We introduce, instead of the eccentricity, angle φ = arc sin e. Equations (37) and (38) of Sec. 12 will then give

$$\left.\begin{aligned}
\frac{di}{dt} &= r\cos u\, W' \\
\frac{d\Omega}{dt} &= r\sin u \operatorname{cosec} i\, W' \\
\frac{d\varphi}{dt} &= a\cos\varphi\sin v\, S' + a\cos\varphi(\cos v + \cos E)\, T' \\
\frac{d\pi}{dt} &= -p\operatorname{cosec}\varphi\cos v\, S' + \operatorname{cosec}\varphi\,(r+p)\sin v\, T' + \operatorname{tg}\frac{i}{2}\, r\sin u\, W' \\
\frac{dn}{dt} &= -\frac{3k}{\sqrt{a}}\sin\varphi\sin v\, S' - \frac{3k}{\sqrt{a}}\frac{p}{r}\, T' \\
\frac{d\epsilon}{dt} &= -\left(2\cos\varphi\, r - p\operatorname{tg}\frac{\varphi}{2}\cos v\right) S' + \operatorname{tg}\frac{\varphi}{2}(r+p)\sin v\, T' + \\
&\qquad + \operatorname{tg}\frac{i}{2}\, r\sin u\, W'.
\end{aligned}\right\} \quad (1)$$

These equations form a system of coupled first-order equations. They can be numerically integrated by any of the methods considered in Chapter VI. In astronomical calculations, the method of quadratures, considered in detail in Secs. 49-51, is usually applied. In the present case, the application of the method of quadratures is particularly easy. In the right-hand sides of equations (1), the coefficients of the perturbing masses are very small. Therefore, they can be sufficiently accurately evaluated by means of the approximate values of the unknown elements i, Ω, φ, π, n and ϵ.

We denote by t_o the moment for which the values of the osculating elements i_o, Ω_o, φ_o, π_o, n_o, ϵ_o and ϵ_o are known, or in other words, the epoch of osculation. Let us assume that

$$i = i_0 + \Delta i, \qquad \Omega = \Omega_0 + \Delta\Omega, \ldots$$

In this case, quantities $\Delta i, \Delta\Omega$, ... will be defined by the differential equations

$$\left.\begin{array}{l} \frac{d\Delta i}{dt} = \text{[illegible]} \\ \ldots\ldots\ldots \end{array}\right\} \qquad (1')$$

as well as the initial conditions $\Delta i = 0, \Delta\Omega = 0, \ldots$ for $t = \tau_o$

We choose a definite interval of integration, say w, and denote the epoch of osculation t_o by $t_o - \frac{w}{2}$. We try to find the values $\Delta i, \Delta\Omega$, ..., and, consequently, the osculating elements for the moments

$$t_k = t_0 + kw,$$

where k is an integer.

Since, on one hand, we have to evaluate the right-hand side of equation (1') multiplied by w in the course of integration of these equations, and on the other hand, it is preferbale to obtain the quantities $\Delta i, \Delta\Omega, \Delta\varphi$, ... we shall put

$$
\left.\begin{aligned}
\delta i &= r \cos u\, W \\
\delta\Omega &= r \sin u \operatorname{cosec} i\, W \\
\delta\varphi &= a \cos\varphi \sin v\, S + a \cos\varphi (\cos v + \cos E)\, T \\
\delta\pi &= -p \operatorname{cosec}\varphi \cos v\, S + \operatorname{cosec}\varphi (r+p) \sin v\, T + \\
&\qquad + \operatorname{tg}\frac{i}{2}\, r \sin u\, W \\
w\,\delta n &= -\frac{3kw}{\sqrt{a}} \sin\varphi \sin v\, S - \frac{3kw}{\sqrt{a}} \frac{p}{r}\, T \\
\delta\varepsilon &= -\left(2\cos\varphi\, r - p \operatorname{tg}\frac{\varphi}{2} \cos v\right) S + \operatorname{tg}\frac{\varphi}{2} (r+p) \sin v\, T + \\
&\qquad + \operatorname{tg}\frac{i}{2}\, r \sin u\, W.
\end{aligned}\right\} \qquad (2)
$$

Taking the values of S', T' and W', obtained in Sec. 11, into consideration, we obtain

$$
\begin{aligned}
S &= \frac{w}{\text{arc } 1''} S' = \frac{w}{k\sqrt{p}\,\text{arc } 1''} \mathrm{S}, \\
T &= \frac{w}{k\sqrt{p}\,\text{arc } 1''} \mathrm{T}, \quad W = \frac{w}{k\sqrt{p}\,\text{arc } 1''} \mathrm{W},
\end{aligned} \qquad (3)
$$

where k is the Gaussian constant expressed in seconds of arc.

We shall now apply the method of quadratures, which has been considered in detail in Sec. 50. We shaol use, in particular, equations (17 bis) and (18) of that section. Taking for f each of the functions δi, $\delta\Omega$, ..., we obtain the following formulae

$$
\left.\begin{aligned}
x_n &= f_n^{-1} - \frac{1}{12} f'_n + \frac{11}{720} f'''_n - \frac{191}{60480} f^{V}_n + \ldots \\
f^{-1}_{-\frac{1}{2}} &= -\frac{1}{24} f'_{-\frac{1}{2}} + \frac{17}{5760} f'''_{-\frac{1}{2}} - \frac{367}{967680} f^{V}_{-\frac{1}{2}} + \ldots
\end{aligned}\right\} \qquad (4)
$$

where x_u denotes the corresponding values of $\Delta i, \Delta\Omega$, ... for the moment $t = t_o + nw$.

The average longitude is calculated by the following equation (sec. 12).

$$
\ldots \int n\,dt.
$$

where its unperturbed value equals $\epsilon_0 + n_0 (t - \tau_0)$. The perturbation of the average longitude will then be given by

$$\Delta\lambda = \lambda - [\epsilon_0 + n_0(t - \tau_0)] = \Delta\epsilon + \int_{\tau_0}^{t} (n - n_0)\, dt.$$

Consequently, the calculation of $\Delta\lambda$ is reduced to the evaluation of the quantity

$$\Delta'\lambda = \int_{\tau_0}^{t} (n - n_0)\, dt,$$

which satisfies the differential equation

$$\frac{d^2\Delta'\lambda}{dt^2} = \frac{dn}{dt} \qquad (5)$$

as well as the initial conditions

$$\Delta'\lambda = 0, \quad \frac{d\Delta'\lambda}{dt} = 0$$

for $t = \tau_0$

The right-hand side of equation (5) is defined by the second to last of equations (1). In order to integrate this equation, we apply formulae (37 bis), (36) and (39) of Chapter VI. This yields the the following formulae

$$\left.\begin{aligned} x_n &= f_n^{-2} + \frac{1}{12} f_n - \frac{1}{240} f_n^2 + \frac{31}{60480} f_n^4 - \dots \\ f^{-1}_{\frac{1}{2}} &= -\frac{1}{24} f^1_{\frac{1}{2}} + \frac{17}{5760} f^3_{\frac{1}{2}} - \frac{367}{967680} f^5_{\frac{1}{2}} + \dots \\ f_0^{-2} &= +\frac{1}{24} f_{-1} - \frac{17}{5760} (2f^2_{-1} + f^2_0) + \frac{367}{967680} (3f^4_{-1} + 2f^4_0) - \dots \end{aligned}\right\} (6)$$

where x_n denotes the quantity $\Delta'\lambda$ for the moment $t = t_0 + nw$. After obtaining the value of $\Delta'\lambda$, we calculate the perturbation of the average longitude λ for the moment t using the following formula

$$\lambda = \varepsilon_0 + n_0(t - \tau_0) + \Delta\varepsilon + \Delta'\lambda. \qquad (7)$$

Once we obtain the perturbed values

$$i = i_0 + \Delta i, \qquad \Omega = \Omega_0 + \Delta\Omega, \ldots$$

of all the elements for this moent, we can calculate the position of the luminary by the conventional formulae of the elliptic motion:

$$\left.\begin{aligned} &E - e\sin E = \lambda - \pi \\ &r\sin v = a\cos\varphi\sin E \\ &r\cos v = a(\cos E - \sin\varphi), \end{aligned}\right\} \qquad (8)$$

where the semimajor axis is obtained by means of the following relation

$$a = \left(\frac{k}{n}\right)^{\frac{2}{3}}, \qquad (9)$$

in which

$$\lg k'' = 3.550\,0066,$$

if n is expressed in seconds of arc, and

$$\lg k^2 = 9.993\,7041,$$

if n is expressed in fractions of a second.

The calculation of the rectangular equatorial coordinates, required when the perturbed position of the luminary is to be compared with the experimental value, is done by the following formulae:

$$\left.\begin{aligned} x &= r\sin a\sin(A' + v) \\ y &= r\sin b\sin(B' + v) \\ z &= r\sin c\sin(C' + v), \end{aligned}\right\} \qquad (10)$$

where the Gaussian constants a, b and c as well as the quantities

$$A' = A + \omega, \qquad B' = B + \omega, \qquad C' = C + \omega$$

are defined by

$$\left.\begin{aligned}
\sin a \sin A &= \cos \Omega \\
\sin a \cos A &= -\cos i \sin \Omega \\
\sin b \sin B &= \sin \Omega \cos \varepsilon \\
\sin b \cos B &= \cos i \cos \Omega \cos \varepsilon - \sin i \sin \varepsilon \\
\sin c \sin C &= \sin \Omega \sin \varepsilon \\
\sin c \cos C &= \cos i \cos \Omega \sin \varepsilon + \sin i \cos \varepsilon \,,
\end{aligned}\right| \qquad (11)$$

where ϵ denotes the slope of the ecliptic with respect to the equator.

If we thus disregard the simple integrations, we see that the calculation of the perturbed values of the elements leads to the calculation of the functions (2), which depend on the components of the perturbing acceleration, S, T and W as well as the coordinates r and v of the perturbing body. We have already considered the calculation of the latter quantities, which can be done by means of formulae (7) and (8). We still have to find the most convenient method for the calculation of the quantities S, T and W, which depend on the components of the perturbing acceleration.

67. Calculation of the components of the perturbing acceleration

Let us consider the following auxiliary coordinate system. We take the axis ξ along the radius vector of the perturbed body in the direction of increase of r, the axis η along the perpendicular to the radius vector in the orbital plane in the direction of increase of the true anomaly, and the axis ζ along the normal to the orbital plane so that the coordinate system becomes right-handed. We denote by S, T, and W the components of the perturbed acceleration in the directions ξ, η and ζ. We can calculate these quantities using different approaches. The most widely used approach is the following.

Let us denote by m_1 and ξ_1, η_1 and ζ_1 the mass and coordinates of one of the perturbing planets. Since the coordinates of the perturbed planet are evidently equal to (r, 0, 0), then the general expression for the component of perturbing acceleration along any axis, namely (sec. 3),

$$k^2 m_1\left(\frac{\xi_1 - x}{\Delta_1^3} - \frac{\xi_1}{r_1^3}\right),$$

yields in the present case

$$S_1 = k^2 m_1\left(\frac{\xi_1 - r}{\Delta_1^3} - \frac{\xi_1}{r_1^3}\right)$$

$$T_1 = k^2 m_1\left(\frac{\eta_1}{\Delta_1^3} - \frac{\eta_1}{r_1^3}\right)$$

$$W_1 = k^2 m_1\left(\frac{\zeta_1}{\Delta_1^3} - \frac{\zeta_1}{r_1^3}\right),$$

where

$$\Delta_1^2 = (\xi_1 - r)^2 + \eta_1^2 + \zeta_1^2,$$

and r_1 is the radius vector of the perturbing planet. Taking equation (3) into consideration, and putting

$$K_1 = \frac{\mu k^2 m_1}{\sqrt{p}}\left(\frac{1}{\Delta_1^3} - \frac{1}{r_1^3}\right), \tag{12}$$

we obtain

$$S_1 = K_1 \xi_1 - \frac{\mu k^2 m_1}{\sqrt{p}}\frac{r}{\Delta_1^3}, \quad T_1 = K_1 \eta_1, \quad W_1 = K_1 \zeta_1. \tag{13}$$

If we consider perturbations caused by other planets having masses m_2, ... and coordinates (ξ_2, η_2, ζ_2), ..., we can then calculate by formulae, similar to equations (12) and (13), the corresponding components (S_2, T_2, W_2), ... and then adopt in equations (2) that

$$S = S_1 + S_2 + \ldots$$

$$T = T_1 + T_2 + \ldots$$

$$W = W_1 + W_2 + \ldots$$

We still have to show how the coordinates of the perturbing planets, (ξ_1, η_1, ζ_1) can be found. We can find in the astronomical annual the ecliptic heliocentric coordinates, namely the radius vector r_1 , longitude ℓ_1 and latitude b_1, for each of the large planets. These quantities

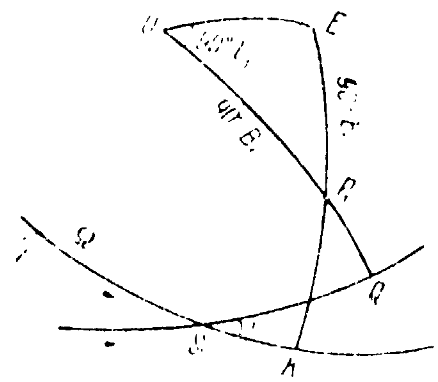

Figure 10

are given with high accuracy in Comrie's table for most of the interesting cases.

Let us denote by L_1 and B_1 the longitude and latitude of the perturbing planet relative to the orbital plane of the perturbed planet. We shall consider that the longitude L_1 is calculated from the ascending node Ω of this orbit (Fig. 10).

In the spherical triangle formed by the position of planet P_1, the pole of the ecliptic E, and the pole of the orbit O, the angles at the apexes O and E are respectively equal to 90-L, and 90- ℓ_1 -Ω. Hence, in the evaluation of L_1 and B_1, we may apply the following relations

Denoting by u the argument of latitude of the perturbed planet, the quantities L_1 - u and B_1 will be the spherical coordinates of the perturbing planet that correspond to the coordinate system ξ, η, ζ .

Therefore

$$\begin{aligned} \xi_1 &= r_1 \cos B_1 \cos (L_1 - u) \\ \eta_1 &= r_1 \cos B_1 \sin (L_1 - u) \\ \zeta_1 &= r_1 \sin B_1 \end{aligned} \qquad (15)$$

Formulae (14) and (15) constitute the solution of the problem of calculating the coordinates ξ_1, η_1 and ζ_1 in terms of the given quantities r_1, ℓ_1 and b_1.

At an earlier date when calculating machines were not widely used, the separation distance between the planets, Δ_1, was calculated not by means of formula

$$\Delta_1^2 = (\xi_1 - r)^2 + \eta_1^2 + \zeta_1^2 = r_1^2 + r^2 - 2r\xi_1,$$

but by means of the following equations

$$\begin{aligned} \Delta_1 \cos q_1 \cos Q_1 &= \xi_1 - r \\ \Delta_1 \cos q_1 \sin Q_1 &= \eta_1 \\ \Delta_1 \sin q_1 &= \zeta_1, \end{aligned}$$

wher[illegible] are auxiliary quantities, unnecessary to calculate further.

68. Another Method for Calculating the Components of the Perturbing Acceleration

We shall here consider the case, when we need to calculate the perturbations that occur after a few rotations of the luminary, provided that these perturbations are small. In this case, we can calculate in another way the coordinates of the perturbing planet, which we shall call Jupiter.

We shall assume that the motion of Jupiter and the motion of the perturbed luminary proceed in the invariable planes defined by the elements i_1, Ω_1 and i_2, Ω_2. We first consider the spherical triangle $\Omega \Omega_1$ N, (Fig. 11), formed by the ascending nodes Ω and Ω_1 of the orbit under consideration relative to the ecliptic, and the ascending node N of the orbit of Jupiter relative to the orbit of the perturbed body. In order to evaluate the angle J between the orbit and the arcs Ω N and Ω_1 N_1 which will be denoted by N and N_1, we shall apply the the following formulae, which can easily be obtained from the study of the triangle under consideration:

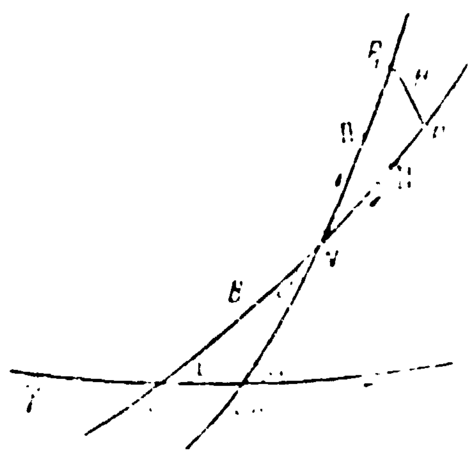

Figure 11

Let us now consider a new coordinate system ξ', η', ζ', which differs from the previous system ξ, η, ζ in that the axis ξ' is directed towards the orbital point B, removed from the node by an angle Ω B $= \beta$, and the axis η' toward a point, removed from the node an angle $\beta + 90^{\circ}$. The coordinates of Jupiter in the new coordinate system can be calculated by means of the conventional formulae (formulae (8) in Sec. 10), assuming in these formulae that the longitude of the node equals to

$$BN = N - \beta,$$

and the argument of the latitude equals to

$$NP_1 = \lambda_1 - N_1 - \Omega_1,$$

where λ_1 is the longitude of Jupiter in the orbit. This yields

$$\begin{aligned}
\xi_1' &= r_1 \{\cos(\lambda_1 - N_1 - \Omega_1)\cos(N - \beta) - \sin(\lambda_1 - N_1 - \Omega_1)\sin(N - \beta)\cos J\} \\
\eta_1' &= r_1 \{\cos(\lambda_1 - N_1 - \Omega_1)\sin(N - \beta) + \sin(\lambda_1 - N_1 - \Omega_1)\cos(N - \beta)\cos J\} \\
\zeta_1' &= r_1 \sin(\lambda_1 - N_1 - \Omega_1)\sin J.
\end{aligned}$$

In analogy with the Gausian constants, we introduce the following quantities

$$\left.\begin{aligned}
A \sin A' &= \cos(N - \beta) \\
A \cos A' &= -\sin(N - \beta)\cos J \\
A' &= A'' - N_1 - \Omega_1
\end{aligned}\;\right|\;
\begin{aligned}
B \sin B' &= \sin(N - \beta) \\
B \cos B' &= \cos(N - \beta)\cos J \\
B' &= B'' - N_1 - \Omega_1
\end{aligned}\;\left|\; \begin{aligned} C &= \sin J, \quad C' = -N_1 - \Omega_1 . \end{aligned}\right. \tag{17}$$

We then finally obtain

$$\left.\begin{aligned}
\xi_1' &= r_1 A \sin(A' + \lambda_1) \\
\eta_1' &= r_1 B \sin(B' + \lambda_1) \\
\zeta_1' &= r_1 C \sin(C' + \lambda_1)
\end{aligned}\right| \tag{18}$$

In order to obtain the required unknown coordinates ξ and η we rotate the coordinate system around the axis ζ. This yields

$$\left.\begin{aligned}
\xi_1 &= \xi_1' \cos(u - \beta) + \eta_1' \sin(u - \beta) \\
\eta_1 &= -\xi_1' \sin(u - \beta) + \eta_1' \cos(u - \beta)
\end{aligned}\right| \tag{19}$$

We can now simplify the calculations performed by means of equations (18), (19) by an appropriate choice of the arbitrary angle β. The most simple formulae[1] are obtained in the following cases

Since, in the second case, angle β = u depends on the position of the perturbed luminary, constants A, A', ... will then also depend on the coordinates of the perturbed luminary. The corresponding formulae are suitably applied only when it is required to calculate the functions (2), subject to integration, of only a few orbital points (cf. Sec. 69).

The application of formulae (18) and (19) rather than the conventional formulae (14) and (15) is useful only if the elements i, Ω, i_1, Ω_1 are approximately constant during a considerable interval of time, and, moreover, when the longitudes λ_1, λ_2, ... of the perturbing planets in the orbit are known.

The influence of the small variations di, dΩ, ... on the constants A, A', ... can evidently be evaluated by differentiation. We shall not consider here the derivation of the corresponding formulae.

69. The tabulation of thecoefficients

The calculation of the expressions given by equations (2) is reduced to the evaluation of the quantities S, T and W, which have been considered in detail in the previous two sections, as well as the coefficients

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(1) These formulae were given by Merton for the case = u:
G. Merton, The periodic comet Grigg (1902 II) = Skjellerup (1922 I) (1902 to 1927), Memoirs of the R. Astr. Society; 64, Part III, 1927.
The formulae that correspond to the case = are given in:
N.I. Idel'son, La comete d'Enke en 1924-1934, Proceedings of the Principle astronomical observatory in Polkov (Izvestija Glavnoj astronomiceskoj observatopii v Pulkove) vol. XV, I, 1935.

obtained by multiplying these quantities. The calculation of these coefficients is reduced to the evaluation of the following quantities

$$
\left.\begin{aligned}
A &= \frac{r}{a}\sin v, & B &= \frac{r}{a}\cos v, \\
C &= \cos\varphi \sin v, & D &= \cos\varphi\,(\cos v - [illegible] \cos E), \\
E &= \operatorname{tg}\varphi \cos\varphi \cos v, & F &= [illegible]\left(\frac{r}{a} [illegible]\right)[illegible], \\
G &= \sin\varphi \sin v, & H &= 2\frac{r}{a}\cos [illegible] - \cos\varphi \operatorname{tg} [illegible] \cos v
\end{aligned}\right\} \qquad (20)
$$

which are functions of only φ and M. Here, we have singled out the multipliers which depend on a, i and w, since their evaluation is simple enough.

The calculation will be significantly simplified by constructing tables for these coefficients. In order to reduce the volume of these table, we shall transform equations (1) by introducing a new independent variable M, instead of the variable t, using the relation

$$[illegible]$$

In this case, it is possible to calculate the quantities (20) for a few round values of M. Instead of tables of two arguments, M and we shall then have a table of a single argument, φ. Such tables have been constructed by Crommelin[1] for values of the argument e = sin φ varying from 0.37 to 0.84 by intervals of 0 01, i.e. corresponding to the orbits of short-periodic comets. The first table gives the values of the coefficients, which slightly differ in form A, B, ..., for M = 0°, 7.5°, 15°, 22.5°, ... with five figures. The second table

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(1) A.C.D. Crommelin, Tables for facilitating the computation of the perturbations of periodic comets by the planets, Memoirs of the R. Astr. Society, 64, Part V, 1929.

gives the logarithms of these coefficients with four figures for $M = 0, 1^{\circ}, 2^{\circ}, \ldots, 25^{\circ}, 26^{\circ}$. The first table is devoted to the calculation of perturbations caused by Jupiter and Saturn, and the second to the calculation of perturbations caused by the four internal planets, which have an appreciable influence only when the planet passes near the perihelion.

In some cases, it is more useful to choose the excentricity, rather than the mean anomaly, as an independent variable. In these cases, equations (1) will be transformed by means of the relation

$$\frac{d}{dE} = \frac{r\sqrt{a}}{k}\,\frac{d}{dt}.$$

The points that correspond to equidistant values of E, are located on the orbit more uniformly than the points that correspond to equidistant values of M. This is particularly perceptible for large values of the eccentricity. Indeed, the series-expansion, considered in Ch. XII, indicates that

$$v = M + 2e\sin M + \ldots ,$$
$$v = E + e\sin E + \ldots ,$$

if we only keep the first power of the eccentricity. Hence, equidistant values of E give "more equidistant" values of v than do equidistant values of M. This can be shown in a more convincing way by the following table, which gives the values of the three anomalies for the case e = 0.85, i.e. for a round value of the eccentricity of Enke's comet:

v	0	20	40	60	80	100	120	140	160	180
E	0.0	5.7	11.8	18.7	26.9	37.5	52.5	76.1	116.5	180.0
M	0.0	0.9	1.8	3.1	4.9	7.9	13.9	28.8	72.9	180.0

We obtain the most homogeneous distribution of the positions of the perturbed luminary in the orbit by choosing the true anomaly as the independent variable by which we carry out the integration. This is easily done by using the relation

$$\frac{d}{dv} = \frac{\sec\varphi}{n}\left(\frac{r}{a}\right)\frac{d}{dt}$$

We note that the choice of the eccentricity or the true anomaly as an independent variable considerably complicates the calculation of the coordinates of the perturbing planet.

70. <u>Comparison between the formulae</u>

As a rule, the perturbations of the elements are calculated to within 0".0001 in the average daily motion and to within 0".001 for all the other elements. In order to obtain such an accuracy, we take the case of a small planet with an interval $w = 40^d$ and perform the calculation to five decimal places. In the case of a comet, it is advisable to change the interval w, depending on whether the comet is near or far from the perturbing planet and also on its distance to the perihelion.

One should pay attention that the elements of the perturbed planets as well as the coordinates evaluated by these elements should be related to the same equator and equinox, as the coordinates of the perturbing planet.

In order to avoid unnecessary interpolations, we have to choose the moments $t_o + kw$ such, that the coordinates of the perturbing planet are known for these moments.

Let us consider the osculating system of elements a, e, i, ... for the epoch τ_o. Choosing the interval w, we define the initial moment t_o by the relation

$$t_0 - \frac{w}{2} = \tau_0$$

We then calculate r, v, u, p for the moments $t_o - 2w$, $t_o - w$, t_o and $t_o + w$ using the given (i.e. unperturbed) elements by means of the formulae

$$\left.\begin{aligned} E - e\sin E &= M \\ r\sin v &= a\cos\varphi\sin E \\ r\cos v &= a(\cos E - \sin\varphi) \\ u = v + \omega, \qquad & p = a\cos^2\varphi. \end{aligned}\right. \tag{I}$$

We then take from the astronomical annual (or from Comrie's tables) the values of the coordinates (r_1, ℓ_1, b_1), (r_2, ℓ_2, b_2), ... of the perturbing planets and calculate the corresponding orbital elements using the following relations:

$$\left.\begin{aligned} \cos B_1\cos L_1 &= \cos(l_1 - \Omega)\cos b_1 \\ \cos B_1\sin L_1 &= \sin i\sin b_1 + \cos i\cos b_1\sin(l_1 - \Omega) \\ \sin B_1 &= \cos i\sin b_1 - \sin i\cos b_1\sin(l_1 - \Omega) \end{aligned}\right| \tag{II}$$

$$\left.\begin{aligned} \xi_1 &= r_1\cos B_1\cos(L_1 - u) \\ \eta_1 &= r_1\cos B_1\sin(L_1 - u) \\ \zeta_1 &= r_1\sin B_1. \end{aligned}\right| \tag{III}$$

We evaluate the distance between the unperturbed body and the perturbing planet under consideration by

$$\begin{aligned} \Delta_1^2 &= (\xi_1 - r)^2 + \eta_1^2 + \zeta_1^2 \\ &= r_1^2 + r^2 - 2r\xi_1, \end{aligned} \tag{IV}$$

We calculate the components of the perturbing acceleration by using the following relations:

$$\left.\begin{aligned} K_1 &= \frac{wk^2m_1}{\sqrt{p}}\left(\frac{1}{\Delta_1^3} - \frac{1}{r_1^3}\right) \\ S_1 = K_1\xi_1 - \frac{wk^2m_1}{\sqrt{p}}\frac{r}{\Delta_1^3}, &\qquad T_1 = K_1\eta_1, \qquad W_1 = K_1\zeta_1 \end{aligned}\right| \tag{V}$$

where the values of wk"m, are given in table IV at the end of this book. We then sum the components of acceleration caused by the action of different planets, and obtain

[illegible] (V...)

We calculate the functions which are subject to integration by means of the following formulae:

$$\delta \pi_{III} = \operatorname{tg}\frac{i}{2}\, r \sin u\, W \quad [\text{partly illegible}]$$

$$\delta \varepsilon_{I} = -\left(2\cos r - p \operatorname{tg}\frac{i}{2}\cos u\right) S$$

$$\delta \varepsilon_{II} = \operatorname{tg}\frac{i}{2}\,(r + p)\sin u\, T \quad [\text{partly illegible}]$$

$$\delta \varepsilon_{III} = \operatorname{tg}\frac{i}{2}\, r \sin u\, W$$

$$w\delta\eta_{I} = -\frac{3kw}{\sqrt{a}}\sin \varphi \sin v\, S \qquad w\delta\eta_{II} = -\frac{3kw}{\sqrt{a}}\frac{p}{r}\, T$$

$$w\delta\eta = w\delta\eta_{I} + w\delta\eta_{II}$$

Where $w = 20^{d}$, $\lg 3kw = 0.013733$,
$w = 40^{d}$, $\lg 3kw = 0.314763$.

We carry the integration of the first five elements using the following formulae:

$$\int \ldots - \frac{1}{24} f \ldots \frac{1}{330} f \ldots \frac{1}{2637} f \ldots + \cdots$$

$$\Sigma_n = f_n^{-1} - \frac{1}{12} f_n \ldots + \left(\frac{1}{65} - 0.00011\right) f_n \ldots \frac{1}{317} f_n \ldots \qquad \text{(VIII)}$$

We construct for w δ n a column of second sums, the initial value of which is chosen as

$$f_0'' = \frac{1}{24} f - \frac{1}{389}\left(2f' + f\right) + \frac{1}{2637}(3f' + 2f') - \dots$$

We carry out a first integration, which yields wn as well as a second integration which yields $\Delta\lambda$ by using the following formula

$$\lambda_n = {}^{II}f_n + \frac{1}{12} f_n - \frac{1}{240} f_n'' + \frac{1}{1951} f_n^{iv} \dots \qquad \text{(IX)}$$

In all these integrations, the unavailable differences are obtained by extrapolation.

After integration, we obtain the perturbations of the elements $\Delta i, \Delta\Omega, \Delta\varphi, \Delta\pi, \Delta\epsilon$, $W\Delta$ n and $\Delta\lambda$, for the four moments mentioned above. Adding these perturbations to the initial osculating elements, we obtain the perturbed elements for these moments. The perturbation of the semimajor axis can be found by the following differential relations

[illegible]

In order to obtain more accurate values for the quantities $\delta i, \delta\Omega, \dots$ we repeat all the above mentioned calculations starting with the perturbed elements and re-integrate For this purpose we use the same formulae, with only one small modification. We calculate the average anomaly using equations

[illegible]

in which ϵ_0 and n_0 are the initial values of the elements ϵ and a which correspond to the epoch τ_0. This calculation repetion is continued until the process is established. When interval w is appropriately chosen, the second approximation may be considered as final. One then continues calculating the perturbations for the next moments $t_o + 2w$, $t_o + 3w$, ... (or $t_o - 3w$, $t_o - 4w$, ..., depending on the direction that should be followed starting from the initial epoch). If the perturbations are large, and the final values of the differences of the functions δi,... are significantly different from the primary values, we then should once more repeat the perturbation calculations for the moments $t_o - 2w$, $t_o - w$, t_o, and $t_o + w$.

The continuation of the calculation is not difficult and since the osculating elements vary slowly, extrapolation gives such accurate values for the perturbations, that it is never necessary to repeat the calculation of $\delta i, \delta \Omega$, ..., provided that the interval w is appropriately chosen.

Calculations have to be done on several separate sheets. The values of r, v, ..., δi, ..., $w\,\delta n$ are obtained on the first sheet. The second sheet is subsediary to the first one. There, we calculate the coordinates of the perturbing planets, ξ_1, η_1, ζ_1, ..., ..., the corresponding components of acceleration S_1, T_1, ... and the quantities S, T and W. The computations that correspond to a given moment should be written in the same vertical column in both sheets. The integration of each of the functions $\delta i, \delta \Omega$, ... has to be done in a separate sheet according to the scheme indicated in chapter VIII.

71. Particular cases for calculation of perturbations of the elements of small planets

Two kinds of difficulties are encountered in the application of the methods considered in the previous sections to small planets.

1- If the slope of the orbit is very small, then the computation of the quantity

[illegible]

will be accompanied by a large decrease in accuracy, and leads to the inaccurate definition of the longitude of the node.

2. If the eccentricity of the orbit is small, then a similar difficulty occurs on calculating the longitude of the perihelion, which requires the computation of

[illegible]

In the first case, the easiest way is to change the basic plane so that the elements i, Ω, π and ϵ, related to the plane of the ecliptic, are transformed into elements i', Ω', π' and ϵ' (instead of the last two, one may take w' and Mo), related to the plane of the equator.

In the cases when only an approximate calculations is required with an accuracy not exceeding the first powers of the mass, it is better to introduce the auxiliary variables p and q instead of i and Ω as defined by

[illegible]

The perturbations of these elements are calculated by

[illegible]

which can easily be deduced from equation (2). The perturbations of i and Ω can be found up to within quantities of the first order by means of the relations

$$\Delta i = \cos\omega \cdot \Delta p - \sin\omega \cdot \Delta q$$
$$\sin i\,\Delta\Omega = \sin\omega\,\Delta p + \cos\omega \cdot \Delta q.$$

We now consider the second case, in which the eccentricity of the orbit is small. Instead of e and π , we introduce the new variables

$$h = e\sin\pi, \qquad l = e\cos\pi. \tag{21}$$

Since,

$$\frac{dh}{dt} = \frac{de}{dt}\sin\pi + l\frac{d\pi}{dt} = \frac{d\varphi}{dt}\cos\varphi\sin\pi + l\frac{d\pi}{dt},$$

$$\frac{dl}{dt} = \frac{de}{dt}\cos\pi - h\frac{d\pi}{dt} = \frac{d\varphi}{dt}\cos\varphi\cos\pi - h\frac{d\pi}{dt}.$$

we then easily see according to equations (2) that

$$\lambda\frac{dh}{dt} = -p\cos u\,S + (p+r)\sin u\,T + rhT + l\,\mathrm{tg}\frac{i}{2}\,r\sin u\,W,$$

$$\lambda\frac{dl}{dt} = p\sin u\,S + (p+r)\cos u\,T + rlT - h\,\mathrm{tg}\frac{i}{2}\,r\sin u\,W,$$

where

$$\lambda = [illegible]\ e.$$

The integration of these equations gives the perturbed values of h and . Equation (21) can then be used to calculate the corresponding values of e and .

72. <u>Some aspects on thecalculation of perturbations of the orbital elements of comets</u>

The calculation of the perturbations of short-periodic elements, for which the eccentricity is not so large that the use of the eccentric anomaly is impossible, is performed by means of the formulae given in Sec. 70. However, for a comet whose eccentricity is near unity, these formulae should be partially transformed in such a way, that instead of elements φ , n and ϵ the perturbation of the elements e, q and τ are obtained (the time of passing by the perihelion). We shall not consider here these transformations. After cowell's and Crommelin's work on

Halley's comet, the perturbations of comets of this type are not calculated for the elements but for the rectangular coordinates by the methods which will be considered in the following chapter.

The comet may pass so close to one of the large planets, that the gravitation of this planet becomes stronger than the gravitation of the sun. In this case, it is advisable, as Laplace (Mechanique Celeste, t.4, Livre IX, Chap. II) pointed out, to consider the planet as the central body and the sun as the perturbing one. Let us denote by x, y and z the heliocentric coordinates of the planet, and by m_1 its mass. We write the differential equations for the comet's motion under the influence of the sum's gravitation and planet:

$$\left.\begin{aligned}
\frac{d^2x}{dt^2} + k^2\frac{x}{r^3} &= k^2m_1\left(\frac{x_1-x}{\Delta^3} - \frac{x_1}{r_1^3}\right)\\
\frac{d^2y}{dt^2} + k^2\frac{y}{r^3} &= k^2m_1\left(\frac{y_1-y}{\Delta^3} - \frac{y_1}{r_1^3}\right)\\
\frac{d^2z}{dt^2} + k^2\frac{z}{r^3} &= k^2m_1\left(\frac{z_1-z}{\Delta^3} - \frac{z_1}{r_1^3}\right)
\end{aligned}\right\} \qquad (22)$$

and the equations of the unperturbed motion of the planet:

$$\left.\begin{aligned}
\frac{d^2x_1}{dt^2} + k^2(1+m_1)\frac{x_1}{r_1^3} &= 0\\
\frac{d^2y_1}{dt^2} + k^2(1+m_1)\frac{y_1}{r_1^3} &= 0\\
\frac{d^2z_1}{dt^2} + k^2(1+m_1)\frac{z_1}{r_1^3} &= 0.
\end{aligned}\right\} \qquad (23)$$

If we take the centre of the planet as the origin of a coordinate system, the axes of which are parallel to the axes of the heliocentric system, and denote the corresponding coordinates of the comet by ξ, η and ζ, we obtain

$$x = x_1 + \xi, \qquad y = y_1 + \eta, \qquad z = z_1 + \zeta.$$

Using equations (22) and (23), we obtain

$$\begin{aligned}
\frac{d^2\xi}{dt^2} &= -k^2 m_1 \frac{\xi}{\Delta^3} - k^2\left(\frac{x_1}{r_1^3} - \frac{x}{r^3}\right),\\
\frac{d^2\eta}{dt^2} &= -k^2 m_1 \frac{\eta}{\Delta^3} - k^2\left(\frac{y_1}{r_1^3} - \frac{y}{r^3}\right),\\
\frac{d^2\zeta}{dt^2} &= -k^2 m_1 \frac{\zeta}{\Delta^3} - k^2\left(\frac{z_1}{r_1^3} - \frac{z}{r^3}\right),
\end{aligned} \qquad (24)$$

where

$$\Delta^2 = \xi^2 + \eta^2 + \zeta^2.$$

When the sun is taken as the central body, the quantity R is the acceleration it imports to the comet, and F is the perturbing acceleration caused by the attraction of the planet. Equation (22) shows that

$$R = \frac{k^2}{r^2}, \qquad F = k^2 m_1 \left[\left(\frac{x_1 - x}{\Delta^3} - \frac{x_1}{r_1^3}\right)^2 + \ldots\right]^{\frac{1}{2}}$$

On the other hand, when the planet is taken as the central body, we denote the acceleration it imparts to the comet by R_1, and the perturbing acceleration caused by the sun by F_1. Then, it follows from equations (24) that

$$R_1 = \frac{k^2 m_1}{\Delta^2}, \qquad F_1 = k^2 \left[\left(\frac{x_1}{r_1^3} - \frac{x}{r^3}\right)^2 + \ldots\right]^{\frac{1}{2}}$$

The region in space, in which it is more useful to take the sun as the central body, is separated from that, in which it is more useful to consider the planet as the central body, by points satisfying the following equation

$$\frac{F}{R} = \frac{F_1}{R_1}.$$

$$m_1 r^4\left[\frac{1}{\Delta^4}+\frac{1}{r_1^4}+2\,\frac{x_1\xi+y_1\eta+z_1\zeta}{\Delta^3 r_1^3}\right]^{\frac{1}{2}}=\Delta^2\left[\frac{1}{r^4}+\frac{1}{r_1^4}-2\,\frac{xx_1+yy_1+zz_1}{r^3 r_1^3}\right]^{\frac{1}{2}}$$

Putting

$$\frac{x_1\xi+y_1\eta+z_1\zeta}{\Delta r_1}=\cos\theta,\quad \frac{\Delta}{r_1}=u,$$

we obtain

$$xx_1+yy_1+zz_1=x_1(x_1+\xi)+y_1(y_1+\eta)+z_1(z_1+\zeta)=$$
$$=r_1^2(1+u\cos\theta)$$
$$r^2=(x_1+\xi)^2+(y_1+\eta)^2+(z_1+\zeta)^2=r_1^2(1+2u\cos\theta+u^2).$$

Substituting these equations into the previous ones, we obtain

$$m_1^2=u^4\frac{(1+2u^3\cos\theta+u^4)^{\frac{1}{2}}}{(1+2u\cos\theta+u^2)^2}\Big[1+(1+2u\cos\theta+u^2)^2-$$
$$-2(1+u\cos\theta)(1+2u\cos\theta+u^2)^{\frac{1}{2}}\Big]^{\frac{1}{2}}.$$

Expanding the right-hand side of this equation in powers of the small quantity u, we obtain

$$m_1^2=u^{10}\sqrt{1+3\cos^2\theta}\left[1-2u\cos\theta\,\frac{1+6\cos^2\theta}{1+3\cos^2\theta}+\ldots\right]$$

Keeping only the first term of this expansion, we obtain

$$\Delta=r_1\left(\frac{m_1^2}{\sqrt{1+3\cos^2\theta}}\right)^{\frac{1}{5}} \qquad (25)$$

This is an approximate equation for the plane under consideration in the polar coordinate system. The surface (25) is evidently a surface of rotation around the polar axis i.e. around the radius vector of the planet. This surface slightly differ from a sphere; the ratio of the maximal to minimal value of Δ is equal to

$$2^{\frac{1}{5}}=1.15.$$

C-4

The sphere, drawn from the centre of the planet with a radius equal to

$$\Delta_0 = r_1 m_1^{\frac{2}{5}},$$

is called the sphere of activity of the planet. The radii of the spheres of activity of some planets aregiven in the following table

Mercury	0.001	Jupiter	0.322
Venus	0.004	Saturn	0.362
Earth	0.006	Uranus	0.339
Mars	0.004	Neptune	0.576

Inside the sphere of activity $F:R > F_1:R_1$. Outside this sphere $F:R < F_1 : R_1$ so that it is more useful to consider the planet as the central body.

It is interesting to evaluate the ratio F:R of the perturbing force to the gravitational force of the sun for points located on the sphere of activity. It is easy to see that for such points

$$\frac{F}{R} = m_1 \frac{r^2}{\Delta^2}\sqrt{1 + \frac{2\Delta^3}{r_1^3}\cos\varphi + \frac{\Delta^6}{r_1^6}} \approx m_1\left(\frac{1+3\cos^2\varphi}{m_1^4}\right)^{\frac{1}{5}}.$$

For Jupiter, the limits within which this ratio varies are from $m_1 = \sqrt[5]{0.25}$ to $\sqrt[5]{16\, m_1} = 0.43$.

We have to deal quite often with comets passing through the sphere of Jupiter's activity. This is explained by the fact that the aphelions of most of the short-periodic comets are grouped near Jupiter's orbit. In order to make a transformation from the heliocentric coordinate system to the Jove-centric coordinate system, in which Jupiter is taken as a central body, it is necessary to find the Jove-centric coordinates at the time t_0 for which the distance Δ is reduced to 0.3. These will be denoted by $\xi_0 = x - x_1, \quad \eta_0 = y - y_1, \quad \zeta_0 = z - z_1$

while their derivatives will be given by

$$\dot{\xi}_0 = \dot{x} - \dot{x}_1, \quad \dot{\eta}_0 = \dot{y} - \dot{y}_1, \quad \dot{\zeta}_0 = \dot{z} - \dot{z}_1.$$

Using the formulae obtained in Chapter IV of Volume I, we can obtain the Jove-centric elements of the comet. The perturbations of these elements are computed by means of conventional formulae. The sun in almost all cases can be considered as the only perturbing body. It is interesting to note that the Jove-centric orbit of the comet is usually a hyperbola of large eccentricity. The eccentric anomaly appearing in the formulae of Sec. 70 will then be imaginary (vol. I, Sec. 15). Consequently, it is worthwile transforming these formulae, adopting that

$$E = H\sqrt{-1}$$

We shall not consider here these complicated transformations. We find that, in the case under consideration when the comet is inside or even near the planet's sphere of activity, it will be more convenient to calculate the perturbations in the coordinates using Cowell's method. This method will be considered in the next chapter. In order to use this method, it is sufficient to find the coordinates of the comet x_o , y_o and z_o and their derivatives x_o , y_o and z_o at the time t_o by means of the osculating elements (Sec. 74).

The computation of the perturbations of the Jove-centric coordinates has been carried out by Kamienski[1] in his study on the motion of Wolf's comet during its approach to Jupiter in 1922. In this case, the eccentricity of the Jove-centric orbit varied from e = 6.457 to e = 6.480, and the

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(1) M. Kamienski, Recherches sur le mouvement de la Comite periodique de Wolf (IX Partie), Publications of the Astr. Obs. of Warsaw University, 2, 1926.

semimajor axis from a = -0.022800 to a = - 0.022912.

When the comet under consideration closely approaches Jupiter, one has to take into account the perturbation caused by the compression of Jupiter. Such an approach took place,for example with Brook's comet (1889V), when the least distance between Jupiter's surface and the comet became approximately 1.14 times Jupiter's radius (approximately 80.000 km). In considering thiscase, the interval w used in integration has to be reduced to 0.25 hours in the vicinity of the time of approach. For details on the effects of the compression of Jupiter on a comet's motion, we refer the reader to the literature quoted herein(1).

73. Approximate Calculation of Perturbations of Small Planets

The exact calculation of the perturbed coordinates of a small planet by a numerical integration requires a large amount of work. This does not depend on whether we are calculating the perturbations in the elements or immediately calculating the purturbed coordinates (Secs. 74, 75). This type of calculation is only carried out for planets, which are interesting in some aspects. The analytical theory of motion of the type developed by Leverrie and Newcome for large planets has been applied to an even smaller number of small planets. It is not, however possible to calculate the perturbations of small planets for, after a few years their actual motions will differ so much from their unperturbed motions, that the planets will hardly be distinguished and would thus be

---

(1) G. Deutschland, Der Eintluss der Abplattung auf die Attraction der Himmelskorper nach der Theorie der speziellen Storungen, mit Anwendung auf den Kometen 1889 V (Brooks) bet seiner Kupiternahe un Jahare 1886 (Diss) Berlin 1909.

The results are partially given in. Astr. Nachr., 181, 1909, 1-8.

completely lost. It is thus extremely important to have on hand a simple method which would allow us to calculate the perturbations of small planets with an accuracy that would identify these planets. Several methods have been suggested for this purpose and amongst them, the following two approximate forms of an accurate method suggested by Starke(1), have been widely practised.

We first of all point out that it is sufficient to only take into account the perturbations caused by Jupiter for the approximate calculations of the perturbations necessary to identify these planet.

We shall limit ourselves to an accuracy within 0.0001 for the perturbations of n and up to within 0.0001 in the perturbations of all the other elements. We shall then be able perform all computations to three decimal places. The interval w can always be taken equal to 80^d, but when the planet is far from Jupiter (e.g., when the heliocentric angular distance between them is greater than 60^o) the interval w may be set equal to 160^d. It is useful to take all the constant factors for the interval 80^d and complete the missing ones by interpolation at $w = 160^d$. Under these conditions, we shall be able to make considerable simplifications in the calculations of the formulae obtained in Sec. 70. Instead of computing v and r using equations (I), we can find their values by consulting the special tables given in Volume 1, Chapter III. We can always substitute in equations (II) $\cos b_1 = 1$, and, if the slope i is small ($i < 8^o$), put

---

(1) G. Starke, Genaherte Storungsrechnung und Behverbesserung, Veroff des Astr. Recheninstituts, Nr. 44, 1921. Berlin; Tafeln zur genaherten Speziellen Storungsrechnung, Veroff. des Astr. Recheninstituts, Nr. 48, 1930, Berlin.

$$\cos B_1 = 1, \quad \sin B_1 = \cos i \sin b_1 + \sin i \sin(l_1 - \Omega), \quad L_1 = l_1 - \Omega.$$

Furthermore, we have to replace the coefficient $wk'' m_1$ in formulae (V) by 80 $k^o m$, where k^o is the Gaussian constant expressed in degrees. For Jupiter,

$$\lg(80\, k\, m_1) = 1.877.$$

In calculating using formulae VII, we use

$$\lg(3\, k w) = 1.616$$

This yields δn in units of $0^o.0001$. The increments of the other elements will be expressed in units of one-thousandth of a degree. We can generally neglect the differences in the calculations of formulae (VIII) and (IV). The integration will then be reduced to a single summation. Finally, we note that the perturbed value of n is not required, since we calculate the average longitude using formula

$$\lambda = \varepsilon + n_0(t - t_0) + \Delta'\lambda,$$

where n_o denotes the unperturbed value of n. However, the quantity $\omega\, \Delta n = 80 \cdot \Delta n$ is necessary for the computation of the perturbation of the semimajor axis. This latter quantity may be calculated by formula

$$\Delta \lg a = -\frac{13.03}{n_0}\, 80\, \Delta n,$$

where n_o is expressed in seconds of the arc.

In 1930, Starke developed another version of this method. It requires the use of subsidiary tables, but once these tables are available(1) the amount of work necessary will be considerably reduced.

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(1) These tables are quoted in the previous foot note.

Here, the average anomaly M is taken as the independent variable. The coefficients in formulae (VII), Sec. 70, are functions of . They are tabulated by the argument φ for the values m = 0°, 12°, 24°, ... 348°. The computation of the components of acceleration is also simplified by special tables. We shall not consider here the construction of these tables.

Finally, we remind the reader of the method of approximate computation of perturbations in the elements, suggested by Stromgren(1). This method is similar to Storke's method mentioned above in that both methods are designed for their use with calculating machines. Strömgren transformed the formulae in such a way, that they can be quickly and easily used to find the perturbed values of the directing cosines P_x, P_y, P_z, Q_x, Q_y and Q_z.

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(1) B. Stromgren, Formelu sur genaherten Srorungsrechnung in Bahnelementen, Publikatione og mindre Meddelelser fra Kobenhavens Observatorium, Nr. 65, 1929.

CHAPTER XI

CALCULATION OF THE PERTURBATIONS IN THE COORDINATES

74. Direct Calculation of the Perturbations in the Coordinates (Cowell's Method).

We consider the motion of a luminary, the mass of which is denoted by m and the heliocentric coordinates by x, y and z. We denote by m_i and x_i, y_i, z_i the masses and coordinates of the perturbing planets. The euqations of relative motion, derived in Sec. 3, yield

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k^2(1+m)\frac{x}{r^3}+F_x \\ \frac{d^2y}{dt^2} &= -k^2(1+m)\frac{y}{r^3}+F_y \\ \frac{d^2z}{dt^2} &= -k^2(1+m)\frac{z}{r^3}+F_z, \end{aligned}\right\} \qquad (1)$$

where

$$F_x = k^2\sum_i m_i\left(\frac{x_i-x}{\Delta_i^3}-\frac{x_i}{r_i^3}\right), \ldots \qquad (2)$$

Here, the summation is over all the perturbing bodies and the quantities r^2, r_i and Δ_i are defined by

$$r^2 = x^2+y^2+z^2, \qquad r_i^2 = x_i^2+y_i^2+z_i^2,$$
$$\Delta_i^2 = (x_i-x)^2+(y_i-y)^2+(z_i-z)^2.$$

If it is required to compute the values of the coordinates x, y and z for a relatively short interval of time for e.g. some decades, the easiest way then is to integrate equations (1) numerically. Any of the numerical integration of differential equations methods that have been considered in chapter VIII, enables us to calculate the values of x, y and z with an arbitrarily high accuracy. The calculation by means of any of these methods is straightforward and elementary. This is extremely important, especially when is doing substantial work, since it enables the use of calculating machines. Another advantage of this particular method of computation of the perturbed coordinates, is its universal character. The analytical methods of finding the perturbations are only valid if perturbations are small. The methods of numerical calculation of the perturbations in the elements, given in the previous chapter, are only preferentially applied if the perturbations are not very large, although they are generally valid for arbitrary perturbations. On the other hand, the question of the magnitude of perturbations is not raised in the numerical integration of equations (1). Consequently, this method is conveniently used for small planets, subject to small perturbations, and for planets which approach Jupiter so closely that their perturbations become particularly large. Similarly in this method, there is not difference between comets which are far removed from planets and comets entering the sphere of activity of a planet. Finally, the numerical integration method enables us to compute in a very simple way the positions of such bodies like Jupiter's eighth satelite, whose magnitude of perturbation is comparable to the value of the central force.

We have seen in Sec. 41 that the most complicated cases of motion can be studied by means of methods of the numerical integration of equations. These methods had been used by Darwin, Tile, Burrau and Stormgren long before Cowell applied numerical integrations to solve astronomical problems. Indeed the direct numerical integration of the equations of motion had been applied before Cowell. Moreover, the method which be first developed was inferior to the method of quadratures suggested by Gauss and applied by Tile and the other previously mentioned authors. This is the reason for our returning later on to this method.

We however, owe Cowell the introduction of the method of direct numerical integration of equations (1) into the field astronomy as a relatively rapid and practical method of obtaining the perturbed ephemeride. For these reasons and for convenience sake, we call the method of calculating the perturbations of the coordinates of a luminary by means of the numerical integration of equations (1) as Cowell's method.

We have already mentioned the advantages of Cowell's method, which in many cases makes this method the most practical way of obtaining the ephemeride of a luminary, taking into account its perturbations. We can also point out, that in this method the trigonometric calculations are completely singled out. The only auxiliary table necessary for these computations is the table that gives the values r^{-3} by values of the argument r^2,

One of the most serious difficulties of Cowell's method consists in that all the intermediate calculations must be carried out to at least the same number of significant figures, the final result is required to have (In practice, one should carry out the intermediate calculations with a large number of significant figures to guarantee against the accumulation of errors). For example, if we have to connect two appearance times of a comet separated by a large interval of time, if we use Cowell's method we then must calculate the perturbed coordinates to seven significant figures for this interval. If we apply the methods developed in the previous chapter, it will then be sufficient to calculate the perturbations of the elements during this interval of time., rom three to five significant figures.

If the luminary closely approaches the sun, which is the case for most of the comets near the perihelion, the value of the interval of integration must then be significantly decreased. The amount of computation required by Cowell's method will then be more substantial than that required, for example, for Euke's method (Sec. 76).

We finally note that Cowell's method is at least convenient for accurate calculation only in the case when the computations will be carried out by means of calculating machines.(1)

75. A Compilation of theformulae used in Cowell's method

We denote by w the interval which will be chosen for the integrations of equ. (1). We have already pointed out in Sec. 58 that this interval should never be chosen very large, otherwise there will be less chance to control the computation accuracy by means of the differences. In addition the calculations would become less practical.

It is advisable to start the computation by using a small value fow w. If the fourth differences are then found not to affect the calculated values of the coordinates, the interval magnitude may then be doubled. For small planets, the interval may be set to equal 20-40-80 days, depending on the required accuracy. We have to take smaller interval values, say 5-10 days, for comets close to the sun. If it

---

(1) The integrations of equations (1) can, of course, be carried out using any method for the numerical integration of differential equations. In particular, instead of applying the method of quadratures, we can use Cowell's method (Sec. 52), supplemented by Numerov's method for the reduction of successive approximations (Sec. 58). We then obtain Numerov's method or the method of extrapolation". A detailed account of this method, as well as an example of its application are given in: Belletin of the Institute of Astronomy (Bjulleten Astronomiceskogo Instituta) No. 12, 1926.

Probably, the method of quadratures (Secs. 52-54, 58) is the best method for integrating equations (1) from the point of view of accuracy of the results and the simplicity of computation.

is required to further reduce the interval, it is then advisable to replace Cowell's method by Enke's method (Sec. 76).

We shall consider that the values of the osculating elements a, e, i, Ω , w and M_o of a luminary at moment t_o are known. In order to find the solution of equations (1) which describes the motion of this luminary, we have to calculate the values of coordinates x_o, y_o and z_o and their derivatives x'_o , y'_o and z'_o at moment t_o. For this purpose we use the following formulae (Vol. I).

$$\left.\begin{aligned} x &= aP_x(\cos E - e) + bQ_x \sin E \\ y &= aP_y(\cos E - e) + bQ_y \sin E \\ z &= aP_z(\cos E - e) + bQ_z \sin E \end{aligned}\right\} \qquad (\mathrm{I})$$

$$\left.\begin{aligned} x' &= \frac{k}{r\sqrt{a}}(-aP_x \sin E + bQ_x \cos E) \\ y' &= \frac{k}{r\sqrt{a}}(-aP_y \sin E + bQ_y \cos E) \\ z' &= \frac{k}{r\sqrt{a}}(-aP_z \sin E + bQ_z \cos E) \end{aligned}\right\} \qquad (\mathrm{II})$$

in which

$$b = a\cos\varphi, \quad r = \sqrt{x^2 + y^2 + z^2} = a(1 - e\cos E),$$

where E is defined by

$$E - e\sin E = M.$$

For checking the calculations, we use the following relation

$$ke\sqrt{a}\sin E = xx' + yy' + zz'.$$

The directing cosines of the orbital axes, P_x , P_y , ..., Q_z , may be calculated by using the following formulae (Vol. I, Sec. 25):

$$
\begin{aligned}
&A_1 = \cos\Omega; && A_2 = -\cos i \sin\Omega\\
&B_1 = \sin\Omega\cos\epsilon; && B_2 = \cos i\cos\Omega\cos\epsilon - \sin i\sin\epsilon\\
&C_1 = \sin\Omega\sin\epsilon; && C_2 = \cos i\cos\Omega\sin\epsilon + \sin i\cos\epsilon\\
&P_x = A_1\cos\omega + A_2\sin\omega; && Q_x = A_2\cos\omega - A_1\sin\omega\\
&P_y = B_1\cos\omega + B_2\sin\omega; && Q_y = B_2\cos\omega - B_1\sin\omega\\
&P_z = C_1\cos\omega + C_2\sin\omega; && Q_z = C_2\cos\omega - C_1\sin\omega,
\end{aligned}
$$

where ϵ denotes the slope of the ecliptic with respect to the equator. For checking, we apply the following relations

$$
\begin{aligned}
&aP_x \cdot bQ_x + aP_y \cdot bQ_y + aP_z \cdot bQ_z = 0\\
&(aP_x)^2 + (aP_y)^2 + (aP_z)^2 = a^2\\
&(bQ_x)^2 + (bQ_y)^2 + (bQ_z)^2 = b^2
\end{aligned}
$$

When the initial values x_o, y_o, ..., z'_o are already calculated, we start integrating equations (1). For this purpose, we first of all calculate for each considered moment $t_n = t_o + nw$ the values of the functions

$$
\left.
\begin{aligned}
f &= w^2\frac{d^2x}{dt^2} = -w^2k^2\frac{x}{r^3} + X\\
g &= w^2\frac{d^2y}{dt^2} = -w^2k^2\frac{y}{r^3} + Y\\
h &= w^2\frac{d^2z}{dt^2} = -w^2k^2\frac{z}{r^3} + Z
\end{aligned}
\right\} \qquad \text{(III)}
$$

$$
\left.
\begin{aligned}
X &= \sum_i \left(w^2k^2m_i\frac{x_i - x}{\Delta_i^3} - X^i\right)\\
Y &= \sum_i \left(w^2k^2m_i\frac{x_i - x}{\Delta_i^3} - Y^i\right)\\
Z &= \sum_i \left(w^2k^2m_i\frac{y_i - y}{\Delta_i^3} - Z\right).
\end{aligned}
\right\} \qquad \text{(IV)}
$$

where the particularly singled-out quantities

$$X^i = -w^2k^2m_i\frac{x_i}{r_i^3}, \quad Y^i = -w^2k^2m_i\frac{y_i}{r_i^3}, \quad Z^i = -w^2k^2m_i\frac{z_i}{r_i^3},$$

depend entirely on the coordinates and masses of the perturbing planets.

The irreplaceable hand-book used for these calculations, are Comrie's tables: "Planetary co-ordinates for the years 1800–1940" which we have mentioned several times before. The extension of these tables to the years 1940-1060 is expected to appear in the near future. Besides the coordinates ofall the large planets (except Mercury and Plutonus), these tables also give the corresponding values of X^i, Y^i and Z^i. Moreover, several other auxiliary tables are given there, and in particular the table by which the quantity r^{-3} can be found from the value of the argument r^2.

If Comrie's tables are not available, we can compute the rectangular coordiates of the perturbing planets by the values of the corresponding ecliptic coordinates obtained from the year-book. For this purpose, we use the following relations

$$\left.\begin{aligned} x_i &= r_i \cos b_i \cos l_i \\ y_i &= r_i \cos b_i (\sin l_i \cos \varepsilon - \operatorname{tg} b_i \sin \varepsilon) \\ z_i &= r_i \cos b_i (\sin l_i \sin \varepsilon + \operatorname{tg} b_i \cos \varepsilon), \end{aligned}\right\} \qquad (3)$$

where r_i, ℓ_i and b_i are the radius vector, the longitude and latitude of the planet respectively.

The constants involved in the expressions f, g and h are given in table IV at the end of this volume.

We apply the following formula during integration:

[illegible]

as well as two similar formulae defining the values of y and z which correspond to the moment $t_0 + nw$. The initial terms of the columns of sums are defined by the following formulae (Sec. 54):

$$
\begin{aligned}
f'_{\frac{1}{2}} &= w x'_0 + \frac{1}{2} f_0 + \frac{1}{12} f^{\mathrm{I}}_{\frac{1}{2}} - \frac{11}{720} f^{\mathrm{III}}_{\frac{1}{2}} + \frac{1}{317} f^{\mathrm{V}}_{\frac{1}{2}} - \dots \\
f''_0 &= x_0 + \frac{1}{12} f_0 - \frac{1}{240} f^{\mathrm{II}}_0 + \frac{1}{1951} f^{\mathrm{IV}}_0 - \frac{1}{12557} f^{\mathrm{VI}}_0 - \dots
\end{aligned}
\qquad \text{(VI)}
$$

In order to avoid some of the successive approximations at the very beginning of the computation (cf. Sec. 55), it is easy to determine, using formulae (I), the values of the unperturbed coordinates for a few moments t_{-2}, t_{-1}, t_1, t_2, t_3, ... near the initial moment t_0. One approximation will then be sufficient to obtain the final (perturbed) values of the coordinates and the moment. For this purpose it is also possible to use the following Taylor expansion.

$$x(t_0 + nw) = x_0 + x'_0(t_n - t_0) + \frac{1}{2} f_0 w^{-2}(t_n - t_0)^2 + \dots$$

When some of the initial values for the coordinates are corrected and the final values of the quantities (VI) are obtained, we further integrate quite easily by means of formulae (V)

Annotation I

When the Gaussian constants a, A, b, B, ... corresponding to the initial osculating elements are known, then formulae (I) and (II) are recommended to be replaced by

$$
\left.
\begin{aligned}
x &= r \sin a \sin(A + u) \\
y &= r \sin b \sin(B + u) \\
z &= r \sin c \sin(C + u)
\end{aligned}
\right\} \qquad \text{(I')}
$$

$$
\left.
\begin{aligned}
x' &= r^{-1}[x r' + k\sqrt{p} \sin a \cos(A + u)] \\
y' &= r^{-1}[y r' + k\sqrt{p} \sin b \cos(B + u)] \\
z' &= r^{-1}[z r' + k\sqrt{p} \sin c \cos(C + u)],
\end{aligned}
\right\} \qquad \text{(II')}
$$

where $p = a\cos^2\varphi$, quantities r, r' and u are calculated by means of the following formulae

$$r = a(1 - e\cos E), \quad rr' = ke\sqrt{a}\sin E,$$

$$\operatorname{tg}\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}, \quad u = v + \omega,$$

and for checking, we use the following auxiliary equation

$$rr' = xx' + yy' + zz'.$$

Annotation II

If the luminary, whose motion is under investigation, exists at a sufficiently largedistance from the sun, then the perturbations that Mercury, Venus, ... prcduce in its motion, are either insensible or quite small. In such cases, it is sufficient to only consider the secular parts of these perturbations. This can simply be done by correspondingly increasing the mass of the sun.

The factor 1 + m, involved in equations (1), will be replaced by unity, since the mass m of a small planet or comet may always be set equal to zero. If we consider that the mass of the sun is unity, and take into consideration the masses of the perturbing planets, then formulae (III) will be replaced by

$$\left.\begin{aligned} f &= -m^2k^2Mxr^{-3} + X \\ g &= -m^2k^2Myr^{-3} + Y \\ h &= -m^2k^2Mzr^{-3} + Z, \end{aligned}\right| \qquad (\text{III}')$$

where factor M has one of the following values

M = 1.00 000 14 (sum of masses of sun and Mercury)

= 1.000 002 60 (sum of masses of sun, Mercury and Venus)

= 1.000 005 64 (sum of masses of sun, Mercury, Venus and Earth).

= 1.000 005 96 (sum of masses of sun, Mercury, Venus, Earth and Mars).

76. Emke's Method

Instead of finding the perturbed coordinates of a luminary by means of the numerical integration of equation (1), it is possible to calculate the differences

$$\xi = x - \bar{x}, \quad \eta = y - \bar{y}, \quad \zeta = z - \bar{z}$$

between the perturbed coordinates (x, y, z) and the unperturbed ($\bar{x}$, $\bar{y}$, $\bar{z}$). Since the unperturbed coordinates satisfy the following equations:

$$\begin{aligned} \frac{d^2\bar{x}}{dt^2} &= -k^2(1+m)\,\bar{x}\,\bar{r}^{-3} \\ \frac{d^2\bar{y}}{dt^2} &= -k^2(1+m)\,\bar{y}\,\bar{r}^{-3} \\ \frac{d^2\bar{z}}{dt^2} &= -k^2(1+m)\,\bar{z}\,\bar{r}^{-3}, \end{aligned}$$

then by subtracting these equations term-wise from equations (1), we obtain'

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= F_x + k^2(1+m)\left(\frac{\bar{x}}{\bar{r}^3} - \frac{x}{r^3}\right) \\ \frac{d^2\eta}{dt^2} &= F_y + k^2(1+m)\left(\frac{\bar{y}}{\bar{r}^3} - \frac{y}{r^3}\right) \\ \frac{d^2\zeta}{dt^2} &= F_z + k^2(1+m)\left(\frac{\bar{z}}{\bar{r}^3} - \frac{z}{r^3}\right). \end{aligned}\right\} \qquad (4)$$

Thus, the calculation of the differences ξ, η and ζ, which are nothing else but the perturbations of the rectangular coordinates, is reduced to the integration of equations (1).

Particular attention should be devoted to the evaluation of the second terms in the right-hand ride of equ. (4). The direct computation of the differences in the brackets is accompanied by a large accuracy loss. It is more convenient from the computational technique point of view, to transform these equations into the following form

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$$\frac{x}{r'^3} - \frac{x}{r^3} = \frac{1}{r'^3}\left[x\left(1-\frac{r'^3}{r^3}\right)-\xi\right].$$

It is thus clear that all three coordinates could be found if the difference $1 - \frac{\bar{r}^3}{r^3}$ is obtained. Since

$$r'^2 = (x+\xi)^2 + (y+\eta)^2 + (z+\zeta)^2 =$$
$$= r^2 + (2x+\xi)\xi + (2y+\eta)\eta + (2z+\zeta)\zeta,$$

we can then write that

$$\frac{r'^2}{r^2} = 1 + 2q,$$

where

$$q = \frac{1}{r^2}\left[\left(x+\frac{1}{2}\xi\right)\xi + \left(y+\frac{1}{2}\eta\right)\eta + \left(z+\frac{1}{2}\zeta\right)\zeta\right]. \qquad (5)$$

Therefore,

$$\frac{r^3}{r'^3} = (1+2q)^{-\frac{3}{2}} = 1 - 3q + \frac{3.5}{1.2}q^2 - \frac{3.5.7}{1.2.3}q^3 + \dots$$

Following Enke, we adopt that

$$f = 3\left(1 - \frac{5}{2}q + \frac{35}{6}q^2 - \frac{315}{24}q^3 + \dots\right); \qquad (6)$$

This yields

$$1 - \frac{r^3}{r'^3} = qf.$$

Hence,

$$\frac{x}{r^3} - \frac{x}{r'^3} = \frac{1}{r'^3}(qfx - \xi).$$

When the differences are presented in this form, they can be computed without any accuracy loss. The reevaluation of the following functions

$$F = w^2\frac{d^2\xi}{dt^2}, \quad G = w^2\frac{d^2\eta}{dt^2}, \quad H = w^2\frac{d^2\zeta}{dt^2}$$

required for the numerical integration of equations (4), will be carried out by means of the following equations

$$\left.\begin{aligned} F &= X + \frac{w^2k^2}{r^3}(qfx - \xi) \\ G &= Y + \frac{w^2k^2}{r^3}(qfy - \eta) \\ H &= Z + \frac{w^2k^2}{r^3}(qfz - \zeta), \end{aligned}\right\} \qquad \text{(VII)}$$

where quantities X, Y and Z are those defined by formulae (IV) of the previous section.

Quantity f, involved here and defined by equation (6), may be determined by using table III at the end of this volume. The value of f is given in sixteen decimal places in Comrie's tables previously mentioned.

In the following, we enumerate the operations that have to be carried out for the application of Comrie's method.

1- Starting with given values for the osculating elements, we calculate the unperturbed equatorial coordinates using equations (I) and (I') for a series of moments $t_h = t_o + hw$, where h = -2, -1, 0, 1, 2, ... It is useful to choose the initial moment so that the epoch of osculation takes place in the moment $t = t_o - \frac{w}{2}$. In the following, this will be assumed to be the case.

2- We use Comrie's tables to find the values of the rectangular coordinates x_i , y_i and z_i of the perturbing planets and the corresponding quantities X^i, Y^i and Z^i for all the moments whose perturbations intend to calculate.

3- In order to start integrating we compute the values of quantities F_{-2}, F_{-1}, F_o , F_1 and F_2 using formulae (IV) and (VII). In the first approximation, we assume that

$$\xi = \eta = \zeta = 0, \quad x_0 = x, \quad y_0 = y, \quad z_0 = z, \quad q = 0.$$

We determine the initial terms of the column of sums by the following formulae (cf. Sec. 54; in the present case we have $\xi = \eta = \zeta = 0$ and $\xi' = \eta' = \zeta' = 0$ for the epoch of osculation $t_o - \frac{w}{2}$):

$$\left.\begin{aligned} F^{-1}_{\frac{1}{2}} &= -\frac{1}{24}F^{1}_{-\frac{1}{2}} + \frac{1}{339}F^{3}_{\frac{1}{2}} - \dots \\ F^{-2}_{0} &= +\frac{1}{24}F_{-1} - \frac{1}{339}(2F^{2}_{-1} + F^{2}_{0}) + \dots, \end{aligned}\right\} \qquad \text{(VIII)}$$

we calculate the perturbations using the following formulae

$$\xi_n = F^{-2}_{n} + \text{Red}, \quad \text{Red} = \frac{1}{12}F_n - \frac{1}{240}F^{2}_{n} + \frac{1}{1951}F^{4}_{n} - \dots \qquad \text{(IX)}$$

We then repeat the calculation of quantities F_{-2}, F_{-1}, F_o, F_1, F_2,... using the values obtained for ξ, η and ζ. We repeat this procedure until we find that these quantities do not improve further.

4- When the above-mentioned calculations give the final values' for the quantities (VIII), we start integrating conventionally using formulae (IX) and substituting therein, the extrapolated values of the differences or the Red correction (Sec. 55). We add the perturbations of the coordinates obtained thisway to the unperturbed coordinates previously obtained (item 1). This leads us to the required values of the perturbed coordinates:

$$x = x + \xi, \quad y = y + \eta, \quad z = z + \zeta.$$

In conclusion, we note that the calculations made by Enke's method are more useful than the calculations made by Cowell's method only in the case when perturbations ξ, η and ζ are small. The values of these perturbations can always be reduced by changing the epoch of osculation of the elements for which we are calculating the unperturbed coordinates. However, the calculation of the new osculating elements by means of the obtained perturbed coordinates (Vol. I, Chap. IV) constitutes additional work which significantly reduces the advantage of Enke's method. Therefore, this method is only applied if the perturbations are small and when it is necessary to evaluate them for small intervals of time. Under these conditions (e.g., for comets only observed once) Enke's method is the best.

Sometimes, in the study of the motion of comets, the two methods, Enke's and Cowell's, can be combined. When a comet is far from the sun and is subject to considerable perturbations from the planets, it is better to apply Cowell's method since this method is the most general and the most independent from the magnitude of perturbations. On the other hand, when the comet is near the perihelion, its perturbations are generally small (because of the high speed of motion and the large distance from planets of large masses), and the unperturbed coordinates vary very rapidly. In this case Cowell s method is not useful because the interval w should be strongly decreased. It is more useful then to replace this method by Enke's method.

Annotation I

We can use Titjen's method for the reduction of the number of successive approximations (Sec. 58) during the integration of equations (4). Indeed, noting that on the basis of equations (VII).

$$F_n = X_n + hqfx_n - h\xi_n,$$

where

$$h = \frac{w^2k^2}{r_n^3}, \qquad (7)$$

we may write equation (IX), by which we compute ξ_n, in the following way

$$\xi_n\left(1+\frac{1}{12}h\right) = S_n^x + \frac{1}{12}hq/x_n, \tag{8}$$

where

$$S^x = \frac{1}{12}X_n + F_n^{-2} - \frac{1}{240}F_n^{2} + \frac{1}{1951}F_n^{4} - \dots$$

Similarly,

$$\eta_n\left(1+\frac{1}{12}h\right) = S_n^y + \frac{1}{12}hq/y_n$$
$$\zeta_n\left(1+\frac{1}{12}h\right) = S_n^z + \frac{1}{12}hq/z_n. \tag{9}$$

We immediately obtain the final values of the quantities S_n^x, S_n^y and S_n^z owing to the smallness of the coefficients $\frac{1}{240}$, ... On the other hand, when we calculate quantity q by means of formula (4)

$$q = \frac{1}{r^2}\left[\left(\bar{x}_n + \frac{1}{2}\xi_n\right)\xi_n + \left(\bar{y}_n + \frac{1}{2}\eta_n\right)\eta_n + \left(\bar{z}_n + \frac{1}{2}\zeta_n\right)\zeta_n\right],$$

we can replace the quantities ½ ξ_n, ½ η_n and ½ ζ_n inside the brackets by their extrapolated values. This will introduce a slight error because these quantities have small multipliers ξ_n, η_n and ζ_n. The quantities ξ_n, ... standing outside the brackets can be replaced by their values which may be obtained by formulae (8) and (9). This yields

$$q = \alpha S_n^x + \beta S_n^y + \gamma S_n^z + \frac{1}{12}hq(\alpha x_n + \beta y_n + \gamma z_n),$$

where

$$\alpha = \frac{\bar{x}_n + \frac{1}{2}\xi_n}{\bar{r}_n^2\left(1+\frac{1}{12}h\right)}, \quad \beta = \frac{\bar{y}_n + \frac{1}{2}\eta_n}{\bar{r}_n^2\left(1+\frac{1}{12}h\right)}, \quad \gamma = \frac{\bar{z}_n + \frac{1}{2}\zeta_n}{\bar{r}_n^2\left(1+\frac{1}{12}h\right)}. \tag{10}$$

Therefore, we finally obtain for the calculation of q the following formula

$$q = \frac{\alpha S_n^{\prime\prime} + \beta S_n^{\prime\prime} + \gamma S_n^{\prime\prime}}{1 - \frac{1}{12} hf(\alpha x_n + \beta y_n + \gamma z_n)} \quad (11)$$

Thus calculating the coefficients α, β and γ using formulae (7) and (8) by means of the extrapolated values, we can obtain q from equations (II). Then, formulae (8) and (9) yield new and more exact values for ξ_n, η_n and ζ_n.

In the calculation of q by means of formula (II), we usually prefer to extrapolate the denominator rather than find the denominator by extrapolating the values of ξ_n, η_n and ζ_n.

The above-mentioned application of Titjen's method to the integration of equations (4) was first suggested by Oppol'cer and was called Oppol'cer's method.

Annotation II

In the absence of Comrie's tables(1) the rectangular equatorial coordinates of the perturbing planets may be calculated by formulae (3). In this case however, it is more useful to calculate the perturbations in the ecliptic coordinates. Forthe perturbing planets, these are calculated by the following simple formulae

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(1) We quote herein the convenient tables, published in:
H.Q. Rasmusen, Hilfstaflen fur die numerische Integration der rechtwinkligen Koordinaten eines Himmelskorpers, Astr. Nachr., 260, 1936, 325-376.
The following article includes a table which simplifies the application of Titjen's method in the computation of the coordinates by Cowell's method.
M. Th. Subbotin, Sur le calcul des coordinnees heliocentriques des planets et des comets au moyen des quadratures, Poulkovo Observatory Circular, No. 9, 1933, 15-25.

$$x_i' = r_i \cos b_i \cos l_i$$
$$y_i' = r_i \cos b_i \sin l_i$$
$$z_i' = r_i \sin b_i .$$

When the perturbations ξ', η' and ζ' of the ecliptic coordinates are obtained, the perturbations of the equatorial coordinates are calculated by the following evident relations

$$\xi = \xi', \quad \eta = \eta' \cos \varepsilon - \zeta' \sin \varepsilon, \quad \zeta = \eta' \sin \varepsilon + \zeta' \cos \varepsilon .$$

PART THREE

ANALYTICAL METHODS FOR STUDYING PERTURBED MOTIONS

CHAPTER XII

THE SERIES EXPANSION OF THE COORDINATES OF THE ELLIPTIC MOTION

77. Introduction

The equations governing perturbed motions are generally complicated. The analytical integration of these equations is only possible when the perturbing accelerations involved in these equations are explicit functions of the independent variables. Usually, the independent variable in the theory of perturbed motion is taken to be time (or, equivalently, the mean anomally of the perturbed luminary), or the eccentric anomaly of the perturbed luminary, or finally, its true anomaly. The true anomaly is often replaced by the true latitude. The perturbing accelerations can be expressed in terms of the perturbation function in a straightforward manner. The task of integrating analytically equations of motion can thus be reduced to the task of expressing the perturbation function, by a function of one of the above-mentioned variables. The first step which has to be performed is to express the coordinates of the elliptic motion by an explicit function of time or, equivalently, the average anomally. This will constitute the topic of the present chapter.

The coordinates of the elliptic motion r and v, and the function of coordinates, F (r, v), are periodic functions f(M) of the average anomaly M with a period of 2π. Therefore, any such function F(r,v) = f(M) can be expanded in a Fourier series

$$f(M) = \frac{1}{2} a_0 + a_1 \cos M + \ldots + a_k \cos kM + \ldots$$
$$+ b_1 \sin M + \ldots + b_k \sin kM + \ldots, \qquad (1)$$

This series converges for all values of M since we are only considering continuous functions $f(M)$ which have continuous derivatives. It is well known that, in such a case, the expansion coefficients a_k and b_k will decay so rapidly that the products $a_k\ k^2$ and $b_k\ k^2$ will tend to zero for any arbitrary constant factor α . These coefficients are given by

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(M) \cos kM\, dM, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(M) \sin kM\, dM. \qquad (2)$$

Series (1) is often replaced by the corresponding Maclaurin series. Indeed, let us put

$$z = e^{iM} = \exp iM,$$

where $i = \sqrt{-1}$, then

$$2\cos kM = \exp ikM + \exp(-ikM)$$
$$2i \sin kM = \exp ikM - \exp(-ikM)$$

Hence,

$$f(M) = \frac{1}{2} a_0 + \frac{1}{2}\sum_1^{\infty} (a_k - ib_k) \exp ikM + \frac{1}{2}\sum_1^{\infty} (a_k + ib_k) \exp(-ikM).$$

We introduce coefficients a_k and b_k for negative indices by adopting that

$$a_{-k} = a_k, \qquad b_{-k} = -b_k;$$

Series (1) will then assume the following final form

$$f(M) = \sum_{-\infty}^{\infty} P_k z^k, \qquad (3)$$

where

$$p_k = \frac{1}{2}(a_k - ib_k).$$

The coefficients of the Maclaurin series (3) are given by the following well-known formula

$$p_k = \frac{1}{2\pi i}\int_C \frac{f(M)\,dz}{z^{k+1}},$$

where the integration is carried over the contour C that encloses the point z = 0 in the plane of the complex variable z. Taking this contour as a sphere of unit radius with centre at the point z = 0, we obtain

$$p_k = \frac{1}{2\pi}\int_0^{2\pi} f(M)\exp(-ikM)\,dM. \qquad (4)$$

This formula is equivalent to equation (2)

For these functions which we will consider later on, the integrals (2) or (4) cannot generally be expressed in terms of elementary functions. They are conveniently expressed in terms of Bessel functions. Hence, we shall start by studying some properties of these functions.

78. The Bessel Functions

Let us consider the following expression

$$\Phi(z) = \exp\left(\frac{x}{2}z - \frac{x}{2}z^{-1}\right).$$

Since

$$\exp\left(\frac{x}{2}z\right) = \sum_0^{\infty}\left(\frac{x}{2}\right)^{\alpha}\frac{z^{\alpha}}{\alpha!}, \qquad \exp\left(-\frac{x}{2}z^{-1}\right) = \sum_0^{\infty}\left(\frac{x}{2}\right)^{\beta}\frac{(-1)^{\beta}}{\beta!}z^{-\beta},$$

then by multiplying these absolutely convergent series, we obtain

$$\Phi(z) = \sum_0^{\infty}\sum_0^{\infty}\left(\frac{x}{2}\right)^{\alpha+\beta}\frac{(-1)^{\beta}}{\alpha!\,\beta!}z^{\alpha-\beta}.$$

We expand this series in powers of z and put $\alpha - \beta = n$.

We then obtain

$$\Phi(z) = \sum J_n(x) z^n, \tag{6}$$

where n varies from $-\infty$ to $+\infty$, and β from 0 to $+\infty$ if $n \geqslant 0$ and from $-n$ to $+\infty$ if $n < 0$. Consequently, the expansion coefficients are given by

$$J_n(x) = \sum_{\beta=0} \frac{(-1)^\beta}{\beta!(\beta+n)!}\left(\frac{x}{2}\right)^{n+2\beta} \tag{7}$$

if $n \geqslant 0$, and

$$J_n(x) = \sum_{\beta=-n} \frac{(-1)^\beta}{\beta!(\beta+n)!}\left(\frac{x}{2}\right)^{n+2\beta}$$

if $n < 0$. These expansion coefficients, $J_n(x)$ are known as the Bessel functions of indices n. The series (7) converges for all values of x, and can be considered as a definition of the Bessel function $J_n(x)$.

Assuming in the latter equation $n = -m$, where $m > 0$, we obtain the following relation

$$J_{-m}(x) = (-1)^m J_m(x), \tag{8}$$

which indicates that it is possible to consider only the Bessel functions that have positive indices. It also follows from equation (7) that

$$J_n(-x) = (-1)^n J_n(x). \tag{9}$$

We can obtain other properties of the Bessel functions by means of equation (6). Differentiating this equation with respect to z, we obtain

$$\frac{x}{2}(1+z^{-2})\Phi(z) = \sum n J_n(x) z^{n-1}.$$

Substituting here the expression (6) of the function $\Phi(z)$, and equating the coefficients of z^{n-1} in both sides, we obtain

$$n J_n(x) = \frac{x}{2} [J_{n-1}(x) + J_{n+1}(x)] \tag{10}$$

On the other hand, differentiating equation (6) with respect to x, we obtain

$$\frac{1}{2} (z - z^{-1}) \sum J_n(x) z^n = \sum J_n'(x) z^n .$$

Equating the coefficients of z^n in both sides of this equation yields

$$J_n'(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)]. \tag{11}$$

This equation enables us to express any derivative of the Bessel function as a linear combination of these functions. For example,

$$J_n''(x) = \frac{1}{2} [J_{n-1}'(x) - J_{n+1}'(x)],$$

which may be reduced to

$$J_n''(x) = \frac{1}{4} [J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]. \tag{12}$$

We shall now show that the Bessel function $J_n(x)$ satisfies a linear second-order differential equation. We consider equation (10), which yields

$$(n-1) J_{n-1}(x) = \frac{x}{2} [J_{n-2}(x) + J_n(x)]$$

$$(n+1) J_{n+1}(x) = \frac{x}{2} [J_n(x) + J_{n+2}(x)].$$

Adding these equations and for simplifying dropping the argument x, we obtain

$$n[J_{n-1} + J_{n+1}] - [J_{n-1} - J_{n+1}] = \frac{x}{2} [J_{n-2} - 2J_n + J_{n+2}] + 2x J_n .$$

We replace the square brackets by the expressions given in equations (10), (11) and (12). We then obtain

$$J_n''(x) + \frac{1}{x} J_n'(x) + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0. \tag{13}$$

This differential equation enables us to study the Bessel functions $J_n(x)$ for both real and complex values of the indices n. It is usually considered as the basic equation in the general theory of Bessel functions.

If we substitute into equation (6)

$$z = \exp i\varphi, \tag{14}$$

we then obtain

$$\exp(ix \sin \varphi) = \sum J_n(x) \exp(in\varphi).$$

Assuming that both φ and x are real, and equating the real and imaginary parts in the equation, we obtain

$$\left.\begin{aligned} \cos(x \sin \varphi) &= J_0(x) + 2J_2(x) \cos 2\varphi + 2J_4(x) \cos 4\varphi + \ldots \\ \sin(x \sin \varphi) &= 2J_1(x) \sin \varphi + 2J_3(x) \sin 3\varphi + \ldots \end{aligned}\right\} \tag{15}$$

where, we have taken equations (8) into consideration. The replacement of φ by $\varphi + \frac{\pi}{2}$ yield

$$\left.\begin{aligned} \cos(x \cos \varphi) &= J_0(x) - 2J_2(x) \cos 2\varphi + 2J_4(x) \cos 4\varphi - \ldots \\ \sin(x \cos \varphi) &= 2J_1(x) \cos \varphi - 2J_3(x) \cos 3\varphi + \ldots \end{aligned}\right\} \tag{16}$$

In conclusion, we derive some simple integral representations of the Bessel function. Applying formulae (9) to the coefficients of the series (6), and taking into account equation (14), we obtain

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \Phi(z) \exp(-in\varphi)\, d\varphi,$$

or,

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ix \sin \varphi - in\varphi)\, d\varphi. \tag{17}$$

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Equating the real parts of this equation yields

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\varphi - x\sin\varphi)\, d\varphi. \qquad (18)$$

This formula is not useful for calculating the Bessel function $J_n(x)$ for large values of the index n, because the integrand will have large maxima and minima values. Moreover, this formula does not stress the property that the Bessel function $J_n(x)$ behaves like x^n for small values of n and x, while this property is very important in astronomy applications. It is thus useful to rewrite equation (7) in the following manner

$$J_n(x) = x^n \sum_{\beta=0}^{\infty} \frac{(-1)^\beta x^{2\beta}}{2.4\ldots(2\beta).2.4\ldots(2n+2\beta)} =$$

$$= \frac{x^n}{1.3.5\ldots(2n-1)} \sum_{0}^{\infty} \frac{1.3.5\ldots(2n-1).1.3.5\ldots(2\beta-1)}{2\cdot4\ldots(2n+2\beta)} \frac{(-1)^\beta x^{2\beta}}{(2\beta)!}.$$

For arbitrary integral values of n and β, the following relation holds

$$\int_0^\pi \sin^{2n}\varphi \cos^{2\beta}\varphi\, d\varphi = \frac{1.3.5\ldots(2n-1).1.3.5\ldots(2\beta-1)}{2.4\ldots(2n+2\beta)}\pi.$$

Therefore,

$$J_n(x) = \frac{x^n}{1\cdot3\cdot5\ldots(2n-1)} \sum_{\beta=0}^{\infty} \frac{1}{\pi}\int_0^\pi \sin^{2n}\varphi \frac{(-1)^\beta x^{2\beta}\cos^{2\beta}\varphi}{(2\beta)!}\, d\varphi,$$

or, finally

$$J_n(x) = \frac{1}{\pi}\frac{x^n}{1.3.5\ldots(2n-1)}\int_0^\pi \sin^{2n}\varphi \cos(x\cos\varphi)\, d\varphi. \qquad (19)$$

This formula isfree from the shortcomings of equation (18) as previously mentioned.

79. The Computation of Bessel Functions

In the problems we are going to consider, we have to compute for a given value of x, all the functions $J_o(x)$, $J_1(x)$, ... that differ from zero up to within the accepted number of decimals. We will now show the simplest and most convenient method of doing this. We first note that formula (19) yields

$$|J_n(x)| < \frac{x^n}{1.3.5\ .\ .\ .\ (2n+1)}.$$

This inequality enables us to find the maximum value for n_1 for which the function $J_n(x)$ differs from zero within the accepted accuracy.

<u>The first method</u>

We can write equation (7) in an unfolded form as follows

$$J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ .\ .\ . \qquad (20)$$

$$J_n(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n\left\{1 - \frac{1}{1!\,(n+1)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,(n+1)(n+2)}\left(\frac{x}{2}\right)^4 - \ .\ .\ .\ \right\}. \qquad (21)$$

These series are convenient for the purpose of the rapid calculation of the Bessel functions when the values of x and n are not large.

It is sufficient to compute the values of only two functions, e.g. $J_o(x)$ and $J_1(x)$, and then find the values of other functions by means of the successive application of formulae (10). For example,

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x), \qquad J_3(x) = \frac{4}{x}J_2(x) - J_1(x),\ .\ .\ .$$

We should however point out, that owing to the presence of factors $\frac{2}{x}$, $\frac{4}{x}$, $\frac{6}{x}$, ..., we will have a progressive loss of accuracy, which will be all the more significant for smaller values of x.

<u>The second method</u>

Let us introduce the ratio p_k of two neighbouring Bessel functions, defined by the relation

$$J_1(x) = p_1 J_0(x), \qquad J_2(x) = p_2 J_1(x),\ .\ .\ .\ , \qquad J_n(x) = p_n J_{n-1}(x).$$

Dropping the argument x for simplifying we obtain

$$\left.\begin{aligned} J_1 &= J_0 p_1 \\ J_2 &= J_0 p_1 p_2 \\ &\cdots \\ J_n &= J_0 p_1 p_2 \cdots p_n . \end{aligned}\right| \qquad (22)$$

Thus, our task is reduced to the computation of $J_o(x)$ on one hand, and to the computation of p_1 , p_2, ... p_n on the other hand. The function $J_o(x)$ can be computed by means of the series (20), or, if the value of x is large, by formula (19). Let us now turn to the computation of p_1, p_2, ..., p_n. Formula (10) leads to

$$\frac{2k}{x} = \frac{J_{k-1}}{J_k} + \frac{J_{k+1}}{J_k},$$

or

$$\frac{2k}{x} = \frac{1}{p_k} + p_{k+1}. \qquad (23)$$

Substituting here for k = n - 1, n - 2, ..., 1, we obtain

$$\left.\begin{aligned} \frac{1}{p_{n-1}} &= \frac{2n-2}{x} - p_n \\ \frac{1}{p_{n-2}} &= \frac{2n-4}{x} - p_{n-1} \\ &\cdots\cdots \\ \frac{1}{p_1} &= \frac{2}{x} - p_2 . \end{aligned}\right| \qquad (24)$$

These formulae allow us to compute p_{n-1} , p_{n-2}, ..., p_1 in a simple manner without any loss of accuracy, provided that p_n is known. However, from the same equation (23), we obtain

$$p_n = \frac{1}{\frac{2n}{x} - p_{n+1}}, \qquad p_{n+1} = \frac{1}{\frac{2n+2}{x} - p_{n+2}},$$

so that p_n may be represented by the following continued fraction

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$$p_n = \cfrac{1}{\cfrac{2n}{x} - \cfrac{1}{\cfrac{2n+2}{x} - \cfrac{1}{\cfrac{2n+4}{x} - \cdots}}} \qquad (25)$$

which converges more rapidly for larger values of n.

<u>The third method</u>

Adding equations (16) term by term, we obtain

$$F(\varphi) = c_0 + c_1 \cos\varphi + c_2 \cos 2\varphi + c_3 \cos 3\varphi + \ldots ,$$

where

$$F(\varphi) = \cos(x \cos\varphi) + \sin(x \cos\varphi)$$

$$c_0 = J_0(x), \quad c_1 = 2J_1(x), \quad c_2 = -2J_2(x), \quad c_3 = -2J_3(x),$$
$$c_4 = 2J_4(x), \quad c_5 = 2J_5(x), \quad c_6 = -2J_6(x), \ldots$$

Computing the function F(φ) for a series of equally spaced values of φ , and applying the usual formulae of the harmonic analysis, we obtain $J_0(x)$, $J_1(x)$, ...

Let us for example assume that J_7 (x), $J_8(x)$, ... is equal to zero. We then introduce the following notation

$y_0 = F(0°)$ [illegible] $y_6 = F(180°)$
[illegible]

Then,

[illegible]
[illegible]

from which we obtain C_o , C_2, C_4 and C_6. Similarly, putting

$$A' = y_0 - y_6, \quad B' = y_1 - y_5, \quad C' = y_2 - y_4,$$

we obtain

$$3c_1 + 3c_5 = A' + C'; \qquad 2c_1 - 2c_3 + 2c_5 = A'$$
$$3c_1 - 3c_5 = \sqrt{3}\, B'; \qquad 6c_3 = A' - 2C'$$

from which we obtain C_1, C_3 and C_5.

For checking, we can apply any particular form of formulae (15) and (16), e.g. any of the following equations

$$1 = J_0(x) + 2J_2(x) + 2J_4(x) + \ldots$$
$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \ldots$$
$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \ldots$$

Ten-figure tables of the functions $J_0(x)$ and $J_1(x)$ were given by Bessel[1] for values of x varying from 0.00 to 3.20 by increments of 0.01. Hansen[2] gave six-figure tables for these functions for values of x varying from 0.0 to 20.0 by increments of 0.1.

80. The Expansion of the Excentric Anomaly and its Functions by Multiples of the Average Anomaly

It follows from the Kepler equation

$$E - e \sin E = M \tag{26}$$

that for all values of the eccentricity satisfying the following condition

$$0 < e < 1,$$

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(1) F.W. Bessel, Untersuchung des Teils der planetarischen Storungen, welcher aus der Bewegung der sonne entsteht, Abhaadlungen des Berliner Akademie 1824.

(2) P.A. Hansen, Ermittelung der absoluten Storungen in Blupsen von Belicbiger Exzentrizitat und Neigung, Schriften der Sternwarte Seeberg (Gotha), 1843.

the eccentric anomaly E is a finite and continuous (as well as all its derivatives) function of the average anomaly M. When M is increased by 2π , the eccentric anomaly also increases by 2π . Consequently, any periodic function of E having a period of 2π will also be a periodic function of M having the same period.

Let us consider thefunction cos mE, where m is an integer. This function is evidently a periodic and even function of M. We can thus assume that

$$\cos mE = \frac{1}{2} a_0^m + a_1^m \cos M + a_2^m \cos 2M + \ldots \tag{27}$$

where, on the basis of equations (2),

$$a_k^m = \frac{2}{\pi}\int_0^\pi \cos mE \cos kM \, dM.$$

In partioular, by excluding M by means of equation (26), we obtain for k = 0

$$\pi a_0^m = \int_0^\pi \cos mE\,(1 - e\cos E)\, dE =$$

$$= \int_0^\pi \cos mE \, dE - e\int_0^\pi \cos mE \cos E \, dE.$$

If m $>$ 1, each of these integrals is equal to zero. Then,

$$a_0^m = 0.$$

If m = 1, it is easy to see that

$$a_0^1 = -e.$$

If k $>$ 0, then partial integration yields

$$\pi a_k^m = -\int_0^\pi \frac{\sin kM}{k} \frac{d\cos mE}{dM} dM =$$

$$= -\frac{1}{k}\int_0^\pi \sin kM \frac{d\cos mE}{dE} dE =$$

$$= \frac{m}{k}\int_0^\pi \sin kM \sin mE \, dE$$

Substituting here for the value of M given by equation (26), we obtain

$$\frac{2k\pi}{m} a_k^m = 2\int_0^{\pi} \sin mE \sin (kE - ke \sin E)\, dE =$$

$$\int_0^{\pi} \cos [(k - m)E - ke \sin E]\, dE -$$

$$\int_0^{\pi} \cos [(k + m)E - ke \sin E]\, dE.$$

Using equation (18), we finally obtain for k $>$ 0

$$a_k^m = \frac{m}{k}[J_{k-m}(ke) - J_{k+m}(ke)].$$

Similarly, we can prove that the coefficients of the series

$$\sin mE = b_1^m \sin M + b_2^m \sin 2M + \ldots \tag{28}$$

are given by

$$b_k^m = \frac{m}{k}[J_{k-m}(ke) + J_{k+m}(ke)]$$ [illegible]

We note that coefficients of the two series, given by equations (27) and (28) can simult·aneously be obtained by considering the expansion of the function exp (imE).

When m $>$ 1, the series (27) and (28) can evidently be represented in the following form

$$\left.\begin{aligned} \cos mE &= m \sum J_{k-m}(ke) \frac{\cos kM}{k} \\ \sin mE &= m \sum J_{k-m}(ke) \frac{\sin kM}{k}. \end{aligned}\right| \tag{29}$$

When m = 1, then the series (27) can be transformed by means of equation (11) into the following form

$$\cos E = -\frac{1}{2} e + \sum_{1} \frac{2}{k} J_k'(ke) \cos kM.$$

Similarly, using equation (10), we obtain

$$\sin E = \frac{1}{e} \sum_{1}^{\infty} \frac{2}{k} J_k(ke) \sin kM. \qquad (30)$$

Substituting these expansions into equation (26) and into the following formula

$$r = a(1 - e \cos E),$$

we obtain'

$$E = M + \sum_{1}^{\infty} \frac{2}{k} J_k(ke) \sin kM \qquad (31)$$

$$\frac{r}{a} = 1 + \frac{1}{2} e^2 - e \sum_{1}^{\infty} \frac{2}{k} J_k'(ke) \cos kM. \qquad (32)$$

We now derive an expansion for the square of the radius vector. Since

$$\left(\frac{r}{a}\right)^2 = 1 + \frac{1}{2} e^2 - 2e \cos E + \frac{1}{2} e^2 \cos 2E$$

and

$$\cos 2E = \sum_{1}^{\infty} \frac{2}{k} [J_{k-2}(ke) - J_{k+2}(ke)] \cos kM =$$

$$= \sum_{1}^{\infty} \frac{2}{k} \left[\frac{2(k-1)}{ke} J_{k-1}(ke) - \frac{2(k+1)}{ke} J_{k+1}(ke)\right] \cos kM =$$

$$= \sum_{1}^{\infty} \left[\frac{8}{ke} J_k(ke) - \frac{8}{ke^2} J_k'(ke)\right] \cos kM,$$

We then obtain

$$\sum \qquad (33)$$

This formula can be derived in a simpler way, if we note that

and make use of the expansion given by equation (30). Similarly, noting that

$$\frac{dE}{dM} = (1 - e\cos E)^{-1} = \frac{a}{r},$$

we then obtain, using formula (31),

$$\frac{a}{r} = 1 + 2\sum J_k(ke)\cos kM \qquad (34)$$

This equation enables us to find the expansion coefficients

$$\left(\frac{a}{r}\right)^2 = \frac{1}{2}g_0 + \sum g_k \cos kM. \qquad (35)$$

We shall not derive the complicated expressions of the coefficients g_1 , g_2 , ... in terms of the Bessel functions. For practical purposes, it is sufficient to expand each of these coefficients in powers of e. These expansions will be given in Section 82. In the following we shall confine ourselves to the evaluation of g_0 only.

It follows from equation (35) that

$$g_0 = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{a}{r}\right)^2 dM.$$

The integral of area

$$r^2\frac{dv}{dt} = k\sqrt{1+m}\sqrt{a(1-e^2)},$$

can be represented in the new form:

$$\frac{dv}{dM} = \left(\frac{a}{r}\right)^2\sqrt{1-e^2}, \qquad (36)$$

because

$$M = k\sqrt{1+m}\,a^{-\frac{3}{2}}(t - t_0) + M_0.$$

Consequently

$$g_0 = (1-e^2)^{-\frac{1}{2}}\frac{1}{\pi}\int_0^{2\pi} dv = 2(1-e^2)^{-\frac{1}{2}},$$

and therefore,

$$\frac{1}{2}K_0 = 1 + \frac{1}{2}e^2 + \frac{1.3}{2.4}e^4 + \ldots + \frac{1.3\ldots(2h-1)}{2.4\ldots(2h)}e^{2h} + \ldots$$

Evaluating this coefficient by the simple squaring of equation (34) and comparing the results, we obtain

$$(1-e^2)^{-\frac{1}{2}} = 1 + 2\sum_{k=1}^{\infty} J_k^2(ke)$$

<u>Annotation</u>

In the expansions derived in this section, and in most of the applications in astronomy, Bessel functions are often encountered in one of the two following forms

$$\frac{2}{e}J_k(ke) = \frac{1}{(k-1)!}\left(\frac{ke}{2}\right)^{k-1}\left[1 - \frac{k^2e^2}{2.(2k+2)} + \frac{k^4e^4}{2.4.(2k+2)(2k+4)} - \ldots\right]$$

$$2J_k'(ke) =$$

$$= \frac{1}{(k-1)!}\left(\frac{ke}{2}\right)^{k-1}\left[1 - \frac{k+2}{k}\,\frac{k^2e^2}{2.(2k+2)} + \frac{k+4}{k}\,\frac{k^4e^4}{2.4.(2k+2)(2k+4)} - \ldots\right]$$

We point out the following particular cases, which are useful to have in a readily available form

$$\frac{2}{e}J_1(e) = 1 - \frac{e^2}{8} + \frac{e^4}{192} - \frac{e^6}{9216} + \ldots$$

$$\frac{2}{e}J_2(2e) = e\left(1 - \frac{e^2}{3} + \frac{e^4}{24} - \frac{e^6}{360} + \ldots\right),$$

$$\frac{2}{e}J_3(3e) = \frac{9e^2}{8}\left(1 - \frac{9e^2}{16} + \frac{81e^4}{640} - \ldots\right)$$

$$\frac{2}{e}J_4(4e) = \frac{4e^3}{3}\left(1 - \frac{4e^2}{5} + \frac{4e^4}{15} - \ldots\right)$$

$$\frac{2}{e}J_5(5e) = \frac{625e^4}{384}\left(1 - \frac{25e^2}{24} + \frac{625e^4}{1344} - \ldots\right)$$

$$\frac{2}{e}J_6(6e) = \frac{81e^5}{40}\left(1 - \frac{9e^2}{7} + \frac{81e^4}{112} - \ldots\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$2J_1'(e) = 1 - \frac{3e^2}{8} + \frac{5e^4}{192} - \frac{7e^6}{9216} + \ldots$$

$$2J_2'(2e) = e\left(1 - \frac{2e^2}{3} + \frac{e^4}{8} - \frac{e^6}{90} + \ldots\right)$$

$$2J_3'(3e) = \frac{9e^2}{8}\left(1 - \frac{15e^2}{16} + \frac{189e^4}{640} - \ldots\right)$$

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$$2J_4(4e) = \frac{4e^4}{3}\left(1 - \frac{6e^2}{5} + \frac{8e^4}{15} - \dots\right)$$

$$2J_5(5e) = \frac{625e^5}{384}\left(1 - \frac{25e^2}{21} + \frac{375e^4}{448} - \dots\right)$$

$$2J_6(6e) = \frac{81e^6}{40}\left(1 - \frac{12e^2}{7} + \frac{135e^4}{112} - \dots\right)$$

81. The Transformation of a series in Multiplies of the Eccentric anomaly into a series in multiples of the average anomaly

Let S be a periodic function of E having a period of 2π. We assume that this function is continuous and has continuous derivatives so that it can be expanded in a Fourier series

$$S = \frac{1}{2}a_0 + a_1 \cos E + a_2 \cos 2E + \dots + b_1 \sin E + \dots \qquad (37)$$

As we have pointed out, S will also be a periodic function of M and will have the same period of 2π. Hence it can also be expanded into the series

$$S = \frac{1}{2}A_0 + A_1 \cos M + A_2 \cos 2M + \dots + B_1 \sin M + \dots \qquad (38)$$

Our problem is to find the expansion (38) in the case when the coefficients of the expansion (37) are known.

Substituting for cos mE and sin mE their expressions given by formulae (27) and (28), we obtain

$$\begin{aligned} A_0 &= a_0 + a_1 a_0^1 \\ A_k &= a_1 a_k^1 + a_2 a_k^2 + \dots \\ B_k &= b_1 b_k^1 + b_2 b_k^2 + \dots \end{aligned}$$

Replacing a_k^m and b_k^m by the values found in the previous section, we obtain

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$$\left.\begin{aligned} A_0 &= a_0 - a_1 e \\ kA_k &= \sum_{m=1} m a_m [J_{k-m}(ke) - J_{k+m}(ke)] \\ kB_k &= \sum_{m=1} m b_m [J_{k-m}(ke) - J_{k+m}(ke)] \,. \end{aligned}\right\} \qquad (39)$$

Therefore, the transformation of the series (37) into the series (38) is reduced to the computation of the following quantities

$$\begin{array}{llll} J_0(e), & J_1(e), & J_2(e), & \ldots \\ J_0(2e), & J_1(2e), & J_2(2e), & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

This can be done algebraically up to within a given power of e, or numerically by means of the formulae given in Secs 78 and 79.

Cauchy had suggested another method for treating the problem under consideration. The method is as follows. Let us introduce the following notation

$$y = \exp iE, \qquad z = \exp iM$$

We replace the expansions (37) and (38) by the corresponding Maclaurin series

$$S = \sum_{-\infty}^{+\infty} p_k y^k$$

and

$$S = \sum_{-\infty}^{+\infty} P_k z^k . \qquad (40)$$

In order to calculate the coefficients P_k , we consider formula (4) which yields

$$2\pi P_k = \int_0^{2\pi} S z^{-k} dM \qquad (41)$$

On the basis of equation (26),

$$z^{-k} = \exp(-ikM) = \exp(-ikE + ike \sin E)$$
$$= y^{-k} \exp\left[\frac{ke}{2}(y - y^{-1})\right]$$
$$\frac{dM}{dE} = 1 - \frac{e}{2}(y + y^{-1}).$$

Consequently

$$2\pi P_k = \int_0^{2\pi} S y^{-k} \exp\left[\frac{ke}{2}(y - y^{-1})\right]\left[1 - \frac{e}{2}(y + y^{-1})\right] dE =$$
$$= \int_0^{2\pi} T y^{-k} dE.$$

The latter equation is nothing else but the result of application of the general formula (4) to the finding of the coefficients of expansion of the function

$$T = S\left[1 - \frac{e}{2}(y + y^{-1})\right] \exp\left[\frac{ke}{2}(y - y^{-1})\right]$$

in powers of y. We thus obtain Cauchy's first rule. In order to obtain the coefficients P_k of the series (40), it is necessary to expand the function T_1 in which S is replaced by the series (37), in powers of y; the coefficients of y^k will be equal to P_k.

On the other hand, since

$$\frac{dz}{dM} = iz,$$

then equation (41) yields

$$2\pi P_k = -i \int_0^{2\pi} S z^{-k-1} \frac{dz}{dM} dM = ik^{-1} \int_0^{2\pi} S \frac{dz^{-k}}{dM} dM =$$
$$= -ik^{-1} \int_0^{2\pi} z^{-k} \frac{dS}{dM} dM = -ik^{-1} \int_0^{2\pi} z^{-k} \frac{dS}{dE} dE.$$

However,

$$\frac{dS}{dE} = \frac{dS}{dy}\frac{dy}{dE} = iy\frac{dS}{dy},$$

Therefore, expressing again z^{-k} in terms of y^{-k}, we obtain

$$2\pi P_k = \int_0^{2\pi} U y^{k+1}\, dE,$$

where

[illegible]

This expression of P_k proves Cauchy's second rule. In order to obtain the coefficients P_k of the series (40), it is necessary to expand the function U in powers of y and take the coefficients of y^{k-1}.

The functions which we usually have to expand are almost in all cases very simply expressed in terms of the combinations $y + y^{-1}$ or $y - y^{-1}$. The application of theabove-mentioned rules leads to the use of the so-called Cauchy's numbers. These are the coefficients $N_{-p,j,q}$ in the expansion

$$(y + y^{-1})^j (y - y^{-1})^q = \sum_p N_{p,j,q}\, y^p,$$

where j and q are non-negative integers.

At the present time, all the expansions applied in celestial mechanics are available in readily available forms. Therefore, we shall not consider here the properties of Cauchy's numbers(1).

82. <u>The expansion of some functions of the coordinates of the elliptic motion</u>

We have already found the expansions of the radius of vector in a series, (32), in multiples of the average anomaly. In the following, we

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(1) See F Tisserand, Traite de Mécanique Celeste, I, Paris, 1889, 234-237, and references cited therein.

obtain a similar expansion for the true anomaly.

We first of all express the true anomaly in terms of the eccentricity by using the following formula

$$\operatorname{tg}\frac{v}{2}=\sqrt{\frac{1+e}{1-e}}\operatorname{tg}\frac{E}{2}. \qquad (42)$$

Putting

$$\frac{1}{2}v=w, \quad \frac{1}{2}E=u, \quad \sqrt{\frac{1+e}{1-e}}=\mu,$$

we obtain

$$\operatorname{tg}w=\mu\operatorname{tg}u,$$

or

$$\frac{\exp(2iw)-1}{\exp(2iw)+1}=\mu\frac{\exp(2iu)-1}{\exp(2iu)+1}.$$

Assuming that $\beta=\frac{\mu-1}{\mu+1}$, we obtain

$$\exp(2iw)=\frac{1-\mu+(1+\mu)\exp(2iu)}{1+\mu+(1-\mu)\exp(2iu)}=\frac{1-\beta\exp(-2iu)}{1-\beta\exp(2iu)}\exp(2iu).$$

Taking the logarithm of both sides yields

$$w=u+\frac{1}{2i}\ln(1-\beta\exp(-2iu))-\frac{1}{2i}\ln(1-\beta\exp(2iu)),$$

or

[illegible]

Applying this formula to equation (42) yields the following expansion

$$v=E+2\left[\beta\sin E+\frac{1}{2}\beta^2\sin 2E+\frac{1}{3}\beta^3\sin 3E+\ldots\right], \qquad (43)$$

where

$$\beta = \frac{\mu - 1}{\mu + 1} = \frac{e}{1 + \sqrt{1 - e^2}}$$

Rewriting equation (42) in the form

$$\operatorname{tg}\frac{E}{2} = \frac{1}{\mu}\operatorname{tg}\frac{v}{2}$$

and applying again formula (43), we obtain

$$E = v - 2\left|\beta \sin v - \frac{1}{2}\beta^2 \sin 2v + \frac{1}{3}\beta^3 \sin 3v - \dots\right|. \qquad (45)$$

Substituting for E, sin E, sin 2E, ... in equation (44) the corresponding expressions given by equatinns (31) and (28), we obtain the important expansions which defines the equation of the centre, namely

$$v = M + H_1 \sin M + H_2 \sin 2M + \dots, \qquad (46)$$

where

$$H_1 = 4\left(\frac{e}{2}\right) - 2\left(\frac{e}{2}\right)^3 + \frac{5}{3}\left(\frac{e}{2}\right)^5 + \frac{107}{36}\left(\frac{e}{2}\right)^7 - \dots$$

$$H_2 = 5\left(\frac{e}{2}\right)^2 - \frac{22}{3}\left(\frac{e}{2}\right)^4 + \frac{17}{3}\left(\frac{e}{2}\right)^6 + \dots$$

$$H_3 = \frac{26}{3}\left(\frac{e}{2}\right)^3 - \frac{43}{2}\left(\frac{e}{2}\right)^5 + \frac{95}{4}\left(\frac{e}{2}\right)^7 - \dots$$

$$H_4 = \frac{103}{6}\left(\frac{e}{2}\right)^4 - \frac{902}{15}\left(\frac{e}{2}\right)^6 + \dots$$

$$H_5 = \frac{1097}{30}\left(\frac{e}{2}\right)^5 - \frac{5957}{36}\left(\frac{e}{2}\right)^7 + \dots$$

$$H_6 = \frac{1223}{15}\left(\frac{e}{2}\right)^6 - \dots \qquad H_7 = \frac{47\,273}{252}\left(\frac{e}{2}\right)^7 - \dots$$

In the following, we give the values of H_1, F_2, ... in seconds of arc where we use the logarithms instead of the numerical coefficients.

$$H_1 = [5.615\,155\,129]\,e - [4.712\,3651]\,e^3 + [4.031\,124]\,e^5 + [3.68\,3]\,e^7 + [3.541]\,e^9 + \ldots$$
$$H_2 = [5.311\,335\,146]\,e^2 - [4.975\,6066]\,e^4 + [4.261\,57]\,e^6 + [3.1875]\,e^8 + \ldots$$
$$H_3 = [5.319\,187\,24]\,e^3 - [5.111\,7136]\,e^5 + [4.582\,88]\,e^7 - [3.514]\,e^9 \ldots$$
$$H_4 = [5.344\,0011]\,e^4 - [5.287\,56]\,e^6 + [4.682]\,e^8 - [\text{[illegible]}]\,e^{10} + \ldots$$
$$H_5 = [5.372\,5605]\,e^5 - [5.125\,93]\,e^7 + [5.120]\,e^9 - [4.551]\,e^{11} + \ldots$$
$$H_6 = [5.410\,\text{[illegible]}]\,e^6 - [\text{[illegible]}]\,e^8 + [5.318]\,e^{10} - \ldots$$
$$H_7 = \text{[illegible]} \ldots$$
$$H_8 = \text{[illegible]} \ldots$$
$$H_9 = \text{[illegible]} \ldots$$
$$H_{10} = \text{[illegible]} \ldots$$
$$H_{11} = [\text{[illegible]}]\,e^{11} - [\text{[illegible]}\,227]\,e^{13} \ldots$$
$$H_{12} = [5.8\text{[illegible]}]\,e^{12} \ldots \qquad H_{13} = [5.992]\,e^{13} \ldots$$

For convenience, we write equation (22) which determines the radius vector in the following unfolded form

$$\frac{r}{a} = 1 + \frac{1}{2}e^2 - \sum_{1} G_k \cos kM, \tag{47}$$

Then

$$G_1 = 2\left(\frac{e}{2}\right) - 3\left(\frac{e}{2}\right)^3 + \frac{5}{6}\left(\frac{e}{2}\right)^5 - \frac{7}{72}\left(\frac{e}{2}\right)^7 \ldots$$
$$G_2 = 2\left(\frac{e}{2}\right)^2 - \frac{16}{3}\left(\frac{e}{2}\right)^4 + 4\left(\frac{e}{2}\right)^6 \ldots$$
$$G_3 = 3\left(\frac{e}{2}\right)^3 - \frac{15}{4}\left(\frac{e}{2}\right)^5 + \frac{567}{40}\left(\frac{e}{2}\right)^7 \ldots$$
$$G_4 = \frac{16}{3}\left(\frac{e}{2}\right)^4 - \frac{128}{5}\left(\frac{e}{2}\right)^6 + \ldots$$
$$G_5 = \frac{125}{12}\left(\frac{e}{2}\right)^5 - \frac{4375}{72}\left(\frac{e}{2}\right)^7 + \ldots$$
$$G_6 = \frac{108}{5}\left(\frac{e}{2}\right)^6 - \ldots, \qquad G_7 = \frac{16807}{360}\left(\frac{e}{2}\right)^7 \ldots$$

Replacing the numerical coefficients by their logarithms, we obtain

$$
\begin{aligned}
G_1 &= e - [9.5740313]e^3 + [8.4156688]e^5 - [6.8\ldots]e^7 + \ldots \\
G_2 &= [9.6989700]e^2 - [9.5228787]e^4 + [8.7958\ldots]e^6 - [7.7117]e^8 + \ldots \\
G_3 &= [9.5710313]e^3 - [9.5360025]e^5 + [9.011315]e^7 - [8.2\ldots]e^9 + \ldots \\
G_4 &= [9.5228787]e^4 - [9.6020600]e^6 + [9.2\ldots8]e^8 - [8.627]e^{10} + \ldots \\
G_5 &= [9.5125788]e^5 - [9.676136]e^7 + [9.4353\ldots]e^9 - [8.938]e^{11} + \ldots \\
G_6 &= [9.5282738]e^6 - [9.762357]e^8 + [10.0\ldots]e^{10} - [9.21\ldots]e^{12} + \ldots \\
G_7 &= [9.5598]e^7 - [9.8517]e^9 + [9.7762]e^{11} + \ldots \\
G_8 &= [9.6089]e^8 - [9.9557]e^{10} + [9.938]e^{12} + \ldots \\
G_9 &= [9.6659]e^9 - [10.0595]e^{11} + [10.0\ldots]e^{13} \ldots \\
G_{10} &= [9.731]e^{10} - [10.167]e^{12} \\
G_{11} &= [9.8025]e^{11} - [10.277]e^{13} \\
G_{12} &= [9.879]e^{12} \ldots \qquad G_{13} = [9.96\ldots]e^{13} - \ldots
\end{aligned}
$$

where we have to subtract 10 from each logarithm given here.

It is useful to note that the expansion (46) may be obtained in another way. Indeed, substituting series (35) into formula (36) and integrating, we obtain

$$v - M = [illegible]\, e^{2} \sum_{1}^{\infty} [illegible] \sin kM \qquad (46')$$

In the series-expansion of the perturbation function, we make use of the expansion of the following functions

$$[illegible]$$

where p, n and m are integers such that p and m either take positive values or are equal to zero. The calculation of the expansion of such functions up to a given power of e is simple enough. We have for example

$$\left(\frac{r}{a} - 1\right)^p \left(\frac{r}{a}\right)^n \sin m(v - M)$$

$$= \left(\frac{r}{a} - 1\right)^p \left(\frac{r}{a}\right)^n \left\{ m(v - M) - \frac{1}{6} m^3 (v - M)^3 + \ldots \right\},$$

Replacing $\frac{r}{a}$ and v-M by the series (47) and (46), we obtain the required result.

The coefficients of the expansions of the functions (48) up to e^7 were given by Leverrier[1]. Cuyley[2] gave the coefficients of expansion of the functions within the same accuracy.

$$\left(\frac{r}{a}-1\right)^p \frac{\cos}{\sin} mv$$

where p = 0, 1, ..., 7 and m = 0, 1, ..., 7, and also for the functions

$$\left(\frac{r}{a}\right)^n \frac{\cos}{\sin} mv$$

where n = - 5, -4, ..., -1, 1, ..., 4, and m = 0, 1, ... 5.

Some of the most commonly used expansions are given in tables I and II at the end of this volume. These tables give the coefficients of different powers of e in the power series for the coefficients $C_k^{n,m}$ and $S_k^{n,m}$ involved in the following expansions

$$\left.\begin{aligned} \left(\frac{r}{a}\right)^n \cos mv &= C_0^{n,m} + C_1^{n,m} \cos M + C_2^{n,m} \cos 2M + \dots \\ \left(\frac{r}{a}\right)^n \sin mv &= S_1^{n,m} \sin M + S_2^{n,m} \sin 2M + \dots \end{aligned}\right\} \quad (49)$$

For exampoe, table II shows that

$$\begin{aligned} \left(\frac{r}{a}\right) \sin v = & \left(1 - \frac{5}{8} e^2 - \frac{11}{192} e^4 - \frac{457}{9216} e^6 - \dots\right) \sin M + \\ & + \left(\frac{1}{2} e - \frac{5}{12} e^3 + \frac{1}{24} e^5 - \frac{1}{16} e^7 + \dots\right) \sin 2M + \\ & + \left(\frac{3}{8} e^2 - \frac{51}{128} e^4 + \frac{543}{5120} e^6 - \dots\right) \sin 3M + \dots \end{aligned}$$

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(1) U.I.J. Leverrier, Recher ches astronomiques, Annales de l'Observatoire de Paris, 1, 1885, 343-365.

(2) A. Cuyley, Tables of the Developments of Functions in the Theory of Elliptic Motion, Memoirs of the R. Astron. Society, 29, 1869, 191-306.

In conclusion, we give the following expansions, which can easily be obtained from equations (47).

Annotation

The coefficients of the series (46) and (46') have a very complex structure. It is much easier to express the coefficients of the expansion of the equation of the centre in multiples of thetrue anomaly. In order to derive this expansion, we consider the following formulae

These formulae lead to

Since

where

then

[illegible]

Therefore,

[illegible]

Replacing E by expression (45), we finally obtain

[illegible]

83. Hansen's Coefficients

Instead of separately considering the two expansions given by equations (49), it is possible to stuedy only the following Maclaurin series

[illegible]

The coefficients $X_k^{n,m}$ of this series are sometimes called the Hansen's coefficients since Hansen was the first to give general expressions of these quantities in the form of series-expansions in powers of β. An alternative and simpler derivation of Hansen's formulae was suggested by Tisserand[1].

The simplest way to obtain these coefficients is to apply Cauchy's first rule. Let us express the function

$$S = \left(\frac{r}{a}\right)^n x^m$$

in terms of y. Since

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(1) F. Tisserand, Traite de Mechanique Celeste, 1, 1889, Ch. XV.

$$\frac{r}{a} = 1 - e\cos E = 1 - \frac{e}{2}(y + y^{-1}),$$

then, according to Cauchy's first rule, the coefficient $X_k^{n,m}$ will be equal to the coefficient of y^k in the expansion of the following expression

$$T = S\left[1 - \frac{e}{2}(y + y^{-1})\right]\exp\left[\frac{ke}{2}(y - y^{-1})\right] = \left(\frac{r}{a}\right)^{n+1} x^m \exp\left[\frac{ke}{2}(y - y^{-1})\right].$$

It is easy to see that

$$\frac{r}{a} = (1 + \beta^2)^{-1}(1 - \beta y)(1 - \beta y^{-1}),$$

where, we denote as previously

$$\beta = \frac{e}{1 + \sqrt{1 - e^2}} = \frac{1 - \sqrt{1 - e^2}}{e}.$$

On the other hand, the relation

$$\operatorname{tg}\frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}}\operatorname{tg}\frac{E}{2}$$

may be rewritten in the following way

$$\frac{x - 1}{x + 1} = \frac{1 + \beta}{1 - \beta}\,\frac{y - 1}{y + 1}.$$

Therefore,

$$x = y(1 - \beta y^{-1})(1 - \beta y)^{-1}.$$

Consequently,

$$T = (1 + \beta^2)^{-n-1} y^m (1 - \beta y)^{n-m+1}(1 - \beta y^{-1})^{n+m+1}\exp\left[\frac{ke}{2}(y - y^{-1})\right].$$

Using the binomial formula, we can easily calculate the expansion coefficients of

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$$(1-\beta y)^{n-m+1}(1-\beta y^{-1})^{n+m+1} = \sum E_{k,p}^{n,m} y^{k-p-m}$$

and obtain them in the following form

$$E_{k,p}^{n,m} = [illegible]$$

where F(a,b,c,x) is the hypergeometric function. Since,

$$\exp\left[\frac{ke}{2}(y - y^{-1})\right] = \sum J_p(ke)\, y^p,$$

then the unknown coefficient of y^k in the expansion of the function T is equal to

$$X_k^{n,m} = (1+\beta^2)^{n+1} \sum E_{k,p}^{n,m} J_p(ke).$$

This formula enables us to obtain the coefficients of the expansion $C_k^{n,m}$ and $S_K^{n,m}$ in powers of eccentricity.

84. <u>On the Convergence of the Series-Expansions of the Coordinates of the Elliptic Motion</u>

In the previous sections we obtained the expansions of different functions of the eccentric anomaly E in Fourier series, developed by multiples of the average anomaly M. On the basis of Dirichlet's theorem, these series converge for all values of M and e only if e < 1 as in this case where the expanded functions and their derivatives are continuous. However, due to the complexity of the expansion coefficients, these coefficients are usually expanded in powers of e in which terms higher than a given power are dropped. Accordingly, we are practically dealing with power series, developed in positive powers of e, the expansion coefficients of which are periodic functions of M with a period of 2 π. The radius of convergence of such a series is some function $\varphi(M)$ of theaverage anomaly. Our task is to find the minimum value of this function, $\varphi(M)$, when the variable M varies from 0 to 2 Let us consider an arbitrary function F(E) of the eccentric anomaly and investigate its dependence on e and M, implied by the Kepler equation.

$$E - e \sin E = M \qquad (51)$$

For a given M, this function is a holomorphic function for all values of e for which the derivative

$$\frac{dF(E)}{de} = F'(E) \frac{\sin E}{1 - e \cos E}$$

is finite. Hence, the general singular points of all the functions F(E) are given by the following equation

$$1 - e \cos E = 0, \qquad (52)$$

which is to be solved simultaneously with equation (51). The only exception are those functions for which the product F(E) sin E is either zero or infinity for values of the variables satisfying condition (52). We shall not consider these functions now. The radius of convergence (M) of all the functions under considerations will be equal to the least of the roots e of equations (51) and (52).

Let us now study the function $\varphi(M)$. We primarily note that

$$\varphi(\pi \pm M) = \varphi(M)$$

Indeed, replacing E, M and e in equations (51) and (52) by $\pi \pm E$, $\pi \pm M$ and $-e$ does not violate these equations. Hence, the above-mentioned change in the variable M will transform each singular point e_0 into the singular point $-e_0$ that has the same modulus. Consequently, the radius of the circle of convergence will not be changed.

It is somewhat more difficult to prove another property of the function $\varphi(M)$, stating that the minimum value of this function is equal to $\varphi(\frac{\pi}{2})$. According to Poincaré, we consider the function

$$F(E) = \exp(2imE),$$

where m is an integer. The derivative of this function satisfies the above mentioned conditions. The expansion of this function in a power series can be done easily using formulae (27) and (7) which yield

$$F(E) = \sum_k^{+\infty} \frac{2m}{k} J_{k-2m}(ke) z^k = \sum_k^{+\infty} \frac{2m}{k} z^k \sum_\beta \frac{(-1)^\beta}{\beta!(\beta+k-2m)!} \left(\frac{ke}{2}\right)^{k-2m+2\beta}.$$

By adopting that

$$F(E) = \Phi(M, e)$$

and considering the sum

$$\Psi(M, e) = \Phi(M, e) + \Phi(\pi + M, e). \qquad (53)$$

Evidently,

$$\Psi(M, e) = \sum_h^{+\infty} \frac{2m}{h} z^{2h} \sum_\beta \frac{(-1)^\beta}{\beta!(\beta+2h-2m)!} (he)^{2h-2m+2\beta}, \qquad (54)$$

since all terms with even powers of z are cancelled. Let $M = M_1$ be an arbitrary given value of the average anomaly, not equal to $\frac{\pi}{2}$. We denote by e_1 a real number which satisfies the following condition

$$\varphi(M_1) < e_1 < 1 \qquad (55)$$

In this case, the series $\Phi(M_1, e_1)$ is evidently divergent. We shall now prove that the sum (53) will also diverge for the values of these variables. Indeed, the particular points e_0 of the function (M_1, e) correspond to the particular points $-e_0$ of the function $\Phi(\pi + M_1, e)$, although $-e_0$ cannot be a particular point of (M_1, e)

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since the substitution of e by -e in equations (51) and (52) replaces M by $\pi \pm M$. Hence, the particular point of one of the terms of the expression (53) will definitely be a singular point of the whole sum since $M_1 \neq \pi \pm M_1$ once $M_1 \neq \frac{\pi}{2}$. Consequently, the series $\Psi(M_1, e_1)$ diverges, if condition (55) applies. Comparing the terms of this series with the corresponding terms in the series

$$\Psi\left(-\frac{\pi}{2}, ie_1\right) = \Phi\left(-\frac{\pi}{2}, ie_1\right) + \Phi\left(+\frac{\pi}{2}, ie_1\right). \tag{56}$$

Evidently, the absolute values of the compared terms will be equal. At the same time, the arguments of the terms of $\Psi(M_1, e_1)$ will be different although the arguments of all terms of the series Ψ $(-\frac{\pi}{2}, i\,e_1)$ are equal. Indeed the arguments of each term of the latter series are equal to

$$-\frac{\pi}{2}\,2h + \pi\beta + \frac{\pi}{2}\,(2h - 2m + 2) = -m\pi.$$

as one can easily see from equation (54). Therefore once the series $\Psi(M_1, e_1)$ diverges, the expansion of the function (56) also diverges. It then follows that for at least one of the functions $\Phi(\pm\frac{\pi}{2}, ie_1)$ the series expansion in powers of the eccentricity diverges, if it diverges for $\Phi(M_1, e_1)$. In other words,

$$\varphi\left(\frac{\pi}{2}\right) = \varphi(M_1).$$

which was required to prove.

Thus, in order to find the minimum value of the function, it is necessary to find the root e_0 of the equations

$$1 - e_0 \cos E_0 = 0$$
$$E_0 - e_0 \sin E_0 = \frac{\pi}{2}.$$

that has the least absolute value. We then obtain

$$\min \varphi(M) = e_0' .$$

These equations yield

$$E_0 - \operatorname{tg} E_0 = \frac{\pi}{2}$$

or, putting $E_0 = \frac{\pi}{2} - \epsilon$,

$$\epsilon + \operatorname{ctg} \epsilon = 0. \tag{56}$$

We shall only consider thecomplex roots of equation (56). The real roots of this equation yields

$$e_0 = \left| \frac{1}{\cos E_0} \right| > 1$$

and are thus not interesting to us. To each root ϵ of equation (56) there will be a corresponding conjugate root ϵ'. This equation must thus have at least two roots. Considering any pair of roots ϵ and ϵ' of equation (56) and construct the auxiliary functions

$$\varphi = \cos \epsilon u, \quad \varphi' = \cos \epsilon' u.$$

These functions satisfy the following equations

$$\frac{d^2\varphi}{du^2} + \epsilon^2 \varphi = 0, \quad \frac{d^2\varphi'}{du^2} + \epsilon'^2 \varphi' = 0.$$

Consequently,

$$\varphi' \frac{d^2\varphi}{du^2} - \varphi \frac{d^2\varphi'}{du^2} = (\epsilon'^2 - \epsilon^2) \varphi \varphi'$$

or

$$\frac{d}{du} \left| \varphi' \frac{d\varphi}{du} - \varphi \frac{d\varphi'}{du} \right| = (\epsilon'^2 - \epsilon^2) \varphi \varphi'$$

We integrate this equation from 0 to 1. Since at u = 0

$$\frac{d\varphi}{du} = 0, \quad \frac{d\varphi'}{du} = 0,$$

and at u = 1

$$\frac{d\varphi}{du} = -\varepsilon \sin\varepsilon = \cos\varepsilon, \quad \frac{d\varphi'}{du} = \cos\varepsilon',$$

then on using equation (56), we obtain

$$(\varepsilon'^2 - \varepsilon^2) \int_0^1 \varphi\varphi' \, du = 0.$$

It follows that for each pair of complex conjugate roots ϵ and ϵ'

$$\varepsilon'^2 - \varepsilon^2 = 0,$$

because in this case φ and φ' will also be complex conjugate numbers and thus

$$\varphi\varphi' = |\varphi|^2 > 0.$$

The latter equation holds true only for either real or imaginary values of ϵ and ϵ'. Thus, equation (56) has only pure and real, and pure and imaginary roots. Since thefirst case is not of interest to us, we shall search for those imaginary roots having least absolute values. Substituting into equation (56)

$$\varepsilon = i\rho,$$

we obtain

$$\mathrm{ch}\,\rho - \rho\,\mathrm{sh}\,\rho = 0$$

which yields

$$\rho = 1.199678640257734 \ldots$$

Since

$$\min \varphi(u) = c_0 \left| \frac{1}{\sin\varepsilon} \right| = \frac{1}{\mathrm{sh}\,\rho},$$

then the radius of convergence of the expansion in powers of the eccentricity will be equal to

$$0.6627434193492\ldots$$

This is the theoretical limit of convergence of the obtained series. Naturally, these series loose their practical value at a much earlier stage.

<u>Annotation</u>

We have proved that the expansions of the functions of the coordinates of the elliptic motion in trigonometric series developed by multiples of the average anomaly are convergent for all values of M and for all values of e satisfying the condition

$$0 \leq e < 1$$

On the other hand we have just seen that when these expansions are developed in powers of e, then they converge only inside the interval

$$0 \leq e < 0.6627\ldots$$

This change in the radius of convergence is related to the fact that when the Bessel functions are expanded in powers of the eccentricity as

$$J_k(ke) = \frac{1}{k!}\left(\frac{ke}{2}\right)^k \left\{1 - \frac{k^2e^2}{2(2k+2)} + \ldots\right\}$$

the accuracy of approximating them by the leading terms of the expansions decreases with increasing values of k. For example the ratio of the second term of this expansion to the first term equals in absolute value to

$$\frac{k^2e^2}{2(2k+2)},$$

and thus increases to infinity with increasing values of k.

85. The Calculation of the Longitude and Latitude of a Planet

Let us denote by w the longitude of a planet in an orbit. This longitude is expressed in terms of the previously used quantities by the following relation

$$w = \pi + v = \Omega + \omega + v = \Omega + u. \qquad (57)$$

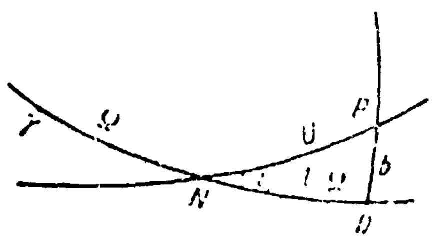

Fig. 12

Let us denote by $l = \gamma Q$ and b = PQ (Fig. 12) the heliocentric longitude and latitude of the planet P. The rectangular spherical triangle PQ yields

$$\operatorname{tg}(l - \Omega) = \cos i \operatorname{tg} u \qquad (58)$$
$$\sin b = \sin i \sin u. \qquad (59)$$

We make use of equation (43) in order to determine l from equation (58). Since in the present case,

$$\beta = \frac{\cos i - 1}{\cos i + 1} = -\operatorname{tg}^2 \frac{i}{2},$$

then

$$l - \Omega = u - \frac{1}{\text{arc } 1''} \operatorname{tg}^2 \frac{i}{2} \sin 2u + \frac{1}{2 \text{ arc } 1''} \operatorname{tg}^4 \frac{i}{2} \sin 4u - \ldots .$$

Taking equation (57) into account, we may write

$$l = w + R,$$

where

$$R = -\frac{1}{\text{arc } 1''} \operatorname{tg}^2 \frac{i}{2} \sin 2u + \frac{1}{\text{arc } 2''} \operatorname{tg}^4 \frac{i}{2} \sin 4u - \ldots$$

We shall call the difference R between the longitude and latitude in the orbit the reduction to the ecliptic. For a constant orbital slope i, this quantity may be tabulated by the argument u. A table which gives the heliocentric latitude b by the argument u may be similarly constructed.

CHAPTER XIII

THE SERIES - EXPANSION OF THE PERTURBATION FUNCTION

86. Introduction. Expansions in Powers of the Mutual Slope

In order that the differential equations which define the perturbations (Sec. 15) can be integrated in a general and not in a particular form, it is necessary to have an analytical expression for the perturbation function

$$R = k^2 m' R_{0,1}, \qquad R' = k^2 m R_{1,0},$$

where

$$R_{0,1} = \frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3}$$

$$R_{1,0} = \frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r^3} \qquad (1)$$

in terms of the orbital elements a, e, ..., a', e', ... In this Chapter, we only consider the most important methods for obtaining these expressions, which will indispensibly have the form of infinite series.

First of all, we shall consider the series expansion of the quantity

$$\Delta^{-1} = (r^2 + r'^2 - 2rr' \cos H)^{-\frac{1}{2}}, \qquad (2)$$

where H is the angle between the radius vectors r and r'. This quantity is known as the principal part of the perturbation function. Its expansion is the most difficult part of our problem. The expansions of the other parts of equations (1) are relatively simple.

The expression of the radius vector in terms of time and orbital elements has been studied in the previous chapter. We now consider how to find the angle H. Referring to figure 11, we find from the triangle Ω, Ω_1 and N the sides N and N_1 and the angle J (we are keeping the notations of Sec. 68) either by means of formulae (16) of Chapter XI, or by means of the following relations

$$\left.\begin{aligned}
\sin J \sin N &= \sin i' \sin(\Omega' - \Omega) \\
\sin J \cos N &= \cos i' \sin i - \sin i' \cos i \cos(\Omega' - \Omega) \\
\cos J &= \cos i' \cos i + \sin i' \sin i \cos(\Omega' - \Omega) \\
\sin J \sin N_1 &= \sin i \sin(\Omega' - \Omega) \\
\sin J \cos N_1 &= -\cos i' \sin i + \sin i' \cos i \cos(\Omega' - \Omega),
\end{aligned}\right\} \qquad (3)$$

which are consequences of the main theorems of spherical trigometry. We can then represent the longitudes w and w' of planets P and P' in their orbits in the following way

$$w = \pi + W, \qquad w' = \pi' + W',$$

assuming that

$$\pi = \Omega + N, \qquad \pi' = \Omega' + N_1$$

and denoting by W and W' the longitudes which are measured from the intersection point of the orbits. Consequently, referring to triangle NPP', we obtain

$$\cos H = \cos W \cos W' + \sin W \sin W' \cos J$$

or

$$\cos H = \cos(W' - W) - 2\sigma^2 \sin W \sin W', \qquad (4)$$

where

$$\sigma = \sin\frac{J}{2}.$$

We substitute this expression of cos H into equation (2) and write Δ^{-1} in the following way

$$[r^2 + r'^2 - 2rr'\cos(W' - W)]^{-\frac{1}{2}}\left\{1 + \frac{4\sigma^2 rr' \sin W \sin W'}{r^2 + r'^2 - 2rr'\cos(W' - W)}\right\}^{-\frac{1}{2}}.$$

We only consider the case in which the second term inside the curved brackets is always less than a proper fraction. Since this term is in absolute values less than

$$\frac{4\sigma^2 rr'}{(r-r')^2}$$

than this condition will be satisfied, as it is easy seen, for all the large planets of the solar system. Indeed, the maximum value of angle J between the planets in orbit (occurring for Mars and Mercury) equals only $12^{\circ}30'$, which gives $\sigma^2 = 0.0118$. On the other hand, the difference r - r' for each pair of planets will always be greater than a given quantity which will be the greater, the greater the product rr'. We shall not consider the cases when the above mentioned condition is not m met.

We thus expand the second factor in a series. Applying the binomial formula and assuming

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(1) A more general method of expansion of the perturbation function, which holds for arbitrary slopes but is in turn much more difficult, has been given by Tisserand: F. Tisserand, Traite de Mechanique Celeste, 1, Ch. XXVIII; H.C. Plummer, An Introductory Treatise on Dynamical Astronomy, 1918, Cambri ge;
O. Backlund, Zur Entwickelung der Storungstunction, Memoirs of the Academy of Science (Memuary Akademii Nauk) VII serie, t. 32, 1884.
A detailed bibliography is given in: H.v. Zeipel, Entwicklung der Storungsfunktion, Encyklopedie der Mathem. Wessenschaften Bd. VI, 2, 1912.

$$\Delta_0 = [r^2 + r'^2 - 2rr' \cos (W' - W)]^{\frac{1}{2}},$$

We finally obtain

$$\left.\begin{aligned} \Delta^{-1} = {} & \Delta_0^{-1} - rr' \Delta_0^{-3} \cdot 2\sigma^2 \sin W' \sin W + \\ & + r^2 r'^2 \Delta_0^{-5} \cdot 6\sigma^4 \sin^2 W' \sin^2 W - \\ & - r^3 r'^3 \Delta_0^{-7} \cdot 20\sigma^6 \sin^3 W' \sin^3 W' + \\ & \cdots\cdots\cdots\cdots \end{aligned}\right\} \quad (5)$$

These terms are sufficient for all of the large planets.

Let us denote by H and H' the perihelion distances of the planets P and P' from Point N at which their orbits intersect. In this case

$$W = H + v, \qquad W' = H' + v',$$

where v and v' are true anomalies.

Formulae (5) defines the expansion of the principal part of the perturbation function in terms of the mutual slopes of the orbits. We now consider the second parts of the functions (1). Since

$$xx' + yy' + zz' = rr' \cos H,$$

the calculation of the second part is then reduced to the computation of expression

$$R_1 = \frac{r \cos H}{r'^2}, \qquad R'_1 = \frac{r' \cos H}{r^2}.$$

Using equation (4), we obtain

$$\left.\begin{aligned} r'R_1 &= \frac{r}{r'} [\cos (W' - W) - \sigma^2 \cos (W' - W) + \sigma^2 \cos (W' + W)] \\ r'R'_1 &= \left(\frac{r'}{r}\right)^2 [\cos (W' - W) - \sigma^2 \cos (W' - W) + \sigma^2 \cos (W' + W)]. \end{aligned}\right\} \quad (6)$$

In order to obtain the expansions of the perturbation function, given by equations (5) and (6), in a final form, it is necessary to express the coordinates of the planets, r, v, r' and v', in terms of the orbital elements. For this purpose, it is necessary to again reconsider the particular case in which the eccentricities of the orbits are equal to zero, i.e. when the motion of planets P and P' proceeds in a circle.

87. The Case of Circular Orbits:

If the eccentricities of the planets under consideration are equal to zero, then

$$r = a, \quad w = \lambda, \qquad r' = a', \quad w' = \lambda',$$

where we denote by λ and λ' the average longitude in the orbit. Putting

$$L = \lambda - \tau, \qquad L' = \lambda' - \tau',$$

we obtain, instead of equation (5),

$$\Delta^{-1} = I - II + III - IV + \ldots, \tag{7}$$

where

$$\left.\begin{aligned}
I &= \Delta_0^{-1} \quad [a^2 + a'^2 - 2aa' \cos(L' - L)] \\
II &= aa' \Delta_0^{-3} \cdot 2\sigma^2 \sin L \sin L' \\
III &= a^2 a'^2 \Delta_0^{-5} \cdot 6\sigma^4 \sin^2 L \sin^2 L' \\
IV &= a^3 a'^3 \Delta_0^{-7} \cdot 20\sigma^6 \sin^3 L \sin^3 L' \\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right| \tag{8}$$

In this way, the problem of further expansion is reduced to the expansion of a trigonometric series of the following type

$$(aa')^{\frac{n-1}{2}} \Delta_0^{-n} = a'^{-1} \alpha^{\frac{n-1}{2}} (1 - 2\alpha \cos S + \alpha^2)^{-\frac{n}{2}},$$

where n = 1, 3, 5, 7, ... and S = L' - L. Moreover,

$$\alpha = \frac{a}{a'}.$$

Choosing our notations such that a < a', we can consider that and write

$$(1 - 2\alpha\cos S + \alpha^2)^{-\frac{n}{2}} = \frac{1}{2}\sum_{-\infty}^{+\infty} b_n^{(i)} \cos iS, \qquad (9)$$

since the function standing on the left-hand side can evidently be expanded in a Fourier series. The coefficients $b_n^{(i)}$ are known as the Laplace coefficients.

Furthermore, putting

$$c_n^{(i)} = \alpha^{\frac{n-1}{2}} b_n^{(i)}, \qquad (10)$$

we obtain

$$(aa')^{\frac{n-1}{2}} \Delta^{-n} = a'^{-1} \cdot \frac{1}{2}\sum_{-\infty}^{+\infty} c_n^{(i)} \cos i(L' - L). \qquad (11)$$

We must note that

$$b_n^{(-i)} = b_n^{(i)}, \qquad c_n^{(-i)} = c_n^{(i)}.$$

Substituting series (11) in equation (7), we have to multiply each of these series (for n = 3, 5, 7, ...) by one of the following expressions

$$\begin{aligned}
2\sin L\sin L' &= \cos(L' - L) - \cos(L' + L),\\
8\sin^2 L\sin^2 L' &= 2 - 2\cos 2L - 2\cos 2L' + \cos(2L' + 2L) +\\
&\quad + \cos(2L' - 2L),\\
32\sin^3 L\sin^3 L' &= 9\cos(L' - L) - 9\cos(L' + L) +\\
&\quad + 3\cos(3L + L') - 3\cos(3L - L') +\\
&\quad + 3\cos(3L' + L) - 3\cos(3L' - L) +\\
&\quad + \cos(3L' - 3L) - \cos(3L' + 3L).\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

In other words, we have to calculate a product of the type

$$\begin{aligned}
\cos\nu \sum c_n^{(i)} \cos i(L' - L) &= \frac{1}{2}\sum c_n^{(i)} \cos[i(L' - L) + \nu] +\\
&\quad + \frac{1}{2}\sum c_n^{(i)} \cos[i(L' - L) - \nu].
\end{aligned}$$

The two sums on the right-hand side of this equation are equal, and they are found to be so when i in one of them is replaced by - i. Therefore

$$\cos\nu \sum c_n^{(i)} \cos i(L'-L) = \sum c_n^{(i)} \cos\left[i(L'-L)+\nu\right].$$

Using this rule, we can easily find

$$a'II = \frac{1}{2}\alpha^2 \sum c_3^{(i)} \cos(i+1)(L'-L) - \frac{1}{2}\alpha^2 \sum c_3^{(i)} \cos\left[(i+1)L'-(i-1)L\right].$$

or, replacing i + 1 in the first sum by i,

$$a'II = \frac{1}{2}\alpha^2 \sum c_3^{(i-1)} \cos i(L'-L) - \frac{1}{2}\alpha^2 \sum c_3^{(i)} \cos\left[(i+1)L'-(i-1)L\right]$$

We calculate III, IV, ... by the same method exactly.

Substituting the resulting expressions into equation (7), and then collecting together the similar terms, we finally obtain

$$
\begin{aligned}
a'\Delta^{-1} = {} & \frac{1}{2}\sum a'A_i \cos i(L'-L) \\
& + \alpha^2 \sum a'B_i \cos\left[(i+1)L'-(i-1)L\right] \\
& + \alpha^4 \sum a'C_i \cos\left[(i+2)L'-(i-2)L\right] \\
& + \alpha^6 \sum a'D_i \cos\left[(i+3)L'-(i-3)L\right] \\
& + \alpha^8 \sum a'E_i \cos\left[(i+4)L'-(i-4)L\right] \\
& \dots\dots\dots\dots\dots\dots
\end{aligned}
\qquad (12)
$$

where the coefficients of these series are given by

$$
\begin{aligned}
a'A_i = {} & c_1^{(i)} - \alpha^2 c_3^{(i-1)} + \frac{3}{4}\alpha^4\left(c_5^{(i-2)} + 2c_5^{(i)}\right) - \\
& \frac{5}{8}\alpha^6\left(c_7^{(i-3)} + 9c_7^{(i-1)}\right) + \frac{35}{64}\alpha^8\left(c_9^{(i-4)} + \right. \\
& \left. + 16c_9^{(i-2)} + 18c_9^{(i)}\right) - \dots
\end{aligned}
$$

$$
\begin{aligned}
a'B_i = {} & \frac{1}{2}c_3^{(i)} - \frac{3}{4}\alpha^2\left(c_5^{(i-1)} + c_5^{(i+1)}\right) + \frac{15}{16}\alpha^4\left(c_7^{(i-2)} + 3c_7^{(i)} + c_7^{(i+2)}\right) - \\
& \frac{35}{32}\alpha^6\left(c_9^{(i-3)} + 6c_9^{(i-1)} + 6c_9^{(i+1)} + c_9^{(i+3)}\right) + \dots
\end{aligned}
$$

$$a'C_i = \frac{3}{8} c_5^{(i)} - \frac{15}{16} \alpha^2 (c_7^{(i-1)} + c_7^{(i+1)}) + \frac{35}{64} \alpha^4 (3 c_9^{(i-2)} + 8 c_9^{(i)} + 3 c_9^{(i+2)}) - \ldots$$
$$a'D_i = \frac{5}{16} c_7^{(i)} - \frac{35}{32} \alpha^2 (c_9^{(i-1)} + c_9^{(i+1)}) + \ldots$$
$$a'E_i = \frac{35}{128} c_9^{(i)} - \ldots$$

Formula (12) defines the expansion of the principal part of each of the perturbation functions (1) for the case of circular orbits. In order to actually perform this expansion, we only need to be able to calculate the coefficients $C_n^{(i)}$, or equivalently the Laplace coefficients for the values of $\alpha = a/a'$ under consideration. The way to do this calculation will be shown in one of the coming sections.

We now consider the second part, of R_1 and R_2, of the perturbation functions, namely

$$R = k^2 m' (\Delta^{-1} - R'_1), \qquad R' = k^2 m' (\Delta^{-1} - R'_2)$$

Consulting formulae (6), we write

$$\left. \begin{aligned} a'R'_1 &= \alpha (1 - \sigma^2) \cos (L' - L) + \alpha \sigma^2 \cos (L' + L) \\ a'R'_2 &= \alpha^{-2} (1 - \sigma^2) \cos (L' - L) + \alpha^{-2} \sigma^2 \cos (L' + L). \end{aligned} \right\} \tag{13}$$

Comparing these expressions with the expansion coefficients of series (12) obtained here, we see that the influence of the second term of the perturbation function R will be completely taken into account if we replace

$a'A_1$ [illegible] $a'A_1 - \alpha(1$ [illegible] $)$
$a'A_1$ [illegible] $a'A_1$ [illegible] $\alpha(1$ [illegible] $)$
$a'B_1$ [illegible] $a'B_1 - \alpha.$

Similarly, in order to obtain R', it is sufficient to replace in expansion (12)

$$aA_1 \quad \text{by} \quad a'A_1 - \alpha\,[illegible]\,(1 - [illegible])$$
$$a\,[illegible] \qquad a'A_1 - \alpha\,[illegible]\,(1 - [illegible])$$
$$a'B \qquad a'B_0 - [illegible].$$

In this way, the problem of the expansion of the perturbation function for the case of circular orbits is perfectly solved.

Annotation:

It is important to note that the second part of the perturbation function consists entirely of periodic terms. This can easily be seen from expressions (13). The perturbation function does not include other secular terms except those which can be obtained from the sum of the expansion (12) for i = 0.

88. Expansion of the perturbation function in powers of the eccentricities. Newcomb's method

We have seen in Sec. 86 that the perturbation function R is a function of r, r', W = π + v and W' = π' + v', and it is thus possible to write

$$R = f(r, r', W, W').$$

In the previous section, we have put e = 0 and e' = 0. Consequently, r, r', W and W' have been respectively transformed into a, a' and

$$L = W = [illegible]\,M, \qquad L' = W' = [illegible]\,M',$$

where M and M' are the average anomalies of the planets under consideration. In the corresponding case, the functions f (a, a', L, L') have been given by formulae (12) and (13) in the form of unfolded expressions.

We shall now consider that the eccentricities e and e' have small values. We shall expand the expression f(r, r', W, W') in powers of e and e'. In order to simplify the application of Taylor's formula, it is better to consider the perturbation function R as a function of log r and log r' and not of r and r'. Actually, the transition from log a to log r will be performed by adding an increment, whereas the transition from a to r is done by multiplying a by some correction factor. Hence, we assume on one hand

$$R = F(\lg r, \lg r', W, W'),$$

and on the other hand

$$\lg r = \lg a + \rho, \qquad \lg r' = \lg a' + \rho',$$
$$W = L + f, \qquad W' = L' + f',$$

where we denote by

$$f = v - M, \qquad f' = v' - M'$$

the equation of the centre for the planets under consideration.

We have proved in Sec. 82 that ρ and f can be expanded in powers of e, in series having the form

$$[illegible] \qquad (14)$$

Our task consists in expanding the expression

$$R = F(\lg a + \rho, \lg a' + \rho', L + f, L' + f')$$

in powers of e and e'. For this purpose, we apply Taylor's formula, which can be represented for the case of expanding a function of several variables in the following symbolic form

$$f(u+\Delta u, v+\Delta v, \dots) = \exp\left(\Delta u \frac{\partial}{\partial u} + \Delta v \frac{\partial}{\partial v} + \dots\right) f(u, v, \dots).$$

To make this formula more compact, we introduce the following notations

$$D = \frac{\partial}{\partial(\lg a)}, \qquad D' = \frac{\partial}{\partial(\lg a')},$$

$$D_1 = \frac{\partial}{\partial L}, \qquad D_1' = \frac{\partial}{\partial L'}.$$

In these notations, we shall have the following operator equation

$$R = \exp(\rho D + \rho' D' + f D_1 + f' D_1') F(\lg a, \lg a', L, L').$$

Since the operator exp ($\rho D + \rho' D' + fD_1 + f'D_1'$) is a product of the following two operators

$$\exp(\rho D + f D_1) \text{ and } \exp(\rho' D' + f' D_1'), \tag{15}$$

then the operator equation under consideration is the product of each of these operators and the function

$$F(\lg a, \lg a', L, L') \tag{16}$$

Putting

$$x = \exp(\sqrt{-1}\, L), \qquad x' = \exp(\sqrt{-1}\, L'),$$

we can write the function (16), represented by formulae (12) and (13), in the following way

$$\sum E(s, s') x^{s} x'^{s'}.$$

We multiply the arbitrary term

$$R' = H(s, s') x^{s} x'^{s'}$$

of this series by the first of the two operators given by equation (15). Since

$$D_1 \ldots V - 1\lambda \ldots \qquad D_1 K \ldots - V - 1 \lambda K^{\mu},$$

then

$$\exp(\ldots) K \ldots \exp(\ldots D \ldots V - 1) K. \qquad (17$$

Let us now consider equations (14). Putting

$$\exp(\ldots D \ldots V - 1) \ldots k_0 + k_1 e + k_2 e \ldots \qquad (18)$$

whre k_o , k_1, ... are functions of D, s and M, and

$$\mu \quad \exp(V - 1 \ldots)$$

we write equation (14) as

$$\rho \quad e\left(-\frac{1}{2}\mu \quad \frac{1}{2}\mu^{-1}\right) \cdot e^{2}\left(\frac{1}{4} - \frac{3}{8}\mu^{2} - \frac{3}{8}\mu \ldots\right) \ldots$$

$$\ldots V - 1 = e(\mu - \mu^{-1}) + \frac{3}{8}e^{2}(\mu - \mu \ldots) +$$

$$+ \frac{1}{8}e^{3}\left(-\mu + \mu^{-1} + \frac{13}{3}\mu^{3} - \frac{13}{3}\mu \ldots\right) + \ldots$$

Substituting these expressions into equation (18), we obtain

$$\left.\begin{array}{l} k_0 = 1 \\ k_1 = \Pi_1^1 \mu + \Pi_{-1}^1 \mu^{-1} \\ k_2 = \Pi_2^2 \mu^2 + \Pi_0^2 + \Pi_{-2}^2 \mu^{-2} \\ \ldots \ldots \ldots \ldots \\ k_n = \Pi_n^n \mu^n + \Pi_{n-2}^n \mu^{n-2} + \ldots + \Pi_{-n}^n \mu^{-n}, \\ \ldots \ldots \ldots \ldots \ldots \ldots \end{array}\right| \qquad (19)$$

where $\Pi_m^n = \Pi_m^n (D, s)$ is an n-order polynomial in S and D. These symbolic polynomials will be called operators. It is easy to see that

$$\Pi^n_m(D, -s) = \Pi^n_{-m}(D, s)$$

Thus, the term of expansion (17) that contains the factor e^n will be of the form

$$R^n_n = e^n \sum_{m=-n}^{n} P^n_m(s, s') \ldots \tag{20}$$

where m = n, n-2, n-4, ..., -n , and

$$P^n_m(s, s') = \Pi^n_m H(s, s').$$

Multiplying expression (20) by thesecond of the operations (15), and denoting the corresponding polynomials by $\Pi^{0m'}_{\omega m'}$, we obtain for the expansion term of function (20) having factor $e^{'n'}$ in the following form

$$R^{n'}_{n, n} = e^n e'^{n'} \sum \Pi^{\ldots} \sum P^n_m(s, s) \ldots \tag{21}$$

The result of acting first by Π^n_{m} and then by $\Pi^{0m'}_{0m'}$ on the coefficient (H (s,S) may be represented as the result of action of the composite operator

$$\Pi^{\ldots} (D, s, s) = \Pi^{\ldots}(D, s) \Pi^{\ldots}(D, s)$$

In short, assuming that

$$\Pi^{\ldots} H(s, s) = P^{\ldots}_{m}(s, s)$$

and separating the real part in expression (21), we finally obtain the expansion of the perturbation function in the following form

$$R = \sum e^n e'^{n'} P^{\ldots}_{m}(s, s) \cos(\ldots M \ldots m M). \tag{22}$$

where the indices n and n' vary from 0 to $+\infty$, while the indices S, S', m and m' vary from $-\infty$ to $+\infty$. The coefficients P depend on the semimajor axes a and a' and on the mutual slope of the orbit J. Equations (12) and (13) indicate that the sum S + S' is always an even integer. Moreover, taking equation (14) into account, it is easy to see that each of the differences n - m and n' - m' is also equal to a non-negative even integer.

Thus, the expansion of the perturbation function in powers of the eccentricities is finally reduced to the calculation of the operators. For the initial values of the indices n, m, n' and m' the calculation of the operators is simple. With increasing values of these indices, complexity of the calculation rapidly increases. In order to calculate the operators in these case it is advisable to use the recurrence relations existing between them[(1)].

We have seen in the previous section that in the case when e = e' = 0, the secular terms in the expansion of the perturbation functions are entirely obtained from the expansion of the principal term Δ^{-1}. The application of the operators can only give periodic terms, as we have already seen in our consideration of the above-ment ioned method of expansion in powers of eccentricities. This enables us to formulate the following theorem.

---

(1) The methods which have been suggested for the calculation of the operators are given in detail in the monograph:
B.A. Orlov, Expansion of the perturbation functions by Newcomb's method, Transactions of the Astronomical Observatory of the University of Leningrad (Razlozenie perturbacionnoj Funkcii po metocud N'jocoma, Trudy Astronomiceskoj obsefvatorii Leningradskogo universiteta) 6, 1936, 82 - 125.

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Theorem:

The secular terms in the expansion of the perturbation functions are obtained only in the expansion of the principal term.

In conclusion, we note that in order to carry out the expansion given by equation (22), it is necessary to not only know the coefficients (10) but also their derivatives with respect to $\log \alpha$. Indeed, since α = a/a' then

$$D = \frac{d}{d(\lg a)} = a\frac{d}{da} = a\frac{d\alpha}{da}\frac{d}{d\alpha} = \alpha\frac{d}{d\alpha} = \frac{d}{d(\lg \alpha)}.$$

Hence, we can consider D as the differentiation symbol with respect to $\log \alpha$ and write

$$D^k c_n^m = \frac{d^k c_n^m}{(d \lg \alpha)^k} \qquad (k = 1, 2, \ldots)$$

Annotation

For an arbitrary homogeneous function Φ(a,a'), the order of which is -1, Euler's theorem gives

$$a\frac{\partial \Phi}{\partial a} + a'\frac{\partial \Phi}{\partial a'} = -\Phi,$$

or

$$D\Phi + D'\Phi = -\Phi.$$

Thus, for any such function, the following symbolic equation holds

$$D + D' = -1.$$

From this equation, it follows that

$$\Pi_{m'}^{n'} = \Pi_{m'}^{n'}(-D-1, s').$$

Hence, the calculation of all operators

$$\Pi_{m,m'}^{n,n'}(D, s, s') = \Pi_m^n(D, s)\,\Pi_{m'}^{n'}(-D-1, s')$$

is reduced to thecalculation of the simple operator $\prod_{m}^{u}$.

8º. The Final Form of the Expansion of the Perturbation Function

In the previous section, we have become aquainted with the methods for obtaining an arbitrary number of terms for the expansion of the perturbation functions in powers of the excentricities and mutu l slope of the orbits. We have shown that this expansion has the following form

$$R = \sum K e^{h} e'^{h'} \gamma^{2f} \cos(sL + s'L' + mM + m'M'), \qquad (23)$$

where h, h' and f run over the values o, 1,2, ... while indices s, s', m and m' take the values 0, ± 1, ± 2, ... The sum m + m' should always be equal to an even integer. The differences h-/m/, h'-/m'/ and 2f-/s+s'/ may be only equal to the even integers 0, 2, 4, ... The coefficients K depend on the indices h, h', s, s', m and m' and are functions of the semimajro axes of the orbits a and a'. The power of each term of the expansion (23), i.e. the sum h + h' + 2f, is equal to or exceeds by an even integer the following quantity

$$|s + s'| + |m| + |m'|.$$

Expansion (23) represents the perturbation function in the form of a trigonometric series of four arguments. It is the simplest of all representations of the perturbation function as an explicit function of time.

In order to integrate in a simpler way the Lagrange euqations which define the osculating elements (sec. 13), it is recommended to slightly modify expansion (23). Noting that

$$M = L - \Pi, \qquad M' = L' - \Pi',$$

then this expansion may be given the following form

$$R = \sum K e^{h} e'^{h'} J^{2l} \cos (pL + p'L' + q\Pi + q'\Pi'). \qquad (24)$$

Indices p, p', q and q' run all the integral values from $-\infty$ $+\infty$,
where the differences

$$h - |q|, \qquad h' - |q'|, \qquad 2l - |p + p' + q + q'|$$

are non-negative even integers. We thus write

$$h + h' + 2l \geq |p + p' + q + q'| + |q| + |q'|$$

From this inequality, it follows that

$$h + h' + 2l \geq |p + p'|.$$

It is easy to see that thedifference between the left and right-hand sides of this inequality is always equal to an even integer. Hence, the power ofeach term of the expansion (24) is either equal to / p + p'/ or exceeds this quantity by an even integer.

We cannot directly apply the expansions (23) or (24) to the integration of Lagrange equations. The reason is that these expressions involve the elements Π , Π' and J' which define the mutual orientation of the orbits, while the differential equations involve the elements i, Ω , π , i', ... In order to obtain the perturbation function R in the form of an explicit function of these orbital elements, we use the following relations

$$\Pi = \pi - \Omega - N, \qquad \Pi' = \pi' - \Omega' - N'$$
$$L = \Pi + M = nt + \varepsilon - \Omega - N, \qquad L' = n't + \varepsilon' - \Omega' - N'.$$

Therefore, the argument of the expansion (24) may be replaced by

$$p(nt + \varepsilon) + p'(n't + \varepsilon') + q\pi + q'\pi' - (p + q)\Omega$$
$$- (p' + q')\Omega' - (p + q)N - (p' + q')N'.$$

or

$$D + \frac{1}{2}\beta(N + N') + \frac{1}{2}\gamma(N - N'), \tag{25}$$

where

$$D = p(nt + \varepsilon) + p'(n't + \varepsilon') + q\pi + q'\pi' - (p + q)\Omega - (p' + q')\Omega'$$
$$\beta = -p - q - p' - q', \qquad \gamma = -p - q + p' + q'.$$

When we are unfolding the cosine functions having arguments of the type (25), we find that the factors of the expressions, which do not depend on Π, Π' and J, have one of the following forms

$$\sigma^{\beta}\,{\sin \atop \cos}\,\frac{1}{2}\beta(N + N'), \qquad \sigma^{\gamma}\,{\sin \atop \cos}\,\frac{1}{2}\gamma(N - N'), \tag{26}$$

where $\sigma = \sin \frac{1}{2} J$. Using formulae (16), Sec. 68, to express these quantities in terms of the elements and using Euler's formulae, we obtain from these equations

$$\left.\begin{aligned} \sin\frac{1}{2}J\cdot\exp\frac{1}{2}(N - N')\sqrt{-1} &= \sin\frac{i'}{2}\cos\frac{i}{2}\cdot\exp\frac{1}{2}(\Omega' - \Omega)\sqrt{-1} - \\ &\quad - \sin\frac{i}{2}\cos\frac{i'}{2}\cdot\exp\frac{1}{2}(\Omega - \Omega')\sqrt{-1} \\ \cos\frac{1}{2}J\cdot\exp\frac{1}{2}(N - N')\sqrt{-1} &= \cos\frac{i}{2}\cos\frac{i'}{2}\cdot\exp\frac{1}{2}(\Omega - \Omega')\sqrt{-1} + \\ &\quad + \sin\frac{i}{2}\sin\frac{i'}{2}\cdot\exp\frac{1}{2}(\Omega - \Omega')\sqrt{-1}. \end{aligned}\right\} \tag{2.}$$

Raising these equations to the power β and making a transition from the exponentional to trigonometric functions, we obtain the following expressions for the quantities

$$\sigma^{\beta}\,[illegible]\,\frac{1}{2}\beta(N + N'), \qquad (1 - \sigma^2)^{[illegible]}\,[illegible]\,\frac{1}{2}\gamma(N - N') \qquad (2$$

in terms of the orbital elements. On the other hand, by applying the same formulae (16), Sec. 68, we obtain after raising them to the second power and adding in pairs

$$\sigma^2 = \sin^2\frac{i'}{2}\cos^2\frac{i}{2} + \sin^2\frac{i}{2}\cos^2\frac{i'}{2} - \frac{1}{2}\sin i \sin i' \cos(\Omega' - \Omega)$$

$$1 - \sigma^2 = \cos^2\frac{i'}{2}\cos^2\frac{i}{2} + \sin^2\frac{i'}{2}\sin^2\frac{i}{2} + \frac{1}{2}\sin i \sin i' \cos(\Omega' - \Omega)$$

Riasing these equations respectively to the powers f - ½β and $-\beta$ and multiplying them term by term by the expressions obtained for the quantities (28), we obtain the final expressions for the quantities (26). In this way, the perturbation function R will be given the following form

$$R = \sum A e^{h} e'^{h'} \left(\sin\frac{i}{2}\right)^{k} \left(\sin\frac{i'}{2}\right)^{k'} \cos D_0, \qquad (29)$$

where

$$D_0 = p(nt + \varepsilon) + p'(n't + \varepsilon') + q\pi + q'\pi' + s\Omega + s'\Omega'.$$

The coefficient A depends only on a and a'. It is easily seen that the indices p, p', ..., s' involved in the argument D_o must always satisfy the relation

$$p + p' + q + q' + s + s' = 0.$$

Indeed, the perturbation function R does not evidently depend on the initial point for calculating the longitude. Yet, if this initial point is displaced by an angle Λ, then the argument D_o will be changed into Λ (p + p' + q + q' + s + s'), and since the function R does not depend on Λ , then this quantity should be set equal to zero. This means that the trigonometric series (29) is developed not by six indices, but only by five indices.

The actual working up of expansion (29) is an extremely tedians job and it has never been actually applied. The theory of motion of large planets, developed by Laverrier, which describes the most commonly applied method of variation of the elements in the analytical calculation of perturbations, is based on expansions occupying an intermediate place between formulae (24) and (29).

Laverrier introduced, istead of the longitudes L and L' measured from the point of intersection of the orbits, the longitudes in the orbits (using his notations):

$$l = nt + \varepsilon = L + \tau, \qquad l' = n't + \varepsilon' = L' + \tau'.$$

Assuming

$$\lambda = l + \tau' - \tau, \qquad (30)$$

yields

$$L = \lambda - \tau', \qquad L' = l' - \tau'. \qquad (31)$$

Similarly, notir that the longitudes of the perihelions are equal to

$$\pi = \tau + \Pi, \qquad \pi' = \tau' + \Pi'$$

and assuming

$$\omega = \pi + \tau' - \tau = \Pi + \tau' \qquad (32)$$

we obtain

$$\Pi = \omega - \tau', \qquad \Pi' = \pi' - \tau'. \qquad (33)$$

Substituting expressions (31) and (33) irto the expansion (24), we finally have the following expansion for the perturbation function

$$R = \Sigma N e^{h} e'^{h'} s^{2g} \cos(j\lambda + j'l' + k\omega + k'\pi' - 2g\tau'). \qquad (34)$$

This expansion constitutes the basis of all Laverrier's work.

It is easily seen that the j, j', k, k' and 2g are related by the following relation

$$j + j' + k + k' - 2g = 0,$$

This relation is a consequence of the independence of the perturbation function R on the initial point for calculating the longitudes.

The quantities λ and ω, defined by equations (30) and (32), differ from ℓ and π by the infinitesimal quantity $\tau' - \tau$. It easy to show (though not undertaken here) that

$$\operatorname{tg} \frac{\tau' - \tau}{2} = \frac{\operatorname{tg} \frac{i}{2} \operatorname{tg} \frac{i'}{2} \sin(\Omega' - \Omega)}{1 + \operatorname{tg} \frac{i}{2} \operatorname{tg} \frac{i'}{2} \cos(\Omega' - \Omega)},$$

Hence,

$$\frac{1}{2}(\tau' - \tau) = \operatorname{tg} \frac{i}{2} \operatorname{tg} \frac{i'}{2} \sin(\Omega' - \Omega) - \frac{1}{2} \operatorname{tg}^2 \frac{i}{2} \operatorname{tg}^2 \frac{i'}{2} \sin 2(\Omega' - \Omega) + \ldots \quad (35)$$

In order to make use of the expansion (34), Leverrier was obliged to introduce some special modification into the Lagrange equations (Sec. 97).

90. The Initial Terms of the Expansion of the Perturbation Function

In order to carry out the expansion (24), it is necessary to express the coefficients K in terms of the functions $c_k^{(j)}$ of the ratio $\alpha = a/a'$ that have been introduced in Sec. 87, and then to compute these functions for a given value of α . We shall consider the latter problem in the next chapter. Here we shall consider the calculation of the coefficients K in an explicit form.

We shall confine ourselves to second-order terms with respect to e, e' and σ . The operations described in the previous section will then apply in a quite simple manner. It is easily seen that these operations will lead to the following expansion for the principal part of the perturbation function

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\begin{aligned}
a'\Delta^{-1} = \sum \Big[& \frac{1}{2} c_1^{(i)} - \frac{1}{2} \sigma^2 c_3^{(i-1)} + \frac{1}{8} (e^2 + e'^2)(-4i^2 + D + D^2) c_1^{(i)} \cos V + \\
& + \frac{1}{2} e \sum (-2i - D)\, c_1^{(i)} \cos(V + M) + \\
& + \frac{1}{2} e' \sum (2i + 1 + D)\, c_1^{(i)} \cos(V + M') + \\
& + \frac{1}{8} e^2 \sum [4i^2 - 5i + (4i - 3) D + D^2]\, c_1^{(i)} \cos (V + 2M) + \\
& + \frac{1}{4} ee' \sum (4i^2 \ldots 2i \ldots D \ldots D^2)\, c_1^{(i)} \cos(V \ldots M \ldots M') \\
& + \frac{1}{4} ee' \sum [\ldots 4i^2 \ldots 2i - (\ldots 4i - 1) D - D^2]\, c_1^{(i)} \cos(V \ldots M \ldots M') + \\
& + \frac{1}{8} e'^2 \sum [4i^2 \ldots 9i + 4 + (4i + 6) D + D^2]\, c_1^{(i)} \cos(V + 2M') + \\
& + \frac{1}{2} \sigma^2 \sum c_3^{(i-1)} \cos(V + 2L) + \ldots
\end{aligned}
\tag{36}
$$

where, the following notation has been adopted

$$V = i(L' - L), \qquad D = \frac{\partial}{\partial(\lg \alpha)};$$

and the summation is to be carried over the values $i = 0, \pm 1, \pm 2 \ldots$.

The second term of the perturbation function can be taken into account by means of the methods given at the end of Sec. 87. In fact, the series-expansion of this term can be immediately written down using formula (36). Indeed, assuming that the eccentricities e and e' are equal to zero, the second part of the perturbation function

$$R_1 = \frac{r \cos H}{r'^2}$$

will be defined, according to the first of formulae (13), by the following relation

$$a' R_1 = \alpha (1 - \sigma^2) \cos(L' - L) + \alpha \sigma^2 \cos(L' + L).$$

Comparing this expression with that given by equation (12), we see that the former can be considered as a particular case of the latter, if we put in that latter expression

$$C_1^{(1)} = C_1^{(-1)} = \alpha, \quad C_1^{(0)} = 2\alpha,$$

and equate all the other quantities $C_k^{(i)}$ to zero.

Applying in a similar manner formula (36), to this particular case, we obtain in terms of the second power of e, e' and σ:

$$\begin{aligned}\alpha^{-1} a' R_1 = & \left(1 - \frac{1}{2} e^2 - \frac{1}{2} e'^2 - \sigma^2\right)\cos(L' - L) - \\ & - \frac{3}{2} e\cos(L' - \Pi) + \frac{1}{2} e\cos(L' - 2L + \Pi) + \\ & 2e'\cos(2L' - L - \Pi') + \\ & + \frac{1}{8} e^2\cos(L' + L - 2\Pi) + \frac{3}{8} e^2\cos(L' - 3L + 2\Pi) + \\ & + ee'\cos(2L' - 2L - \Pi' + \Pi) - 3ee'\cos(2L' - \Pi' - \Pi) + \\ & + \frac{1}{8} e'^2\cos(L' + L - 2\Pi') + \frac{27}{8} e'^2\cos(3L' - L - 2\Pi') + \\ & + \sigma^2\cos(L' + L) + \dots\end{aligned}$$

...ary, the second of formulae (13), which yields

$$a'R_1 = a'\frac{r\cos H}{r'^2} = \alpha(1 - \sigma^2)\cos(L' - L) + \alpha\sigma^2\cos(L' + L),$$

indicates that the second part of the perturbation function R' can be obtained from expression (12) by putting

$$[illegible]$$

Again using formula (24), we obtain

$$
\begin{aligned}
a'R_1 &= \left(1 - \frac{1}{2} e^2 - \frac{1}{2} e'^2 - \gamma^2\right) \cos(L' - L) + \\
&+ 2e \cos(L' - 2L + \Pi) - \\
&- \frac{3}{2} e' \cos(L - \Pi') + \frac{1}{2} e' \cos(2L' - L - \Pi') + \\
&+ \frac{1}{8} e^2 \cos(L' + L - 2\Pi) + \frac{27}{8} e^2 \cos(L' - 3L + 2\Pi) + \\
&+ ee' \cos(2L' - 2L - \Pi' + \Pi) - 3ee' \cos(2L - \Pi' - \Pi) + \\
&+ \frac{1}{8} e'^2 \cos(L' + L - 2\Pi') + \frac{3}{8} e'^2 \cos(3L' - L - 2\Pi') + \\
&+ \gamma^2 \cos(L' + L) + \dots
\end{aligned}
$$

This is the expansion of the perturbation function in the case when terms of the third-order with respect to the eccentricities and slopes can be neglected. This expansion has already been previously obtained by Lagrange and Laplace to within the same accuracy.

In order to obtain the perturbation of the coordinates of large planets with an accuracy corresponding to that of the recent observations, it is necessary to carry out the expansion of the perturbation function up to terms of the 7-th order inclusively. The possibility of doing such expansions was shown by Burckhardt. His calculations were corrected and developed by Binet and de Ponteocoulant, who gave expansions that included a considerable part with sixth order terms. The complete expansion of the perturbation function up to 6-order terms inclusively, was first obtained in Pierce's work in 1849. Finally in 1855, Leverrier(1) published the expansion of the perturbation function up to the 7-order terms inclusively. The accurate expressions which he obtained for all the terms up to this limit did not loose their

---

(1) Annales de l'Observatoire de Paris, t.1, 1855.

values, even at the present time. Boquet[1] included all the 8-th order terms into the expansion preserving Laverrier's method and notations.

The most convenient way to obtain the expansion of the perturbation function in the form given by equation (23) is by Newcomb's method[2], based on the application of operators as defined in Sec. 88.

Annotation

Using the formulae obtained in the previous chapter, it is easy to obtain the expansion of the second part of the perturbation in a general form with coefficients expressed in terms of Bessel functions.

91. Numerical Method for the Expansion of the Perturbation Function

In the previous sections, we studied the methods of the accurate calculation of the perturbation functions in the form of a series. Each term of such series is an explicit function of the orbital elements and average anomalies M and M' of the planets under consideration. This form of expansions gives the most general solution to the problem. It allows us to obtain the perturbations as explicit functions of the orbital elements. The methods of obtaining such expansions, where all the elements enter as letters (except the semimajor axes, which are given numerical values in order to be able to compute the Laplace coefficients), are known as the analytical methods of expansion of the perturbation function.

---

(1) F. Boquet, Developpment de la fonction perturbatrice, Annales de l'Observatoire de Paris, t. 19, 1885.

(2) S. Newcomb, A Development of the perturbation function etc., Astronomical Papers, Vol. V, 1895.

The analytical methods of expansion give the perturbation function in the form of a series in powers of eccentricities. They can be applied practically for only small values of eccentricities (not exceeding 0.15 or 0.20). If this condition does not apply, then the expansion can in practice be only obtained by means of numerical methods although it may converge quite rapidly (cf. the annotation to Sec. 84). In this case, one has to apply the numerical methods of expansion in which the elements enter from the very beginning using their numerical values.

If the orbital elements of the planets under consideration are given using their numerical values, the perturbation function may then be represented by the series

$$R = \sum_{i,i'} \{A_{i,i'} \cos(iM + i'M') + B_{i,i'} \sin(iM + i'M')\},$$

developed by multiples of the average anomalies M and M'. The coefficients A and B of this series are expressed by the following well-known formulae

$$A_{i,i'} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R \cos(iM + i'M')\, dM\, dM',$$

$$B_{i,i'} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} R \sin(iM + i'M')\, dM\, dM'.$$

These coefficients can be obtained by means of the approximate formulae which replace each integral by a sum of the values of the integrand for various values of the argument. These formulae may be put together in a single formula in the following way.

$$A_{i,i'} + \sqrt{-1}\, B_{i,i'} = \frac{1}{mm'} \sum_{k=0}^{m-1} \sum_{k'=0}^{m'-1} R_{k,k'} \exp\left\{-\sqrt{-1}\left(ik\frac{2\pi}{m} + i'k'\frac{2\pi}{m'}\right)\right\}.$$

where $R_{k,k'}$ is the value of thefunction R for $M = k \frac{2\pi}{m}$ and $M' = k' \frac{2\pi}{m'}$. Evaluating the function R for a sufficiently large number of specific values of the average anomaly, we can compute the coefficients of the expansion (37) to an arbitrarily high accuracy.

Naturally, the expansion (37) with numerical coefficients cannot be used in calculating the derivatives of the perturbation function with respect to the elements. Hence, it is not possible to apply this expansion to compute the perturbation of the elements by means of Lagrange's formulae (Sec. 15). However, this kind of expansion is quite useful for the purpose ofthe direct computation of the perturbations of the coordinates (Chapter XVI). In this case, it is sufficient to have the expansions of Δ^{-1} and Δ^{-3} if only first order perturbations are required. For higher-order perturbations, it is necessary to also have the expansions of Δ^{-5}, Δ^{-7}, ...

Hansen was the first to publish an application of the numerical method of expansion of the perturbation function, which he had been applying to the study of the mutual perturbations of Jupiter and Saturn (1831). He used as an argument the difference M-M' between the average anomalies and the average anomaly of Saturn M'. In order to obtain the expansion coefficients by means of the harmonic analysis forMulae, he computed Δ^{-1}, Δ^{-3}, ... for all the combinations of the following values of the argument

$$M - M' = 11^\circ 15' \cdot k, \qquad k = 0, 1, \ldots, 31$$
$$M' = 22^\circ 30' \cdot k', \qquad k' = 0, 1, \ldots, 15$$

By this method he had to compute 32 x 16 = 512 particular values of the above mentioned functions.

92. Hansen's Method

The numerical method mentioned in the previous section enables us to obtain the expansion of the perturbation function with an arbitrarily high accuracy by means of simple and easily mechanized computations. The only inconvenience in these computations is in their extensiveness.

In applying numerical methods, we do not make use of the properties of the function to be expanded into a series. It is quite natural that the following question crops up: can we reduce the calculation work by the appropriate use of the properties which we know on the analytical structure of the expanded function?

Cauchy was the first to apply a semianalytical method for the expansion of the perturbation function (1844). He carried out the expansion analytically by one argument and numerically by the other. This idea was further developed by Hansen (1857) who gave an expansion method which had been widely applied. Hill, in particular, applied this method to the construction of the theory of motion of Jupiter and Saturn. The problem consists in expanding the quantities Δ^{-1}, Δ^{-3}, $\Delta^{-5}, \ldots$, where

$$\Delta^2 = r^2 + r'^2 - 2rr'\cos H,$$
$$\cos H = \cos(v + \Pi)\cos(v' + \Pi') + \sin(v + \Pi)\sin(v' + \Pi')\cos J.$$

in a double trigonometric series. Considering again $a < a'$, and putting $\alpha = a/a'$, we write this equation as

$$\left(\frac{\Delta}{a'}\right)^2 = \left(\frac{r'}{a'}\right)^2 + \left(\frac{r}{a}\right)^2 \alpha^2 - 2\,\frac{r}{a}\,\frac{r'}{a'}\,\alpha\cos H. \qquad (38)$$

On the other hand, the function cos H equals in an unfolded form, to

$$C_1\cos v\cos v' + C_2\cos v\sin v' + C_3\sin v\cos v' + C_4\sin v\sin v',$$

where

$$
\begin{aligned}
C_1 &= \cos\Pi\cos\Pi' + \sin\Pi\sin\Pi'\cos J\\
C_2 &= -\cos\Pi\sin\Pi' + \sin\Pi\cos\Pi'\cos J\\
C_3 &= \sin\Pi\cos\Pi' + \cos\Pi\sin\Pi'\cos J\\
C_4 &= \sin\Pi\sin\Pi' + \cos\Pi\cos\Pi'\cos J.
\end{aligned}
$$

Introducing the auxiliary quantities k, K, k_1 and K_1 by means of the following relations

$$
\left.
\begin{aligned}
k\sin(\Pi' + K) &= \alpha\sin\Pi\cos J,\\
k\cos(\Pi' + K) &= \alpha\cos\Pi\\
k_1\sin(\Pi' - K_1) &= \alpha\sin\Pi\\
k_1\cos(\Pi' - K_1) &= \alpha\cos\Pi\cos J,
\end{aligned}
\right\}
$$

we easilv f:nd that

$$
\alpha\cos H = k\cos v\cos(v' + K) + k_1\sin v\sin(v' + K_1)
$$

Substituting this expression into equation (38), and using the following well-known formulae

$$
\begin{aligned}
r &= a(1 - e\cos E), & r' &= a'(1 - e'\cos E'),\\
r\cos v &= a(\cos E - e), & r'\cos v' &= a'(\cos E' - e'),\\
r\sin v &= a\cos\varphi\sin E, & r'\sin v' &= a'\cos\varphi'\sin E',
\end{aligned}
$$

we obtain the following expression for the square of the separation distance of two planets in terms of their excentric anom

$$
\left(\frac{\Delta}{a'}\right)^2 = D - f\cos(E - F) + \frac{1}{2}\gamma_2\cos 2E, \tag{41}
$$

where

$$
\left.
\begin{aligned}
\gamma_2 &= \alpha^2 e^2\\
D &= D_0 + D_1\cos E' + D_2\sin E' + e'^2\cos^2 E'\\
f\sin F &= G_0 + G_1\cos E' + G_2\sin E'\\
f\cos F &= H_0 + H_1\cos E' + H_2\sin E',
\end{aligned}
\right\} \tag{42}
$$

where the constant coefficients have the following values

$$
\begin{aligned}
D_0 &= 1 + e'^2 + \frac{1}{2} k^2 e^2 - 2kee' \cos K, \\
D_1 &= 2(ek \cos K - e') \\
D_2 &= -2ek \sin K \cos \varphi \\
G_0 &= -2e k_1 \sin K_1 \cos \varphi \\
G_1 &= 2k_1 \sin K_1 \cos \varphi \\
G_2 &= 2k_1 \cos K_1 \cos \varphi \cos \varphi' \\
H_0 &= -2(e' - ek \cos K) \\
H_1 &= 2k \cos K \\
H_2 &= -2k \sin K \cos \varphi'
\end{aligned}
\qquad (43)
$$

We note that if the quantities D, f and F are known, it is then easy to compute the functions E' and M'. When M' is increased by 2 , the angle F also increases by 2π as it is easily seen from the above equations.

The detailed examination of formulae (42) indicates that when the excentricities e and e' are small, the differences E - E' and consequently F-M' remain within sufficiently close limits whatever the change in M'.

In all practical cases, the last term of expression (41) is very small as compared to the sum of the first two terms. This situation enables us to write, on the basis of the binomial formula, the following rapidly converging expansion

$$\Delta^{-1} = \Delta_0^{-1} - \ldots \cos \ldots \quad (44)$$

where

$$\Delta_0 = [D - f\cos(E - F)]^{\frac{1}{2}}.$$

The present problem should be reduced to the expansion of the quantities Δ_0^{-1}, Δ_0^{-3}, ... Each of these quantities can be represented by a

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Fourier series

$$\Delta_0^{-n} = \alpha_n^{(0)} + 2\alpha_n^{(1)}\cos(E-F) + 2\alpha_n^{(2)}\cos 2(E-F) + \dots \tag{45}$$

The coefficients of this series can easily be expressed in terms of Laplace's coefficients. In fact, setting

$$D = \mathfrak{M}(1+b^2), \quad f = 2\mathfrak{M}b, \tag{46}$$

we obtain

$$\Delta_0^{-n} = \mathfrak{M}^{-\frac{n}{2}}\left[1+b^2-2b\cos(E-F)\right]^{-\frac{n}{2}} =$$
$$= \frac{1}{2}\mathfrak{M}^{-\frac{n}{2}}\sum b_n^{(i)}\cos i(E-F),$$

consequently,

$$\alpha_n^{(i)} = \frac{1}{2}\mathfrak{M}^{-\frac{n}{2}} b_n^{(i)} \tag{47}$$

Replacing the argument of the series (45) by the difference

$$E - F = E - M' - (F - M'),$$

we will be able to transform this series in the following way

$$\Delta_0^{-n} = \beta_n^{(0)} + 2\beta_n^{(1)}\cos(E-M') + 2\beta_n^{(2)}\cos 2(E-M') + \dots$$
$$+ 2\gamma_n^{(1)}\sin(E-M') + 2\gamma_n^{(2)}\sin 2(E-M') + \dots \tag{48}$$

where

$$\left.\begin{aligned} \beta_n^{(i)} &= \alpha_n^{(i)}\cos i(F-M') \\ \gamma_n^{(i)} &= \alpha_n^{(i)}\sin i(F-M'). \end{aligned}\right\} \tag{49}$$

As we have already pointed out we can calculate the quantities D, f and F for any values of M', and subsequently compute the coefficients   and by means of formulae (46), (47) and (49). Computing each of these coefficients for a series of equally-spaced values of M', and applying the conventional methods of harmonic analysis, we can expand each of the coefficients β and γ in a series of the type

$$\frac{1}{2} c_0 + \sum c_i \cos iM + \sum s_i \sin iM$$

Substituting such a series for each of the coefficients of expansions (48), and introducing these latter expansions into formula (44) and then unfolding the resulting, product of trigonometric functions, we finally obtain series of the type

$$\binom{\alpha}{\Delta} = \sum (n, i, i')_c \cos (iE - i'M') + \sum (n, i, i')_s \sin (iE - i'M') \quad (50)$$

where n = 1, 3, 5, ..., while the indices i and i' are set equal to $i = 0, \pm 1, \pm 2, \ldots$, $i' = 0, 1, 2, \ldots$.

The expansion (50) involves two variables, E and M', and hence cannot be directly applied to the integration of the equations which define the perturbations. Hansen expressed M' in terms of the eccentric anomaly E, which he considered as an independent variable. Since,

$$M = nt + M_0, \quad M' = n't + M_0',$$

then

$$M' = \mu M + C,$$

where $\mu = n'/n$ and C is some constant. Hence, using Kepler's equation, we obtain

$$M' = \mu E - \mu e \sin E + C$$

Substituting this expression instead of M' in formula (50) and unfolding the functions

$$\sin(\mu e \sin E), \quad \cos(\mu e \sin E)$$

involved in the trigonometric series by means of formulae (15), Sec. 78, we obtain an expansion of the type

$$\left(\frac{a'}{\Delta}\right)^n = \sum [i, i', c]_n \cos(iE - i'\mu E) + \sum [i, i', s]_n \sin(iE - i'\mu E). \quad (51)$$

In the theory described above on Jupiter and Saturn, Hill transformed expansion (50) into an expansion of the type

$$\left(\frac{a'}{\Delta}\right)^n = \sum [i, i', c]_n \cos(iM - i'M') + \sum [i, i', s]_n \sin(iM - i'M'), \quad (52)$$

which could easily becarried out by means of the methods indicated in Sec. 81.

Annotation

In order to compute Θ and $\mathfrak{M}$ by means of formulae (16), it is recommended to introduce the auxiliary angle ψ, defined by the relation

$$\sin\psi = \frac{J}{D}$$

and the condition $0 < \psi < 90^\circ$

The equation

$$\frac{2\eta}{1+\eta^2} = \sin\psi$$

has the following two roots

$$\eta_1 = \operatorname{tg}\frac{\psi}{2}, \quad \eta_2 = \operatorname{ctg}\frac{\psi}{2}.$$

Taking the first of these roots, we will have the following equalities

$$\eta = \operatorname{tg}\frac{\psi}{2}, \quad \mathfrak{M} = D\cos^2\frac{\psi}{2}.$$

CHAPTER XIV

THE LAPLACE COEFFICIENTS

93. Calculation of the Laplace Coefficients by Means of Series

In order to end the question on the expansion of the perturbation functions into series, it remains for us to consider the methods of calculating the quantities $C_n^{(i)}$. It follows from equation (10), Sec. 87, that the computation of these quantities is equivalent to the computation of the quantities $b_b^{(i)}$, defined by the following relation

$$(1 - 2\alpha\cos S + \alpha^2)^{-n/2} = \frac{1}{2}\sum b_n^{(i)}\cos iS \qquad (1)$$

and known as the Laplace coefficients. We shall prove that the Laplace coefficients can be computed by means of infinite series. We put

$$z = \exp(S\sqrt{-1}).$$

Then

$$1 - 2\alpha\cos S + \alpha^2 = 1 + \alpha^2 - \alpha(z + z^{-1}) = (1 - \alpha z)(1 - \alpha z^{-1}).$$

Consequently, equation (1) may be replaced by

$$(1 - \alpha z)^{-n/2}(1 - \alpha z^{-1})^{-n/2} = \frac{1}{2}\sum b_n^{(i)} z^i. \qquad (2)$$

Since

$$(1 - \alpha z^{\pm 1})^{-n/2} = 1 + \frac{n}{2}\alpha z^{\pm 1} + \frac{n(n+2)}{2\cdot 4}\alpha^2 z^{\pm 2} + \frac{n(n+2)(n+4)}{2\cdot 4\cdot 6}\alpha^3 z^{\pm 3} + \ldots$$

then, we can easily obtain, by eouating the coefficients of z^i on the right- and left-hand sides, the following formula

$$\frac{1}{2} b_n^i = \frac{n(n+2)\dots(n+2i-2)}{2\cdot 4\dots(2i)} \alpha^i \left| 1 + \frac{n}{2}\,\frac{n+2i}{2i+2}\,\alpha^2 + \frac{n(n+2)(n+2i)(n+2i+2)}{2\cdot 4\,(2i+2)(2i+4)}\,\alpha^4 + \dots \right| . \qquad (3)$$

Taking i = 0, we obtain

$$\frac{1}{2} b_n = 1 + \left(\frac{n}{2}\right)^2 \alpha^2 + \left(\frac{n(n+2)}{2\cdot 4}\right)^2 \alpha^4 + \dots \qquad (4)$$

It is clear that these series converge for all positive values of α which satisfy the condition

$$\alpha < 1$$

However, the convergence of these series is very slow for all values of α even for the case in which n = 1.

We make use of the conventional notation of the hypergeometric function

$$F(A, B, C, x) = 1 + \frac{AB}{1\cdot C}\,x + \frac{A(A+1)B(B+1)}{1\cdot 2\cdot C(C+1)}\,x^2 + \dots \qquad (5)$$

and also introduce the symbol

$$(\lambda, i) = \lambda(\lambda+1)(\lambda+2)\dots(\lambda+i-1), \quad (\lambda, 0) = 1.$$

We then write formula (3) in the following way

$$\frac{1}{2} b_n^i = \frac{\left(\frac{n}{2}, i\right)}{(1, i)}\,\alpha^i F\left(\frac{n}{2}, \frac{n}{2}+i, i+1; \alpha^2\right). \qquad (6)$$

It is well known that the hypergeometric function, defined by series (5), satisfies the following relation

$$F(A, B, C, x) = (1-x)^{-A} F\left(A, C-B, C; \frac{x}{x-1}\right).$$

Hence

$$\frac{1}{2} b_n^i = \frac{\left(\frac{n}{2}, i\right)}{(1, i)}\,\alpha^i (1-\alpha^2)^{-\frac{n}{2}} F\left(\frac{n}{2}, 1-\frac{n}{2}, i+1, \frac{-\alpha^2}{1-\alpha^2}\right).$$

or, in an unfolded form,

$$\frac{1}{2} b_r^{(i)} = \frac{n(n-2)\ldots(n+2i-2)}{2\cdot 4\cdot 6\ldots(2i)} \alpha^{2i}(1-\alpha^2)^{-\frac{n}{2}} \left[1 + \frac{n}{2}\,\frac{n-2}{2i+2}\,p + \frac{n(n+2)}{2\cdot 4}\,\frac{(n-2)(n-4)}{(2i+2)(2i+4)}\,p^2 + \ldots \right], \qquad (7)$$

where

$$p = \frac{\alpha^4}{1-\alpha^4}.$$

This series converges when p $<$ 1, i.e. when the value of α satisfies the following condition

$$\alpha < \frac{1}{\sqrt{2}} = 0.707\ldots$$

The advantage of applying series (7) rather than series (3) is more apparent for larger values of i. Indeed, the ratios of the corresponding coefficients in these two series are

$$1, \quad \frac{n}{n+2i}, \quad \frac{(n-2)(n-4)}{(n+2i)(n+2i+2)}, \quad \ldots$$

This ratio tends to zero when i tends to infinity.

If $\alpha \geqslant$ 0.707 ..., then series (7) diverges. However, even in this case we can apply this series for computing $b_r^{(i)}$ if i is sufficiently large. As a matter of fact, it is possible to show that in this case series (7) becomes an asymptotic series. The application of the divergent series (7) for large values of i will be even more practical than the application of the convergent series (3).

Series (7) may be transformed into a more practical and at the same time as a convergent series by means of an analytical continuation. Indeed, the branch of the function (7) under consideration has no singular points except α = 1 and $\alpha = \infty$ consequently the only singular points of the corresponding branch of the function

$$f(p) = [illegible]$$

will be the points $p = -1$ and $p = \infty$. Hence, applying Taylor's formula to the function $F(p)$,

$$F(p) = F(p_0) + (p - p_0)F'(p_0) + \frac{(p - p_0)^2}{2!} F''(p_0) + \ldots$$

where p_0 is real and positive, we obtain a series which has a circle of convergence with radius $1 + p_0$.

We shall now give the numerical coefficients of the series which may be used for computing $b_1^{(10)}$ and $b_1^{(11)}$ in the cases, when $p_0 = 0$, $\frac{1}{2}$ and 1.

The first of these Laplace coefficients is equal to

$$b_1^{(10)} = \frac{1.3.5 \ldots 19}{2.4.6 \ldots 20} 2^{10} (1 - \alpha^2)^{-\frac{1}{2}} F_{10}(p) = \frac{46189}{262144} \alpha^{10} \sqrt{p}\, F_{10}(p), \qquad (8)$$

where, for the function $F_{10}(p)$, we have the following expansion

$p_0 = 0$	$p_0 = \frac{1}{2}$	$p_0 = 1$
$+1.0000000$	$+0.[illegible]$	$+0.9[illegible]$
$[illegible]p$	$[illegible]\left(p - \frac{1}{2}\right)$	$[illegible](p - 1)$
$[illegible]p^2$	$+0.00[illegible]\left(p - \frac{1}{2}\right)^2$	$[illegible](p - 1)^2$
$[illegible]p^3$	$[illegible]\left(p - \frac{1}{2}\right)^3$	$-0.000[illegible](p - 1)^3$
$+0.00[illegible]p^4$	$+[illegible]\left(p - \frac{1}{2}\right)^4$	$[illegible](p - 1)^4$
$[illegible]p^5$	$[illegible]\left(p - \frac{1}{2}\right)^5$	$-0.0000[illegible]451(p - 1)^5$
$+[illegible]p^6$	$+0.00[illegible]\left(p - \frac{1}{2}\right)^6$	$[illegible](p - 1)^6$
		$[illegible](p - 1)^7$
		$[illegible](p - 1)^8$
		$[illegible](p - 1)^9$

For $b_1^{(11)}$, equation (7) yields

$$b_1^{(11)} = \frac{88179}{524288} \alpha^{11} \sqrt{p}\, F_{11}(p), \qquad (9)$$

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where one of the following expansions may be used for the function $F_{11}(p)$:

$p_0 = 0$	$p_0 = \frac{1}{2}$	$p_0 = 1$
$+1.00000\,000$	$0.99000\,356$	$+0.98071\,527$
$-0.02083\,333p$	$-0.01920\,796\left(p-\frac{1}{2}\right)$	$-0.01786\,120(p-1)$
$+0.00180\,288p^2$	$+0.00146\,840\left(p-\frac{1}{2}\right)^2$	$+0.00123\,022(p-1)^2$
$-0.00026\,830p^3$	$-0.00018\,51\left(p-\frac{1}{2}\right)^3$	$-0.00013\,597(p-1)^3$
$+0.00005\,48p^4$	$+0.00003\,11\left(p-\frac{1}{2}\right)^4$	$+0.00001\,956(p-1)^4$
$-0.00001\,4p^5$	$-0.00000\,7\left(p-\frac{1}{2}\right)^5$	$-0.00000\,335(p-1)^5$
$+0.00000\,1p^6$	$+0.00000\,2\left(p-\frac{1}{2}\right)^6$	$+0.00000\,065(p-1)^6$
		$-0.00000\,014(p-1)^7$
		$+0.00000\,003(p-1)^8$
		$-0.00000\,001(p-1)^9$

The choice for the most convenient series may be obtained by consulting the following table

α	0	0.15	0.61	0.66	0.71	0.78	0.82	0.88
p	0	0.25	0.50	0.75	1.00	1.50	2.00	3.00

94. <u>The recurrence relations between the Laplace coefficients</u>

We consider equation (2), which may be used for the determination of the Laplace coefficients. This equation may be written as

$$[1+\alpha^2-\alpha(z+z^{-1})]^{-\frac{n}{2}} = \frac{1}{2}\sum b_n^{(i)} z^i. \qquad (9)$$

Differentiating this equation with respect to z, we obtain

$$\frac{1}{2}n\alpha(1-z^{-2})[1+\alpha^2-\alpha(z+z^{-1})]^{-\frac{n}{2}-1} = \frac{1}{2}\sum i b_n^{(i)} z^{i-1}.$$

Owing to equation (9), we may rewrite this previous equation in the following two forms

$$\frac{1}{2}\, n\alpha(1-z^{-2})\sum b_n^{(i)} z^i = [1+\alpha^2-\alpha(z+z^{-1})]\sum i b_n^{(i)} z^{i-1} \qquad (10)$$

$$\frac{1}{2}\, n\alpha(1-z^{-2})\sum b_{n+2}^{(i)} z^i = \sum i b_n^{(i)} z^{i-1}. \qquad (11)$$

Equating the coefficients of z^{i-1} in both sides of equation (10), we obtain

$$b_n^{(i+1)} = \frac{2i}{n+2i}(\alpha+\alpha^{-1})\, b_n^{(i)} - \frac{2i-n}{2i-n+2}\, b_n^{(i-1)}. \qquad (12)$$

This relation enables us to know all the coefficients $b_n^{(i)}$ if two of them, say $b_n^{(0)}$ and $b_n^{(1)}$, we known . Similarly, equating the coefficients of z^{i-1} on both sides of equation (11), we obtain

$$\frac{1}{2}\, n\alpha(b_{n+2}^{(i-1)} - b_{n+2}^{(i+1)}) = i b_n^{(i)}. \qquad (13)$$

On the other hand, it follows from equation (9) that if n is replaced by n + 2, then

$$[1+\alpha^2-\alpha(z+z^{-1})]\sum b_{n+2}^{(i)} z^i = \sum b_n^{(i)} z^i,$$

from which we easily obtain

$$(1+\alpha^2)\, b_{n+2}^{(i)} - \alpha(b_{n+2}^{(i-1)} + b_{n+2}^{(i+1)}) = b_n^{(i)}. \qquad (14)$$

Eliminating $b_{n+2}^{(i-1)}$ from equations (13) and (14), we obtain

$$n(1+\alpha^2)\, b_{n+2}^{(i)} - 2\alpha n b_{n+2}^{(i+1)} = (n+2i)\, b_n^{(i)}.$$

Similarly, eliminating $b_{n+2}^{(i+1)}$ from both equations, we obtain

$$n(1+\alpha^2)\, b_{n+2}^{(i)} - 2\alpha n b_{n+2}^{(i-1)} = (n-2i)\, b_n^{(i)}.$$

Replacing here i by i + 1 and simultaneously solving the resulting equations with the previous one, we obtain

$$\begin{aligned} n(1-\alpha^2)\left(b_{n+2}^{(i)}+b_{n+2}^{(i+1)}\right) &= (2i+n)\,b_n^{(i)}-(2i-n+2)\,b_n^{(i+1)} \\ n(1-\alpha^2)\left(b_{n+2}^{(i)}-b_{n+2}^{(i+1)}\right) &= -(2i+n)\,b_n^{(i)}+(2i-n+2)\,b_n^{(i+1)}. \end{aligned} \qquad (15)$$

In this way, if we obtain coefficients $b_1^{(i)}$, calculate all the coefficients $b_3^{(i)}$, and subsequently find all $b_5^{(i)}$ etc. Combining this result with the result obtained from equation (12), we conclude that it is sufficient to directly calculate only two of the Laplace coefficients, for example $b_1^{(0)}$ and $b_1^{(1)}$, and to find the other coefficients by applying the relations (12) and (15). Instead of $b_1^{(0)}$ and $b_1^{(1)}$, the computation of which will be considered in the following section, we can take as the initial quantities the coefficients $b_1^{(10)}$ and $b_1^{(11)}$ which can easily be found by means of the formulae of the preceding section.

The application of the recurrence relation (12) is not convenient for small values of α since in this case it is accompanied by a considerable loss of accuracy. The same may be said on the application of formulae (15) for large values of α. However, it is necessary to point out that at present, one is rarely in need of computing Laplace coefficients for there are several published tables which give the values of these coefficients. The best of these table is by Brown and Brouwer[1]. Putting

$$b_n^{(i)} = \frac{\alpha^i}{\sqrt{1-\alpha^2}}\, G_n^{(i)},$$

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(1) E.W. Brown and D. Brouwer, Tables for the development of the disturbing functions with schedules for harmonic analysis. Cambridge, 1933.

these authors computed $\lg G^{(i)}_{\frac{n}{2}}$ (for n = 1, 3, 5 with eight decimals and for n = 7 with seven decimals) for the argument $p = \alpha^2 : (1 - \alpha^2)$, varying from 0.00 to 2.50. They took i = 0, 1, 2, ..., 11.

95. The expression of the Laplace Coefficients in Terms of Definite Integrals

Applying the well-known Euler's formula for the computations of the coefficients of the Fourier series to expression (1), we obtain

$$b_n^{(i)} = \frac{2}{\pi}\int_0^\pi (1+\alpha^2-2\alpha\cos x)^{-\frac{n}{2}}\cos ix\,dx. \qquad (16)$$

This formula is not useful forcomputing the Laplace coefficients with Large values of i, because in this case the function cos ix changes sign many times. Moreover, for small values of α the coefficient $b_n^{(i)}$ behaves like α^{ℓ}. This important property is not clear in formula (16). A more convenient formula can easily be obtained from formula (16). The well known relation

$$\int_0^\pi \cos^p x\cos ix\,dx = \frac{p(p-1)\ldots(p-i+1)}{1.3\ldots(2i-1)}\int_0^\pi \cos^{p-i}x\sin^{2i}x\,dx$$

enables us to write for any function f(t) which can be expanded by the following uniformly convergent series in the interval $-1 < t < 1$

$$f(t) = \sum_0 a_p t^p,$$

the following relation

$$\int_0^\pi f(\cos x)\cos ix\,dx = \frac{1}{1.3.5\ldots(2i-1)}\int_0^\pi f^{(i)}(\cos x)\sin^{2i}x\,dx,$$

which has been indicated by Jacobi. Applying this relation to the integral (16), we obtain

$$b_n^{(i)} = \frac{n(n+2)\ldots(n+2i-2)}{1.3.5\ldots(2i-1)}\,\frac{2}{\pi}\,\alpha^i \int_0^\pi (1+\alpha^2-2\alpha\cos x)^{-\frac{n}{2}-i}\sin^{2i} x\,dx. \quad (17)$$

This integral can be computed using the formula of quadratures, shown in Sec. 56. This is almost the best method of calculating the Laplace coefficients, especially when high accuracy is required.

We can apply Landen's transformation

[illegible]

which yields

$$\sin\varphi = \frac{\sin x}{\sqrt{1+\alpha^2-2\alpha\cos x}}, \quad d\varphi = \frac{1-\alpha\cos x}{1+\alpha^2-2\alpha\cos x}\,dx,$$

to equation (17). We then easily obtain

$$b_n^{(i)} = \frac{n(n+2)\ldots(n+2i-2)}{1.3\ldots(2i-1)}\,\frac{2}{\pi}\,\alpha^i \cdot \int_0^\pi \left\{\frac{\alpha\cos\varphi+\sqrt{1-\alpha^2\sin^2\varphi}}{1-\alpha^2}\right\}^{n-1} \frac{\sin^{2i}\varphi\,d\varphi}{\sqrt{1-\alpha^2\sin^2\varphi}}. \quad (18)$$

In particular, we obtain for n = 1

$$b_1^{(i)} = \frac{2}{\pi}\,\alpha^i \int_0^\pi \frac{\sin^{2i}\varphi\,d\varphi}{\sqrt{1-\alpha^2\sin^2\varphi}},$$

or, as we can easily see,

$$b_1^{(i)} = \frac{4}{\pi}\,\alpha^i \int_0^{\frac{\pi}{2}} \frac{\sin^{2i}\varphi\,d\varphi}{\sqrt{1-\alpha^2\sin^2\varphi}}, \quad (19)$$

whence,

$$b_1^{(0)} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-\alpha^2\sin^2\varphi}} = \frac{4}{\pi}\,F\left(\alpha, \frac{\pi}{2}\right), \quad (20)$$

where $F(\alpha, \frac{\pi}{2})$ is the complete elliptic integral of the first kind.

For i = 1, equation (19) yields

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$$b_1^{(1)} = \frac{4\alpha}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2\varphi}{\sqrt{1-\alpha^2\sin^2\varphi}}\,d\varphi = \frac{4}{\pi\alpha}\left[F\left(\alpha,\frac{\pi}{2}\right) - E\left(\alpha,\frac{\pi}{2}\right)\right], \qquad (21)$$

where

$$E\left(\alpha,\frac{\pi}{2}\right) = \int_0^{\frac{\pi}{2}} \sqrt{1-\alpha^2\sin^2\varphi}\,d\varphi$$

is the complete elliptic integral of the second kind.

The existance of many detailed tables on the complete elliptic integrals makes the application of formulae (20) and (21) particularly simple. However, the complete elliptic integrals can be easily calculated in a simple manner. For example, the complete elliptic integral of the first kind can be evaluated by means of Gauss' formula

$$F\left(\alpha,\frac{\pi}{2}\right) = \frac{\pi}{2}\,\frac{1}{M(1,\sqrt{1-\alpha^2})}, \qquad (22)$$

where M(a,b) is the arithmetic-geometric mean of a and b, i.e. the quantity defined by thefollowing limiting transition

$$\begin{aligned}
a_1 &= \frac{1}{2}(a+b), & b_1 &= \sqrt{ab}\\
a_2 &= \frac{1}{2}(a_1+b_1), & b_2 &= \sqrt{a_1 b_1}\\
&\cdots\cdots & &\cdots\cdots\\
a_n &= \frac{1}{2}(a_{n-1}+b_{n-1}), & b_n &= \sqrt{a_{n-1}b_{n-1}}\\
M(a,b) &= \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n. & &
\end{aligned}$$

96. Calculation of the Derivatives of the Laplace Coefficients. Newcomb's Method

We have seen in the previous chapter that in order to calculate the perturbation function by a series expansion, we have to not only know the Laplace coefficients but also their derivatives with respect to

These derivatives can be calculated by means of the series which results from differentiating series (3) term by term. However, these series converge more slowly than series (3). Hence, this method cannot be of any practical value.

Differentiating equation (9) term by term with respect to α, we obtain

$$-\frac{n}{2}\left[2\alpha-(z+z^{-1})\right]\left[1+\alpha^2-\alpha(z+z^{-1})\right]^{-\frac{n}{2}-1}=\frac{1}{2}\sum\frac{db_n^{(i)}}{d\alpha}z^i$$

or

$$\frac{n}{2}(z+z^{-1}-2\alpha)\sum b_{n+2}^{(i)}z^i=\sum\frac{db_n^{(i)}}{d\alpha}z^i.$$

Consequently,

$$\frac{db_n^{(i)}}{d\alpha}=\frac{n}{2}\left(b_{n+2}^{(i-1)}+b_{n+2}^{(i+1)}-2\alpha b_{n+2}^{(i)}\right).$$

Differentiating this relation and combining it with formulae (12) and (15), we easily find a series of recurrence relations which enables us to define the derivative of any order. These formulae, which have been used by Leverrier[1], are however not very practical.

On the other hand, as we have already pointed out in Sec. 88, what enters the expansion of the perturbation function is not the Laplace coefficients, but the quantities

$$c_n^{(i)}=\alpha^{\frac{n-1}{2}}b_n^{(i)} \tag{23}$$

and their derivatives with respect to $\lg\alpha$, i.e.

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(1) Annales de l'Observatoire de Paris, t. 2, 1856.

$$D^i c_n^{(i)} = \frac{d^i c_n^{(i)}}{(d \lg \alpha)^i} \,. \qquad (24)$$

The most convenient and the most accurate method for the computation of quantities (23) and (24) is the method suggested by Newcomb[1]. It consists in the development and the improvement of the method of computing Laplace coefficients which had been previously suggested by Laplace. In the following, we give a brief account of this method.

It follows from equation (6) that

$$c_n^{(i)} = 2\,\frac{\left(\frac{n}{2}, i\right)}{(1, i)}\,\alpha^{\frac{1}{2}(n+2i-n)}\,F\left(\frac{n}{2}, \frac{n}{2}+i, i+1; \alpha^2\right).$$

Newcomb introduced the following, more general function

$$c_n^{i,j} = 2^{j+1}\left(\frac{n}{2}, j\right)\frac{\left(\frac{n}{2}, i+j\right)}{(1, i+j)}\,\alpha^{\frac{1}{2}(n+2i+2j-n)} \cdot$$

$$\cdot F\left(\frac{n}{2}+j, \frac{n}{2}+i+j, i+j+1, \alpha^2\right), \qquad (25)$$

so that

$$c_n^{(i)} = c_n^{i,0} \,.$$

We note that equation (5) yields

$$\frac{d}{dx} F(A, B, C; x) = \frac{AB}{C} F(A+1, B+1, C+1; x),$$

from which it follows that

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(1) S. Newcomb, Development of the Perturbative Function and its Derivatives in sines and cosines of multiples of the eccentric anomalies, and in powers of the eccentricities, Astronomical papers, 3, Washington, 1891.

$$DF(A, B, C; \alpha^2) = 2\alpha^2 \frac{AB}{C} F(A+1, B+1, C+1; \alpha^2).$$

Applying this formula, we easily represent the derivative of the function (25) with respect to log α in the following form:

$$Dc_n^{i,j} = \frac{1}{2}(n+2i+4j-1)\, c_n^{i,j} + c_n^{i,j+1}.$$

Applying to both sides of the latter equation the operation D^k, we obtain the following relation

$$D^{k+1} c_n^{i,j} = \frac{1}{2}(n+2i+4j-1)\, D^k c_n^{i,j} + D^k c_n^{i,j+1}, \qquad (26)$$
$$(n = 1, 3, 5, \ldots, \; i, j, k = 0, 1, 2, \ldots)$$

which enables us to first obtain $D\, C_n^{i,j}$, then $D^2\, C_n^{i,j}$, and so on, if we know the quantities (25). In this way, the quantities (24) become known.

In this way the problem under consideration is reduced to the computation of the quantities (25). We divide this problem into three parts and solve them using the linear relations occurring between any three hypergeometric functions F (A, B, C, x), the parameters of which differ by integral values. For example, using series (5) it is easy to obtain that

$$C(C+1)F(A, B, C, x) - [C - (B - A + 1)x]\, F(A, B+1, C+1; x) - (B+1)(C - A + 1)x\, F(A, B+2, C+2, x) = 0. \qquad (27)$$

Formula (25) yields

$$c_n^{i-1,j} = L F\left(\frac{n}{2} + i, \frac{n}{2} + i + j - 1, i + j, \alpha^2\right)$$
$$c_n^{i,j} = L \frac{n + 2i + 2j - 2}{2i + 2j} \alpha^2 F\left(\frac{n}{2} + i, \frac{n}{2} + i + j, i + j + 1, \alpha^2\right)$$
$$c_n^{i+1,j} = L \frac{(n + 2i + 2j - 2)(n + 2i + 2j)}{(2i + 2j)(2i + 2j - 2)} \alpha^4 F\left(\frac{n}{2} + i, \frac{n}{2} + i + j + 1, i + j + 2; \alpha^2\right),$$

where

$$L = 2^{j+1}\left(\frac{n}{2}, j\right)\frac{\left(\frac{n}{2}, i+j-1\right)}{(1, i+j-1)}\,\alpha^{\frac{1}{2}(n+2i+\ldots)}$$

Therefore, substituting into equation (27)

$$A = \frac{n}{2}+j,\ B = \frac{n}{2}+i+j-1,\ C = i+j,\ x = -\alpha^2,$$

we obtain

$$(2i+2j+n-2)\,c_n^{i-1,j} - 2(i+j+i\alpha^2)\,c_n^{i,j} + (2i-n+2)\,\alpha\, c_n^{i+1,j} = 0. \quad (28)$$

Putting

$$p_n^{i,j} = \frac{c_n^{i,j}}{c_n^{i-1,j}}, \quad (29)$$

we can rewrite this equation as

$$p_n^{i,j} = \frac{P_n^{i,j}}{1 - Q_n^{i,j}\,p_n^{i+1,j}}, \quad (30)$$

where

$$P_n^{i,j} = \frac{(2i+2j+n-2)\alpha}{2i(1+\alpha^2)+2j}, \quad Q_n^{i,j} = \frac{(2i-n+2)\alpha}{2i(1+\alpha^2)+2j}.$$

Hence, if we know the quantity (29) for $i = k$, we can find its values for $i = k-1, k-2, \ldots, 2, 1$. In order to compute $p_n^{k,j}$, we can use the following continued fraction which immediately follows from this same formula (30):

$$p_n^{k,j} = \cfrac{P_k}{1 - \cfrac{Q_k P_{k+1}}{1 - \cfrac{Q_{k+1} P_{k+2}}{1 - \ldots}}}, \quad (31)$$

where for simplification it has been adopted that

$$P_k = P_n^{k,j}, \quad Q_k = Q_n^{k,j}.$$

Computing in this manner quantities (29) for $i = 1, 2, \ldots, k$ and knowing the value $C_n^{o,j}$, we can easily use equation (29) to obtain the values of $C_n^{i,j}$ for all of values of i under consideration.

We shall now calculate the quantity $C_n^{o,j}$ by first noting that the quantity

$$C_1^{0,0} \quad C_1^{0,1} \quad E_1^{1}$$

is very practically computed by means of formulae (20) and (22). Once we know $C_1^{(0,0)}$, $C_1^{(0,2)}$, ... by means of the recurrence relations which we are going to deduce.

On the basis of formula (25), we obtain

$$c_n^{1,j} = MF\left(\frac{n}{2}+j, \frac{n}{2}+j+1, j+2; \alpha^2\right)$$

$$c_n^{1,j+1} = M(n+2j)\frac{n+2j+2}{2j+4}\alpha^2 F\left(\frac{n}{2}+j+1, \frac{n}{2}+j+2, j+3; \alpha^2\right)$$

$$c_n^{0,j+1} = M(n+2j)\alpha F\left(\frac{n}{2}+j+1, \frac{n}{2}+j+1, j+2; \alpha^2\right),$$

where

$$M = 2^{j+1}\left(\frac{n}{2}, j\right)\frac{\left(\frac{n}{2}, j+1\right)}{(1, j+1)}\alpha^{\frac{1}{2}(n+2j+1)}.$$

Hence, putting

$$A = \frac{n}{2}+j, \quad B = \frac{n}{2}+j+1, \quad C = j+2, \quad x = \alpha^2$$

and, noting that

$$CF(A, B, C; x) - CF(A+1, B, C; x) + BxF(A+1, B+1, C+1; x) = 0,$$

we obtain

$$(2j+n)\alpha c_n^{1,j} - c_n^{0,j+1} + \alpha c_n^{1,j+1} = 0,$$

or

$$C_1^{0,j+1} = \frac{(2j+1) z p_1^{1,j} C_1^{0,j}}{1 - z p_1^{1,j+1}} . \qquad (32)$$

This is the required relation, which enables us to find $C_1^{0,1}$, $C_1^{0,2}$, ... when the quantities $C_1^{0,0}$ and $p_1^{1,j}$ are already computed.

It remains for us to consider the computation of the quantities in terms of $C_1^{0,j}$, we derive a new recurrence relation. From equation (25), we obtain

$$C_n^{0,j} = N F\left(\frac{n}{2}+j, \frac{n}{2}+j, j+1; z^2\right)$$

$$C_{n+2}^{0,j} = N\binom{n+2j}{n}^2 z F\left(\frac{n}{2}+j+1, \frac{n}{2}+j+1, j+1; z^2\right)$$

$$C_{n+2}^{1,j} = N\binom{n+2j}{2}\frac{n+2j+2}{2j+2} z^2 F\left(\frac{n}{2}+j+1, \frac{n}{2}+j+2, j+2; z^2\right),$$

where

[illegible]

Applying the following properties of the hypergeometric functions, which can be easily checked by means of equations (5),

[illegible]

we obtain

$$(n+2j) z C [illegible] n [n-(n+2j) z] C [illegible] z C_n^{1,j} [illegible] 0$$

from which it follows that

$$C_n = \frac{(n+2j) z C_n}{n [n-(n+2j) z] - 2n(n [illegible]) z p_{n+2}^{1,j}} \qquad ([illegible])$$

This formula completely solves the problem of computing the quantities $C_3^{0,j}$, $C_5^{0,j}$, ... by the values $C_1^{0,j}$ which have already been found.

Hence, in applying the Newcomb's method, we have to carry out the following operations.

(1) We compute the quantities $p_n^{i,j}$ for the largest of the values of $i = k$ and for all the required values of n and j by means of the continued fraction given by equation (31).

(2) We find all the other values of $p_n^{i,j}$ by means the relation (30).

(3) We find the quantity $C_1^{0,0} = b_1^{(0)}$ by means of equations (20) and (22).

(4) We compute all the values $C_1^{0,j}$ by using equation (32)..

(5) We obtain $C_3^{0,j}$, $C_5^{0,j}$, ... by means of equation (33).

(6) We calculate all the $C_n^{j,j}$ values by means of equation (29).

(7) We finally use equation (26) to find the quantities required for expanding the perturbation function, namely,

$$D^k c_n^{i,j} = \frac{d^k c_n^{i,j}}{(d \lg \alpha)^k} .$$

Annotation:

The continued fraction (30) rapidly converges only for small values of α . For this reasons, Hansen, suggested that this fraction should be replaced by

$$p_n^{i,j} = \cfrac{p}{1 - a_1 - \cfrac{b_1}{1 - a_2 - \cfrac{b_2}{1 - a_3 - \cfrac{\cdots}{1 - \cdots}}}}$$

where

$$[illegible]$$

$$a_1 = \frac{1}{4}\frac{(n \; \ldots)(2 \; n)}{(n \; \ldots)(n+j+1)}x^2, \qquad b_1 = \frac{1}{4}\frac{(2j+2j+n)(2j+2-n)}{(n+j+1)(n+j+2)}x^2,$$

and a_{m+1} and b_{m+1} are obtained from a_m and b_m by the replacement of j and n into j + 2 and n-2. This formula is a particular case of the following expansion which has been obtained by Gauss:

$$\frac{F(A,B+1,C+1;x)}{F(A,B,C;x)} = \cfrac{1}{1 - \cfrac{\alpha_1 x}{1 - \cfrac{\beta_1 x}{1 - \cfrac{\alpha_2 x}{1 - \cfrac{\beta_2 x}{1 - \ldots}}}}}$$

where

$$\alpha_1 = \frac{A}{C}\frac{C-B}{C+1}, \qquad \beta_1 = \frac{B+1}{C+1}\frac{C+1-A}{C+2}$$

$$\alpha_2 = \frac{A+1}{C+2}\frac{C+1-B}{C+3}, \qquad \beta_2 = \frac{B+2}{C+3}\frac{C+2-A}{C+4}$$

. . . .

CHAPTER XV

ANALYTICAL METHODS FOR OBTAINING THE PERTURBATIONS OF THE ELEMENTS

97. Transformation of the Differential equations which define the Orbital elements

The perturbation functions that correspond to the case, in which the motion of two planets is being considered, are given by

$$R = k^2 m' R_{0,1}, \qquad R' = k^2 m R_{1,0},$$

where R_{o1} and R_{o2} are defined by equation (1), Sec. 86.

Leverrier noted that

$$n^2 a^3 = k^2(1 + m), \qquad n'^2 a'^3 = k^2(1 + m')$$

and, hence, expressed the perturbation functions as

$$R = \frac{m'}{1+m} n^2 a^3 R_{0,1}, \qquad R' = \frac{m}{1+m'} n'^2 a'^3 R_{1,0}. \tag{1}$$

We already mentioned in Sec. 89 that Leverrier had applied the following expansion of the perturbation function(1)

$$a' R_{0,1} = \sum N e^h e'^{h'} \gamma^{2l} \cos D$$
$$a' R_{1,0} = \sum N e^h e'^{h'} \gamma^{2l} \cos D, \tag{2}$$

where

$$D = D_0 + j'\lambda' + k\varpi + k'\varpi' - 2g\tau'.$$

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(1) In this chapter, we shall keep as much as possible Leverrier's notations. We only note that he denotes the quantities J, , , and by , , , and respectively.

Since,

$$\varepsilon_0 = \lambda + \tau' - \tau, \qquad \omega = \pi + \tau' - \tau, \qquad (3)$$

where the difference $\tau' - \tau$ depends only on i, i' and $\Omega' - \Omega$ as can be seen from formula (35) in Chapter XIII, then

$$\frac{\partial R}{\partial \varepsilon} = \frac{\partial R}{\partial \lambda_0}, \qquad \frac{\partial R}{\partial \pi} = \frac{\partial R}{\partial \omega}.$$

Therefore, the Lagrange equations (41) given in Sec. 13 may be written in the following form

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2m'}{1+m} na \frac{\partial R_{0,1}}{\partial \varepsilon_0} \\
\frac{d\varepsilon}{dt} &= -\frac{2m'}{1+m} na \frac{\partial R_{0,1}}{\partial a} + \frac{m'}{1+m} na \cos\varphi \operatorname{tg}\frac{\varphi}{2} \frac{\partial R_{0,1}}{\partial e} + \operatorname{tg}\frac{i}{2} \sin i \frac{d\Omega}{dt} \\
\frac{de}{dt} &= -\frac{m'}{1+m} na \operatorname{ctg}\varphi \frac{\partial R_{0,1}}{\partial \omega} - \frac{m'}{1+m} na \cos\varphi \operatorname{tg}\frac{\varphi}{2} \frac{\partial R_{0,1}}{\partial \varepsilon_0} \\
\frac{d\pi}{dt} &= \frac{m'}{1+m} na \operatorname{ctg}\varphi \frac{\partial R_{0,1}}{\partial e} + \operatorname{tg}\frac{i}{2} \sin i \frac{d\Omega}{dt} \\
\frac{d\Omega}{dt} &= \frac{m'}{1+m} na \sec\varphi \operatorname{cosec} i \frac{\partial R_{0,1}}{\partial i} \\
\frac{di}{dt} &= -\frac{m'}{1+m} na \sec\varphi \operatorname{cosec} i \frac{\partial R_{0,1}}{\partial \Omega} - \frac{m'}{1+m} na \sec\varphi \operatorname{tg}\frac{i}{2} \left(\frac{\partial R_{0,1}}{\partial \varepsilon_0} + \frac{\partial R_{0,1}}{\partial \omega}\right),
\end{aligned}\right\} \quad (4)$$

where, we denote as usual the angle of eccentricity by φ. Since the expressions given by equation (2) do not explicitly involve i and Ω, we then have to eliminate the derivatives of $R_{0,1}$ with respect to these quantities by replacing them by derivatives with respect to τ, $\tau' - \tau$ and σ. Taking into account equation (3), we obtain

$$\left.\begin{aligned}\frac{\partial R}{\partial i} &= \frac{\partial R}{\partial \tau'}\frac{\partial \tau'}{\partial i} + \left(\frac{\partial R}{\partial \iota} + \frac{\partial R}{\partial \omega}\right)\frac{\partial(\tau'-\tau)}{\partial i} + \frac{1}{2}\frac{\partial R}{\partial \tau}\cos\frac{J}{2}\frac{\partial J}{\partial i}\\ \frac{\partial R}{\partial \Omega} &= \frac{\partial R}{\partial \tau'}\frac{\partial \tau'}{\partial \Omega} + \left(\frac{\partial R}{\partial \iota} + \frac{\partial R}{\partial \omega}\right)\frac{\partial(\tau'-\tau)}{\partial \Omega} + \frac{1}{2}\frac{\partial R}{\partial \tau}\cos\frac{J}{2}\frac{\partial J}{\partial \Omega}\end{aligned}\right| \quad (5)$$

In order to calculate the derivatives of τ', $\tau'-\tau$ and J with respect to i and Ω it is sufficient to apply the differential formulae of spherical trigonometry to the triangle $\Omega N \Omega_1$ (fig. 11) formed by three nodes. This yields the following set of equations

$$\begin{aligned} dJ &= -\cos(\tau-\Omega)\,di - \cos(\tau'-\Omega')\,di' - \sin i'\sin(\tau'-\Omega')\,d(\Omega'-\Omega)\\ \sin J\,d(\tau-\Omega) &= -\cos J\sin(\tau-\Omega)\,di + \sin(\tau'-\Omega')\,di' - \sin i'\cos(\tau'-\Omega')\,d(\Omega'-\Omega)\\ \sin J\,d(\tau'-\Omega') &= -\sin(\tau-\Omega)\,di + \cos J\sin(\tau'-\Omega')\,di' - \sin i'\cos(\tau'-\Omega')\,d(\Omega'-\Omega), \end{aligned}$$

from which the required partial derivatives are easily obtained. For example,

$$\frac{\partial(\tau'-\tau)}{\partial i} = -\frac{\sin(\tau-\Omega)}{\sin J} + \frac{\cos J\sin(\tau-\Omega)}{\sin J} = -\operatorname{tg}\frac{J}{2}\sin(\tau-\Omega).$$

Substituting these partial derivatives into expressions (5), and then substituting the resulting expressions into equations (4), we obtain a set of integrable differential equations in their final form.

Leverrier introduced the auxiliary quantities L, Λ, P_1, ... T and V by means of the following relations

$$\left.\begin{aligned} \frac{dL}{dt} &= \frac{2m'}{1+m}na^2\frac{\partial R_1}{\partial \varepsilon_0}, & \frac{d\Lambda}{dt} &= \frac{3m'}{1+m}n^2a\frac{\partial R_{\cdot}}{\partial \varepsilon},\\ \frac{dP_1}{dt} &= -\frac{2m'}{1+m}na\frac{\partial R_{\cdot 1}}{\partial a}, & \frac{dP_2}{dt} &= \frac{m'}{1+m}na\cos\varphi\frac{\partial R_{\cdot}}{\partial \varepsilon},\\ \frac{dP_3}{dt} &= \frac{m'}{1+m}na\operatorname{tg}\varphi\frac{\partial R_{\cdot 1}}{\partial \omega}, & \frac{dP_4}{dt} &= \frac{1}{2}\frac{m'}{1+m}\frac{na}{\cos\varphi}\cos\frac{J}{2}\frac{\partial R}{\partial \tau},\\ & & \frac{dT}{dt} = -\frac{1}{2}\frac{m'}{1+m}\frac{na}{\varphi}\sec\varphi\sec\frac{J}{2}\frac{\partial R_{0,1}}{\partial \tau}, & \\ & & \frac{dV}{dt} = \frac{m'}{1+m}na\sec\varphi\operatorname{tg}\frac{J}{2}\left(\frac{\partial R_{0,1}}{\partial \varepsilon_0} + \frac{\partial R_{0,1}}{\partial \omega}\right). & \end{aligned}\right| \quad (6)$$

This enables us to give to the final equations that define the elements the following forms

$$
\left.
\begin{aligned}
\frac{da}{dt} &= \frac{dL}{dt}, \qquad \frac{d^2\rho}{dt^2} = \frac{d^2\Lambda}{dt^2} \\
\frac{d\varepsilon}{dt} &= \frac{dP_1}{dt} + \operatorname{tg}\frac{\varphi}{2}\frac{dP_2}{dt} + \operatorname{tg}\frac{i}{2}\sin i\,\frac{d\Omega}{dt} \\
\frac{de}{dt} &= \frac{dP_3}{dt} - \frac{1}{2a}\operatorname{tg}\frac{\varphi}{2}\cos\varphi\,\frac{dL}{dt} \\
\frac{d\pi}{dt} &= \frac{dP_2}{dt} + e\operatorname{tg}\frac{i}{2}\sin i\,\frac{d\Omega}{dt} \\
\frac{di}{dt} &= -\sin(\tau-\Omega)\frac{dP_4}{dt} + \cos(\tau-\Omega)\left(\frac{dT}{dt} + \frac{dV}{dt}\right) \\
\sin i\,\frac{d\Omega}{dt} &= \cos(\tau-\Omega)\frac{dP_4}{dt} + \sin(\tau-\Omega)\left(\frac{dT}{dt} + \frac{dV}{dt}\right).
\end{aligned}
\right\} \quad (7)
$$

Because the slopes of the planetary orbits i, i', ... are small, Leverrier introduced instead of i and Ω the following elements

$$p = \operatorname{tg} i \sin\Omega, \qquad q = \operatorname{tg} i \cos\Omega.$$

The differential equations which define these elements can be written in the following way

$$
\left.
\begin{aligned}
\cos i\,\frac{dp}{dt} &= \cos\tau\,\frac{dP_4}{dt} + \sin\tau\left(\frac{dT}{dt} + \frac{dV}{dt}\right) + p\operatorname{tg}\frac{i}{2}\frac{di}{dt} \\
\cos i\,\frac{dq}{dt} &= \sin\tau\,\frac{dP_4}{dt} + \cos\tau\left(\frac{dT}{dt} + \frac{dV}{dt}\right) + q\operatorname{tg}\frac{i}{2}\frac{di}{dt}.
\end{aligned}
\right\} \quad (8)
$$

The use of the elements p and q is convenient not only because their perturbations are small while the perturbations of Ω may increase as much as possible due to the presence of a small factor in the denominator of the last of equations (7), but also because these quantities can in particular, be easily expressed in terms of the perturbations of the heliocentric latitudes (Sec. 100).

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98. <u>The Perturbations of the Elements</u>

Let us denote by $\delta_1\lambda$, $\delta_1 a$, $\delta_1 e$, ... the first order perturbations of the mean longitude λ and the elements a, e, ... Assuming that the elements involved in the right-hand side of equations (7) are constants and that they properly define the integration constants we obtain

$$\left.\begin{aligned}
&\delta_1 a = L, \qquad \delta_1\rho = \Lambda \\
&\delta_1\varepsilon = P_1 + P_2 \operatorname{tg}\frac{\varphi}{2} + \operatorname{tg}\frac{i}{2}\sin i\,\delta_1\Omega \\
&\delta_1 l = \delta_1\rho + \delta_1\varepsilon \\
&\delta_1 e = P_3 - \frac{1}{2a}\operatorname{tg}\frac{\varphi}{2}\cos\varphi\, L \\
&e\delta_1\pi = P_4 + e\operatorname{tg}\frac{i}{2}\sin i\,\delta_1\Omega, \\
&\dots\dots\dots\dots\dots
\end{aligned}\right| \qquad (9)$$

Substituting expressions (2) for $R_{0,1}$ into equation (6), and integrating, we obtain

$$\begin{aligned}
L &= \frac{2m'a^2}{(1+m)a'}\sum\frac{j}{j+j'\mu}Ne^n e'^{n'}\gamma^q\cos D \\
\Lambda &= -\frac{3m'a}{(1+m)a'}\sum\frac{j}{(j+j'\mu)^2}Ne^n e'^{n'}\gamma^q\sin D, \\
&\dots\dots\dots\dots\dots
\end{aligned}$$

where we denote by μ the ratio of the mean motions n/n'. Introducing these expansions into equations (9), we obtain the first-order perturbations of the elements in theform

$$\left.\begin{aligned}
&\delta_1 a = \sum A\cos D \qquad && \delta_1\rho = \sum L\sin D \\
&\delta_1 e = \sum E\cos D \qquad && e\delta_1\pi = \sum P\sin D \\
&\dots\dots\dots && \dots\dots\dots
\end{aligned}\right| \qquad (10)$$

where

$$D = jl + j'l' + k\omega + k'\pi' + 2q\theta .$$

Combining in these series the terms for which j = j' = 0, we obtain the secular perturbations. The remaining terms give the periodic inequalities. Denoting the secular part of each of the perturbations, say $\delta_1 a$, by $[\delta_1 a]$, we obtain

$$[\delta_1 a] = 0, \qquad [\delta_1 e] = 0,$$

$$[\delta_1 i] = [\delta_1 \ldots] \quad [P_1] \ldots [P_2] \operatorname{tg} \frac{i}{2} \ldots \operatorname{tg} \frac{i}{2} \sin i \, [\delta_1 \Omega],$$

. . . .

Let the secular term in the mean longitude be equal to

$$[\delta_1 \varepsilon] = \nu t.$$

Then, the mean longitude will be calculated up to within the first powers in mass by the formula

$$l = nt + \varepsilon_0 + \nu t + \text{periodic terms}$$

In this way, if we define the mean motion of the planets by means of the longitudes obtained from the observations in two epochs which are divided by a long interval of time, as it usually occurs in practice, we then do not obtain n but the quantity

$$n_1 = n + \nu \qquad (1)$$

We calculate a_1 by means of the relation

$$n_1^2 a_1^3 = k^2 (1 + m),$$

which is similar to equation

$$n^2 a^3 = k^2 (1 + m),$$

that relates the unperturbed mean motion to the semimajor axis. Since

$$a^3 = \frac{n_1^2 a_1^3}{(n_1 - \nu)^2},$$

then, disregarding errors in the order of m'^2, we obtain the following equation

$$a = a_1\left(1 + \frac{2}{3}\frac{\varkappa}{n_1}\right). \qquad (13)$$

Thus, defining n_1 from the results of these observations, we determine n and a by means of equations (12) and (13). These quantities should be substituted into equations (10), which determine the first order perturbations.

In the calculation of the second- and higher-order perturbations of the mean longitude, we have to use the quantity defined by equation (11). In other words, n should be replaced by $n + \varkappa = n_1$. Hence, it is better to write from the very beginning n_1 instead of n in all the arguments of D than to take into consideration a considerable part of the second-order perturbations in the first approximation. In doing this, we must be aware that the replacement of n by n_1 can only be done in the argument of D. As far as the coefficient of equations (4) and (6) are concerned, the quantity a will always have values given by equation (13), while n will only be a notation for the quantity

$$k\sqrt{1+m}\,a^{-3/2} = n_1 - \varkappa.$$

It is useful to point out that the mean motion of a planet which is given by the tables of the elements is n_1. This means that the tabular values of the mean motion includes the constant parts of the perturbations. Conforming with this, the value of the semimajor axis that are given by the tables and which are equal to a_1 in our notations should be replaced during the computation of the perturbations by

$$a = a_1 + \Delta a.$$

where

$$\Delta a_1 = \frac{2}{3}\frac{a_1}{n_1}x.$$

Within the limits of the accuracy accepted,

$$\lg a = \lg(a_1 + \Delta a_1) = \lg a_1 + M\frac{\Delta a_1}{a_1}, \qquad M = 0 \cdot 43429\ .\ .\ .,$$

Hence, the corresponding correction to lg a_1 is equal to

$$\Delta \lg a_1 = \frac{2}{3} M \frac{x}{n_1}.$$

Carrying out the calculation of the first terms of formulae (10), it is easy to find that

$$\Delta \lg a_1 = \frac{1}{6} Mm\, \alpha \frac{dc_1^{(0)}}{d\alpha} = \frac{1}{6} Mm\, \alpha\, Dc_1^{(0)} \tag{14}$$

Similarly, we obtain for the other planet

$$\Delta \lg a_1' = \frac{1}{6} Mm\, (c_1^{(0)} + Dc_1^{(0)}). \tag{15}$$

It is assumed in the derivation of these formulae that the ratio α = a_1/a_1' is less than unity. For each planet, we should take the sum of all those corrections, which correspond to all perturbing planets.

In order to illustrate the influence and character of the periodic perturbations, we list the perturbations produced by Venus on the motion of the sun, and calculated by Leverrier by means of equation (10).

j	j'	k	k'	[illegible]	A	L	[illegible]	P
1	1	0	0	0	+ 1.[illegible]	− 3.49	− 0.01	0.0[illegible]
2	− 2	0	0	0	− 0.86	− 2.0[illegible]	–	+ 0.0[illegible]
3	3	0	0	0	− 0.53	1.27	– –	+ 0.0[illegible]
4	− 4	0	0	0	0.[illegible]	− 0.[illegible]	– –	+ 0.01
5	− 5	0	0	0	− 0.22	− 0.11		+ 0.01
6	6	0	0	0	0.15	0.[illegible]5	–	0.01
7	7	0	0	0	0.1[illegible]	− 0.16		+ 0.0[illegible]
8	8	0	0	0	− 0.07	0.10		+ 0.0[illegible]
9	[illegible]	0	0	0	0.0[illegible]	− 0.07	–	+ 0.01
2	1	0	1	0	0.07	− 0.13		
3	− 2	0	1	0	+ 0.1[illegible]	1.02		—
4	3	0	1	0	0.04	− 0.14		
5	− 4	0	1	0	+ 0.[illegible]2	+ 0.0[illegible]		—
0	1	− 1	0	0	–		+ 1.01	+ 1.01
1	0	1	0	0	+ 0.0[illegible]	+ 0.01	+ 0.91	0.91
2	1	− 1	0	0	0.12	0.0[illegible]	+ 1.[illegible]2	+ 1.[illegible]
3	2	1	0	0	0.12	+ 1.[illegible]	1.17	1.17
4	[illegible]	1	0	0	0.11	[illegible]	1.0[illegible]	1.01
5	4	− 1	0	0	0.[illegible]	− [illegible]18	0.[illegible]	− 0.[illegible]
6	− 5	1	0	0	0.0[illegible]	0.[illegible]	0.27	0.27
7	− 6	1	0	0	0.04	− 0.0[illegible]	0.16	− 0.1[illegible]
8	− 7	1	0	0	0.03	0.[illegible]	− 0.10	0.10
9	− 8	1	0	0	0.02	− 0.0[illegible]	− 0.07	− 0.0[illegible]
5	[illegible]	0	2	0	+ 0.01	− 0.07		–
5	− 4	1	1	0	0.01	+ 0.18	0.06	0.[illegible]6
6	1	1	1	0	+ 0.01	+ 0.[illegible]	+ 0.06	0.06

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	i	[illegible]	P
[illegible]		2	0	0	0 07	[illegible]	[illegible]	[illegible]
[illegible]	1	2	[illegible]	0	[illegible]	0 0[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]	0	0 01	[illegible]	[illegible]	[illegible]
5	-3	0	0	2	[illegible]	0 02		
6	1	0	0	2	-0 01	0 07		
8	[illegible]	2	2	0	-	-0 08	+0 [illegible]	[illegible]
8	5	3	0	0	0.01	0 08	0 07	
8	5	-1	0	2	-0 01	-0 16	0 01	0 01
13	8	-2	[illegible]	0		+0 11	+0 0[illegible]	[illegible]
13	-8	1	-2	0	-	0 16	-0 01	0 [illegible]
13	-8	-4	-1	0	—	+0 71	0 02	0 [illegible]
13	-8	5	0	0	—	-0 46	-0 0[illegible]	-0 02
13	8	0	-3	2	—	+0.06	-	-
13	-8	-1	2	-2	—	0 58	-0 00	0 00
13	-8	-2	1	-2		+1 07	+0 02	0 02
13	-8	-3	0	2	—	-2.21	0 0[illegible]	-0 01
13	-8	0	-1	1	-	+0 33	-	
13	-8	1	0	-1	-	-1.18	0 01	-0 01

We have only listed in this table the 44 arguments, for which at least one term exceeds 0".05. Leverrier computed all the terms exceeding 0".001 and his table included 123 arguments.(1)

We note that in the case when Earth is one of the planets under consideration,

[illegible]

In this case, argument D becomes

---

(1) The coefficients whose values were less than 0".001 were replaced by dashes.

The perturbations of the Elements (Second Order with Respect to Masses):

We have been able to express by means of equations (10) the first order perturbations for planet P as well as for all planets P', P", ... under consideration. We now consider the calculation of the second order corrections. For this purpose, we have to replace the elements a, e, ..., a', e', ..., a", e", ... of all the planets in the expression (1) of the perturbation function by

$$
\begin{array}{ll}
a + \delta_1 a, & e + \delta_1 e, \ \ldots \\
a' + \delta_1 a', & e' + \delta_1 e', \ \ldots \\
a'' + \delta_1 a'', & e'' + \delta_1 e'', \ \ldots
\end{array}
$$

In this way, equations (4) lead to the following equations for the calculation of the perturbations of the second order

$$
\frac{d\delta_2 a}{dt} = \frac{2m'}{1+m} n a^2 \left[\frac{\partial^2 R_{01}}{\partial \varepsilon\, \partial a} \delta_1 a + \ldots + \frac{\partial^2 R}{\partial \varepsilon\, \partial a'} \delta_1 a' + \ldots \right] + \frac{2m'}{1+m} a^2 \frac{\partial R_{01}}{\partial \varepsilon} \delta_1 n + \frac{4m'}{1+m} n a \frac{\partial R_{01}}{\partial \varepsilon} \delta_1 a. \qquad (16)
$$

Substituting here expressions (2) and (10), and doing the necessary multiplication of the series, we obtain on the right-hand side a series developed in cosines and sines of arguments of the type

$$
j\lambda_0 + j'\lambda' + j''\lambda'' + \ldots + k\omega + k'\pi' + \ldots
$$

or

$$
(jn + j'n' + j''n'' + \ldots)t + \text{const.}
$$

Consequently, the integration of equations of the type (16) introduces divisors of the form $jn + j'n' + j''n'' + \ldots$. If these exists a group of integers $j, j', j'', \ldots$, the absolute value of each is not large, for which the sum $jn + j'n' + j''n'' + \ldots$ is small, then the corresponding second-order perturbation will be particularly large. The period of this perturbation, which is equal to $360/(jn + j'n' + j''n'' + \ldots)$, will be quite great.

As a result of the second approximation, we obtain for each element an expression of the type

$$A + A't + A''t^2 + \sum B\cos(ht + q) + t\sum B'\cos(h't + q').$$

The computation of the second-order perturbations of the elements is quite tedious since it involves a large number of terms in equations of the type (16). This difficulty becomes more significant when we carry out the calculation of third-order corrections. In the following chapter, we shall see that it is much easier to calculate second- and higher-order perturbations in the coordinates.

When Leverrier studied the motion of Mercury, Venus, Earth and Mars, he could confine himself to the calculation of a few second-order terms. For other planets, and in particular for Jupiter and Saturn, one has to take into consideration not only a large number of second-order perturbations, but also some third order perturbations. Leverrier's work on the calculation of the perturbations of these planets has been continued by Gaillot.

100. <u>The transformation of perturbations of the elements into perturbations of coordinates. Construction of Tables.</u>

After calculating the perturbations $\delta_1 a_1, \ldots, \delta_1\lambda$ of the elements and the mean longitude, we can derive general expressions for the coordinates of the planet. We first consider the longitude in the orbit w. It is equal for the unperturbed motion, to

$$w = \lambda + f, \tag{17}$$

where we denote by f, the equation of the centre (Sec. 82)

$$f = H_1 \sin M + H_2 \sin 2M + \ldots \tag{18}$$

Here,

$$H_1 = 2e - \frac{1}{4}e^3 + \ldots, \quad H_2 = \frac{5}{4}e^2 - \frac{11}{24}e^4 + \ldots$$

The mean anomaly is equal to

$$M = \lambda - \pi; \tag{19}$$

Hence, replacing in equation (17) λ , π and e by

$$\lambda + \delta_1\lambda, \quad \pi + \delta_1\pi, \quad e + \delta_1 e,$$

and confining ourselves to quantities of the first order with respect to mass, we obtain in the first approximation the following expression for the perturbation of the longitude

$$\begin{aligned} \delta_1 w = \delta_1\lambda + \{H_1 \cos(\lambda - \pi) + 2H_2 \cos 2(\lambda - \pi) + \ldots\}\delta_1\lambda - \\ - \{H_1 \cos(\lambda - \pi) + 2H_2 \cos 2(\lambda - \pi) + \ldots\}\delta_1\pi + \\ + \left\{\frac{dH_1}{de}\sin(\lambda - \pi) + \frac{dH_2}{de}\sin 2(\lambda - \pi) + \ldots\right\}\delta_1 e. \end{aligned} \tag{20}$$

This formula is usually only applied to the inclusion of the short-periodic perturbation of λ and the periodic perturbations of π and e . The other perturbations of these elements will be best of all taken into account in the following way.

Formula (18) enables us to compute a table giving the equation of the centre f by the argument M for some definite values of e. For example, computations by Newcomb show that for Earth (for the mean contiguous midday 0 January 1900), we have

$$f = 6910''.057 \sin M + 72''.338 \sin 2M + 1''.054 \sin 3M + 0''.018 \sin 4M + \ldots$$

When we use this table, we make the argument (19) out of the values of λ, that have already been corrected for the long-periodic perturbations, and the values of π, in which the secular parts of the perturbations have been included. In this way, only the periodic perturbations of π and the short-periodic perturbations of λ remain to the share of the corrections $\delta_1\lambda$ and $\delta_1\pi$, involved in formula (20). On the other hand, the influence of the secular part of $\delta_1 e$, which we have denoted by $[\delta_1 e]$, is expressed by the following relation

$$[f] = \left\{\frac{df_1}{de} \sin M + \frac{df_2}{de} \sin 2M + \ldots\right\}[\delta_1 e]$$

This can be taken into account quite separately by means of special tables, which give these quantities in terms of the argument M. For Earth

$$e = 0,01675104 - 0,00004180\,T - 0,000000126\,T^2 = 3455''.150 - 8''.617\,T - 0''.0260\,T^2,$$

where T is time given in centuries and measured from the above-mentioned initial moment. Hence the influence of the secular perturbations of the eccentricity on the equation of the centre is taken into consideration by the quantity

$$[f] = (-17''.2107 - 0''.0527\,T)\,T \sin M - 0''.3617\,T \sin 2M - 0''.0017\,T \sin 3M.$$

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In Newcomb's tables for the motion of the earth (Astronomical Papers, Vol. VI), the quotient obtained by dividing this quantity by $T + 0.0030\ T^2$ is given by the argument M. It is worthwhile noting that the term proportional to T^2 expresses the contribution of the second-order secular perturbations.

Thus, the quantity $\delta_1 e$ appearing in equation (20) may be understood as the aggregate of only the periodic terms.

The computation of the sum of the periodic terms involved in equation (20) is simplified by constructing special tables, each of which give the sum of theterms that depend on a given argument $j\lambda + j'\lambda'$. The most important terms will be found by these tables. The sum of the remaining terms of equation (20) may be obtained by means of a table with two entrances corresponding to the arguments λ and λ'

Let us now consider the perturbations of the logarithm of the radius vector. For the perturbed motion, this logarithm is given by (Sec. 82)

$$\lg r = \lg a + A_0 + A_1 \cos M + A_2 \cos 2M + \dots \qquad (21)$$

where A_0, A_1, ... are functions of e. Consequently,

$$\begin{aligned}\delta_1 \lg r = \delta_1 \lg a &- \{A_1 \sin(l-\pi) + 2A_2 \sin 2(l-\pi) + \dots\}\delta_1 l + \\ &+ \{A_1 \sin(l-\pi) + 2A_2 \sin 2(l-\pi) + \dots\}\delta_1 \pi + \\ &+ \left\{\frac{dA_0}{de} + \frac{dA_1}{de}\cos M + \frac{dA_2}{de}\cos 2M + \dots\right\}\delta_1 e. \qquad (22)\end{aligned}$$

Formula (21) enables us to construct a table, which gives the values of the logarithm of the radius-vector for given values of e by the argument

$$M = l - \pi.$$

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For Earth, such a table has been constructed by Newcomb on the basis of the following formulae

$$\begin{aligned} \lg r = 0.00003057 &- 0.00727112 \cos M \\ &- 0.00009138 \cos 2M \\ &- 0.00000145 \cos 3M \\ &- 0.00000002 \cos 4M. \end{aligned}$$

In constructing this table, we agree to include in M the long-periodic part $\delta_1 \lambda_1$ for which a special table may be constructed together with the secular part of $\delta_1 \pi$. The influence of the secular part of $\delta_1 \lambda$ or ℓ_{gr} can easily be taken into account by means of a special table. If we take the sun as an example, we find that the sum of the terms of equation (22), which correspond to the secular part of $\delta_1 e$, are equal to

[illegible]

Newcomb constructed a table, from which we find the quotient resulting from the division of this quantity by $T + 0.0030\ T^2$, by the argument M. The sum of the periodic terms, which remain in formula (22) after having made all the simplifications, is partially computed for each of the perturbing planets by means of tables having a single entrance. The remainder can be taken from a table having two entrances.

We shall finally consider the determination of the heliocentric longitude ℓ and latitude b. We have seen in Sec. 85 that

$$l = w + R, \qquad \sin b = \sin i \sin u, \tag{23}$$

where

$$R = -\frac{1}{\operatorname{arc} 1''} \operatorname{tg}^2 \frac{i}{2} \sin 2u + \frac{1}{\operatorname{arc} 2''} \operatorname{tg}^4 \frac{i}{2} \sin 4u - \ldots \tag{24}$$

is the reduction to the ecliptic, and whereby u is devoted as the argument of latitude

$$u = w - \Omega. \tag{25}$$

In this way, taking an arbitrarily given value of i, we can construct two tables, which give by the argument u, the values of R and b. Putting , as before,

$$p = \operatorname{tg} i \sin \Omega, \qquad q = \operatorname{tg} i \cos \Omega,$$

we obtain

$$\sin b = \cos i (q \sin w - p \cos w).$$

According to the generally accepted rules, we take within argument (25) the values of w which include all the perturbations. Therefore, when we compute $\delta_1 b$, we can consider that in the latter equation, increments are only given to i, p and q. Whence,

$$\delta_1 b = \frac{\cos i}{\cos b}(\sin w \cdot \delta_1 q - \cos w \cdot \delta_1 p) - \operatorname{tg} b \operatorname{tg} i \cdot \delta_1 i. \tag{26}$$

The secular perturbations of Ω are taken into account by including them in the argument (25). The influence of the secular perturbations of i are are evaluated by means of the following formula

$$[\delta_1 b] = \operatorname{tg} b \operatorname{tg} i [\delta_1 i],$$

which can be reduced for all the large planets to the following form

$$[\delta_1 b] = A \sin u \cdot T$$

This formula can be replaced by a table having u as an argument. It is hence necessary to include by means of equation (26) the influence of only the periodic perturbations of i, p and q. This can be done in analogy to the previous cases.

We can take into consideration the influence of the secular perturbation of i on the reduction to the ecliptic, given by equation (24), in a very simple manner. The periodic perturbations of i do not significantly change R. In order to show this, we write down for Mars, the expressions of these quantities

[illegible]

These expressions constitute the basis of the corresponding tables given by Newcomb.

Annotation

We have considered the derivation of the formulae, that define the first-order terms in the perturbations of the coordinates. The second-order terms can be obtained exactly in a similar way. However this requires a great deal of tedious work.

The tables constructed by Leverrier give the perturbations of the coordinates r, ℓ and b for Mars, Earth, Venus and Mercury. As regards the other planets, for which second-order perturbations play an important role, Leverrier has only given tables which enable us to find the osculating elements for these planets at any moment. In order to find their coordinates, we have to apply the conventional formulae of the elliptic motion.

101. The computation of the secular terms by Gauss' method

The coefficients of the secular terms in the equations, which define the perturbations of the elements of a given planet, must be calculated more accurately than the coefficients of the other terms because the influence of the secular terms increases with time. The method of computation, developed in the previous sections, produces the coefficients of all perturbations with thesame accuracy. This accuracy is defined by the greatest powers of eccentricities and mutual slopes of the orbits which are kept in the expansion. Gauss suggested (1818) an alternative method for calculating the first-order perturbations, which enables us to find them independently from the other perturbations. This method does not imply the expansion of the perturbation function in a series. Hence, it can be equally applied for any eccentricity and for any slope of the orbit.

In section 12, we obtained equations (37) which express the derivatives of the elements in terms of the components of the perturbing accelerations. These formulae are of the components of the perturbing accelerations. These formulae are of the form

$$\left.\begin{aligned} \frac{di}{dt} &= r\cos u\, W_1 \\ \frac{de}{dt} &= p\sin v\, S_1 + (\cos v + \cos E)\, T_1 \\ &\dots\dots\dots\dots \end{aligned}\right\} \qquad (27)$$

where

$$S_1 = \frac{1}{k\sqrt{p}}\, S, \qquad T_1 = \frac{1}{k\sqrt{p}}\, T, \qquad W_1 = \frac{1}{k\sqrt{p}}\, W,$$

and S, T and W are the components of the perturbing accelerations.

When the perturbing acceleration is caused by the gravitation of a single planet, P', the components S, T and W are equal to the derivatives of the perturbation function in the direction of the radius vector and in the two directions perpendicular to the radius vector. One of these perpendicular directions is taken in the orbital plane and the other in the normal direction to this plane (Sec. 11).

We have already pointed out in section 88 that the second part of the perturbation function does not produce secular terms. We can therefore replace the perturbation function by its principal part

$$m'\Delta^{-1}, \qquad (28)$$

where Δ is the distance between planets P and P' and m the mass of planet P'. Using the series expansion of the perturbation function which we have already studied, we can write each of equations (27) in the following way

$$\frac{di}{dt} = \left[\frac{di}{dt}\right] + \sum A_{j,j'} \cos(jM + j'M' + Q), \qquad (29)$$

where we denote by $\left[\frac{di}{dt}\right]$ the constant term of the expansion. Assuming as before that the mean motions n and n' are incommensurable, we can set the quantity jn + j'n' equal to zero only in the case in which j = j' = 0. After integrating, we obtain

$$i = i_0 + \left[\frac{di}{dt}\right] t + \sum \frac{A_{j,j'}}{jn + j'n'} \sin(jM + j'M' + Q).$$

This means that the computation of the first-order secular perturbations is equivalent to the computation of the constant terms in expansions of the type (29).

It follows from equation (29) that

$$\left[\frac{di}{dt}\right] = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{di}{dt}\, dM\, dM';$$

In other words, the unknown constant term is obtained by averaging the quantities (27) over the variables M and M'. The variable M' which appears in expressions (27) depends only on S_1 , T_1 , W_1 . Hence, we shall first integrate with respect to M' and then with respect to M. Putting

$$S_0 = \frac{1}{2\pi}\int_0^{2\pi} S_1\, dM' = \frac{1}{2\pi}\,\frac{1}{k\sqrt{p}}\int_0^{2\pi} \mathbf{S}\, dM' \tag{30}$$

and similarly defining T_o and W_o, we finally obtain

$$\left.\begin{aligned} \left|\frac{di}{dt}\right| &= \frac{1}{2\pi}\int_0^{2\pi} r\cos u\, W_0\, dM \\ \left|\frac{d\varphi}{dt}\right| &= \frac{p}{2\pi}\int_0^{2\pi} \sin v\, S_0\, dM + \frac{p}{2\pi}\int_0^{2\pi} (\cos v + \cos E)\, T_0\, dM \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{31}$$

We shall first of all consider the computation of integrals (30). On the basis of the above-mentioned arguments concerning the replacement of the perturbation function by its principal part given by equation (28), we can consider integrals

$$\frac{1}{2\pi}\int_0^{2\pi} \mathbf{S}\, dM', \quad \frac{1}{2\pi}\int_0^{2\pi} \mathbf{T}\, dM', \quad \frac{1}{2\pi}\int_0^{2\pi} \mathbf{W}\, dM', \tag{32}$$

as components of some force of gravitation, which corresponds to the potential

$$\frac{m'}{2\pi}\int_0^{2\pi} \frac{dM'}{\Delta}.$$

This potential has a quite simple mechanical interpretation. As a matter of fact, let us imagine that the mass of planet P' is distributed over the orbit of this planet in such a way, that each element of mass dm' which will be distributed over one of the linear elements of the orbit, will be proportional to the time interval dt during which the planet passes this linear element. Accordingly,

$$\frac{dm'}{m'} = \frac{dt}{P},$$

where we denote by P' the period of rotation of planet P'.

However

$$\frac{dt}{P'} = \frac{n'\,dt}{n'P'} = \frac{dM'}{2\pi},$$

and, consequently

$$\frac{m'}{2\pi}\int_0^{2\pi}\frac{dM'}{\Delta} = \int\frac{dm'}{\Delta}.$$

This is nothing else but the potential of the elliptic ring produced by distributing the mass of planet P' along its orbit in the above-mentioned way. In this manner, our problem is reduced to the calculation of the components of the force of gravitation induced by a material elliptic ring, the density of which is defined by Kepler's law. We shall not do these calculations here. We only point out that the unknown integrals (32) can easily be expressed in terms of elliptic functions.

We have thus found the way to calculate the values of integrals (32) and consequently the quantities S_o, T_o and W_o in any arbitrary point of space. We shall now consider the computation of the quantities given by equations (31). The integrals involved in these equations can be calculated numerically. We calculate each of the integrands, for example r cos u W_o, for different values of M, and then take the mean values of the given quantities. Let us, for example, consider the expansion

$$\begin{aligned} r\cos u\, W_0 = a_0 + a_1\cos M + a_2\cos 2M + \ldots \\ + b_1\sin M + b_2\sin 2M + \ldots \end{aligned} \qquad (33)$$

We apply to it the conventional methods of harmonic analysis. We denote by Φ_0, Φ_1, ..., Φ_{k-1} the values of the functions (33) that correspond to

$$M = 0,\ \frac{2\pi}{k},\ 2\frac{2\pi}{k},\ \ldots,\ (k-1)\frac{2\pi}{k}. \tag{34}$$

In this case,

$$\left|\frac{di}{dt}\right| = a_0 = \frac{1}{k}(\Phi_0 + \Phi_1 + \ldots + \Phi_{k-1}) - a_k - a_{2k} - a_{3k} - \ldots$$

The series (33) converges so rapidly, that even for small values of k, the secular perturbations are obtained with a high accuracy. It is easy to show that the error obtained in calculating the eccentricities and mutual slopes of the orbits will be of the order of k-1 for the secular perturbations of λ, Ω, e and π, and of the order of k for the secular perturbations of the mean longitude of the epoch.

Instead of the variable M in the integrals (31), the eccentric anomaly is often introduced by means of the following relation

$$dM = (1 - e\cos E)\,dE.$$

That is to say, the averaging over M is replaced by the averaging over E. As we have already pointed out in Sec. 69, points corresponding to equidistant values of E are distributed along the orbit more uniformly than points (34) in the case in which the value of the eccentricity e is significant. We should note, however, that the advantages of applying E as an independent variable are not above reproach. Considering the uniform distribution of points along the orbits, the parts of the orbit over which the planet rapidly passes and those over which it passes slowly, have equal weights. Hence, we should not conclude that the replacement of M by the variable E will significantly reduce the value of k.

An interesting generalization of the restricted three body problem was given by Fatou(1). He considered the motion of the material point P,

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(1) P. Fatou, Sur le mouvement d'un point materiel dans un champ de gravitation fixe, Acta Astronomica, Ser a, Vol. 2, 1931, 401-462.

having an infinitesimally small mass, in the field of gravitation of the central body S and some material elliptic rings having the above-mentioned density distributions.

102. Lagrange's Differential Equations for the Determination of Secular Perturbations.

When we study the motion of a planet during a relatively short period of time, in the order of a few centuries, we can confine ourselves to first-order secular perturbations. In this case, the secular perturbations are best of all calculated by means of Gauss' method. This method enables us to obtain the secular perturbations of the eccentricities and of the slopes in a simple way with an arbitrarily high accuracy. If we are interested in longer periods of time, we have to apply the methods developed in sections 98 and 99. By these methods, we are able to obtain the second-, third-, ... etc. order secular perturbations. Naturally, the amount of work required by these calculations rapidly increases with increasing order of perturbations. Practically, we can hardly calculate the perturbations of orders higher than the third. Finally, if it is necessary to describe the motion of a planet during a considerably large interval of time, the decisive role will then be played by zero-rank terms. These are the terms in which the perturbing masses are raised to some power appear as multiplying factors (Sec. 15). Lagrange suggested an alternative method for calculating the secular perturbations of the eccentricities and the mutual slopes of the orbits. The main point of this method is that it can immediately lead tc us to just the zero-rank terms but will a relatively low accuracy. In the following, we shall give a brief account of the method.

Lagrange investigated the possibility of integrating equations (41) of section 13 under the condition of neglecting all the periodic terms in the perturbation functions appearing on the right-hand side of these equations. It is difficult to say without special considerations, that the integration of these diminished equations could lead us to the precise values of the perturbations, which would have been obtained in the form of secular terms if the exact equations were integrated by the method of successive approximations. However, we still believe that the result of integrating these diminished equations will satisfactorily elucidate the character of motion of a planet during an extremely long period of time. The results obtained will thus be of particular interest for cosmological investigations.

Replacing the perturbation function involved in the right hand side of the Lagrange equations by its secular terms, we can confine ourselves to the series expansion of this function in which terms involving higher second powers of the eccentricities and of the slopes are neglected. Only under this limitation, can we exactly carry out the integration of the Lagrange equations.

Let us now consider the expansion (36), Sec. 90. Noting that the second part of the perturbation function does not lead to secular terms, we obtain within the limits of accuracy, that the secular part part of the perturbation function will be given by

$$R = \frac{m'}{a'}\Big\{\frac{1}{2}C_0 - \frac{1}{4}[illegible]C_1 + \frac{1}{8}(e^2 + e'^2)(D - D')C_1 - \frac{1}{4}ee'(D + D'')C_1''\cos(\Pi' - \Pi)\Big\}.$$

Since

$$[illegible] = \Pi - \Pi'$$

and, as formula (35) of section 89 indicates, the difference $\tau' - \tau$ is a small quantity being of the second-order with respect to the slopes, we then may replace cos ($\Pi' - \Pi$) by cos ($\varpi' - \varpi$) always remaining within the limits of accepted accuracy. On the other hand, and within the same accuracy,

$$2\gamma^2 = 2\sin^2\frac{J}{2} = 1 - \cos J = 1 - \cos i\cos i' - \sin i\sin i'\cos(\Omega' - \Omega)$$
$$= 2\sin^2\frac{i}{2} + 2\sin^2\frac{i'}{2} - \sin i\sin i'\cos(\Omega' - \Omega)$$
$$= \frac{1}{2}\,\mathrm{tg}^2 i + \frac{1}{2}\,\mathrm{tg}^2 i' - \mathrm{tg}\, i\,\mathrm{tg}\, i'\cos(\Omega' - \Omega)$$

Hence, we can write

$$R_{0,1} = fm'[M_{0,1} + N_{0,1}(e^2 + e'^2 - \mathrm{tg}^2 i - \mathrm{tg}^2 i' + 2\,\mathrm{tg}\, i\,\mathrm{tg}\, i'\cos(\Omega' - \Omega)) + 2P_{0,1}ee'\cos(\varpi' - \varpi)], \tag{35}$$

where we denote by $M_{0,1}$, $N_{0,1}$ and $P_{0,1}$ coefficients which only depend on a and a'. According to the formulae developed in section 88 we can consider that these coefficients are symmetric functions of a and a'.

According to Lagrange, we replace the elements e, ϖ and i, Ω by the following variables

$$h = e\sin\varpi, \qquad l = e\cos\varpi, \tag{36}$$
$$p = \mathrm{tg}\, i\sin\Omega, \qquad q = \mathrm{tg}\, i\cos\Omega. \tag{37}$$

We can then rewrite expression (35) in the following manner

$$R_{0,1} = fm'[M_{0,1} + N_{0,1}(h^2 + l^2 + h'^2 + l'^2 - p^2 - q^2 - p'^2 - q'^2 + 2(pp' + qq')) + 2P_{0,1}(hh' + ll')], \tag{38}$$

where the variables h', ℓ' , p' and q' are defined by equations similar to equations (36) and (37).

We now deduce the differential equations that define the new elements given by equations (36) and (37). Since

$$[illegible]$$

then, using equation (41) of section 13, and noting that

$$\frac{\partial R}{\partial e} = \frac{\partial R}{\partial h}\sin\pi + \frac{\partial R}{\partial l}\cos\pi$$

$$\frac{\partial R}{\partial \pi} = \frac{\partial R}{\partial h} l - \frac{\partial R}{\partial l} h,$$

we easily obtain

$$\left.\begin{aligned}\frac{dh}{dt} &= \frac{\sqrt{1-h^2-l^2}}{na^2}\left(\frac{\partial R}{\partial l} - \frac{h}{1+\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \varepsilon}\right) + \frac{l\,\mathrm{tg}\,\frac{i}{2}}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i}\\ \frac{dl}{dt} &= -\frac{\sqrt{1-h^2-l^2}}{na^2}\left(\frac{\partial R}{\partial h} - \frac{l}{1+\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial \varepsilon}\right) - \frac{h\,\mathrm{tg}\,\frac{i}{2}}{na^2\sqrt{1-h^2-l^2}}\frac{\partial R}{\partial i}.\end{aligned}\right\} \quad (39)$$

Differentiating expressions (38) with respect to h, $\mathcal{l}$ and i yields first-order powers of those small quantities, in terms of which the expansion is carried out. Hence, neglecting the terms involving third powers, we obtain

$$\frac{dh}{dt} = \frac{1}{na^2}\frac{\partial R}{\partial l}, \quad \frac{dl}{dt} = -\frac{1}{na^2}\frac{\partial R}{\partial h}. \quad (40)$$

When we are interested in the secular perturbation produced by the interaction of the two planets P and P', we have to replace the quantity R in equation (40) by the expression given by equation (38). We shall however consider the general problem which involves an arbitrary number of interacting elements and try to find the secular terms of their elements. We denote by m_0, m_1, m_2, ... the masses of planets P, P', P", ..., and by a_0, a_1, ...; e_0, e_1, ... their elements.

Applying equation (40) to planet $P^{(\mu)}$, we obtain

$$\frac{dh_\mu}{dt} = \frac{1}{n_\mu a_\mu^2}\frac{\partial R_\mu}{\partial l_\mu}, \quad \frac{dl_\mu}{dt} = -\frac{1}{n_\mu a_\mu^2}\frac{\partial R_\mu}{\partial h_\mu}, \quad (41)$$

where, due to equations (38),

$$R = \sum k^2 \ldots \sum k^2 m\, M \ldots \sum k^2 m N_{\ldots} (h^2 \ldots) \ldots$$
$$\ldots h \ldots l \ldots p \ldots q \ldots p \ldots q \ldots 2(p\, p \ldots q\, q) \ldots$$
$$\sum k^2 m P_{\ldots} (h\, h \ldots l\, l), \qquad (42)$$

In order to avoid making exclusions during the summation, we consider that

$$N_{\mu,\mu} = P_{\mu,\mu} = 0$$

Denoting, in short,

$$\frac{2k^2 m_\nu}{n_\mu a_\mu^2} N_{\mu,\nu} = (\mu,\nu) \qquad \frac{2k^2 m_\nu}{n_\mu a_\mu^2} P_{\mu,\nu} = [\mu,\nu]$$

the result of substituting expression (42) in equation (41), is rewritten in the following way:

$$\left.\begin{aligned} \frac{dh_\mu}{dt} - A_\mu l_\mu + [\mu,0] l_0 + [\mu,1] l_1 + \ldots &= 0 \\ \frac{dl_\mu}{dt} + A_\mu h_\mu - [\mu,0] h_0 - [\mu,1] h_1 - \ldots &= 0 \end{aligned}\right\} \qquad (43)$$

where

$$A_\mu = (\mu,0) + (\mu,1) + (\mu,2) + \ldots \qquad (44)$$

Since each of the coefficients $N_{\mu,\nu}$ and $P_{\mu,\nu}$ is a symmetric function of a_μ and a_ν, then the following relations hold

$$\left.\begin{aligned} m_\mu n_\mu a_\mu^2 (\mu,\nu) &= m_\nu n_\nu a_\nu^2 (\nu,\mu) \\ m_\mu n_\mu a_\mu^2 [\mu,\nu] &= m_\nu n_\nu a_\nu^2 [\nu,\mu]. \end{aligned}\right\} \qquad (45)$$

In this way, the determination of the secular perturbations of the eccentricities e_0, e_1, e_2, ... and perihelion longitudes π_0, π_1, π_2, ... is reduced, within the accuracy accepted to the integration of equations (43).

We now consider the variables defined by equation (37), which determine the position of the orbits. We, first of all, find that

$$\frac{dp}{dt} = \operatorname{tg} i \cos\Omega \frac{d\Omega}{dt} + \sin\Omega \sec^2 i \frac{di}{dt}$$
$$\frac{dq}{dt} = -\operatorname{tg} i \sin\Omega \frac{d\Omega}{dt} + \cos\Omega \sec^2 i \frac{di}{dt}$$
$$\frac{\partial R}{\partial \Omega} = \operatorname{tg} i \cos\Omega \frac{\partial R}{\partial p} - \operatorname{tg} i \sin\Omega \frac{\partial R}{\partial q}$$
$$\frac{\partial R}{\partial i} = \sec^2 i \sin\Omega \frac{\partial R}{\partial p} + \sec^2 i \cos\Omega \frac{\partial R}{\partial q}.$$

Taking into account equations (41) in section 13, we easily obtain

$$\left.\begin{aligned} \frac{dp}{dt} &= \frac{\sec^3 i}{na^2\sqrt{1-e^2}} \frac{\partial R}{\partial q} - \frac{p \sec i \sec^2 \frac{i}{2}}{2na^2\sqrt{1-e^2}} \left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \varepsilon}\right) \\ \frac{dq}{dt} &= -\frac{\sec^3 i}{na^2\sqrt{1-e^2}} \frac{\partial R}{\partial p} - \frac{q \sec i \sec^2 \frac{i}{2}}{2na^2\sqrt{1-e^2}} \left(\frac{\partial R}{\partial \pi} + \frac{\partial R}{\partial \varepsilon}\right). \end{aligned}\right\} \quad (46)$$

In the present case, since the equantity R is expressed by equation (42), then $\frac{\partial R}{\partial \epsilon} = 0$ and $\frac{\partial R}{\partial \pi}$ is a second-order quantity. Hence, neglecting terms involving third order powers, we obtain

$$\frac{dp}{dt} = \frac{1}{na^2} \frac{\partial R}{\partial q}, \quad \frac{dq}{dt} = -\frac{1}{na^2} \frac{\partial R}{\partial p} \quad (47)$$

Substituting expression (42) into similar equations that define p_μ and q_μ, we finally obtain the following system

[illegible] (48)

We shall consider the solution of systems (43) and (48) in the next section. We here confine ourselves to obtain the outstanding first integrals of these systems that were first discovered by Laplace.

We multiply equations (13) by $m_\mu \eta_\mu a_\mu^2 h_\mu$ and $m_\mu n_\mu a_\mu^2$ respectively, add them term by term, then sum the results. This yields, due to relation (45),

$$\sum m_\mu n_\mu a_\mu \left(h_\mu \frac{dh_\mu}{dt} + l_\mu \frac{dl_\mu}{dt} \right) = 0,$$

or

$$\sum m_\mu n_\mu a_\mu^2 \left(h_\mu^2 + l_\mu^2 \right) = C,$$

where C is an arbitrary constant which may be defined by means of the initial conditions of motion. Taking equation (36) into account, we finally obtain

$$\sum m_\mu n_\mu a_\mu^2 e_\mu^2 = C. \tag{49}$$

Similarly, we obtain the following first integral of equations (48):

$$\sum m_\mu n_\mu a_\mu^2 \operatorname{tg}^2 i_\mu = C' \tag{50}$$

At present, the excentricities and the slopes of the orbits have small values. Hence, constants C and C' are also small. Due to the fact that all the terms which constitute the sums (49) and (50) are positive, Laplace considered it possible to conclude that in the future, e_μ and i_μ will always remain positive quantities. This conclusion is only valid as far as it concerns planets, whose mass constitute a large part of the sum of the planetary masses. If the mass of a planet is very small, its excentricity and slope can be sufficiently large without violating equations (49) and (50).

Since

$$n_\mu = k\sqrt{1 + m_\mu}\, a_\mu^{-\frac{3}{2}},$$

then, neglecting second-order quantities relative to the masses, we can replace equations (49) and (50) by

$$\sum m_\nu \sqrt{a_\nu}\, e_\nu^2 = \text{const}$$
$$\sum m_\nu \sqrt{a_\nu}\, \mathrm{tg}^2 i_\nu = \text{const} .$$

103. Trigonometric Expressions of the Secular Perturbations

We shall now consider the solution of equations (43). According to the general theory for integrating systems of linear differential equations with constant coefficients, we search for the particular solutions in the form of

$$h_\nu = \frac{L^{(\nu)}}{a_\nu \sqrt{m_\nu n_\nu}} \sin(st+\beta), \quad l_\nu = \frac{L^{(\nu)}}{a_\nu \sqrt{m_\nu n_\nu}} \cos(st+\beta), \tag{51}$$

where S, β and $L^{(\mu)}$ are constants. Substituting these expressions into equations (43), we obtain the following system for determining S and L :

$$\left.\begin{array}{l}
(A_{0,0}-s)L^{(0)}+A_{0,1}L^{(1)}+\ldots+A_{0,m}L^{(m)}=0 \\
A_{1,0}L^{(0)}+(A_{1,1}-s)L^{(1)}+\ldots+A_{1,m}L^{(m)}=0 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
A_{m,0}L^{(0)}+A_{m,1}L^{(1)}+\ldots+(A_{m,m}-s)L^{(m)}=0,
\end{array}\right\} \tag{52}$$

where m + 1 is the number of planets, and

$$A_{\mu,\nu} = -\frac{a_\mu \sqrt{m_\mu n_\mu}}{a_\nu \sqrt{m_\nu n_\nu}}[\mu,\nu], \quad \mu \neq \nu. \tag{53}$$

Equations (45) show that

$$A_{\mu,\nu} = A_{\nu,\mu}.$$

Hence, the determinant of the previous system

$$D(s) = \begin{vmatrix} A_{0,0} - s, & A_{0,1}, & \dots, & A_{0,m} \\ A_{1,0} & A_{1,1} - s, & \dots & A_{1,m} \\ \dots & \dots & \dots & \dots \\ A_{m,0} & A_{m,1}, & \dots, & A_{m,m} - s \end{vmatrix} \qquad (54)$$

is symmetric with respect to the main diagonal. Let us denote by S_o , S_1, ..., S_m the roots of the equation

$$D(s) = 0, \qquad (55)$$

where $L_i^{(\mu)}$ are the values of the coefficients, which will be obtained if we put $S = S_i$. One of these coefficients will remain arbitrary. Hence, we can put

$$L_i^{(\mu)} = C_i q_i^{(\mu)},$$

where C_o , C_1 , ..., C_m are arbitrary constants, and $q_\lambda^{(\mu)}$ are known numbers.

Since the parameter β involved in each of equations (51) remains arbitrary, then denoting by β_o , β_1 , ..., β_m another m + 1 arbitrary constant, we can write the general solution of system (43) in the following way

$$\begin{aligned} h &= \sum M_i^{(\mu)} \sin(s_i t + \beta_i) \\ l &= \sum M_i^{(\mu)} \cos(s_i t + \beta_i), \end{aligned} \qquad (56)$$

where we have put

$$M_i^{(\mu)} = \frac{L_i^{(\mu)}}{a_\mu \sqrt{m_\mu n_\mu}}. \qquad (57)$$

These equations are known as the secular equations. The properties of their solution depends strongly on the nature of the roots of equation (55).

Lagrange confined himself to the calculation of the roots of the secular equations using the values of constants that are obtained for the solar system. He found that these roots were real and unequal. Laplace was able to prove by means of integral (49) that equation (55) could not have any complex roots whatever the values of the constants were. Indeed, if there were any complex roots amongst the roots of equation (55), then the correspond term of equation (56) would include an exponential function. In thiscase the sum $h^2 + \ell^2$ would tend to infinity when $t \longrightarrow \pm\infty$. This would violate relation (49).

Laplace tried to prove the absence of equal roots of the secular equations by using similar argments. In doing this he made an error. He considered that in the presence of equal roots, there should be in the general integral (56), certain polynomials of t multiplying the trigonometric functions, which would certainly violate equation (46). However, it was almost simultaneously pointed out by Weierstrass (1858) and Somov (1859) that for equal roots, it was not at all necessary for t to appear outside the signs of the trigonometric functions.

When only two planets are considered, the absence of equal roots of th the secular equations is established quite simply by means of direct verification. The impossibility of the presence of equal roots was shown by Seelinger in 1878 for the three body problem.

In order to obtain the limiting values of the eccentricity e_{μ}, we square expressions (56) and add. This procedure yields.

$$
\begin{aligned}
e^2 = {} & M_0^{(\mu)2} + M_1^{(\mu)2} + M_2^{(\mu)2} + \dots \\
& + 2M_0^{(\mu)} M_1^{(\mu)} \cos[(s_0 - s_1)t + \beta_0 - \beta_1] \\
& + 2M_0^{(\mu)} M_2^{(\mu)} \cos[(s_0 - s_2)t + \beta_0 - \beta_2] \\
& \dots\dots\dots\dots\dots\dots
\end{aligned}
$$

from which it is clear that

$$e_\mu \le |M_0^{(\mu)}| + |M_1^{(\mu)}| + \dots \qquad (58)$$

We note that if one of the coefficients (57) in formulae (56), say $M_\rho^{(\mu)}$, exceeds by an absolute value the sum of all of the other terms, then the perihelions of planet $P^{(\mu)}$ will have a translational motion with an average velocity S_ρ. Indeed, combining together equations (56), we obtain

$$\begin{aligned} e_\mu \sin(\pi_\mu - s_\rho t - \beta_\rho) &= \sum M_i^{(\mu)} \sin[(s_i - s_\rho)t + \beta_i - \beta_\rho] \\ e_\mu \cos(\pi_\mu - s_\rho t - \beta_\rho) &= M_\rho^{(\mu)} + \sum M_i^{(\mu)} \cos[(s_i - s_\rho)t + \beta_i - \beta_\rho]. \end{aligned}$$

By condition,

$$|M_\rho^{(\mu)}| > \sum_{i \ne \rho} |M_i^{(\mu)}|.$$

Hence, $\cos(\pi_\mu - S_\rho t - \beta_\rho)$ will never be zero, andthus

$$\pi_\mu - s_\rho t - \beta_\rho = k \cdot 180^\circ + \delta_\mu(t),$$

where k is an integer, and the last term satisfies the condition

$$-90^\circ < \delta_\mu(t) < +90^\circ.$$

Hence, the perihelion of the planet under consideration will never be displaced by an angle more than 90° from a point moving with a uniform velocity S_ρ.

Considering equations (48) which define p_μ and q_μ, since they have the same form as equation (43), we can then immediately write their general solutions in the following way

$$\left.\begin{aligned} p_\mu &= \sum N_i^{(\mu)} \sin(\sigma_i t + \gamma_i) \\ q_\mu &= \sum N_i^{(\mu)} \cos(\sigma_i t + \gamma_i). \end{aligned}\right\} \qquad (59)$$

where, we denote by $\sigma_0, \sigma_1, \ldots, \sigma_m$ the roots of the secular equation

$$D'(\sigma) = 0, \tag{60}$$

which has the same structure as equation (55), with the only exception that in the present case

$$A'_{\mu\nu} = \frac{a_\mu \sqrt{m_\nu n_\mu}}{a_\nu \sqrt{m_\mu n_\nu}} (\mu,\nu) \tag{61}$$

for $\mu \neq \nu$.

Annotation:

Since all the quantities $A_{\mu\mu}$ and $A_{\mu\nu}$ involved in the secular equations (55) and (60) are of the order of the planetary masses, then the roots of these equations will be first-order quantities relative to these masses. Hence, expanding expressions (56) and (59) in powers of S_λ t and σ_λ t, we obtain zero-rank-terms.

104. Secular Perturbations of Large Planets

The numerical values of the coefficients of formulae (56) and (59) were for the first time obtained by Lagrange himself. His results are only of historical interest because Uranus could not be involved in these computations, and moreover, hypothetical values were taken for the masses of Mercury, Venus and Mars, obtained by means of multiplying the valumes of these planets by some assumed density.

In 1839, Leverrier repeated these computations with better values for the constants and taking Uranus as an example. However, the influence of Neptune, which was subsequently discovered by Leverrier in 1846, was not yet taken into account. The most complete results obtained in this field are those obtained by Stockwell[1], who suggested

(1) J.N. Stockwell, Smithsonian Contributions to Knowledge, Vol. 18, 1870, Washington 1873.

on the bases of his computations the following values for the masses, mean anual motions and semimajor axes

Table 1

Planet	1 : m	n	a
Mercury	4 865 751	5 381 016.″2	0.387 0987
Venus	390 000	2 106 641.138	0.723 3325
Earth	368 689	1 295 977.110	1.000 0000
Mars	2 680 637	689 050.9023	1.523 6878
Jupiter	1 047.879	109 256.719	5.202 798
Saturn	3 501.6	43 996.127	9.538 852
Uranus	24 905	15 424.5694	19.183 581
Neptune	18 780	7 873.993	30.033 86

Furthermore, he used the values of the required elements for 1.0 january 1850 and used as a basis the ecliptic and equinox of 1850.0. These values are given in table 2.

Table 2

μ	Planet	e	π	i	Ω
0	Mercury . . .	0.205 6179	75 7′ 0.″0	7° 0′ 8.″2	16° 33′ 3.″2
1	Venus	0.006 8118	129 28 51.7	3 23 31.1	75 20 12.9
2	Earth	0.016 7712	100 21 41.0	0 0 0.0	0 0 0.0
3	Mars	0.093 1324	333 17 47.8	1 51 2.3	48 23 36.8
4	Jupiter	0.048 2388	11 54 53.1	1 18 40.3	98 54 20.5
5	Saturn	0.055 9956	90 6 12.0	2 29 22.4	112 19 20.5
6	Uranus	0.046 2149	170 31 17.6	0 46 29.9	73 14 11.4
7	Neptune	0.009 1740	50 16 39.1	1 47 0.9	130 7 45.3

He obtained values for the parameters involved in equations (56) and (59) and which are given in tables 3 and 4.

The values obtained by Stockwell enables us to determine the limits, within which the mean values of the eccentricities and slopes vary. By the term, mean values, we mean as usual the values of the elements

Table 3

$\lambda =$	0	1	2	3	4	5	6	7
$s_\lambda =$	5".464 503	7".248 427	17".014 373	17".784 456	0".616 685	2".727 639	3".716 607	[illegible]
$\beta_\lambda =$	88° 0' 38"	20° 50' 19"	335° 11' 31"	137° 6' 36".5	67° 56' 35"	105° 3' 53"	28° 8' 46"	[illegible]
$M_\lambda^{(0)} =$	+0.176 6064	+0.026 8538	+0.001 4673	+0.001 5934	+0.000 0077	+0.000 5685	+0.024 4939	[illegible]
$M_\lambda^{(1)} =$	+0.008 5906	−0.020 1444	−0.011 2171	−0.013 1892	+0.000 0117	+0.000 5571	−0.016 6053	[illegible]
$M_\lambda^{(2)} =$	−0.005 4825	−0.015 3619	+0.011 3105	+0.016 2641	+0.000 0136	+0.000 5832	+0.015 3113	[illegible]
$M_\lambda^{(3)} =$	+0.000 8418	−0.002 5451	+0.022 5719	−0.079 0650	+0.000 0219	+0.000 7765	+0.018 7954	[illegible]
$M_\lambda^{(4)} =$	−0.000 0090	+0.000 0106	−0.000 1011	+0.000 0011	+0.000 0636	−0.001 9436	+0.043 1601	[illegible]
$M_\lambda^{(5)} =$	−0.000 0080	+0.000 0109	−0.000 0061	+0.000 0110	+0.000 0717	+0.001 7694	+0.034 1011	[illegible]
$M_\lambda^{(6)} =$	+0.000 0035	−0.000 0027	+0.000 0004	−0.000 0006	−0.001 5578	+0.029 7330	−0.044 8614	[illegible]
$M_\lambda^{(7)} =$	+0.000 0001	−0.000 0001	+0.000 0000	−0.000 0000	+0.010 0389	+0.002 9105	−0.001 42[illegible]5	[illegible]

Table 4

$\lambda =$	0	1	2	3	4	5	6
$\sigma_\lambda =$	$-5''.126\,112$	$-6''.592\,128$	$-17''.393\,390$	$-18''.408\,914$	$-0''.661\,606$	$-2''.916\,082$	[illegible]
$\gamma_\lambda =$	$21°\,6'\,26''.8$	$132°\,4[illegible]'\,57''.8$	$292°\,49'\,53''.2$	$251°\,45'\,8''.6$	$20°\,31'\,24''.6$	$133°\,56'\,10''.8$	[illegible]
$N_\lambda^{(0)} =$	$+0.121\,0760$	$+0.028\,3520$	$+0.001\,5240$	$+0.003\,6775$	$+0.001\,4778$	$+0.003\,1283$	[illegible]
$N_\lambda^{(1)} =$	$+0.014\,8670$	$-0.007\,83[illegible]$	$-0.008\,4783$	$-0.022\,4278$	$+0.001\,3565$	$+0.001\,8108$	[illegible]
$N_\lambda^{(2)} =$	$+0.010\,6550$	$-0.006\,5210$	$+0.006\,9546$	$+0.024\,4768$	$+0.001\,32[illegible]1$	$+0.001\,6228$	[illegible]
$N_\lambda^{(3)} =$	$+0.002\,1280$	$-0.001\,3250$	$+0.050\,6672$	$-0.037\,5951$	$+0.001\,2586$	$-0.001\,1557$	[illegible]
$N_\lambda^{(4)} =$	$-0.000\,0252$	$-0.000\,0095$	$-0.000\,0025$	$-0.000\,0001$	$+0.001\,1993$	$+0.000\,879[illegible]$	[illegible]
$N_\lambda^{(5)} =$	$-0.000\,0320$	$+0.000\,0[illegible]$	$-0.000\,0214$	$-0.000\,0006$	$+0.001\,1577$	$+0.000\,7150$	[illegible]
$N_\lambda^{(6)} =$	$+0.000\,0280$	$-0.000\,0070$	$-0.000\,0021$	$+0.000\,0000$	$-0.001\,1248$	$-0.017\,6572$	[illegible]
$N_\lambda^{(7)} =$	$-0.000\,0008$	$-0.000\,0004$	$-0.000\,0002$	$+0.000\,0000$	$-0.011\,7[illegible]82$	$-0.001\,9[illegible]$	[illegible]

in which both the secular and the periodic perturbations are taken into consideration. These limits are given by the following table

Planet.	Eccentricity Min.	Eccentricity Max.	Slope Min	Slope Max.
Mercury	0.121	0.232	4 11'	9 11'
Venus	0	0.071	0	3 16
Earth	0	0.068	0	3 6
Mars	0.018	0.140	0	5 5[illegible]
Jupiter	0.025	0.061	0°14'	0 29
Saturn	0.012	0.081	0 [illegible]	1 1
Uranus	0.012	0.078	0 51	1 7
Neptune	0.006	0.015	0 31	0 47

In this table, the slope is measured relative to the invariable plane. On the basis of the above-mentioned arguments, Stockwell defined this plane by thefollowing values of the elements

$$i = 1^\circ 35' 19''.376, \qquad \Omega = 106^\circ 14' 6''.00,$$

which are measured relative to the ecliptic and equinox of 1850.0. For all the planets except Venus and Earth, one of the coefficients $M_\lambda^{(\mu)}$ exceeds in absolute value the sum of the absolute values of the other coefficients. Thus, the perihelions of all of these planets have mean motions. It is interesting to note that the mean motions of the perihelions of Jupiter and Uranus are equal and the longitudes of these perihelions differ by exactly 180°. The perihelion of Jupiter vibrates about its mean position within the limits $\pm 24^\circ 10'$, while the perihelion of Uranus vibrates within the limits $\pm 47^\circ 33'$. Therefore, the distance between the perihelions of these planets at their closest point of approach is given by

$$180^\circ - (24^\circ 10' + 47^\circ 33') = 108^\circ 17'.$$

<u>Annotation</u>

The solution of equations (55) and (60), the right hand side of which are expressed by determinants of the type (54), has always been considered as a difficult problem. For this reason, Leverrier (1839) used a special method for solving these equations. Later on, Jacobi (1845) suggested an alternative method also based on the properties of determinants of the type (54). The best of such methods is that suggested by Krylov[1], which makes use of the specific properties of the secular equations. We shall not consider these methods here, since the conventional methods of unfolding determinants together with the Lobacevskij-Greffe method for numerical solution of equations leads to the required solution in a sufficiently simple way.

105. Secular Perturbations of Small Planets

We consider the case in which planet P has an infinitesimal mass such that the influence of this planet on the other planets P', P", ... $P^{(m)}$ can be neglected. In this case, the system of equations (43) is devided into two systems, one consisting of the equations obtained by putting μ = 1, 2, ..., m which do not involve quantities that concern P, and the other consists of the following two equations:

$$\left.\begin{aligned} \frac{dh}{dt} &= l\sum(0,\mu) - \sum l_\mu [0,\mu] \\ \frac{dl}{dt} &= -h\sum(0,\mu) + \sum h_\mu [0,\mu], \end{aligned}\right|$$

which define the secular perturbations of the small planet under

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(1) A.N. Krylov, On the numerical solution of the equations, by which the frequencies of small vibrations of material systems are defined. Transactions of the Academy of Science of the USSR, 1931 (Ocislennom resenii uravnenija, kotorym v tehniceskih voprosah opredeljajutsa castoty malyh kolebanija materialnyh sistem, Izvestija Akademii nauk SSSR, 1931).

investigation. When the first system which defines the mutual perturbations of the large planets is solved, we substitute the resulting expressions into equations (56) and (61). This yields a system of two non-homogeneous equations of the type

$$\left.\begin{aligned} \frac{dh}{dt} &= l\sum(0,\mu) - \sum A_\lambda \cos(s_\lambda t + \beta_\lambda) \\ \frac{dl}{dt} &= -h\sum(0,\mu) + \sum A_\lambda \sin(s_\lambda t + \beta_\lambda), \end{aligned}\right\} \quad (63)$$

where A_λ are constant coefficients. The solution of this system is given by the following equations

$$\left.\begin{aligned} h &= M\sin\left(t\Sigma(0,\mu)+\beta\right) + \sum \frac{A_\lambda}{\Sigma(0,\mu)-s_\lambda}\sin(s_\lambda t+\beta_\lambda) \\ l &= M\cos\left(t\Sigma(0,\mu)+\beta\right) + \sum \frac{A_\lambda}{\Sigma(0,\mu)-s_\lambda}\cos(s_\lambda t+\beta_\lambda), \end{aligned}\right\} \quad (64)$$

in which M and β are arbitrary constant s. Similar results can be obtained for the elements p and q.

The question on whether the sum $\sum(0,\mu)$ can be equal to one, of the quantities S_λ or not, which would simplify the simplification of the variables of integration has been investigated by Charlier(1).

The constants M and β characterize the motion of the small planet better than the variable elements e and π . For this reason, Hirayama called them the proper eccentricities and the proper perihelion longitudes.

Computing these quantities for a large number of small planets, Hirayama could separate several families of small planets. To each family, he related the planets having close values for the proper elements M and and semimajor axes.

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(1) C.L. Charlier, Die Mechanik des Himmels, 1, 424.
(2) Kiyotsugu Hirayama, Families of Astroids, Japanese Journal of Astronomy and Geophysics, Vol. I, 1923; Vol. V, 1928.

CHAPTER XVI

ANALYTICAL METHODS FOR OBTAINING THE PERTURBATIONS OF THE COORDINATES

106. Equations of the Perturbed Motion in Hansen's Coordinates

Let us consider a fixed heliocentric system of rectangular coordinates, e.g. the ecliptic system of a given epoch, and denote by x, y and z the coordinates of planet P, the motion of which is being investigated, and by x', y' and z' the coordinates of the perturbing planet P'. Let m and m' be the masses of these planets, r and r' their radius-vectors and Δ the distance between them. The equations of motion of planet P are thus given by (Sec. 3).

$$\left.\begin{aligned}\ddot{x}+k^2(1+m)\frac{x}{r^3}&=\frac{\partial R}{\partial x}\\ \ddot{y}+k^2(1+m)\frac{y}{r^3}&=\frac{\partial R}{\partial y}\\ \ddot{z}+k^2(1+m)\frac{z}{r^3}&=\frac{\partial R}{\partial z},\end{aligned}\right\} \qquad (1)$$

where

$$R=k^2m'\left(\frac{1}{\Delta}-\frac{xx'+yy'+zz'}{r'^3}\right) \qquad (2)$$

is the perturbation function.

We introduce a new moving rectangular system of axes by means of the following equations

$$\left.\begin{aligned}X&=\alpha x+\alpha_1 y+\alpha_2 z\\ Y&=\beta x+\beta_1 y+\beta_2 z\\ Z&=\gamma x+\gamma_1 y+\gamma_2 z\end{aligned}\right\} \qquad (3)$$

The angular coefficients α, α_1, ... γ_2 are functions of time. They satisfy the following relations

$$\left.\begin{array}{ll} \alpha^2+\alpha_1^2+\alpha_2^2=1; & \alpha\beta+\alpha_1\beta_1+\alpha_2\beta_2=0 \\ \beta^2+\beta_1^2+\beta_2^2=1; & \alpha\gamma+\alpha_1\gamma_1+\alpha_2\gamma_2=0 \\ \gamma^2+\gamma_1^2+\gamma_2^2=1; & \beta\gamma+\beta_1\gamma_1+\beta_2\gamma_2=0 \end{array}\right\} \quad (4)$$

from which it follows that

$$\left.\begin{array}{ll} \alpha^2+\beta^2+\gamma^2=1; & \alpha\alpha_1+\beta\beta_1+\gamma\gamma_1=0 \\ \alpha_1^2+\beta_1^2+\gamma_1^2=1, & \alpha\alpha_2+\beta\beta_2+\gamma\gamma_2=0 \\ \alpha_2^2+\beta_2^2+\gamma_2^2=1, & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0 \end{array}\right\} \quad (5)$$

Using these relations, we obtain from equations (3) that

$$\left.\begin{array}{l} x=\alpha X+\beta Y+\gamma Z \\ y=\alpha_1 X+\beta_1 Y+\gamma_1 Z \\ z=\alpha_2 X+\beta_2 Y+\gamma_2 Z \end{array}\right\} \quad (6)$$

The nine angular coefficients are already connected by means of six relations. We imply that these coefficients satisfy another auxiliary conditions, namely

$$\left.\begin{array}{l} x\dot{\alpha}+y\dot{\alpha}_1+z\dot{\alpha}_2=0 \\ x\dot{\beta}+y\dot{\beta}_1+z\dot{\beta}_2=0 \\ x\dot{\gamma}+y\dot{\gamma}_1+z\dot{\gamma}_2=0, \end{array}\right\} \quad (7)$$

Because of these conditions, the derivatives of the new moving coordinates are expressed by means of the following formulae

$$\left.\begin{array}{l} \dot{X}=\alpha\dot{x}+\alpha_1\dot{y}+\alpha_2\dot{z} \\ \dot{Y}=\beta\dot{x}+\beta_1\dot{y}+\beta_2\dot{z} \\ \dot{Z}=\gamma\dot{x}+\gamma_1\dot{y}+\gamma_2\dot{z}, \end{array}\right\} \quad (8)$$

exactly as if this coordinate system was not moving. By virtue of equations (6), one of equations (7) is a consequence of the other two.

Therefore, the nine angular coefficients α, α'_1, ... are related only by eight equations. Accordingly, there is an infinite number of moving coordinate systems which satisfy all the conditions that we have imposed here.

Hansen called the rectangular coordinates (X, Y, Z), the derivatives of which satisfy equations (8), the ideal coordinates. As we have already seen, these coordinates are expressed in terms of the fixed coordinates by means of formulae (3) together with the subsidiary conditions (4) and (7).

The choice of the ideal coordinate system is still not completely defined. We remove this arbitrariness by implying that these coordinates should satisfy condition

$$\gamma x + \gamma_1 y + \gamma_2 z = 0. \qquad (9)$$

This condition produces a coordinate system in which Z is always equal to zero. In other words, the plane XY will always pass by the radius vector of planet P. We shall call such a coordinate system a Hansen coordinate system. It is easy to see that the Hansen coordinate system is completely defined by conditions (4), (7) and (9), if we disregard the two arbitrary constants resulting from the integration of equations (7).

Our problem now is to obtain the equations of motion in terms of the Hansen coordinates. For this purpose we introduce a couple of subsidiary conditions. We multiply equations (3) by $\dot{\alpha}$, $\dot{\beta}$ and $\dot{\gamma}$ and add. This procedure gives

$$\begin{aligned} X\dot{\alpha} + Y\dot{\beta} = {} & x(\alpha\dot{\alpha} + \beta\dot{\beta} + \gamma\dot{\gamma}) + \\ & + y(\alpha_1\dot{\alpha} + \beta_1\dot{\beta} + \gamma_1\dot{\gamma}) + \\ & + z(\alpha_2\dot{\alpha} + \beta_2\dot{\beta} + \gamma_2\dot{\gamma}). \end{aligned}$$

Taking into account the relations which can be obtained from equations (5) by means of differentiation, we rewrite this equation as

$$X\dot{\alpha} + Y\dot{\beta} = -y(\alpha\dot{\alpha}_1 + \beta\dot{\beta}_1 + \gamma\dot{\gamma}_1) - \\ - z(\alpha\dot{\alpha}_2 + \beta\dot{\beta}_2 + \gamma\dot{\gamma}_2).$$

The expression found on the right-hand side of this equation disappears. This can be proved by multiplying equation (7) respectively by α, β and γ and adding the resulting equations, term by term. In this way, we obtain the first group of relations

$$\left.\begin{aligned} X\dot{\alpha} + Y\dot{\beta} &= 0 \\ X\dot{\alpha}_1 + Y\dot{\beta}_1 &= 0 \\ X\dot{\alpha}_2 + Y\dot{\beta}_2 &= 0; \end{aligned}\right\} \qquad (10)$$

Another two groups can be obtained in a similar way. Hence, equations (6) will have for the Hansen coordinates the following form:

$$x = \alpha X + \beta Y; \quad y = \alpha_1 X + \beta_1 Y; \quad z = \alpha_2 X + \beta_2 Y, \qquad (11)$$

Differentiating these equations twice, we obtain

$$\begin{aligned} \ddot{x} &= \alpha\ddot{X} + \beta\ddot{Y} + \dot{\alpha}\dot{X} + \dot{\beta}\dot{Y} \\ \ddot{y} &= \alpha_1\ddot{X} + \beta_1\ddot{Y} + \dot{\alpha}_1\dot{X} + \dot{\beta}_1\dot{Y} \\ \ddot{z} &= \alpha_2\ddot{X} + \beta_2\ddot{Y} + \dot{\alpha}_2\dot{X} + \dot{\beta}_2\dot{Y}. \end{aligned}$$

Multiplying these equations firstly by α, α_1 and α_2 secondly by β, β_1, and β_2 and thirdly by γ, γ_1 and γ_2 and adding after each multiplication, we finally obtain

$$\begin{aligned} \alpha\ddot{x} + \alpha_1\ddot{y} + \alpha_2\ddot{z} &= \ddot{X} \\ \beta\ddot{x} + \beta_1\ddot{y} + \beta_2\ddot{z} &= \ddot{Y} \end{aligned} \qquad (12)$$

$$\begin{aligned} \gamma\ddot{x} + \gamma_1\ddot{y} + \gamma_2\ddot{z} &= (\gamma\dot{\alpha} + \gamma_1\dot{\alpha}_1 + \gamma_2\dot{\alpha}_2)\dot{X} + \\ &+ (\gamma\dot{\beta} + \gamma_1\dot{\beta}_1 + \gamma_2\dot{\beta}_2)\dot{Y}. \end{aligned} \qquad (13)$$

Substituting expression (11) into the first of equations (7) yields.

$$(\alpha\dot{\alpha} + \alpha_1\dot{\alpha}_1 + \alpha_2\dot{\alpha}_2)X + (\beta\dot{\alpha} + \beta_1\dot{\alpha}_1 + \beta_2\dot{\alpha}_2)Y = 0,$$

Taking equation (4) into consideration, we obtain

$$\alpha\dot{\alpha} + \alpha_1\dot{\alpha}_1 + \alpha_2\dot{\alpha}_2 = 0, \quad \alpha\dot{\beta} + \alpha_1\dot{\beta}_1 + \alpha_2\dot{\beta}_2 + \dot{\alpha}\beta + \dot{\alpha}_1\beta_1 + \dot{\alpha}_2\beta_2 = 0,$$

Comparing the last two equations, we obtain

$$\beta\dot{\alpha} + \beta_1\dot{\alpha}_1 + \beta_2\dot{\alpha}_2 = 0, \quad \alpha\dot{\beta} + \alpha_1\dot{\beta}_1 + \alpha_2\dot{\beta}_2 = 0.$$

These equations yield

$$\frac{\dot{\alpha}}{\beta_1\alpha_2 - \beta_2\alpha_1} = \frac{\dot{\alpha}_1}{\beta_2\alpha - \beta\alpha_2} = \frac{\dot{\alpha}_2}{\beta\alpha_1 - \beta_1\alpha}.$$

However, since it follows from conditions (4) that

$$\frac{\gamma}{\beta_1\alpha_2 - \beta_2\alpha_1} = \frac{\gamma_1}{\beta_2\alpha - \beta\alpha_2} = \frac{\gamma_2}{\beta\alpha_1 - \beta_1\alpha},$$

we finally obtain

$$\frac{\dot{\alpha}}{\gamma} = \frac{\dot{\alpha}_1}{\gamma_1} = \frac{\dot{\alpha}_2}{\gamma_2}.$$

We can in a similar way prove the following relations

$$\frac{\dot{\beta}}{\gamma} = \frac{\dot{\beta}_1}{\gamma_1} = \frac{\dot{\beta}_2}{\gamma_2}.$$

By taking all these relations into account, we can replace equation (13) by

$$\gamma\ddot{x} + \gamma_1\ddot{y} + \gamma_2\ddot{z} = \frac{1}{\gamma_2}(\dot{\alpha}_2\dot{X} + \dot{\beta}_2\dot{Y}). \tag{14}$$

We shall now consider equations (1). Multiplying them firstly by α, α_1 and α_2, secondly by β, β_1 and β_2 and thirdly by γ, γ_1 and γ_2, and adding after each multiplication, we obtain considering equations (12) and (14)

$$\left.\begin{aligned} \ddot{X} + k^2(1+m)\frac{X}{r^3} &= \frac{\partial R}{\partial X} \\ \ddot{Y} + k^2(1+m)\frac{Y}{r^3} &= \frac{\partial R}{\partial Y} \end{aligned}\right\} \tag{15}$$

$$\dot{\alpha}_2\dot{X} + \dot{\beta}_2\dot{Y} = \gamma_2\frac{\partial R}{\partial Z}. \tag{16}$$

In fact, it follows from equations (6) that

$$\alpha\frac{\partial R}{\partial x}+\alpha_1\frac{\partial R}{\partial y}+\alpha_2\frac{\partial R}{\partial z}=\frac{\partial R}{\partial X}$$
$$\beta\frac{\partial R}{\partial x}+\beta_1\frac{\partial R}{\partial y}+\beta_2\frac{\partial R}{\partial z}=\frac{\partial R}{\partial Y}$$
$$\gamma\frac{\partial R}{\partial x}+\gamma_1\frac{\partial R}{\partial y}+\gamma_2\frac{\partial R}{\partial z}=\frac{\partial R}{\partial Z}$$

and, moreover,

$$r^2=x^2+y^2+z^2=X^2+Y^2.$$

Equation (16) and the last of equations (10) yield

$$\left.\begin{aligned}\frac{\dot\alpha_2}{\gamma_2}&=-h^{-1}Y\frac{\partial R}{\partial Z}\\ \frac{\dot\beta_2}{\gamma_2}&=+h^{-1}X\frac{\partial R}{\partial Z}\end{aligned}\right\}\qquad(17)$$

where

$$h=X\frac{dY}{dt}-Y\frac{dX}{dt}\qquad(18)$$

since,

$$\gamma_2=\sqrt{1-\alpha_2^2-\beta_2^2},$$

then equations (15) and (17) perfectly define X, Y, α_2 and β_2. After having done this, we can now obtain α, β, α_1 and β_1 by means of equations (4). In this way, we are able to find all the quantities, which enter expressions (11) defining the coordinates of planet P.

107. Transformation to the Polar Coordinates in the Plane of Osculating Orbits

Let us first prove that the coordinate plane of the Hansen coordinate system is the plane of the osculating orbits. For this purpose, it is sufficient to prove that the product

$$Z=\gamma\dot x+\gamma_1\dot y+\gamma_2\dot z$$

is equal to zero. Using relations (9) and (11), this product can be written in thefollowing way

$$\begin{aligned} Z &= -x\dot{\gamma} - y\dot{\gamma}_1 - z\dot{\gamma}_2 = \\ &= -X(\alpha\dot{\gamma} + \alpha_1\dot{\gamma}_1 + \alpha_2\dot{\gamma}_2) - \\ &\quad - Y(\beta\dot{\gamma} + \beta_1\dot{\gamma}_1 + \beta_2\dot{\gamma}_2). \end{aligned}$$

Substituting expressions (11) into the last of relations (7), we see that this quantity is really equal to zero.

We now introduce the polar coordinates in the plane of osculating orbits by putting

$$X = r\cos w, \quad Y = r\sin w. \tag{19}$$

since,

$$\begin{aligned} \frac{\partial R}{\partial w} &= -\frac{\partial R}{\partial X} r\sin w + \frac{\partial R}{\partial Y} r\cos w \\ \frac{\partial R}{\partial r} &= +\frac{\partial R}{\partial X}\cos w + \frac{\partial R}{\partial Y}\sin w, \end{aligned}$$

then, instead of equations (15), we obtain

$$\left.\begin{aligned} &\frac{d}{dt}\left(r^2\frac{dw}{dt}\right) = \frac{\partial R}{\partial w} \\ &\frac{d^2r}{dt^2} - r\left(\frac{dw}{dt}\right)^2 + \frac{k^2(1+m)}{r^2} = \frac{\partial R}{\partial r}. \end{aligned}\right\} \tag{20}$$

Equation (18) yields

$$h = r^2\frac{dw}{dt}, \tag{21}$$

from which it follows that the first of equations (20) may be replaced by

$$\frac{dh}{dt} = \frac{\partial R}{\partial w}. \tag{22}$$

108. The case of unperturbed motions.

If the perturbation function R is equal to zero, then equations (20) are reduced into the well-known equations of the two-body problem. In this case, their general solution will be given by

$$\left.\begin{array}{ll} E - e\sin E = n_0 t + M_0; & \operatorname{tg}\frac{1}{2}v = \sqrt{\frac{1+e_0}{1-e_0}}\operatorname{tg}\frac{1}{2}E \\ n_0^2 a_0^3 = k^2(1+m); & p_0 = a_0(1-e_0^2) \\ w = v + \chi_0, & r = \frac{p_0}{1+e_0\cos v} \end{array}\right\}, \tag{23}$$

which includes the four arbitrary constants a_o , e_o , M_o and X_o Equation (7) can in this case be considered separately from equations (20). They lead to constant values for α_2 and β_2. The remaining angular coefficients required, α , β , α_1 and β_1 can be defined by either equations (4) or (5). It is clear that one of these coefficients will remain arbitrary.

The final equations of motion of planet P are

$$\left.\begin{array}{l} x = \alpha r\cos w + \beta r\sin w \\ y = \alpha_1 r\cos w + \beta_1 r\sin w \\ z = \alpha_2 r\cos w + \beta_2 r\sin w, \end{array}\right| \tag{24}$$

They involve seven arbitrary constants. t is, however, easy to see that two of these constants, namely x_o and one of the coefficients α , β , α_1 and β_1 which remains arbitrary, define one and the same one thing, namely the position of the X axis in the XY plane. Hence, the value of one of these two coefficients may be fixed.

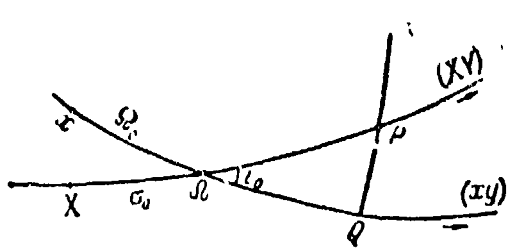

Fig. 13

Instead of the integration constants α_2 and β_2 defining the position of the orbital plane, we introduce another pair of more convenient parameters, namely the longitude of the ascending node Ω_0 and the slope i (Fig. 13).

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The position of the X axis in the orbital plane may be defined by the angle σ_0 between this axis and the ascending node of the XY plane relative to xy plane. Denoting by l and b the heliocentric longitude xQ and latitude QP of planet P, we obtain

$$\left.\begin{aligned} x &= r\cos b\cos l \\ y &= r\cos b\sin l \\ z &= r\sin b. \end{aligned}\right| \qquad (25)$$

Referring to triangle ΩQP, we see that since ΩP = XP - XΩ = $\omega - \sigma_0$, then

$$\left.\begin{aligned} \cos b\cos(l-\Omega_0) &= \cos(\omega-\sigma_0) \\ \cos b\sin(l-\Omega_0) &= \sin(\omega-\sigma_0)\cos i_0 \\ \sin b &= \sin(\omega-\sigma_0)\sin i_0, \end{aligned}\right| \qquad (26)$$

from which it follows that

$$\begin{aligned} \cos b\cos l &= \cos(\omega-\sigma_0)\cos\Omega_0 - \sin(\omega-\sigma_0)\sin\Omega_0\cos i_0 \\ \cos b\sin l &= \cos(\omega-\sigma_0)\sin\Omega_0 + \sin(\omega-\sigma_0)\cos\Omega_0\cos i_0 \end{aligned}$$

Comparing equations (24) and (25), we obtain

$$\left.\begin{aligned} \alpha &= \cos\sigma_0\cos\Omega_0 + \sin\sigma_0\sin\Omega_0\cos i_0 \\ \beta &= -\sin\sigma_0\cos\Omega_0 - \cos\sigma_0\sin\Omega_0\cos i_0 \\ \alpha_1 &= \cos\sigma_0\sin\Omega_0 - \sin\sigma_0\cos\Omega_0\cos i_0 \\ \beta_1 &= \sin\sigma_0\sin\Omega_0 + \cos\sigma_0\cos\Omega_0\cos i_0 \\ \alpha_2 &= -\sin\sigma_0\sin i_0 \\ \beta_2 &= \cos\sigma_0\sin i_0. \end{aligned}\right\} \qquad (27)$$

Equations (23), (24) and (27) establish the complete solution of the differential equations (1) for the case in which R = C. This wolution involves seven arbitrary constants, two of which are equivalent to a single constant. The fact lies in that euquations (26) deperd only on

$$w - \pi_0 = v + \chi_0 - \sigma_0,$$

i.e., on the difference $x_0 - \sigma_0$ which represents the distance between the perrihelion and the node. The presence of two constants instead of one in the above formulae is made use of in Hansen's method for the calculation of perturbations.

109. The Laplace-Newcomb Method. The perturbation of the Radius Vector

Laplace was the first to develop analytical methods for calculating the perturbations of the semimajor axis. He developed methods which made it possible to calculate the perturbations of the radius vector r, the longitudes w in the osculating orbits and the sines of the longitudes of the planets relative to the plane of the unperturbed orbits. Laplace confined himself to the calculation of the perturbations of only the first order and involving no more than the third powers of the eccentricities and the slopes.

In the eighties of the last century, Newcomb aimed to develop a theory for the motion of all the large planets. The reviewed Laplace's method, which considered to be the most practical method. All the tables computed according to Newcomb are to-day the basis of all of the annuals on astronomy. The construction of these tables is via the computation of the perturbations of the coordinates. The perturbations of Jupiter and Saturn are computed by Hansen's method, while the perturbations of the other planets are calculated by methods developed by Laplace. This shows that Laplace's methods still do not loose their partical value.

In the following, we give a brief description of Laplace's methods taking into account the improvements made by Newcomb in order to simplify the computation of the second- and higher-order perturbations.

Let us reconsider equation (20). Putting

$$\rho = \ln r \tag{28}$$

and, noting that

$$r\frac{\partial R}{\partial r} = \frac{\partial R}{\partial \rho},$$

we can rewrite these equations in the following manner

$$\left.\begin{aligned} r\frac{d^2 r}{dt^2} - r^2\left(\frac{dw}{dt}\right)^2 + \frac{k^2(1+m)}{r} &= \frac{\partial R}{\partial \rho} \\ r^2\frac{d^2 w}{dt^2} + 2r\frac{dr}{dt}\frac{dw}{dt} &= \frac{\partial R}{\partial w}. \end{aligned}\right\} \tag{29}$$

Multiplying the first of these equations by $2\frac{d\rho}{dt} = \frac{2}{r}\frac{dr}{dt}$ and the second by $2\frac{dw}{dt}$, adding and integrating, we obtain

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{dw}{dt}\right)^2 - \frac{2k_1^2}{r} = 2\left(C + \int d'R\right).$$

In order to simplify we have put

$$\left(\frac{\partial R}{\partial \rho}\frac{d\rho}{dt} + \frac{\partial R}{\partial w}\frac{dw}{dt}\right)dt = d'R, \quad k^2(1+m) = k_1^2,$$

and by C we have denoted an arbitrary constant. Adding this equation term by term to the first of equations (29), we obtain

$$\frac{1}{2}\frac{d^2(r^2)}{dt^2} - \frac{k_1^2}{r} = 2C + 2\int d'R + \frac{\partial R}{\partial \rho}. \tag{30}$$

We denote by r_0 the radius vector of the unperturbed motion which satisfies the following condition

$$\frac{1}{2}\frac{d^2(r_0^2)}{dt^2} - \frac{k_1^2}{r_0} = 2C.$$

Subtracting this equation term by term from equation (30), we obtain

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2 - r_0^2) - k_1^2(r^{-1} - r_0^{-1}) = 2\int d'R + \frac{\partial R}{\partial \rho}, \tag{31}$$

where the integral is defined by the condition that both the right- and left-hand sides are equal to zero when R = 0.

Newcomb considered it more useful not to determine the perturbation of the radius vector but of its logarithm. Hence, putting in analogy with equation (28)

$$\rho_0 = \ln r_0$$

and introducing the following notation

$$\delta\rho = \rho - \rho_0,$$

we obtain

$$r^2 = \exp(2\rho_0 + 2\delta\rho) = r_0^2 \exp(2\delta\rho) = \\ = r_0^2\left(1 + 2\delta\rho + \frac{4}{1.2}\,\delta\rho^2 + \ldots\right),$$

from which it follows that

$$\frac{1}{2}(r^2 - r_0^2) = r_0^2\delta\rho + r_0^2\delta\rho^2 + \ldots$$
$$r^{-1} - r_0^{-1} = -r_0^{-1}\delta\rho + \frac{1}{2}r_0^{-1}\delta\rho^2 + \ldots$$

We substitute these expressions into equation (31). We keep in the right-hand side only terms of the first order together with the second order terms whilest neglecting cerms of the higher orders. We then obtain

$$\frac{d^2(r_0^2\delta\rho)}{dt^2} + \frac{k_1^2}{r_0^3}(r_0^2\delta\rho) = 2\int d'R + \frac{\partial R}{\partial\rho} - \\ - \frac{d^2(r_0^2\delta\rho^2)}{dt^2} + \frac{k_1^2\delta\rho^2}{2r_0}, \qquad (32)$$

which will be the basic equation for calculating the perturbations of the radius vector. In this way, our task is turned into the integration of a differential equation of the following type

$$\frac{d^2q}{dt^2} + \frac{k_1^2}{r_0^3} q = Q, \tag{33}$$

where $k_1^2 r_0^{-3}$ and Q are unknown functions of t. Indeed, in order to obtain the first-order perturbations, it is necessary to neglect on the right-hand side of equation (32) the terms which involve $\delta\rho^2$ and to calculate the perturbation function R using the unperturbed values of the coordinates of the planets. This will lead us to the expression of the quantity Q by a well defined function of time. Similarly, by means of the computation of the second-order perturbations, we find the value of the right-hand side of equation (32) within any required accuracy in terms of the already known first order perturbations, and so on.

It remains for us to consider the integration of equation (33), which is a linear non-homogeneous second-order differential equation. Instead of applying the conventional form of the variation method of arbitrary constants, we proceed in a different manner. We suppose that the two linearily independent solutions q_1 and q_2 of the corresponding homogeneous equation

$$\ddot{q} + k_1^2 r_0^{-3} q = 0, \tag{34}$$

are known, so that

$$\ddot{q}_1 + k_1^2 r_0^{-3} q_1 = 0, \quad \ddot{q}_2 + k_1^2 r_0^{-3} q_2 = 0. \tag{35}$$

Eliminating $k_1^2 r_c^{-3}$ from these equations, we obtain

$$q_1\ddot{q}_2 - q_2\ddot{q}_1 = 0,$$

from which it follows that

$$q_1\dot{q}_2 - q_2\dot{q}_1 = \text{const.}$$

On the other hand, eliminating the same quantity from equations (33) by means of each of equations (33), we obtain

$$\dot{q}_1 q - q\dot{q}_1 = Qq_1, \quad \dot{q}_2 q - q\dot{q}_2 = Qq_2.$$

Denoting the arbitrary constants by K_1 and K_2, these equations become

$$q_1\dot{q} - q\dot{q}_1 = K_2 + \int q_1 Q\, dt,$$
$$q_2\dot{q} - q\dot{q}_2 = -K_1 + \int q_2 Q\, dt.$$

Multiplying the first equation by q_2 and the second by $-q_1$ and adding, we obtain

$$q(q_1\dot{q}_2 - q_2\dot{q}_1) = K_1 q_1 + K_2 q_2 + q_2 \int q_1 Q\, dt - q_1 \int q_2 Q\, dt. \qquad (\ldots)$$

In order to obtain the required particular solutions q_1 and q_2, we note that the orbital coordinates

$$\xi = a(\cos E - e), \quad \eta = a \cos\varphi \sin E$$

satisfy condition (34). Consequently, it is possible to write

$$q_1 = \cos E - e, \quad q_2 = \sin E,$$

These solutions can be expressed in the form of explicit functions of time as follows

$$q_1 = \frac{1}{2} \sum c_i \cos iM, \quad q_2 = \frac{1}{2} \sum s_i \sin iM, \qquad (37)$$

where, using the results obtained in section 88,

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$$c_0 = -3e; \qquad c_1 = 1 - \frac{3}{8}e^2 + \frac{5}{192}e^4 - \frac{7}{9216}e^6 + \ldots$$

$$c_2 = \frac{1}{2}e - \frac{1}{3}e^3 + \frac{1}{16}e^5 + \ldots; \quad c_3 = \frac{3}{8}e^2 - \frac{45}{128}e^4 + \frac{567}{5120}e^6 - \ldots$$

$$c_4 = \frac{1}{3}e^3 - \frac{6}{15}e^5 + \ldots; \qquad c_5 = \frac{125}{384}e^4 - \frac{4375}{9216}e^6 + \ldots$$

$$c_6 = \frac{27}{80}e^5 - \ldots; \qquad c_7 = \frac{16807}{46080}e^6 - \ldots$$

$$s_1 = 1 - \frac{1}{8}e^2 + \frac{1}{192}e^4 - \frac{1}{9216}e^6 + \ldots; \quad s_2 = \frac{1}{2}e - \frac{1}{6}e^3 + \frac{1}{48}e^5 - \ldots$$

$$s_3 = \frac{3}{8}e^2 - \frac{27}{128}e^4 + \frac{243}{5120}e^6 - \ldots; \qquad s_4 = \frac{1}{3}e - \frac{4}{15}e^5 + \ldots$$

$$s_5 = \frac{125}{384}e^4 - \frac{3125}{9216}e^6 + \ldots; \qquad s_6 = \frac{27}{80}e^5 - \ldots$$

$$s_7 = \frac{16807}{46080}e^6 - \ldots$$

The particular solutions, which we have chosen, satisfy the following relation

$$q_1\dot{q}_2 - q_2\dot{q}_1 = (1 - e\cos E)\frac{dE}{dt} = n,$$

which is implied by the Kepler equation.

We now apply formula (36) to the solution of equation (32), the right-hand side of which we denote by Q. We obtain

$$r\delta = n^{-1}q_2\int q_1 Q\,dt - n^{-1}q_1\int q_2 Q\,dt, \tag{38}$$

where q_1 and q_2 are defined by equations (37). The coefficient K_1 and K_2 will in this case be equal to zero because the right-hand side should be the same order relative to the perturbing mass as the left-hand side. When the expansion of the perturbation function R by multiples of the mean anomalies is known, the application of equation (38) to the calculation of the first-as well as the second order perturbations is quite simple. In evaluating second-order perturbations, the last two terms of equation (32) should be taken into account.

We note that it is better to evaluate the secular perturbations of the radius vector separately, by means of the secular perturbations of the elements which for example can be computed by Gauss' method, and then included in r_0. In this way, $\delta\rho$ will consist of only periodic terms. This will simplify the calculation of the second order perturbations.

110. <u>The Laplace-Newcomb Method. Computation of Longitudes. Computation of Heliocentric Coordinates</u>

According to Newcomb, we use the second of equations (20) to find the perturbed longitudes. We obtain

$$\frac{dw}{dt} = r^{-2}\left\{C + \int \frac{\partial R}{\partial w}\, dt\right\}.$$

We put

$$w = w_0 + \delta w,$$

and denote by w_0 the longitude which corresponds to the ecliptic motion. Since

$$\frac{dw_0}{dt} = r_0^{-2} a^2 n \cos\varphi, \tag{39}$$

and, consequently,

$$C = a^2 n \cos\varphi,$$

then

$$\frac{d\delta w}{dt} = r^{-2}\int \frac{\partial R}{\partial w}\, dt + (r^{-2} - r_0^{-2})\, a^2 n \cos\varphi.$$

Using the following expansion:

$$r^{-2} = r_0^{-2}(1 - 2\delta\rho + 2\delta\rho^2 - \ldots)$$

and confining ourselves to second-order terms, we obtain

$$r_0^2 \frac{d\delta w}{dt} = (1 - 2\delta\rho) \int \frac{\partial R}{\partial w} dt - 2a^2 n \cos\varphi (\delta\rho - \delta\rho^2) \qquad (40)$$

From this, we conclude that once we know the perturbations of the radius vector, we can obtain the perturbed longitudes by means of a quadrature.

Particular attention must be paid to the long-periodic terms of d'R. They yield for both $\delta\rho$ and δw, terms having coefficients involving the squares of small divisors appearing as a result of the double integration. Laplace suggested that these terms might be taken into consideration by calculating the elliptic coordinates r_0 and w_0 in terms of the mean anomaly, which could be found by

$$M = \int n_0 dt + \varepsilon_0 - \dots,$$

where the long-periodic expressions

$$3a^{-\frac{1}{2}} (1 + m)^{\frac{1}{2}} \int d'R$$

had been added to n_0. We are not going to consider here the development of this approach(1).

After obtaining first-order perturbations of the radius vector and latitude by means of equations (32) and (40), Newcomb calculated the perturbations of the elements i and Ω by using convential formulae. This enables us to perform the final steps in the calculation of the heliocentric coordinates by the method developed in section 100. Having calculated the first-order perturbations for all the three coordinates,

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(1) P.S. Lapalce, Traite de mechanique celeste, 1, 1799, 292.

we can find the second order perturbations in equations (32) and (40) by means of the same formulae. Only the right hand sides are changed.

111. <u>The initial form of Laplace's method</u>

We have already pointed out that Newcomb introduced some important modifications in the Laplace method whilst studying the motion of large planets. However, the initial form of this method, as described by Laplace in the volume I of "Mecanique Celeste", is not without interest. This method is even advantageous when we confine ourselves to first order perturbations. Laplace gave

$$r = r_0 + \delta r, \qquad w = w_0 + \delta w,$$

which allows us to write equation (30) and the first of equations (29) in the following manner

$$\frac{d^2(r_0\delta r)}{dt^2} + \frac{k_1^2 r_0 \delta r}{r_0^3} = 2\int d R + r_0 \frac{\partial R}{\partial r_0} + G_2 \qquad (41)$$

$$2r_0^2 \frac{dw_0}{dt}\frac{d\delta w}{dt} + \frac{d^2 r_0}{dt^2}\delta r - r_0 \frac{d^2 \delta r}{dt^2} + \frac{3k_1^2 r_0 \delta r}{r_0^3} = - r_0 \frac{\partial R}{\partial r_0} + H_2, \qquad (42)$$

where we denote by G_2 and H_2, the second- and higher order terms. Excluding quantity $k_1^2 r_0 \delta r / r_0^3$ from these equations, Laplace obtained on the basis of equation (39) the following equation for calculating the perturbations of the longitude:

$$a^{1/2} \cos\varphi \frac{d\delta w}{dt} = \frac{d}{dt}\left(2r_0 \frac{d\delta r}{dt} + \delta r \frac{dr_0}{dt}\right) - 3\int d R - 2r_0 \frac{\partial R}{\partial r_0} + l_2, \qquad (43)$$

where l_2 denotes the aggregate of the second- and higher-order terms.

Equation (41) is equivalent to equation (32). It allows us to obtain first-order perturbations of the radius vector and subsequently to find the second-order perturbations by means of equation (43). Equation (43) has the advantage over equation (40) that it does not involve the derivative $\frac{\partial R}{\partial \omega}$. Hence it does not require the calculations needed for obtaining this derivative. However this equation becomes less interesting when we wish to evaluate second-order perturbations. In this case we also have to calculate not only $\frac{\partial R}{\partial \omega}$ but the second derivatives of R.

Laplace suggested to simplify the integration of equation (41) in the following way. Since (Sec. 82)

$$\left(\frac{a_0}{r_0}\right) = 1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \ldots + \left(3e + \frac{27}{8}e^3 + \ldots\right)\cos M +$$
$$+ \left(\frac{9}{2}e^2 + \frac{7}{2}e^4 + \ldots\right)\cos 2M + \ldots,$$

then, equation (41) may be written as

$$\frac{d^2(r_0\delta r)}{dt^2} + n_1^2(r_0\delta r) = 2\int dR + r_0\frac{\partial R}{\partial r_0} -$$
$$- n_0^2(r_0\delta r)\left\{\left(3e + \frac{27}{8}e^3 + \ldots\right)\cos M + \left(\frac{9}{2}e^2 + \frac{7}{2}e^4 + \ldots\right)\cos 2M + \ldots\right\}, \quad (44)$$

where

$$n_1^2 = n_0^2\left(1 + \frac{3}{2}e^2 + \frac{15}{8}e^4 + \ldots\right).$$

The latter equation can easily be integrated by the method of successive approximations.

If we denote by

$$A\cos(\nu t + \beta) \qquad (45)$$

one of the terms of the right-hand side of this last equation, then the corresponding term in the expression of $r_0\,\delta r$ will be

$$\frac{A}{n_1^2 - \nu^2}\cos(\nu t + \beta), \qquad (46)$$

if $\nu \neq n$, and

$$\frac{At}{2n_1}\sin(n_1 t + \beta)$$

if $\nu = n_1$. We have already pointed out that in order to avoid secular terms in δr, we have to calculate r_0 by means of terms which already include secular perturbations. In this case, the quantities A, and β involved in the terms (45) behave as variables consequently, we shall have an expression of thetype

$$\frac{A}{n_1^2-\nu^2}\cos(\nu t+\beta) + \cos\nu t\left\{\frac{2\nu}{(n_1^2-\nu^2)^2}\frac{d(A\sin\beta)}{dt}+\dots\right\} - \sin\nu t\left\{\frac{2\nu}{(n_1^2-\nu^2)^2}\frac{d(A\cos\beta)}{dt}+\dots\right\}$$

in r_0 δr, instead of the terms (46).

In calculating the perturbations of the longitude, we will similarly have, instead of the integrals

$$\int A\cos(\nu t+\beta)\,dt = \frac{A}{\nu}\sin(\nu t+\beta)$$

$$\int dt\int A\sin(\nu t+\beta)\,dt = -\frac{A}{\nu^2}\sin(\nu t+\beta)$$

the following integrals

$$\int A\cos(\nu t+\beta)\,dt = \frac{A}{\nu}\sin(\nu t+\beta) + \sin\nu t\left\{-\frac{1}{\nu^2}\frac{d(A\sin\beta)}{dt} - \frac{1}{\nu^3}\frac{d^2(A\cos\beta)}{dt^2}+\dots\right\} + \cos\nu t\left\{\frac{1}{\nu^2}\frac{d(A\cos\beta)}{dt} - \frac{1}{\nu^3}\frac{d^2(A\sin\beta)}{dt^2}+\dots\right\}$$

$$\int dt\int A\sin(\nu t+\beta)\,dt = -\frac{A}{\nu^2}\sin(\nu t+\beta) + \sin\nu t\left\{\frac{2}{\nu^3}\frac{d(A\sin\beta)}{dt} + \frac{3}{\nu^4}\frac{d^2(A\cos\beta)}{dt^2}+\dots\right\} + \cos\nu t\left\{-\frac{2}{\nu^3}\frac{d(A\cos\beta)}{dt} + \frac{3}{\nu^4}\frac{d^2(A\sin\beta)}{dt^2}+\dots\right\}$$

112. Calculation of the Coordinates of the Rectangular Coordinates

The conventional form of the equations of motion of the perturbed planet P whose mass is m, is the following

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k_1^2 x r^{-3} + \frac{\partial R}{\partial x} \\ \frac{d^2y}{dt^2} &= -k_1^2 y r^{-3} + \frac{\partial R}{\partial y} \\ \frac{d^2z}{dt^2} &= -k_1^2 z r^{-3} + \frac{\partial R}{\partial z}, \end{aligned}\right| \qquad (47)$$

where $k_1^2 = k^2\ (1 + m)$ and R is the perturbation function defined by

$$R = \sum k^2 m' \left[\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3}\right],$$

where the summation is over all the perturbing planets. Let us replace the masses of the perturbing planets m', m", ... by

$$m' = \mu m_0', \qquad m'' = \mu m_0'', \ldots,$$

where m_0', m_0'', ... are constants and is a parameter varying from 0 to 1. If we take the initial values of the coordinates x, y, z, x', y', z', x", ... at the moment $t = t_0$ in that field, in which the right-hand side of the equations of motion are finite, there then exists such numbers as τ and μ_0, such that in the field

$$|t - t_0| \le \tau, \qquad 0 \le \mu \le \mu_0 \qquad (48)$$

functions $x(t, \mu)$, $y\ (t, \mu)$ and $z\ (t, \mu)$ will exist, which satisfy equations (47) and can be expanded in series of the type

$$x(t, \mu) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

which converge inside the field (48). The terms $x_0(t)$, $y_0(t)$, ... are responsible for the unperturbed motions. The terms

$$\delta_n x = \mu^n \xi_n(t), \qquad \delta_n y = \mu^n \eta_n(t), \qquad \delta_n z = \mu^n \zeta_n(t)$$

represent the n-order perturbations.

The order to obtain equations for the determination of first-order equations, we substitute into equations (47) the following expressions

$$\begin{aligned} x &= x_0 + \delta_1 x + \delta_2 x + \ldots \\ y &= y_0 + \delta_1 y + \delta_2 y + \ldots \\ z &= z_0 + \delta_1 z + \delta_2 z + \ldots \\ r &= r_0 + \delta_1 r + \delta_2 r + \ldots \end{aligned}$$

and then equate those terms having a first power with respect to μ. We obtain

$$\frac{x_0 + \delta_1 x + \ldots}{(r_0 + \delta_1 r + \ldots)^3} = \frac{x_0}{r_0^3} + \frac{\delta_1 x}{r_0^3} - 3\frac{x_0 \delta_1 r}{r_0^4} + \ldots,$$

On the other hand,

$$r = [(x_0 + \delta x + \ldots)^2 + (y_0 + \delta y + \ldots)^2 + (z_0 + \delta z + \ldots)^2]^{\frac{1}{2}} = \\ = r_0 + \frac{x_0 \delta_1 x + y_0 \delta_1 y + z_0 \delta_1 z}{r_0} + \ldots,$$

from which we find that

$$\delta_1 r = \frac{1}{r_0}(x_0 \delta_1 x + y_0 \delta_1 y + z_0 \delta_1 z).$$

Therefore, we finally obtain

$$\frac{d^2 \delta_1 x}{dt^2} = -\frac{k_1^2}{r_0^3}\delta_1 x + \frac{3k_1 x_0}{r_0^5}(x_0 \delta_1 x + y_0 \delta_1 y + z_0 \delta_1 z) + \frac{\partial R}{\partial x_0}$$

as well as two similar equations for the other two coordinates.

Similarly, by equating terms involving the factor μ^2, we obtain the following equations:

$$\left.\begin{aligned} \frac{d^2\delta_n x}{dt^2} + \frac{k_1^2}{r_0^3}\delta_n x &= \frac{3k_1^2 x_0}{r_0^5}(x_0\delta_n x + y_0\delta_n y + z_0\delta_n z) + X_n \\ \frac{d^2\delta_n y}{dt^2} + \frac{k_1^2}{r_0^3}\delta_n y &= \frac{3k_1^2 y_0}{r_0^5}(x_0\delta_n x + y_0\delta_n y + z_0\delta_n z) + Y_n \\ \frac{d^2\delta_n z}{dt^2} + \frac{k_1^2}{r_0^3}\delta_n z &= \frac{3k_1^2 z_0}{r_0^5}(x_0\delta_n x + y_0\delta_n y + z_0\delta_n z) + Z_n \end{aligned}\right\} \quad (49)$$

for the calculation of the n-order perturbations. Here, we denote by X_n, Y_n and Z_n expressions involving $x_o, \ldots, \delta_1 x, \ldots, \delta_{n-1} x$, $\delta_{n-1} y$ and $\delta_{n-1} z$.

Using the series-expansion of the perturbation function, we can successively obtain the first- and second- order perturbations by means of equations (49).

Enke improved this method by suggesting the calculation of the perturbations of the radius vector by means of equations (41) in parallel with the calculation of the perturbations of the coordinates, when $\delta_n r$ is found, then following equation:

$$x_0\delta_n x + y_0\delta_n y + z_0\delta_n z = r_0\delta_n r + R_n,$$

in which R_n denotes the aggregate of terms not higher than the (n-1)-th order, enables us to compute the first terms in the right-hand side of equations (49). Subsequently, these equations are reduced to the form (33), the integration of which is done quite easily as we have already seen.

The present method immediately gives the coordinates x, y and z required for the calculation of these ephemeride. In spite of this, the method was not widely recognized since perturbations in rectangular coordinates are large and their computation is tedious. Moreover, the perturbations of all the three coordinates obtained by this method are equally large. On the other hand, other methods based on the application of polar coordinates, thecalculation of relatively small perturbations of the third coordinate is quite simple.

113. Hill's Method.

We have just seen that the calculation of the perturbation of any order of the radius vector and the rectangular coordinates is reduced to the integration of the following equations

$$\left.\begin{aligned} \frac{d^2(r_0\delta r)}{dt^2} + \frac{k_1^2}{r_0^3} r_0\delta r &= Q_r \\ \frac{d^2\delta x}{dt^2} + \frac{k_1^2}{r_0^3}\delta x &= Q_x \\ \frac{d^2\delta y}{dt^2} + \frac{k_1^2}{r_0^3}\delta y &= Q_y \\ \frac{d^2\delta z}{dt^2} + \frac{k_1^2}{r_0^3}\delta z &= Q_z \end{aligned}\right\} \qquad (50)$$

When the first of these equations is solved, each of the other three equations is integrated independently from the other two. According to section 109 on the integration of such equations, we can write

$$(q_1\dot q_2 - q_2\dot q_1)\, r_0\delta r = q_2 \int q_1 Q_r\, dt - q_1 \int q_2 Q_r\, dt$$
$$(q_1\dot q_2 - q_2\dot q_1)\, \delta x = q_2 \int q_1 Q_x\, dt - q_1 \int q_2 Q_x\, dt$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

In order to represent these equations in a more convenient form, we agree to lable by a dash the functions of t in which t is replaced by .

Putting

$$N = q_2 q_1 - q_1 q_2,$$

we obtain

$$\left.\begin{aligned} r_0 \delta r &= \int N Q_r \, dt, \\ \delta x &= \int N Q_x \, dt, \; . \; . \; . \end{aligned}\right| \qquad (51)$$

if we again replace τ after each integration by t. Taking once more in accordance to section 109

$$q_1 = \cos E - e = \frac{r}{a} \cos v$$
$$q_2 = \sin E \qquad = \frac{r}{a} \sin v;$$

we then obtain

$$N = \sin(\bar{E} - E) - e(\sin \bar{E} - \sin E) = \frac{\bar{r} r}{a^2 \cos \varphi} \sin(\bar{v} - v)$$
$$q_1 \dot{q}_2 - q_2 \dot{q}_1 = n.$$

Hill took as an independent variable, the true anomaly v. Since

$$dt = \frac{r_0^2 \, dv}{a^2 n \cos \varphi},$$

formulae (51) finally is reduced to the following form

$$\left.\begin{aligned} \delta r &= \frac{1}{n^2 a^4 \cos^2 \varphi} \int^v Q_r r_0^3 \sin(\bar{v} - v) \, dv \\ \delta x &= \frac{r_0}{n^2 a^4 \cos^2 \varphi} \int^v Q_x r_0^3 \sin(\bar{v} - v) \, dv \\ \delta y &= \frac{r_0}{n^2 a^4 \cos^2 \varphi} \int^v Q_y r_0^3 \sin(\bar{v} - v) \, dv \\ \delta z &= \frac{r_0}{n^2 a^4 \cos^2 \varphi} \int^v Q_z r_0^3 \sin(\bar{v} - v) \, dv. \end{aligned}\right| \qquad (52)$$

After integration, $\bar{v}$ should be replaced by v. System (52) involves one equation more than what is required for the calculation of the perturbations. For this reason, Hill replaced the second and third equations by a single equation. This was easily done by the introduction of polar coordinates. Putting

$$x = r \cos l \cos \beta, \qquad y = r \sin l \cos \beta, \qquad z = r \sin \beta,$$

we obtain

$$\operatorname{tg} \lambda = \frac{y}{x},$$

from which it follows that

$$\frac{d\lambda}{dt} = \frac{1}{x^2 + y^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right).$$

On the other hand, it follows from equations (47) that

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h_0 + \int \left(x\frac{\partial R}{\partial y} - y\frac{\partial R}{\partial x}\right)dt.$$

The calculation of the constant h_0 is done by considering the case of unperturbed motions in which

$$h_0 = x\frac{dy}{dt} - y\frac{dx}{dt} = k\sqrt{p}\cos i.$$

Hence, assuming

$$Q_\lambda = x\frac{\partial R}{\partial y} - y\frac{\partial R}{\partial x},$$

we finally obtain

$$(r^2 - z^2)\frac{d\lambda}{dt} = h_0 + \int Q_\lambda\, dt.$$

In this way, putting $\lambda = \lambda_0 + \delta\lambda$ and noting that

$$h_0 - (r^2 - z^2)\frac{d\lambda_0}{dt} = [illegible] \frac{d\lambda_0}{dt} = \frac{(r + r_0)\delta r - (z + z_0)\delta z}{r_0^2 - z_0^2} h_0,$$

we obtain

$$\delta\lambda = \int \left[\int Q_\lambda^{*} dt + k_1\sqrt{p}\cos i \frac{(r + r_0)\delta r - (z + z_0)\delta z}{r_0^2 - z_0^2}\right]\frac{dt}{r^2 - z^2}.$$

Finally ending with a transition to the variable v, and obtaining the following equations for the determination of the perturbations of the coordinates r, λ and z:

$$\left.\begin{aligned} \delta r &= \frac{1}{k_1^2 p}\int_0^v Q_r r_0^3 \sin(v - v)\,dv \\ \delta z &= \frac{r_0}{k_1^2 p}\int_0^v Q_z r_0^3 \sin(v - v)\,dv \\ \delta\lambda &= \int_0^v \left[\frac{1}{k^2 p}\int_0 Q_v r_0^2\,dv - \cos i\,\frac{(r + r_0)\,\delta r - (z + z_0)\,\delta z}{r_0^2 - z_0^2}\right]\frac{r_0\,dv}{r^2 - z^2} \end{aligned}\right\} \quad (51)$$

Hill particularly stressed the fact that these equations were accurate and that they could be applied for the calculation of the perturbations of any order.

Hill used the plane of the elliptic orbit of the planet under consideration as the xy plane. This yields an accuracy of the third-order inclusively

$$z = \delta z \quad [illegible]$$

Putting

$$T = \frac{r^3}{k_1^2 p} Q_r = \frac{r^3}{k_1^2 p}\left(r\frac{dR}{dr} + 2\int dR\right),$$

$$Y = \frac{r^3}{k_1^2 p} Q_v, \qquad Z = \frac{r^3}{k_1^2 p}\frac{dR}{dz},$$

we obtain

$$\delta r = \int_0 T \sin(v - v)\,dv$$

[illegible]

[illegible]

Hill introduced the following auxiliary quantity

$$\Lambda = \frac{r^3}{k_1^2 p}\frac{dR}{dv}.$$

By means of this quantity we can reduce the equations obtained above to the form

$$\left.\begin{aligned} \delta r &= \int \left[X - 2r \int r \left(\frac{e \sin v}{r} X + Y \right) dv \right] \sin (\bar{v} - v)\, dv \\ \delta \omega &= \int \left[\int Y dv - 2 \frac{\delta r}{r} \right] dv \\ \delta \beta &= \int Z \sin (\bar{v} - v)\, dv . \end{aligned}\right\} \qquad (55)$$

It is important to note that the application of these formulae require the computation of only three derivatives of theperturbation function, namely

$$\frac{\partial R}{\partial r}, \qquad \frac{\partial R}{\partial v}, \qquad \frac{\partial R}{\partial z}$$

Indeed, equation (53) shows that

$$Q_v = \frac{\partial R}{\partial v} .$$

We shall not go further than this derivation of equations (54) and (55) which constitute the basis of Hill's method. For details on the application of these equations to the computation of perturbation, we refer the reader to Hill's original work.(1), where the queries on the application of the method to thecomputation of first- and second-order perturbations, are thoroughly examined.

We finally point out that in Hill's method the perturbation function

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(1) G.W. Hill, A Method of Computing Absolute Perturbations. Astr. Nachr. 83, 1874, 209-224 = Works, 1, 151-166;
G.W. Hill, Jupiter Perturbations of ceres of the First Order and the Derivation of the Mean elements,,Astr. Journal, 16, 1896, 57-62 = Works 4, 111-122.

is expanded in a series by multiples of the true anomalies v and v', v' being expressed in terms of v. The first operation is easily carried out if the coefficients of expansion are to be found by means of numerical methods, which considerably simplify the calculations when v's are chosen as independent variables.(1) It is clear that the expressions of the perturbations as functions of v, obtained this way, are less convenient than the expressions of the perturbations in terms of explicit functions of time.

114. The Main Ideas of Hansen's Method

The choice of the coordinates, in terms of which the perturbations are calculated, is of significant value. We have already seen in the previous section that the perturbations are more easily calculated in terms of polar coordinates rather than in terms of rectangular ones. One naturally raises the question on whether it is possible to find another system of variables in terms of which the calculation of the perturbations would be even easier. This question was investigated by Hansen.

We have already pointed out in section 110 that Laplace suggested to include long-periodic perturbations into the mean anomaly serving in the calculations of the radius vectors and the longitudes. Hansen

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(1) In the case of analytical expansion of the perturbation function, the choice of v as an independent variable leads to more complicated results, than does the application of Leverrier and Newcomb's method developed in sections 86-89. On theexpansion in multiples of the true anomaly, we quote H. Gylden, Traite analytique des orbites absolues des huit planetes principales, 1, Stockholm 1893; G.W. Hill, Development in Terms of the True Anomaly of odd Negative Powers of the Distance between two Planets Moving in the Same Plane, Works, 4, 398-407.

developed this idea further and suggested to take the mean anomaly as one of the variables in terms of which the perturbations would be evaluated.

Let us first consider the motion in an orbital plane, defined in the absence of perturbations by equation (23). According to Hansen we assume that the perturbed values of the orbital coordinates r and w are defined by the similar formulae:

$$\left.\begin{array}{ll} w = \bar{v} + \pi_0, & r = \bar{r}(1 + v), \\ E - e_0 \sin E = n_0 z + M_0, & \operatorname{tg} \frac{1}{2}\bar{v} = \sqrt{\frac{1+e_0}{1-e_0}}\, \operatorname{tg} \frac{1}{2} E, \\ n_0^2 a_0^3 = k^2(1 + m), & p_0 = a_0(1 - e_0^2), \\ \bar{r} = a_0(1 - e_0 \cos E) = p_0(1 + e_0 \cos \bar{v})^{-1}, & \end{array}\right\} \qquad (56)$$

where z and v are the corresponding functions of time t. The equations which define the unknown functions z and v are deduced by substituting expressions of w and r, given by formulae (56), into equations (20). We shall not give here the method for the deduction of these equations.

For the unperturbed motion z = t. It is hence natural to search for z in the form

$$\delta z + t = z$$

where δz is a small quantity having the same order of magnitude as the perturbing masses. When δz and v are obtained, we can calculate the coordinates w and r that determine the position of the planet in the XY plane. What remains after this is to show how the positions of the axes SX and SY could be specified at an arbitrary moment of time.

In considering the perturbed motion, we have to replace in formulae (27), the constant elements i_0, Ω_0 and σ_0 by their osculating elements at the moment t under consideration. This yields

$$\alpha_2' = -\sin\sigma \sin i, \qquad \beta_2 = \cos\sigma \sin i, \qquad \gamma_2 = \cos i.$$

Hence, returning to equations (17) and taking equations (19) into consideration, we obtain

$$\left.\begin{aligned} \frac{di}{dt} &= h^{-1} r \cos(u - \sigma) \frac{\partial R}{\partial Z} \\ \operatorname{tg} i \frac{d\sigma}{dt} &= h^{-1} r \sin(u - \sigma) \frac{\partial R}{\partial Z} \end{aligned}\right| \tag{57}$$

On the other hand, substituting into the following equation (Sec. 106):

$$\alpha \frac{d\alpha}{dt} + \beta \frac{d\beta}{dt} + \gamma \frac{d\gamma}{dt} = 0$$

the values of the angular coefficients defined by equation (27) of section 106, we obtain

$$d\chi = \cos i \, d\Omega. \tag{58}$$

Consequently

$$\sin i \frac{d\Omega}{dt} = h^{-1} r \sin(u - \sigma) \frac{\partial R}{\partial Z}$$

Integrating equations (57) and (58), we obtain i, σ and Ω. Since we have obtained a superfluous integration constant, then according to Hansen we require, that the initial values of the elements corresponding to the moment t = 0 satisfy the following condition

$$\chi_0 = \Omega_0$$

In this case, the quantity x_0 will be nothing else but the longitude of the perihelion.

It is easy to find the heliocentric coordinates ℓ and b when the integration of the equations that define the quantities δ_z, ν, Ω, i and σ is already carried out. For this purpose, we apply formulae

$$\left.\begin{aligned} \cos b \sin(l - \Omega) &= \cos i \sin(w - \sigma) \\ \cos b \cos(l - \Omega) &= \cos(w - \sigma) \\ \sin b &= \sin i \sin(w - \sigma), \end{aligned}\right| \qquad (59)$$

which are similar to equations (26) corresponding to the unperturbed motion.

Formulae (59) completely solves the problem of obtaining the perturbed motion. They are only convenient when we are interested in the calculation of a small number of separate positions for a planet. Nevertheless, the determination of the perturbations by analytical methods usually ends by constructing tables for the motion of the planet under consideration which simplifies as much as possible the computation of its coordinates. In this case, it is better to avoid the use of tables having two entrances and for this reason, Hansen suggested to replace formulae (59) by other formulae found more convenient on tabulating. Hansen proved the transformation these formulae to the following form

$$\left.\begin{aligned} \cos b \sin(l - \Omega_0 - \Gamma) &= \cos i_0 \sin(w - \Omega_0) + \psi \\ \cos b \cos(l - \Omega_0 - \Gamma) &= \cos(w - \Omega_0) + \psi' \\ \sin b &= \sin i_0 \sin(w - \Omega_0) + s, \end{aligned}\right| \qquad (60)$$

wheree the first terms on the right hand side could be conveniently tabulated by the argument $w - \Omega_0$ while the small quantities Γ, ψ, ψ' and s could be determined without a special effort. Without going into their proof, the final results are given by

$$\left.\begin{aligned} \psi &= -\frac{s}{k}\left[\sin i_0 \cos i - \cos i_0 \sin i \cos(\sigma - \Omega_0)\right] \\ \psi' &= \frac{s}{k} \sin i \sin(\sigma - \Omega_0) \\ s &= Q \sin(w - \Omega_0) - P \cos(w - \Omega_0) \\ k &= \cos i_0 (\cos i_0 + \cos i) + Q \sin i, \end{aligned}\right| \qquad (61)$$

where Γ, P and Q are defined by the following differential equations:

$$\frac{d\Gamma}{dt} = \text{[illegible]},$$

$$\frac{dP}{dt} = \text{[illegible]},$$

$$\frac{dQ}{dt} = \text{[illegible]}.$$

If we confine ourselves to first-order perturbations, then

$$\Gamma = 0, \quad \text{[illegible]} \quad = 0.$$

In this case, the application of formulae (60) requires the construction of only one tabale with two entrances giving the values of S. Due to the smallness of this quantity, the construction of such a table is quite simple.

It is useful to note, that in the differential equations that we have to solve in the application of Hansen's method, the perturbation function appears only in the form of the partial derivatives

$$\frac{\partial R}{\partial r}, \quad \frac{\partial R}{\partial v}, \quad \frac{\partial R}{\partial Z}.$$

When Hansen gave the final account of his method[1], he closely related his method to the expansion of the perturbation function(or more exactly, the partial derivatives which we have just mentioned) by multiples of the eccentric anomalies, using the eccentric anomaly of the perturbed planet as an independent variable. This method however, depends on neither the choice of the independent variable nor the way by which the perturbation

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(1) P.A. Hansen, Auseinandersetzung einer zweckmassigen Methode zur Berechnung der absoluten Storungen der kleinen Planeten, Abh. I-II-III, Leipzig 1857-1861.

function is expanded.

Hansen's method has been widely applied to the calculation of the perturbations of small planets. This is explained, on one hand, by the practicality of this method which actually reduces the magnitude of the perturbations to a minimum, and on the other hand, by the fact that Hasen explained his method with the fullest of details.

115. <u>The calculation of the Derivatives of Perturbation Functions with Respect to the Coordinates</u>

In the calculation of perturbations in the coordinates, it is necessary to expand the partial derivatives

$$\frac{\partial R}{\partial w} = \frac{\partial R}{\partial v}, \quad \frac{\partial R}{\partial r}, \quad \frac{\partial R}{\partial Z} \qquad (62)$$

of the perturbation function in series. We shall now indicate the way by which this expansion could be performed. We have already found that for each of the perturbing planets

$$R = k^2 m' \left(\frac{1}{\Delta} - \frac{r \cos H}{r'^2}\right),$$

where

Differentiating this perturbation function with respect to v, we obtain

It is clear that

On the other hand, the identity

$$\frac{\partial}{\partial M}\left(\frac{r^2}{\Delta}\right)=\frac{\partial r}{\partial M}\frac{\partial}{\partial r}\left(\frac{r^2}{\Delta}\right)+\frac{\partial v}{\partial M}\frac{\partial}{\partial v}\left(\frac{r^2}{\Delta}\right)$$

and the well-known formulae

$$\frac{\partial r}{\partial M}=\frac{ae\sin v}{\cos\varphi},\qquad \frac{\partial v}{\partial M}=\left(\frac{a}{r}\right)^2\cos\varphi$$

allows us to write

$$\frac{\partial}{\partial v}\left(\frac{r^2}{\Delta}\right)=\frac{r^2}{a^2\cos\varphi}\frac{\partial}{\partial M}\left(\frac{r^2}{\Delta}\right)-\frac{er^2}{p}\sin v\frac{\partial}{\partial r}\left(\frac{r^2}{\Delta}\right),$$

from which we can easily obtain

$$\frac{\partial}{\partial v}\left(\frac{1}{\Delta}\right)=\frac{1}{a^2\cos\varphi}\frac{\partial}{\partial M}\left(\frac{r^2}{\Delta}\right)-\frac{e\sin v}{pr}\left[\frac{3}{2}\frac{r^2}{\Delta}+\frac{1}{2}\frac{r^2(r'^2-r^2)}{\Delta^3}\right]. \qquad (65)$$

Formulae (63), (64) and (65) lead us to the first of the derivatives (62). We then consider the calculation of the derivative with respect to the radius vector. Evidently,

$$\frac{\partial R}{\partial r}=k^2m'\left[-\frac{r-r'\cos H}{\Delta^3}-\frac{\cos H}{r'^2}\right];$$

Since

$$2rr'\cos H=r^2+r'^2-\Delta^2,$$

then

$$\frac{\partial R}{\partial r}=k^2m'\left[-\frac{1}{2\Delta}+\frac{r'^2-r^2}{2\Delta^3}-\frac{r\cos H}{r'^2}\right] \qquad (66)$$

The last of the derivatives (62) can be regarded as the component of the perturbing acceleration along the normal to the orbital plane. Therefore, as we have already seen in section 67,

$$\frac{\partial R}{\partial Z}=k^2m'\left(\frac{1}{\Delta^3}-\frac{1}{r'^3}\right)\zeta'.$$

where we denote by ζ' the z-coordinate of the perturbing planet. Evidently,

$$\zeta' = r' \sin(v' - \Pi') \sin I,$$

Consequently

$$\frac{\partial R}{\partial Z} = k^2 m' r' \left(\frac{1}{\Delta^3} - \frac{1}{r'^3}\right) \sin(v' - \Pi') \sin I$$

It is thus sufficient for the calculation of the partial derivatives (62) to expand the quantities Δ^{-1} and Δ^{-3} in double trigonometric series by multiples of the mean anomalies. For all the other quantities involved in formulae (64) - (67), we have already obtained in section 82 in a general form their expansion series.

We have developed a series expansion by multiples of the mean anomalies. The same formulae can be used when the seires expansion is carried out by multiples of the eccentric anomaly. We only have to substitute in equation (65)

$$\frac{\partial}{\partial M}\left(\frac{r^2}{\Delta}\right) = \frac{\partial E}{\partial M} \frac{\partial}{\partial E}\left(\frac{r^2}{\Delta}\right).$$

We note that it is more useful to expand the quantity $r^2 \Delta^{-1}$ and not the quantity Δ^{-1} in a double series, since the former quantity is the one involved in the final equations.

PART FOUR

THEORY OF LUNAR MOTION

CHAPTER XVII

PRINCIPLES OF THE THEORY OF LUNAR MOTION

LAPLACE'S THEORY

116. General Properties of the Lunar Motion

The position of the moon is always determined relative to the centre of the earth which, in this case, is chosen as the central body. The motion that the moon would have in the absence of celestial bodies other than earth is considered to be the basic unperturbed motion. The modifications that thesun and the other planets introduce in this motion are called perturbations or inequalities.

The perturbations produced by the sun into the motion of the moon are of particular interest. These perturbations are quite different in character as compared to those we deal with in the study of the planets' motions. The perturbations produced by all the planets, except earth, onto the lunar motion are small due to the smallness of the perturbing masses as compared to the mass of the sun, although these planets are often much nearer to the moon than to the sun. On the other hand, the sun is considered as a perturbing body in the theory of lunar motion inspite the fact that its mass is 331950 times larger than the mass of the earth, which is considered as the central body. The reason for this choice is that the sun is at a distance almost 400 times larger than the distance from the earth. Taking the mean distances of the moon and the sun from the earth to be 384400 km and 149450000 km respectively, we find that the ratio of these distances is equal to 1/389. It then follows that the acceleration induced by the sun to the moon is on the average

$$\frac{331\,950}{389^2} \simeq 2.2$$

times greater than the acceleration caused by earth to the moon. However, we usually study the motion of the moon relative to earth, we are thus interested in the difference in accelerations caused by the sun on the motions of the moon and the earth. It is easy to see that the perturbing acceleration, occurring this way, is equal on the average to

$$\frac{331\,950}{389^3} \simeq \frac{1}{177}$$

of the acceleration induced by earth. Taking the eccentricities of the terrestial and lunar orbits into consideration, it is easy to show that the previous ratio can at most reach the value 1/80. Hence, we can conclude that the perturbations produced by the sun on the motion of the moon is by two orders of magnitudes larger than those we ordinarily deal with in the theory of planets.

The nearness of the moon to the earth, on one hand, simplifies the investigation of lunar motion by rendering the perturbatioins produced by all of the other planets, quite small. On the other hand, due to this nearness, one has to take into account the influence of the deviation of the earth's structure from the spherical symmetry upon the lunar motion. Taking these reasons into consideration, we find that the theory of lunar motion is naturally divided into the following items

(1) The investigation of the motion of the three material points T, L and S, one of which S (Sun) moves along a Kepler ellipse around the centre of gravity G of theother two points T (Earth) and L (Moon). This is the basic problem of the theory of lunar motion.

(2) The calculation of the perturbations, that the deviation of the earth and moon's structure from a spherical symmetry, causes to the lunar motion.

(3) The calculation of the perturbations produced by the direct attraction of the planets.

(4) The calculation of the perturbations produced by the deviation of the motion of the sun S around point G according to Kepler's law, i.e. the perturbations which indirectly depend on the interaction of the other planets.

(5) The calculation of thesecond- and higher-order perturbations occurring due to the combined influence of the factors indicated in items 2, 3 and 4.

(6) The calculation of the contributions of all of the other factors that can influence the lunar motion (e.g. sea tides, increase in the moon's and the earth's masses due to the accumulation of meteorites).

Only the first of these items involves serious difficulties of the first magnitude. The methods applied to this example are of general interest and the remaining five problems, which sometimes require a great amount of work, can always be solved by applying the method of successive approximations in its conventional form. Taking this fact into consideration, we shall in thefuture confine ourselves entirely to the consideration of the basic problems, which is sometimes called the solar theory of the Moon's motion (theorie solaire du mouvement de la Lune).

We note that the sharp division of the theory of lunar motion into the items given above is not always recommended. For example, it may be useful to take into account part of the perturbations of the sun while solving the basic problem, such as thesecular motion of the perihelion and the secular decrease of the eccentricity of the terrestrial orbit.

We conclude these introductory remarks by citing the most important perturbations produced by the sun onto the lunar motion. The unperturbed orbit of the moon may be taken to be an ellipse with an eccentricity equal to 0.05490, lying in a plane inclined to an ecliptic of 1850.0 by an angle of $5^{\circ}\ 9'$. The perihelion of the moon's orbit has a translational motion. It performs a full rotation in 8.8503 years on the average. The influence of the sun consists, first of all, in adding periodic inequalities to the uniform motion of the perihelion of the moon. The largest of these inequalities has an amplitude of $8^{\circ}\ 41'$. The eccentricity is slightly changed and oscillates around the above-mentioned mean value. On the other hand, the line of nodes moves backwards making a full rotation in 18.5995 years on the average. The most significant of the periodic inequalities, which added to this uniform motion, will have an amplitude of $1^{\circ}\ 26'$. The slope of the orbit will have a periodic inequality as a consequence of which, it will vary within the limits of $4^{\circ}\ 57'$ to $5^{\circ}\ 20'$.

Now, considering the periodic inequalities of the longitude, the following formula gives an estimate of the most significant quantities:

[illegible]

We denote here by v the true longitude of the moon, by λ the mean magnitude, by M and M' the mean anomalies of the moon and the sun respectively, where M is calculated from the mean position of the perihelion, and finally, by D the difference between the mean longitudes of the moon and the sun. The terms whose arguments are M, 2M, ... are called the elliptic terms. Their sum defines the equation of the centre. The term having the argument 2D-M is called the evection. It is easy to see that the period of this perturbation is equal to 31.8 days. The inequality produced by the term, the argument of which is 2D, is called the variation. The period of the variation is evidently equal to one half of a synodic month, i.e. to 14.76 days. The variation does not change the position of the moon in the syzygies or quadrants, it produces a large displacement of the moon in the octants. The term having the argument M' produces a perturbation, the period of which is one year. This perturbation is called the annual inequality. It is caused by the ellipticity of the terrestrial orbit, leading to some changes in the distance to the sun and consequently in the magnitude of the perturbing force. Finally, the terms of arguments D, 3D, ... are responsible for the parallactic inequalities. The amplitude of each of these inequalities is proportional to the ratio a/a' of the mean distances of the moon and the sun. Since the parallax of the moon is easily obtained from the results of the observations, the comparison between the observed and theoretical values of the parallactic inequalities makes it possible to determine the parallax of the sun. This is one of the most accurate methods for the determination of the parallax of the sun.

There are terms having similar arguments in the expansions of the radius vector and latitude of the moon. The series-expansion of the parallax of the moon is easily deduced from the series-expansion of the radius vector. It has the following form

$$\begin{aligned} P_{☾} = 3424'' + 187'' \cos M + 10'' \cos 2M + \ldots \\ + 34'' \cos(2D - M) + 28'' \cos 2D + \ldots \quad , \end{aligned}$$

where the first line involves the mean value of the parallax and the elliptic terms, while the second line involves the most important perturbations.

At the present time, each of the above-mentioned names denotes a group of terms which are similar to the corresponding principal term given above. For example, the following superposition of terms

$$4607.77'' \sin(2D - M) + 171.87'' \sin(2D + M) + \dots;$$

are called an evection. The superposition of terms having arguments 2D, 4D, 6D, ..., i.e.

$$2106.25'' \sin 2D + 8.75'' \sin 4D + \dots,$$

are called a variation. The annual inequality is the name of the group of terms that depend on the mean anomaly of the sun, namely

$$-668.[illegible] \sin M' [illegible] \sin(2D [illegible]) [illegible] \sin(2D + M') \dots .$$

Finally, the parallactic inequality is given by

$$-127.62'' \sin D - 0.84'' \sin 3D + 0.01'' \sin 5D \dots .$$

117. <u>A Brief Historical Survey of the Development of the Lunar Motion Theory</u>

The modern lunar motion theory began after the discovery of the universal law of gravity. Newton proved that the variation, the motion of the perihelion, the motion of the node and the observed changes in the slope and eccentricity can be interpreted within the framework of the universal law of gravity. Newton did not aim to develop a complete theory which could reproduce lunar motion. Nevertheless, he could determine a number of separate inequalities with sufficiently high accuracy. We have the right to think that Newton obtained his results by means of a general method, namely the methods of variation of elements, although he published his results in the form of fragmentary theorems.

The way to construct a theory, capable of describing all the particularities of the moon's motion was indicated by Kepler. He was able to express this problem in terms of a differential equation system and started in 1747 to solve this system by using the method of successive approximations. Clero was first to suggest that the first approximation to be made on the lunar motion should be the taking of an ellipse having a uniformly rotating line of apses instead of a fixed ellipse as Kepler suggested. D'Alembert (1754 - 1756) developed a method similar to that of Clero's, which was much more systematic. While Clero's theory adopted from the very beginning numerical values for the parameters, D'Alembert gave the first example of an algebraic theory in which the parameters were allowed to have arbitrary values. The common factor in both Clero and D'Alembert's works was the choice of the true longitude of the moon as an independent variable.

An alternative method, based on the same ideas, was later developed by Laplace, who studied lunar motion for more than thirty years. The results he obtained were included in the third volume of his book "Mecanique Celeste" published in 1802. Apart from working out a general method for obtaining all the perturbations produced by the attraction of the sun, Laplace could for the first time determine the inequalities produced by the nonsphericity of the earth and the attraction of the other planets. The latter problem is concerned with one of Laplace's most outstanding discoveries, namely, the interpretation of the secular acceleration of the moon's mean motion (See section 125). He was also able to prove that similar accelerations depending on the secular decrease of the eccentricity of the terrestrial orbit took place into the motion of the perihelion and the node.

Laplace calculated the lunar perturbations upto the second- and partically, third-order powers of those parameters, by which the series expansions were developed. Later, in 1827, Damoiseau applied Laplace's method for obtaining the numerical values of the inequalities to a much higher accuracy. In 1832, Plana repeated the same work algebraically, but his results involved several errors. In 1846, de Pontecoulant published a new theory on lunar motion. In analogy with the theories cited above, de Pontecoulant's theory was based on the application of the polar coordinates. The only difference was that he chose time as the independent variable. The corresponding differential equations (Sec. 7) were given by Laplace. Thesame method was simultaneously developed by Lubbock, who published his results in 1834. However, this author only confined himself to the calculation of the second-order approximations.

A new approach to the theory of lunar motion was introduced in 1753 in Euler's book entitled: "Theoria motus lunae exhibens omnes ejus inequalitates". The extensive "Additamentum", by which he concluded his book, actually included a method for the variation of the elliptic elements. A further, yet very rough development of Euler's ideas was given by the method of integration of the perturbed motion equations suggested by Delaunay in 1846. By this method, Delaunay developed a most perfect analytical theory for solar inequalities. After twenty years of work, he succeeded in obtaining general expressions for all the perturbations in the perturbing forces up to the seventh order inclusively. Delaunay's theory was reconsidered by Radau and Andoyer. The revised version of this theory serves as a basis for the extensive tables on lunar theory which have been constructed by Radau.

The problem raised by Euler on the determination of the osculating elements was further developed by Poisson (1835), Puiseaux (1864) and Vil'ev (1919). It should, however, be pointed out that the determination of the osculating elements is not very useful for achieving the main target of lunar theory, namely the construction of tables for the moon's motion.

Euler suggested another important idea in the book, he published in 1772 under the title: "Theoria motuum lunae nova methodo pertractataa una cum tabalis astronomicis, unde ad quodvis tempus loca lunae expedite computari possunt"(1). This idea consists in expanding the unknown functions of the lunar coordinates into a series of the type

$$A + eB_{10} + e^2B_{20} + \ldots + e'B_{01} + e'^2B_{02} + \ldots + ee'B_{11} + \ldots + i^2C_2 + \ldots$$

where e and e' are the eccentricities of the lunar and solar orbits, i the slope of the lunar orbit and A, B_{10}, ..., C_2, ... are periodic functions. Euler obtained systems of differential euquations for the consequent determination of the cofficients.

Amongst the numerous interesting improvements introduced by Euler in the lunar theory, we record the application of uniformly-rotating rectangular coordinate systems. This idea did not find any application for a flong time, in contrast to Euler's other ideas. Only after more' than a hundred years, in 1877, did Hill show in his well known work(2) the advantage of combining this idea with the above-mentioned method for the successive calculation of inequalities of different powers of eccentricities

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(1) These exists a Russian translation for the most important sections of this book, which has been made by Academician A.N. Krylov, and complemented with several interesting comments and addenda. This translation constitutes the material of the book: Leonard Euler, New theory of lunar motion (Leonard Ejler, Novaja Teorija dvizenija Luny) Leningrad 1934.

(2) G.W. Hill, On the Part of Lunar Perigee which is a Function of the Mean Motion of the Sun and Moon, Acta Math., 8, 1886, 1-36 (Works, 1, 243-270); Researches on Lunar Theory, American Journal of Math., 1, 1877 (Works, 1, 281-335).

and slopes, which may be considered as the start of modern celestial mechanics. The theory of lunar motion developed by Hill in this work, as well as in other subsequent works, was later developed by Brown(2) to its final stage. At the present time this theory is considered to be the best theory available on lunar motion sinpite that it is mainly based on Euler's very old ideas.

It is worthwhile mentioning that the development of Euler's ideas in the above-mentioned direction was at the same time started by Hill and simultaneously by Adams who studied the behaviour of separate lunar inequalities.

A slightly different approach to the study of the moon's motion was suggested by Hansen which consisted in applying his method for the study of perturbed motions (Chapter XVI). The work of this author which continued during the period from 1829 to 1864 led to the construction of the tables of lunar motion, which were published in 1857. Until recently, these tables have been considered as one of the most accurate tables available.

In this chapter, we shall give an account of the theory developed by Laplace. This theory gives us a rapid but sufficiently thorough aquaintance with the main features of the lunar motion. The methods developed by Laplace are also quite interesting by themselves for they can be successfully applied to other problems, such as the study of the

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(2) W.E. Brown, Investigations on lunar theory, American Journal of Math., 17, 1895, 318-358, The Theory of the motion of the moon, etc., Memoirs of the R. Astr. Society 53, 54, 57, 59 (1897-1908).

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motion in the systems of triplet stars.(1)

116. Differential equations for the basic problem

Let us adopt the centre of the earth as the origin of a rectangular ecliptic coordinate system. Denoting by v the longitude of the moon, by S the tangent to its latitude and by u the projection of the radius vector on the ecliptic plane, we obtain

$$x = \frac{\cos v}{u}, \quad y = \frac{\sin v}{u}, \quad z = \frac{s}{u},$$

from which it follows that

$$r = \frac{\sqrt{1+s^2}}{u};$$

consequently, we can write the equations of motion in the following way (see section 8):

$$\frac{d^2u}{dv^2} + u = -h^{-2}\frac{\partial U}{\partial u} - su^{-1}h^{-2}\frac{\partial U}{\partial s} - u^{-2}h^{-2}\frac{du}{dv}\frac{\partial U}{\partial v} - 2h^{-2}\left(\frac{d^2u}{dv^2} + u\right)\int u^{-2}\frac{\partial U}{\partial v}\,dv, \quad (1)$$

$$\frac{d^2s}{dv^2} + s = su^{-1}h^{-2}\frac{\partial U}{\partial u} - u^{-2}h^{-2}(1+s^2)\frac{\partial U}{\partial s} - u^{-2}h^{-2}\frac{ds}{dv}\frac{\partial U}{\partial v} - 2h^{-2}\left(\frac{d^2s}{dv^2} + s\right)\int u^{-2}\frac{\partial U}{\partial v}\,dv, \quad (2)$$

[illegible]

where h is a constant of integration.

Denoting by T, L and m' the masses of the earth, moon and sun, and

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(1) A detailed bibliography on the theory of the motion of the moon is given in the paper: E.W. Brown, Theorie des Erdmondes, Encyklopedie der Mathem. Wiessenschaften, Bd VI, 2, 1915.

by x', y', z' and r' the heliocentric coordinates and radius vector of the sun, we obtain the following expression for the force function (Section 3):

$$U = \frac{k^2(T+L)}{r} + k^2 m'\left(\frac{1}{\Delta} - \frac{xx' + yy' + zz'}{r'^3}\right), \qquad (4)$$

where Δ is the distance between the moon and the sun.

Furthermore, denoting by H the angle between the radius vectors r and r', we write

$$xx' + yy' + zz' = rr' \cos H.$$

In order to obtain the expansion of the function U in powers of the ratio r/r', we apply the following well-known formula

$$\frac{1}{\Delta} = \frac{1}{\sqrt{r'^2 + r^2 - 2rr' \cos H}} = \frac{1}{r'}\sum_{n=0}\left(\frac{r}{r'}\right)^n P_n(\cos H), \qquad (5)$$

in which, we denote by

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x, \ \ldots$$

the Legendre polynomials. Substituting this expansion into equation (4), and dropping the term k^2m'/r' which does not depend on the lunar coordinates, and consequently does not affect the partial derivatives which we are interested in, we obtain

$$U = \frac{k^2(T+L)}{r} + \frac{k^2 m'}{r'}\sum_{n=2}\left(\frac{r}{r'}\right)^n P_n(\cos H). \qquad (6)$$

This series converges rapidly since the ratio r/r' is of the order of 1/400. In order to reduce the size of these formulae, we choose the units of time and mass in such a way, that

$$k^2 = 1, \quad T + L = 1.$$

Confining ourselves to only the necessary terms, we finally obtain

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$$U = \frac{\mu}{\sqrt{1+s^2}} + m'\frac{r^2}{r'^3}\left(\frac{3}{2}\cos^2 H - \frac{1}{2}\right) + m'\frac{r^3}{r'^4}\left(\frac{5}{2}\cos^3 H - \frac{3}{2}\cos H\right) + \ldots$$

Denoting by u', v' and s' the coordinates of the sun in the adopted coordinate system, we write

$$x' = \frac{\cos v'}{u'}, \quad y' = \frac{\sin v'}{u'}, \quad z' = \frac{s'}{u'}.$$

Accordingly, neglecting the coordinate s',

$$r = \frac{\sqrt{1+s^2}}{u}, \quad r' = \frac{1}{u'},$$

$$\cos H = \frac{\cos(v-v') + ss'}{\sqrt{1+s^2}\sqrt{1+s'^2}} = \frac{\cos(v-v')}{\sqrt{1+s^2}}.$$

Substituting these expressions into U, and then replacing the powers of cos (v - v') by the cosines of multiples of this arc, we finally obtain

$$U = \frac{\mu}{\sqrt{1+s^2}} + \frac{m'u'^3}{4u^2}\left[1 + 3\cos 2(v-v') - 2s^2 + 12ss'\cos(v-v')\right]$$
$$+ \frac{m'u'^4}{8u^3}\left[(3 - 15s^2)\cos(v-v') + 5\cos 3(v-v')\right], \quad (7)$$

where we have dropped the term m'/r' which will vanish after differentiation.

We note that the quantity s' is very small because the position of the orbit of the earth changes very slightly and very slowly. Laplace always put s' = 0 and considered that this would not cause any considerable violation to the motion of the moon. In 1848, while Airy was observing the latitude of the moon, he discovered a small periodic deviation from the theoretical values. Hansen proved that this deviation could be namely explained by the influence of the term involving s' in the expansion (7). Indeed, this term produces in the parameter s a perturbation equal to

[illegible]

All of the term's observable influence involving s' is exhausted by this perturbation. Taking this into consideration, we shall always set the quantity s' equal to zero in our following discussions.

We note that the first group of terms, causing perturbations in the expansion (7), have the multiplying factor u'^3. If we denote by a' the semimajor axis of the terrestrial orbit, then, since r' is proportional to a', the group of terms under considerations will have the multiplying factor

$$\frac{n'^2}{1+m'}$$

where n' is the mean motion of the sun. Hence, those terms, which cause the largest perturbations of the motion of the moon, depend essentially not on a' but on n'. This latter quantity can be very accurately determined by combining different observations of the sun, separated by sufficiently large intervals of time. On the other hand, the terms of the second group that include the multiplying factor u'^4 will have the following factor after the expansion in powers of the eccentricity e':

$$\frac{n'^2}{(1+m')a'}$$

The comparison between the theoretical values of the perturbations, predicted by these expressions, with the values obtained by the observations gives us the possibility of determining the parallax a' of the sun. For this reason, the corresponding perturbation is called the parallactic inequality.

The next terms, which we have dropped in the expansion (7), have a negligible influence on the moon's motion. This can be judged by the smallness of the amplitudes of the inequalities produced by these terms.

After these general considerations, we start the integration of equations (1), (2) and (3), in which U is replaced by expression (7). We shall use the method of successive approximations.

119. The First Approximation

If the perturbations caused by the sun were absent, or equivalently, if the mass m' of the sun could be set equal to zero, then equations (1) and (2) would change into the following forms

$$\left.\begin{aligned} \frac{d^2u}{dv^2} + u &= \frac{1}{h^2}(1+s^2)^{-\frac{3}{2}}, \\ \frac{d^2s}{dv^2} + s &= 0. \end{aligned}\right\} \qquad (8)$$

The second equation gives

$$s = \gamma \sin (v - \theta), \qquad (9)$$

where γ and θ are integration constants. It is easy to see that the general solution of the first equation can be written as

$$u = \frac{\sqrt{1+s^2} + e \cos (v - \pi)}{h^2(1+\gamma^2)}, \qquad (10)$$

where e and π are new arbitrary constants.

We are dealing in the present case with a two-body problem. Hence, we could have obtained expressions (9) and (10) starting with the well-known elliptical motion formulae. Let look at figure 14, which illustrates the heliocentric celestail sphere. Let x NL' be the ecliptic, and NL the lunar orbit. Denoting by i and θ the slope and longitude of the node of the the lunar orbit, we obtain from the triangle NLL'.

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$$\operatorname{tg} LL' = \operatorname{tg} LNL' \sin NL$$

or

$$s = \operatorname{tg} i \sin(v - \theta)$$

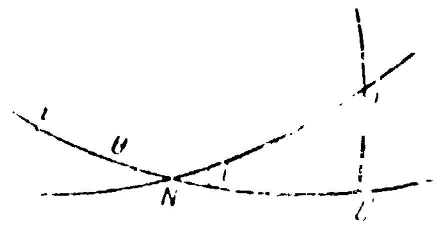

Fig. 14

Comparing with equation (9), we find that

$$\gamma = \operatorname{tg} i,$$

while the constant θ involved in equation (9) is nothing else but the longitude of the node.

In order to obtain the relation between the constants e and with the elliptical elements, we rewrite equation (10) in the following manner

$$r = \frac{\sqrt{1+s^2}}{u} = \frac{h^2(1+\gamma^2)}{1 + e\,\dfrac{\cos(v-\theta)\cos(\pi-\theta) + \sin(v-\theta)\sin(\pi-\theta)}{\sqrt{1+s^2}}}.$$

The triangle NLL' then gives

$$\begin{aligned} \sin NL' \cos LL' &= \sin NL \cos \\ \cos NL' \cos LL' &= \cos NL. \end{aligned}$$

Denoting by w the longitude in the orbit xN + NL, we obtain

[illegible]

Consequently,

$$r = \frac{h^2(1+\gamma^2)}{1 + e\left[\cos(w-\theta)\cos(\pi-\theta) + \dfrac{\sin(w-\theta)\sin(\pi-\theta)}{\sqrt{1+\gamma^2}}\right]}$$

On the other hand, we have

$$r = \frac{a(1-e_0^2)}{1 + e_0 \cos(w - \pi_0)}.$$

where we denote by a, e_o and π_o the semi-major axis, the eccentricity and the longitude of the perihelion. Comparing these expressions, we find

$$e_0 \sin(\pi_0 - \theta) = e(1 + \gamma^2)^{-1} \sin(\pi - \theta)$$
$$e_0 \cos(\pi_0 - \theta) = e \cos(\pi - \theta)$$
$$a(1 - e_0^2) = h^2(1 + \gamma^2)$$

from which it follows that

$$a = \frac{h^2(1 + \gamma^2)}{1 - e^2 \left[1 - \frac{\gamma^2 \cos^2(\pi - \theta)}{1 + \gamma^2}\right]}$$

The first two of these equations indicate that the quantities e and e' differ only by a quantity having the order of magnitude of γ^2. The last equation yields

$$h^2 = a(1 - e^2 - \gamma^2 \ldots) \qquad (11)$$

when fourth order quantities are neglected.

It is useful to note that

$$e \simeq \frac{1}{20} \qquad \gamma \simeq \frac{1}{11}$$

The motion of the perihelion and node of the lunar orbit proceeds so rapidly that it is not useful to adopt expressions (9) and (10) as first approximations. The perihelion and apogee of the lunar orbit interchange their positions each four and a half years. Therefore, if we wish to investigate the motion of the moon during a long interval of time, the fixed ellipse will be as bad an approximation to the real orbit as the circle is.

These arguments led Clero to take, as a first approximation, an invariable ellipse, rotating in its own plane. Laplace developed this idea by taking, as a first approximation, an orbit defined by

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$$s = \gamma \sin(gv - \theta) \qquad (12)$$

$$u = \frac{\sqrt{1+s^2} + e\cos(cv - \pi)}{h^2(1-\gamma^2)} \qquad (13)$$

where g and c are constants slightly differing from unity. Since

[illegible]

then the longitude of the node and the perihelion will be equal to $\theta + (1-g)v$ and $\pi + (1-c)\,v$ respectively. With each rotation, they will be changed by $(1-g)\,360^\circ$ and $(1-c)\,360^\circ$ respectively.

Naturally, expressions (12) and (13) cannot exactly satisfy equations (12) and (13) if g and c are not equal to unity. However, it will be proved further on that when expressions (12) and (13) are substituted into equation (8), second-order quantities relative to the small quantities γ, e, 1-g and 1-c are obtained. Thus, adopting expressions (12) and (13) as a first approximation, we already take into consideration some part of the perturbations.

In order to find the dependence of the coordinates on time in the orbit, defined by equations (12) and (13), we consider equation (3) which gives for the case m' = 0.

[illegible]

Neglecting the fourth powers of the small quantities e and γ, we write expression (13) as follows

[illegible]

or

$$u = h^{-2}\left[1 + \ldots\right] \quad (11)$$ [illegible]

from which it follows that

$$dt = h\left[1 + \frac{3}{2}(e^2 \ldots) \ldots \cos(v - \varpi) \ldots \frac{3}{2}e^2\cos(2v - 2\varpi) + \frac{1}{2}\gamma^2\cos(2gv - 2\theta)\right]dv,$$ [illegible]

or, after integration,

$$t - \text{const} = h\left(1 + \frac{3}{2}e^2 + \frac{3}{2}\gamma^2\right)v - \ldots(1 - \ldots^2)\sin(v - \varpi) + \ldots\sin(2v - 2\varpi) + \ldots\sin(2gv - 2\theta)$$ [illegible]

Denoting by n the mean motion of the moon, related to the semi-major axis by

[illegible]

we write equation (11) in the following manner

$$h = \ldots\left(1 - \frac{1}{2}e^2 - \frac{1}{2}\gamma^2 \cdots\right). \quad (15)$$

from which we obtain

$$n^{-1} = h\left(1 + \frac{3}{2}e^2 + \frac{3}{2}\gamma^2 + \cdots\right)$$

The coefficient of v in the relations between t and v just obtained, must be exactly equal to n^{-1}. Therefore, taking equation (15) into consideration, and neglecting the third order term, we rewrite these relations as follows

[illegible]

where ϵ is an integration constant.

We finally note that equation (14) may be written in the following manner

[illegible]

if we substitute for h its expression given by equation (15).

120. Calculation of the coordinates of the sun

Since we adopt the longitude v of the moon as the independent variable, we have to express the coordinates of the sun by explicit functions of v. The motion of the sun relative to the centre of gravity of the system earth-moon can be considered as an elliptical motion if we neglect the influence of the other planets and only study, as we are going to do now, the solar inequalities of the motion of the moon. On the contrary, the motion of the sun relative to the centre of the earth significantly differs from an elliptical motion. However, the corresponding correction can easily be introduced, as it will be shown at the end of this section.

Hence, we shall assume that the sun moves relative to the earth according to Kepler's laws. Accordingly, we represent the motion of the sun by means of the formulae derived in the preceding section.

Since for the sun s' = 0 and γ = 0, then we obtain the following equations of motion

[illegible]

[illegible]

Here, we kept the coefficient c', depending on the secular motion of the perihelion of the terrestrial orbit inside the arguments. By this method, we have taken a part of the planetary perturbations into account without introducing additional complications to the calculation. Since c' differs very slightly from unity, we can then set in the coefficients c' = 1. Eliminating t from equations (16) and (19), we obtain

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$$v' = 2e' \sin(\xi' v' - \pi') + \frac{5}{4} e'^2 \sin(2\xi' v' - 2\pi')$$
$$+ \mu v + \epsilon' - \mu^2 - 2\mu e \sin(\xi v - \pi) + \ldots$$

where

$$\mu = \frac{n}{n}$$

Since this ratio is a small quantity in order of 1/10, and since third-order terms are dropped from the left-hand side of the last equations we then neglect in the right-hand side, the terms which have the multiplying factors μe^2 and $\mu \gamma^2$. For this reason we have replaced in the previous equations the quantity $\mu e c^{-1}$ by μe.

The equations obtained above can be solved with respect to v' by means of the successive-approximations method. Writing μ v instead of μ v $+ \epsilon' - \mu^2$, we obtain

[illegible]

Substituting this value for v' into equations (18), we obtain, within the taken accuracy,

[illegible]

since,

[illegible]

where c' is not multiplied by an indefinitely increasing quantity and, therefore can be replaced by unity.

Annotation I

We wish to know, how accurately we can approximate the motion of the sun relative to the centre of gravity G of the system earth-moon by an elliptical motion. We denote by x_1, y_1 and z_1 the coordinates of the sun in a coordinate system, the axes of which are parallel to the previous axes, and the origin is in point G. We then obtain (cf. sec. 4).

$$x_1 = x' - \frac{L}{T+L}x, \quad y_1 = y' - \frac{L}{T+L}y, \quad z_1 = z' - \frac{L}{T+L}z.$$

Therefore

$$r'^2 = \left(x_1 + \frac{L}{T+L}x\right)^2 + \left(y_1 + \frac{L}{T+L}y\right)^2 + \left(z_1 + \frac{L}{T+L}z\right)^2$$
$$\Delta'^2 = \left(x_1 - \frac{T}{T+L}x\right)^2 + \left(y_1 - \frac{T}{T+L}y\right)^2 + \left(z_1 - \frac{T}{T+L}z\right)^2$$

or, denoting by H_1, the angle between the vectors Gm' and GL and putting $r_1^2 = x_1^2 + y_1^2 + z_1^2$,

$$r'^2 = r_1^2 + 2\frac{L}{T+L}rr_1\cos H_1 + \left(\frac{L}{T+L}\right)^2 r^2$$
$$\Delta'^2 = r_1^2 - 2\frac{T}{T+L}rr_1\cos H_1 + \left(\frac{T}{T+L}\right)^2 r^2$$

Due to the formulae derived in Sec. 4, the equations of motion of the sun relative to G read

$$\frac{d^2x_1}{dt^2} = \mu'\frac{\partial u_1}{\partial x_1}, \quad \frac{d^2y_1}{dt^2} = \mu'\frac{\partial u_1}{\partial y_1}, \quad \frac{d^2z_1}{dt^2} = \mu'\frac{\partial u_1}{\partial z_1}$$

where

$$\mu' = \frac{T+L+m'}{m'(T+L)}$$
$$u_1 = \mu\left(\frac{TL}{r} + \frac{Tm'}{r'} + \frac{Lm'}{\Delta'}\right)$$

The first term will not contribute to the derivatives and can thus be neglected. Using expansion (5), we obtain

In this way, we can even drop the second term of this expansion which will have an upper limit equal to

Annotation II

We have applied the formulae, derived for the elliptical motion to describe the motion of the sun. In order to be quite objective, we have to start by the equations of motion deduced in section 4 in order to study the motion of the moon. In other words, instead of using the force function U, defined by equation (4), we have to use the force function U_L, given just above. Taking into consideration factor

$$\frac{m_o + m_1}{m_o\, m_1} = \frac{T + L}{TL}$$

, which is multiplied by the force function in the lunar theory, we see that function U has to be replaced by

Expanding the last two terms in series, we obtain

where

and the term $k^2 m'/r_1$, which does not depend on the coordinates of the moon has been neglected.

We now compare this exact expression for the force function, with the expression given by equation (6). This expression will not be very accurate if we make use of the elliptical motion formulae for the determination of the heliocentric coordinantes of the sun. Since the difference between cos H and cos H_1 is not significant, then the transformation from equation (6) to equation (22) can be carried out by multiplying the terms of the former expansion by the following correcting factors

[illegible]

which differ slightly from unity.

121. <u>On the integration of the equations of perturbed motion in the second and higher approximations</u>

Let us substitute the expressions of the coordinates of the moon and the sun, which we have found to be given by equations (12), (14) and (20) in the first approximation, into equations (1) and (2) in which the force function U is replaced by expression (7). In this manner, we obtained the following equations for calculating more accurate values of the coordinates u and s:

[illegible]

where the summation on the right-hand side consists of a finite number of terms, and where p is a constant not necessarily having integral values. Integrating equation (23) as well as a similar equation for the coordinate s, we obtain the values of u and s in the second approximation. Substituting these values into equations (1) and (2), we obtain equations for finding the third approximation, and so on. It is import ant to note that in each of these successive approximations, we have to deal entirely with euqations of the form (23).

If one of the constants p is not equal to unity, then the general solution of equation (23) will be

$$u = A + C_1 \cos v + C_2 \sin v + \sum \frac{P}{1-p^2} \cos(pv + K) \qquad (24)$$

where C_1 and C_2 are arbitrary constants.

If these existed a term of the type P cos (v + K) on the right-hand side of that equation, then the corresponding term in the general solution would be

$$\frac{1}{2} Pv \sin(v + K). \qquad (25)$$

The presence of such a term in the expansions of u and s must be prohibited, since the coordinates of the moon always remain finite, while a term of the type (25) can adopt indefinitely large values. In order to prevent terms of this type from appearing, it is necessary to set in each approximation, the constants C_1 and C_2 equal to zero in the general solution (24).

Formula (24) shows that the coefficients P of the term for which the constant p is close to unity must be calculated with a much higher accuracy than actuall used in calculating the other coefficients. Evidently, this is due to the presence of the small divisor $1-p^2$ which will lead to a loss in accuracy on integrating these terms. Care should also be paid to the computation of terms for which the coefficients p are near zero. These terms vary slowly when we calculate coordinates u and s, but can lead to a considerable loss in accuracy on calculating t using equation (3). Indeed, a term of the type P cos (pv + K) involved in the expression dt/dv will give after integration

$$\frac{P}{p}\sin(pv+K).$$

If this term is involved in theexpression, found in equation (3) behind an integration sign, we then obtain after the second integration a term of the type

$$\frac{P}{p^2}\cos(pv+K)$$

It is thus clear that the coefficients of these terms must be computed with a much higher accuracy, especially in the second case. Such terms will be called critical.

122. Equations for the most periodical inequalities

Let us first of all consider the inequalities, caused by terms having the multiplying factor $u'^3\ u^{-2}$. For this purpose, we put

$$U = \frac{u}{\sqrt{1+s^2}} + \frac{1}{4} m' u'^3 u^{-2}\left[1+3\cos 2(v-v') - 2s^2\right]$$

in equations (1) and (2). We then obtain

$$\frac{d^2u}{dv^2} + u = h^{-2}(1+s^2)^{-\frac{3}{2}} + \mathrm{I} + \mathrm{II} + \mathrm{III} \qquad (26)$$

$$\frac{d^2s}{dv^2} + s = \mathrm{I}' + \mathrm{II}' + \mathrm{III}', \qquad (27)$$

Where

$$\mathrm{I} = -\frac{m'}{2h^2}\frac{u'^3}{u^4}\left[1+3\cos^2(v-v')\right]$$

$$\mathrm{II} = \frac{3m'}{2h^2}\frac{u'^3}{u^4}\frac{du}{dv}\sin 2(v-v')$$

$$\mathrm{III} = \frac{3m'}{h^2}\left(\frac{d^2u}{dv^2}+u\right)\int \frac{u'^3}{u^4}\sin 2(v-v')\,dv$$

$$\mathrm{I}' = -\frac{3m'}{2h^2}\frac{u'^3}{u^4}s\left[1+\cos 2(v-v')\right]$$

$$\mathrm{II}' = \frac{3m'}{2h^2}\frac{u'^3}{u^4}\frac{ds}{dv}\sin 2(v-v')$$

$$\mathrm{III}' = \frac{3m'}{h^2}\left(\frac{d^2s}{dv^2}+s\right)\int \frac{u'^3}{u^4}\sin 2(v-v')\,dv.$$

We denote the values of u and s that correspond to the second approximation by

$$u = u_0 + \delta u, \quad s = s_0 + \delta s,$$

where u_o and s_o are the quantities obtained in the first approximation, i.e.

$$\begin{aligned} s_0 &= \gamma \sin (gv - \theta) \\ u_0 &= a^{-1}\{1 + e^2 + \frac{1}{4}\gamma^2 + (e + e^3)\cos(cv - \pi) - \\ &\quad - \frac{1}{4}\gamma^2 \cos(2gv - 2\theta) + \dots\}. \end{aligned} \tag{28}$$

where the quantity a has been defined in section 119 as the semi-major axis, related to the observed mean motion by the relation

$$n^2 a^3 = 1. \tag{29}$$

By the influence of the perturbing action of the sun, the constant part u will be changed and will no longer be equal to the expression involved in equation (28) (cf. Sec. 95). Following Laplace, it is understood that in the future, symbol a would denote a number which would render the constant part of u have the same expression as u_o in each approximation, namely

$$a^{-1}\left[1 + e^2 + \frac{1}{4}\gamma^2 + \dots\right].$$

This new constant a will no longer be related to n by equation (29), and for this reason equation (15) will no longer hold true. Therefore, we define the new quantity a_1 by

$$n^2 a_1^3 = 1, \text{ or } n = a_1^{-1}\left(1 - \frac{1}{2}e^2 - \frac{1}{2}\gamma^2 \dots\right) \tag{30}$$

in analogy with equation (15).

We now calculate the quantities, denoted by I, II, ... and start by calculating the expression

$$\frac{m'}{2h^2}\frac{u'^3}{u^3} = \frac{m\,a'^3}{2h^2 a^3}\frac{(a'u')^3}{(au)^3}$$

Since

$$n'^2 a'^3 = 1 + m' \simeq m',$$

then, this expression will be equal to

$$\frac{n'^2 a^3}{2a_1(1 - e^2 - \gamma^2 \ldots)}\frac{(a'u')^3}{(au)^3}.$$

Furthermore,

$$(a'u')^3 = 1 + \frac{3}{2}e'^2 + 3e'\cos(c'\mu v - \pi') + \frac{9}{2}e'^2\cos 2(c'\mu v - \pi') + \ldots$$

$$(au)^{-3} = 1 - \frac{3}{4}\gamma^2 - 3e\left(1 - \frac{1}{2}e^2 - \gamma^2\right)\cos(cv - \pi) + 3e^2\cos 2(cv - \pi) + \frac{3}{4}\gamma^2\cos 2(gv - \theta) + \ldots$$

We confine ourselves tc second order terms.

However, on the basis of the arguments of the previcus section, we also keep the third-order terms in the coefficients of the critical terms.

Putting

$$n'^2 a^3 = \mu_1^2,$$

we finally obtain

$$\frac{m'}{2h^2}\frac{u'^3}{u^3} = \frac{\mu_1^2}{2a_1}\left[1 + e^2 + \frac{1}{4}\gamma^2 + \frac{3}{2}e'^2 - 3e\left(1 + \frac{1}{2}e^2 + \frac{3}{2}e'^2\right)\cos(cv - \pi) + 3e'\left(1 + e^2 + \frac{1}{4}\gamma^2\right)\cos(c'\mu v - \pi') + \ldots\right]. \tag{31}$$

On the other hand, within the adopted accuracy,

$$\cos 2(v - v') = \cos(2v - 2\mu v) - 2\mu e'\cos(2v - 2\mu v - c'\mu v + \pi') \ldots$$

Therefore, assuming that

$$[illegible]$$

we obtain

$$\ldots 3\cos(\ldots) \ldots \frac{\ldots}{2a}\Big[\cos\lambda v - \frac{\ldots}{2} e\cos(\lambda v - cv + \pi) + \ldots\Big].$$

Adding this expression to equation (31) we obtain I. In order to obtain expression II, we put

$$u = a^{-1}, \quad \frac{du}{dv} = \frac{e}{a}\sin(cv - \ldots)$$
$$\ldots = a'^{-1}, \quad \ldots$$

This yields

$$II = \frac{\ldots}{4a_1} e[\cos(\lambda v - cv + \pi) - \cos(\lambda v + cv - \pi)] + \ldots$$

It is sufficiant here to only keep the first of the two written terms since the term $\lambda + c$ in the argument of the second cosine differs significantly from unity.

It remains for us to obtain the term III and it is sufficient here to adopt that

$$\frac{d^2u}{dv^2} + u = \frac{1}{a}, \quad u = \frac{1}{a}$$
$$u^{-4} = a^4[1 - 4e\cos(cv - \pi)]$$
$$\sin\lambda(v - v') = \sin\lambda v - 2\mu e\sin(\lambda v - cv + \pi).$$

We then finally obtain

$$III = -\frac{3m_1^2}{a_1}\left[\frac{\cos\lambda v}{2 - 2\mu} - \frac{2 + 2\mu}{1 - 2\mu} e\cos(\lambda v - cv + \pi) + \ldots\right].$$

The first term in equation (26) can be represented in the following form

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$$p^{-2}(1+s^2)^{-\frac{3}{2}} = p^{-2}\left|1 - \frac{3}{2}(s_0+\delta s)^2 + \dots\right|$$

$$= a_1^{-1}\left|1 + e^2 + \frac{1}{4}\gamma^2 + \frac{3}{4}\gamma^2\cos(2gv - 2\theta) -\right.$$

$$\left. 3\gamma\sin(gv - \theta)\delta s + \dots\right|.$$

The calculation of the expressions I' II' and III' is carried out in the same manner. For lack of space, we shall not give this calculation here. After all the necessary substitutions, the final form of equations (26) and (27) will be

$$\frac{d^2u}{dv^2} + u = \frac{1}{a_1}\left(1 + e^2 + \frac{1}{4}\gamma^2\right) - \frac{\mu_1^2}{2a_1}\left(1 + e^2 + \frac{1}{4}\gamma^2 + \frac{3}{2}e'^2\right) +$$

$$+ \frac{3\mu_1^2}{4a_1}(2 + e^2 + 3e'^2)\,e\cos(cv - \pi) - \frac{3\mu_1^2}{2a_1}\,[\text{illegible}]\cos[\text{illegible}] + \qquad (32)$$

$$[\text{illegible}]\,e\cos(2v - [\text{illegible}]) + \frac{3}{a_1}[\text{illegible}]$$

$$\frac{3}{a_1}\gamma\sin(gv - \theta)[\text{illegible}]$$

$$\frac{d^2s}{dv^2} + s = \frac{3}{2}\mu_1^2\left(1 + 2e^2 + \frac{1}{2}\gamma^2 + \frac{3}{2}e'^2\right)[\text{illegible}]$$

$$\frac{3}{4}\mu_1^2[\text{illegible}]\sin(2v - gv - \theta)[\text{illegible}] \qquad (33)$$

In order to estimate correctly the order of magnitude of the coefficients on the right-hand side of these equations, it is necessary to take into account that μ_1 is a quantity of the first order. Indeed, for the unperturbed motion

$$[\text{illegible}]$$

and, therefore, the quantity μ_1 is of the order of $\frac{1}{13}$. In this manner, the coefficients of the trigonometric functions in equations (32) and (33) are given with an accuracy up to terms of the second order. The coefficients of the terms with critical arguments are found with an even greater accuracy. Fourth-order quantities are kept in the constant term equation (32) since we shall need them for future discussions.

12. The Second Approximation

We shall now integrate the equations that we have derived in the previous section using the method of undefinite coefficients. We first consider equation (33). Since

$$\frac{d^2 s_0}{dv^2} + s_0 = \gamma(1 - g^2)\sin(gv + \theta),$$

then, substituting $S = s_0 + \delta s$ into this equation, we may take

$$\delta s = A\sin(2v - gv + \theta),$$

where A is an arbitrary coefficient. Substituting this value into the equation under investigation and equating the coefficients of both of the cosines, we obtain two equations for the determination of the coefficients g and A, from which it follows that

$$g^2 = 1 + \frac{3}{2}\mu_1\left(1 + 2e^2 - \frac{1}{2}\gamma^2 + \frac{3}{2}e'^2\right)$$
$$A = \frac{3}{2}\,\frac{\mu_1\gamma}{1 - (2 - 2\mu - g)^2}.$$

In the latter equation, we may take

$$g = 1, \quad \mu_1 = \mu,$$

hence we will finally obtain as a result of the second approximation

$$\delta s = \frac{3}{8}\mu\gamma\sin(2v - gv + \theta) \tag{34}$$
$$g = 1 + \frac{3}{4}\mu\left(1 + 2e^2 - \frac{1}{2}\gamma^2 + \frac{3}{2}e'^2\right) + \cdots \tag{35}$$

Considering equation (32), the substitution of the value for which we have just found into this equation yields a third-order term. We drop this term because the coefficient of v in its argument differs from unity by a large quantity. Since,

$$\frac{d^2 u_0}{dv^2} + h u_0 = a^{-1}\left[1 + \ldots + \frac{1}{4}\gamma^2\right. \ldots$$

$$+ (e + e^3)(1 - c^2)\cos(cv - \varpi) + \frac{1}{4}\gamma^2 (1 - 4g^2)\cos(2gv - 2\theta) + \ldots$$

we can put

$$\delta u = B_0 \cos \nu v + B_1 \cos(\nu v - cv + \varpi) + B_2 \cos(2gv - 2\theta)$$

In this way, we obtain for the determination of the constants a and c, introduced above, and for the coefficients B_0 , B_1 and B_2 the following equations:

$$\frac{1}{a}\left(1 + e^2 + \frac{1}{4}\gamma^2\right) = \frac{1}{a_1}\left(1 + e^2 + \frac{1}{4}\gamma^2\right) - \frac{\mu_1^2}{2a_1}\left(1 + e^2 + \frac{1}{4}\gamma^2 + \frac{3}{2}e'^2\right)$$

$$\frac{e}{a}(1 + e^2)(1 - c^2) - \frac{3\mu_1^2}{4a_1} e(2 + e^2 + 3e'^2) = 0$$

$$B_0\left[1 - (2 - 2\mu)^2\right] + \frac{3\mu_1^2}{2a_1}\frac{2 - \mu}{1 - \mu} = 0$$

$$B_1\left[1 - (2 - 2\mu - c)^2\right] - \frac{3\mu_1^2}{2a_1}\frac{5 + 4\mu}{1 - 2\mu} e = 0$$

$$B_2\left[1 - 4g^2\right] + \frac{\gamma^2}{4a}(4g^2 - 1) - \frac{3\gamma^2}{4a_1} = 0$$

We obtain from the first equation

$$\frac{1}{a} = \frac{1}{a_1} - \frac{\mu_1^2}{2a_1}\left(1 + \frac{3}{2}e'^2\right) + \ldots \tag{36}$$

from which it can be seen that the quantity a differs from a_1 by a second-order quantity. The second equation enables us to obtain the value of c. It is easy to see that

$$c = 1 - \frac{3}{4}\mu_1^2\left(1 - \frac{1}{2}e^2 + \frac{3}{2}e'^2\right) \ldots \tag{37}$$

The remaining equations yield

$$B_0 = \frac{\mu_1^2}{a_1}, \qquad B_1 = \frac{15}{8}\frac{\mu_1^2 e}{\mu a_1}$$

$$B_2 = \frac{\gamma^2}{3a}(g^2 - 1) + \frac{1}{4}\gamma^2\left(\frac{1}{a} - \frac{1}{a_1}\right),$$

or, neglecting the third and higher-order quantities,

$$B_0 = \frac{\mu^2}{a}, \quad B_1 = \frac{15}{8}\frac{\mu e}{a}, \quad B_2 = 0.$$

Thus, as a result of the second approximation, we obtain, up to second-order terms, the following equations for the orbit

$$u = a^{-1}\Big[1 + e \ldots \cos \ldots - \tfrac{1}{4} \ldots \cos(2g \ldots) \ldots \qquad [illegible]$$

$$\ldots \cos \ldots \tfrac{1}{8} \ldots \cos(\ldots)\Big] \qquad (38)$$

$$\ldots \sin(gv - \theta) + \tfrac{3}{8} \ldots \sin(\ldots - gv - \theta) \qquad (39)$$

The quantity a involved here can be expressed in terms of the mean motion u of the moon by means of equations (36) and (30).

Annotation:

For simplicity and clarity we have confined ourselves to the calculation of u up to second order terms only. According the arguments stated in section 121, we shall in the future need some of the third-order terms, namely the arguments which have small coefficients of v. The computation of these terms is, in principle, not difficult. Hence, we shall only give here the final result which has the following form

$$au = 1 + e^2 + \frac{1}{4}\gamma^2 + e\cos(cv - \pi) - \frac{1}{4}\gamma^2\cos(2gv - 2\theta) +$$
$$+ \mu^2\cos\lambda v + \frac{15}{8}\mu e\cos(\lambda v - cv + \pi) +$$
$$+ \frac{15}{16}\mu e^2\cos(\lambda v - 2cv + 2\pi) + \frac{3}{16}\mu\gamma^2\cos(\lambda v - 2gv + 2\theta) -$$
$$- \frac{3}{2}\mu^2 e'\cos(c'\mu v - \pi') + \ldots \qquad (38\text{bis})$$

124. The Relation Between the Longitude and Time in the Second Approximation

Let us consider the integration of equation (3), namely

$$\frac{dt}{dv} = h^{-1} u^{-2} \left[1 + 2h^{-2} \int u^{-2} \frac{\partial U}{\partial v} dv \right]^{-\frac{1}{2}}.$$

We must use for u expression (38 bis) in order to obtain the second approximation. Using the value of h, given by equation (30), we obtain

$$\begin{aligned} h^{-1}u^{-2} = \frac{a^2}{\sqrt{a_1}} \Big[& 1 - 2e\cos(cv - \varpi) + \frac{3}{2} e^2 \cos(2cv - 2\varpi) + \\ & + \frac{1}{2}\gamma^2 \cos(2gv - 2\theta) - 2\mu^2 \cos \nu v - \frac{15}{4}\mu e \cos(\nu v - cv + \varpi) - \\ & - \frac{15}{8}\mu e^2 \cos(\nu v - 2cv + 2\varpi) - \frac{3}{8}\mu\gamma^2 \cos(\nu v - 2gv + 2\theta) + \\ & + 3\mu^2 e' \cos(c'\mu v - \varpi') + \ldots \Big] \end{aligned} \qquad (40)$$

We only keep the third-order terms, in whose argument the coefficient of v is small.

It is useful to note, that the unperiodic part of this expression must be equal exactly to $a^2/\sqrt{a_1}$ because it should be reduced to $a^{3/2} = a_1^{3/2}$ in the case of the unperturbed motion. On the other hand, in the derivation of equation (26), we meet the following equality

$$2h^{-2} \int u^{-2} \frac{\partial U}{\partial v} dv = -3m' h^{-2} \int \frac{u'^3}{u^4} \sin 2(v - v') dv.$$

Consequently,

$$\left[1 + 2h^{-2} \int u^{-2} \frac{\partial U}{\partial v} dv \right]^{-\frac{1}{2}} = 1 + \frac{3a}{2h^2} \mu^2 \int \frac{(a u')^3}{(a u)^4} \sin 2(v - v') dv + \ldots$$

because

$$\frac{m' a}{a'^3} = \frac{n'^2 a}{a} = \mu^2.$$

However,

$$\frac{a_1}{h^2} = 1 + e^2 + \gamma^2 + \ldots$$

$$(a'u')^3 = 1 + \frac{3}{2} e'^2 + 3e' \cos(c'\mu v - \pi') + \frac{9}{2} e'^2 \cos(2c'\mu' v - 2\pi') + \ldots$$

$$(au)^{-4} = 1 + e^2 - \gamma^2 - 4e \cos(cv - \pi) + \gamma^2 \cos(2gv - 2\theta) + 5e^2 \cos(2cv - 2\pi) + \ldots$$

$$\sin 2(v - v') = \sin \lambda v - 2\mu e \sin(\lambda v - cv + \pi) + \ldots$$

Noting that a_1 differs from a by a second-order quantity, and μ_1^2 from μ^2 by a fourth order quantity, and keeping among the third- and fourth-order only the terms which have very small coefficients of v in their arguments, we finally obtain

$$\left[1 + 2h^{-2} \int u^{-2} \frac{dU}{dv} dv\right]^{-\frac{1}{2}} = 1 - \frac{3}{4} \mu^2 \cos \lambda v - \frac{15}{8} \mu e \cos(\lambda v - 2cv + 2\pi) + \frac{3}{8} \mu \gamma^2 \cos(\lambda v - 2gv + 2\theta) + \ldots \qquad (40')$$

Multiplying this series by the expressions we obtained previously for $h^{-1} u^{-2}$, we finally obtain the following equation, which determines time as a function of longitude:

$$\frac{dt}{dv} = \frac{a^2}{\sqrt{a_1}} \Big[1 - 2e \cos(cv - \pi) + \frac{3}{2} e^2 \cos(2cv - 2\pi) + \frac{1}{2} \gamma^2 \cos(2gv - 2\theta) - \frac{11}{4} \mu^2 \cos \lambda v - \frac{15}{4} \mu e \cos(\lambda v - cv + \pi) + 3\mu e' \cos(c'\mu v - \pi) + \ldots \Big]. \qquad (41)$$

It is of great importance to note that the unperiodic part of the right-hand side of this equation is equal to

$$\frac{a^2}{\sqrt{a_1}} = a_1^{\frac{3}{2}} \left(1 + \mu^2 + \frac{3}{2} \mu^2 e'^2 + \ldots\right).$$

where use is made of equation (36) and where terms higher than the second power of μ are neglected. Indeed, the multiplication of the terms in series (40), the arguments of which do not involve λ, by the terms in series (40'), cannot result in unperiodic expressions. The reason is that the arguments of all the terms in series (40') involve λv

which cannot disappear as a result of the multiplication of the series. On the other hand, the terms in series (40), the arguments of which involve have a multiplying factor μ, whereas all the terms in series (40') except the third and fourth terms are multiplied by μ^2. We may hence write

$$\frac{dt}{dv} = a_1^{\frac{3}{2}}\left(1 + \mu + \frac{3}{2}\mu e'^2 \quad . \ . \ . \right) \text{ periodic terms.}$$

Let us first of all consider the unperiodic part of this expression and postpone the periodic term to section 126. Since the eccentricity e of the terrestrial orbit varies with time, we denote by e_0 the value of e' at some given epoch t = 0. Separating the constant and the variable parts, we may write

$$\frac{dt}{dv} = a_1^{\frac{3}{2}}\left(1 + \mu_1 + \frac{3}{2}\mu_1 e_0^2 \quad . \ . \ . \right) + \frac{3}{2} a_1^{\frac{3}{2}} \mu_1 (e'^2 - e_0^2) + \text{periodic terms.}$$

Integrating and retaining the fact that we have agreed to denote the coefficients of v by n^{-1}, we obtain

$$t = \frac{v}{n_0} + \frac{3}{2} a_1^{\frac{3}{2}} \mu_1^2 \int (e'^2 - e_0^2)\, dv + \text{const} + \text{период. члены}$$

where

$$\frac{1}{n_0} = a_1^{\frac{3}{2}}\left(1 + \mu_1 + \frac{3}{2}\mu_1 e_0^2 + \ . \ . \ . \right).$$

In this manner, confining ourselves to second powers relative to ,
we obtain

$$v = n_0 t + \varepsilon_0 - \frac{3}{2}\mu^2 \int_0^t (e'^2 - e_0^2)\, n_0\, dt + \text{periodic terms} \qquad (42)$$

T' periodic inequalities of e' only lead to periodic terms. Hence, we shall not consider these inequalities here but only confine ourselves to secular perturbations. As a consequence of these secular perturbations, eccentricity e' is found to decrease at the present time. During the next few centrueis, the rule by which e' decreases can be well approximated by the following relation

$$e' = e'_0 - \alpha T - \alpha' T^2,$$

where

$$e'_0 = 0.016\,751\,01, \quad \alpha = 0.000\,011\,80, \quad \alpha' = 0.000\,000\,126,$$

and T is time measured in Julian centuries starting from the central border midday 0 January 1900. Substituting this value for e' into equation (42) and multiplying the terms by α^2 and α^1, we finally obtain the following expression for the longitude of the moon

$$v = n_0 t + \varepsilon + \sigma T^2 + \text{periodic terms.}$$

where

[illegible] (43)

and n and n' are the annual mean motion of the moon and the sun. Since,

$$n = 17\,325\,591.0608\ldots, \quad n' = 1\,[illegible]\,77.11516$$
$$m = \frac{n'}{n} = 0.0718[illegible]\ldots$$

then

$$\sigma = 10.3$$

Thus, the mean annual motion of the moon increases each century by $2\sigma \approx$ 20". The coefficient σ is called the secular acceleration of the mean motion of the moon. At the present time, the most accurate value of may be considered to be the following

$$\sigma = 6''.02 \pm 0''.002,$$

which has been given by Brown (Sec. 117).

Annotation

The second approximation developed here enables us to establish the presence of secular accelerations in the motion of the perihelion and the node. As a matter of fact, equation (38) indicates that the instantaneous longitude of the perihelion is equal to

$$\Pi = v - (cv - \pi) = \pi + (1 - c)v,$$

where c is given by equation (37). Hence, the instantaneous speed of the motion of the perihelion is

$$\frac{d\Pi}{dv} = \frac{3}{4}\mu_1^2\left(1 - \frac{1}{2}e^2 + \frac{3}{2}e_0'^2\right) + \frac{9}{8}\mu_1^2(e'^2 - e_0'^2).$$

Integrating, we obtain the longitude of the perihelion at any arbitrary moment

$$\Pi = \pi + \frac{3}{4}\mu_1^2\left(1 - \frac{1}{2}e^2 + \frac{3}{2}e_0'^2\right)v + \frac{9}{8}\mu_1^2\int(e'^2 - e_0'^2)\,dv.$$

In this way, we will obtain, after expressing the longitude in terms of time, a term which is proportional to the square of time. A similar discussion can be applied to equations (39) and (35) to show that the longitude of the node is expressed by

$$\Omega = \theta - \frac{3}{4}\mu_1^2\left(1 + 2e^2 - \frac{1}{2}\gamma^2 + \frac{3}{2}e_0'^2\right) - \frac{9}{8}\mu_1^2\int(e'^2 - e_0'^2)\,dv.$$

According to Brown's calculations, the secular increments of the perihelion and the node's motions are respectively equal to

$$-38'.2 \pm 0''.1 \quad \text{and} \quad 6''.36 \pm 0''.02.$$

125. Secular acceleration of the mean motion of the moon

Let us discuss in detail the question on the secular acceleration of the mean motion of the moon. The volution of this equation is one of the m most interesting chapters in the history of celestial mechanics.

The secular perturbation was discovered by Halley in 1963. He made an attempt to determine the mean motion of the moon, i.e. the quantity n, by combining the observations of the darkenings, taken in different ages. He used the results of the observations made in Almageste and of those made by Arab astronomers as well as the results of more modern observations. Having determined the longitudes v_1 , v_2 and v_3 of the moon in three different epoches t_1 , t_2 and t_3 ($t_1 < t_2 < t_3$), Halley was able to write the following relations:

$$v_1 = nt_1 + \epsilon + \rho_1, \quad v_2 = nt_2 + \epsilon + \rho_2, \quad v_3 = nt_3 + \epsilon + \rho_3 \qquad (44)$$

where ρ_1 , ρ_2 and ρ_3 are the sums of periodic terms. Solving these equations, we can obtain the two following values:

$$n = \frac{v_2 - v_1 - \rho_2 + \rho_1}{t_2 - t_1}, \qquad n = \frac{v_3 - v_2 - \rho_3 + \rho_2}{t_3 - t_2},$$

which are expected to be equal within the adopted accuracy. However, Halley proved that the second value was definitely larger than the first one. This would suggest that the mean motion of the moon increased with time.

Replacing equations (44) by

$$v_i = nt_i + \epsilon + \sigma\left(\frac{t_i}{100}\right)^2 + \rho_i \qquad (i = 1, 2, 3)$$

we are able to obtain from these equations, not only the values of ϵ and n, but also the values of the acceleration σ . The first reliable numerical determination of σ was performed considerably later, because of the difficulty in using the ancient observations. In 1742, Dunthorne obtained $\sigma = 10''$. Tobias Mayer adopted the value $\sigma = 6.''7$ in the first edition of his tables on lunar motion (1752). In the second edition (1770), he took the value $\sigma = 9.''0$. Finally, Landan (1757) gave the value $\sigma = 9''.886$.

Even at the present time, the accurate determination of the value of σ derived from the results of these observations is still regarded as a difficult problem. Hansen treated this problem several times and obtained the following values successively: 11".93, 11".47, 12".18 and 12".56. The values σ = 8" obtained by Newcomb in 1909 and σ = 10".3 found by Fotheringham in 1915 may be considered as the best.

The theoretical interpretation of the secular acceleration of the moon was considered as one of the most intellectual problems of the eighteenth century. Tremendeous cosmological investigations were devoted to this problem. The reason for this strong interest was the following: In the presence of an acceleration in the mean motion of the moon, the distance between earth and moon decreases. This means that whatever the decrease in rate (approximately 3 cm per year) the moon would eventually collapse on the earth.

After a series of unsuccessful attempts to find the origin of the secular acceleration of the moon, Lagrange became persuaded by the idea that this acceleration was not real and that it probably appeared as a consequence of using wrong information on darkenings occurring in ancient times. On the other hand, Laplace unsuccessfully tried to explain the secular acceleration by introducing a hypothesis on the finite velocity of propagation of gravity. The correct solution was found by Largrange in 1783. He was the first to raise the question on the influence of the secular inequalities of the eccentricity and slope of one planet on the longitude of another. Being convinced that this influence was negligible in the case of Jupiter and Saturn, he made a hasty conclusion that this should be the same for the other cases. Later on, whilst studying the theory of Jupiter's satellites, Laplace discovered that the secular increments of the eccentricity of Jupiter's orbit produced accelerations in the mean motion of these satellites. He hurriedly transferred this idea to the lunar theory and in thisway, he finally uncovered the secret of the secular acceleration of the moon's motion in 1787. At the same time, he discovered that the secular increment of theeccentricity of the terrestrial orbit also produced a secular perturbation in the motion of the perihelion and the node. By this way he gave a new and brilliant proof to the character of the universal law of gravity. At the same time, he was able to give a new guarantee for the stability of the solar system. The fact remains that the theory of secular perturbations proves that the eccentricity of terrestrial orbits varies periodically. This induces the increment of the mean motion of the moon to also vary periodically. At the present time, the eccentricity of the moon is decreasing and hence its mean motion is accelerated. This situation is expected to continue for about 24000 years, after which, the eccentricity will start to increase and consequently the mean motion of the moon will start to decelerate.

Laplace pointed out in his nowfamous "Account on the system of the World" that it could be proved without any calculations, by using simple geometrical considerations, that the decrease in the eccentricity of the terrestrial orbit would produce an acceleration in the lunar motion. He commented on this outcome by stating that one should wonder why this simple interpretation always escaped geometers only "if it was not clear that the simplest ideas were almost always the last to reach the heads of people".

For the investigation of the secular increments of the anogee and the node, Laplace introduced further approximations which took into account the second-order perturbations. He obtained results which were quite different from those obtained by using the first approximations.

The first approximation, however, led Laplace to the value $\sigma = 10''.18$ for the secular acceleration of the mean motion of the moon. This value was in good agreement with the findings of the observations. Laplace was deceived by the corroboration of results between theory and observation. He concluded without further investigation that the first approximation could lead to a result of sufficient accuracy. As it was shown later on, this conclusion was incorrect.

The first attempt towards improving the theoretical value of obtained by Laplace was almost simultaneously made by Plana and Damoiseau. They obtained the values $\sigma = 10''.58$ and $\sigma = 10''.72$ respectively which were essentially in agreement with Laplace's findings. However, in 1853, Adams showed that these authors had made a basic mistake. They fixed the value of e' in the integration of the differential equation and replaced the fixed value of e' in the resulting solution by its expression as a function of time. This procedure could be only made in the first approximation. In order to integrate correctly the differential equations in the second approximation, Adams suggested that the coefficient $\frac{3}{2}\mu^2$ in the expression

$$\frac{3}{2}\mu^2 \int_0^t (e_0'^2 - e'^2)\, n_0\, dt.$$

of the acceleration, given by equation (42), should be replaced by

$$\frac{3}{2}\mu^2 - \frac{3771}{64}\mu^4 - \frac{34047}{64}\mu^5$$

This result was supported by Delaunay[1], who deduced a general expression for the secular acceleration. Delaunay's method was essentially improved by Newcomb and Brown, who found that the acceleration would be equal to

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(1) Comptes rendus de l'Academie des Science de Paris, 68 (1869); 72 (1871).

$$\sigma = 6.02 \pm 0.02,$$

This result may be considered as final. An error ± 0".02 is obtained by taking into account the effect of the inaccuracies in the deffinition of all the quantities on which σ depends.

Thus at the present time, there is a discrepancy between the theoretical value 6".0 for the acceleration and the value 8".0 obtained from the analysis of the observations. The origin of this discrepancy is not yet clear. All attempts to eliminate this difference by improving the theoretical value of   have not been successful. The most widely accepted suggestion is that the difference between the theoretical and the observed values is due to the deceleration of the earth, caused by tidal friction. The shortening of the days required to eliminate this effect is too small to have an observable effect on the motion of other luminaries at the present time. Jeffreys however, estimated theoretically the influence of the tidal friction and found that it could account for an acceleration of the order of 2" in the mean motion of the moon.

126. Periodic Inequalities of the Longitude

Let us once more consider equation (41) and this time concentrate our attention on the periodic terms. Taking into account equation (42), we obtain the following relation

$$nt + \varepsilon + \delta T = v - 2e \sin(cv - \varpi) + \frac{3}{4} e^2 \sin(2cv - 2\varpi) +$$
$$+ \frac{1}{4} \gamma^2 \sin(2gv - 2\theta) - \frac{11}{8} m^2 \sin 2Dv -$$
$$- \frac{15}{4} me \sin(\ell v - cv + \varpi) - 3me' \sin(c'mv - \varpi') + \dots$$

We remind the reader that we have agreed in section 120 to replace the quantity $\mu v + \epsilon - \mu \epsilon$ in the arguments by μe. Hence, by putting $\epsilon - \mu \epsilon = \beta$ and choosing the starting point for counting the time so that $\epsilon = 0$, we finally obtain

$$nt = v - 2e\sin(cv - \pi) + \frac{3}{4}e^2\sin(2cv - 2\pi) +$$
$$+ \frac{1}{4}\gamma^2\sin(2gv - 2\theta) - \frac{11}{8}\mu^2\sin(\lambda v - 2\beta)$$
$$- \frac{15}{4}\mu e\sin(\lambda v - cv - 2\beta + \pi) + 3\mu e'\sin(c'\varphi v + \beta - \pi') + \ldots,$$

where

$$nt = n_0 t + \alpha t^2,$$

or

$$n = n_0 + 10^{-4}\alpha t.$$

We solve this equation with respect to v. For this purpose, we construct the following successive approximations:

$$v_0 = nt$$
$$v_1 = nt + 2e\sin(cnt - \pi)$$
$$v_2 = nt + 2e\sin[cnt - \pi + 2e\sin(cnt - \pi)]$$
$$- \frac{3}{4}e^2\sin(2cnt - 2\pi) + \frac{1}{4}\gamma^2\sin(2gnt - 2\theta) +$$
$$+ \frac{11}{8}\mu^2\sin(\lambda nt - 2\beta) + \frac{15}{4}\mu e\sin[(\lambda - c)nt - 2\beta + \pi]$$
$$- 3\mu e'\sin(c'\varphi nt + \beta - \pi') + \ldots,$$

from which, taking into consideration that $\varphi n = n'$ and $\lambda n = 2n - 2n'$, we obtain

$$v = nt + 2e\sin(cnt - \pi) + \frac{5}{4}e^2\sin(2cnt - 2\pi) +$$
$$+ \frac{1}{4}\gamma^2\sin(2gnt - 2\theta) + \frac{11}{8}\mu^2\sin[2(n - n')t - 2\beta] + \qquad (15)$$
$$+ \frac{15}{4}\mu e\sin[(2n - 2n' - cn)t - 2\beta + \pi] - 3\mu e'\sin(c'n't + \beta - \pi') + \ldots$$

This is the final expression for the longitude of the moon up to second-order terms inclusively. We now consider this expression in detail. Let

$$M = cnt - \pi = nt - \Pi,$$
$$\Pi = \pi + (1 - c)nt,$$

where M is the mean anomaly of the moon, counted from a prihelion of longitude Π. The first two periodic terms of expression (45) give us the leading terms of the series-expansion of the equation of the centre (section 82), while the expression

$$w = nt + 2e \sin M + \frac{5}{4} e^2 \sin 2M + \ldots$$

is nothing else but the longitude in the orbit. These two terms define the elliptic inequalities of the motion of the moon. The next term

$$-\frac{1}{4} \gamma^2 \sin (2gnt - 2\theta) = -\frac{1}{4} \gamma^2 \sin 2 (w - \Omega),$$

where

$$\Omega = \theta + (1 - g) nt,$$

gives the reduction to theecliptic (section 85) calculated to within the second powers of the small quantities γ and e. The term

$$\frac{11}{8} m^2 \sin [2(n - n')t - 2\xi],$$

has a period of $\frac{360}{2(n - n')}$, which is equal to half a synodic month (11.765 days) and gives the variation. The term

[illegible]

gives the evection. The period of this inequality equals to

[illegible]

which is equal to one astral day (27.3166 days), divided by

[illegible]

i.e. approximately 32 days. Finally, the last term in equation (45) gives the annual inequality.

If we had kept one more term, namely

$$\frac{m a^{3}}{8 a'^{4}}\left[3(1-4s)\cos(v-v')+5\cos 3(v-v')\right],$$

in the force function U while expanding equation (7) in section 122, we would have obtained one more inequality in the longitude. The main part of this inequality would have been

$$-\frac{15}{8}\mu\frac{a}{a'}\frac{T-L}{T+L}\sin(v-v') = -\frac{15}{8}\mu\frac{a}{a'}\frac{T-L}{T+L}\sin\left[(n-n')t-\beta\right]+\ldots$$

Such an equation is called a parallactic inequality As we have already pointed out, this inequality enables us to determine the ratio $\frac{a}{a'}$ from the observation.

127. Expression of the Radius Vector and the Latitude as Functions of Time

In order to express the radius vector by a function of time, it is necessary to substitute the expression of the longitude, given by equation (45), into equation (38). Since

$$\begin{aligned} e\cos(cv-\pi) &= e\cos\left[cnt-\pi+2ce\sin(cnt-\pi)\right]+\ldots \\ &= e\cos(cnt-\pi)-e^{2}+e^{2}\cos(2cnt-2\pi)+\ldots, \end{aligned}$$

we then obtain, within the adopted accuracy,

$$\begin{aligned} au = 1 &+ \frac{1}{4}\gamma^{2}+e\cos(cnt-\pi)+e^{2}\cos 2(cnt-\pi)- \\ &-\frac{1}{4}\gamma^{2}\cos 2(gnt-\theta)+\mu''\cos\left[2(n-n')t-2\beta\right]+ \\ &+\frac{15}{8}\mu e\cos\left[(2n-2n'-cn)t-2\beta+\pi\right]+\ldots \end{aligned} \qquad (46)$$

Similarly, noting that

[illegible]

We obtain from equation (39)

$$s = \gamma \sin(gnt - \theta) + \gamma e \sin[(g - c)nt - \theta + \varpi] + \gamma e \sin[(g + c)nt - \theta - \varpi] + \frac{3}{8}\gamma\mu \sin[(2 - 2\nu - g)nt - \theta - 2\beta] + \ldots$$

In further approximations, the quantities c and g, which only depend on , will be given by[1]

$$c = 1 - \frac{3}{4}\mu^2 - \frac{225}{32}\mu^3 - \frac{4071}{128}\mu^4 - \frac{265493}{2048}\mu^5 - \frac{12822631}{24576}\mu^6 - \ldots$$

$$g = 1 + \frac{3}{4}\mu^2 - \frac{9}{32}\mu^3 - \frac{273}{128}\mu^4 - \ldots$$

In conclusion, we derive an expression for the equatorial horizontal parallax of the moon for the moment t. Denoting this parallax by P and the equatorial radius of the earth by A, we find

$$\sin P_{☾} = \frac{A}{r} = \frac{Au}{\sqrt{1 + s^2}} = Au\left(1 - \frac{1}{2}s^2 + \ldots\right).$$

Then, within the adopted accuracy,

$$P_{☾} = p_{☾}\Big\{1 + e\cos(cnt - \varpi) + e^2\cos(2cnt - 2\varpi) +$$
$$+ \mu^2\cos[2(n - n')t - 2\beta] + \frac{15}{8}\mu e\cos[(2n - 2n' - cn)t - 2\beta + \varpi] + \ldots\Big\},$$

where

$$p_{☾} = Aa^{-1}$$

is the parallax that corresponds to the mean distance between the earth and the moon.

128. Further Development of Laplace's Theory

In the previous sections, we have completed the calculations for the second-order inequalities by using Laplace's method. By this same method

(1) The simplest way to calculate the coefficients of these series is given by Hill's theory (cf: sections 140, 142).

Laplace could obtain all the third-order inequalities. The most extensive application of this method was given by Damoiseau[1], who had the aim of finding the coefficients of the inequalities to within 0".1 by using the method of indefinite coefficients.

Damoiseau put

[illegible]

where

[illegible]

where the indices α, α', β, $\varkappa$ and ν, on which the coefficients A, B and C depend, run over all positive and negative integral values. By u_o, the value of u which corresponds to the elliptic motion is denoted. In contrast to what we have presented in section 123, we include in the expression of u_o all terms up to the sixth order inclusively.

Damoiseau considered that it was necessary to keep in the expressions given above 85 coefficients of A, 37 coefficients of B and 85 coefficients of C. Expressing the coordinates v' and u' of the sun in terms of the coefficients C, he substituted these expressions into the differential equations. He then obtained 209 equations for defining the coefficients c, g, A, B and C by equating the coefficients of the trigonometric functions.

(1) M.C.T. Damoiseau, Memoire sur la theorie de la Lune, Memoires pres par divers savants, Paris, 3-e ser. 1, 1827, 315-598;
Tables de la Lune, forme par la seule theorie de l'attraction et suivant la diversion de la circomference en 360 degres, Paris 1828.

The two equations that determine the coefficients c and g could be solved separately. Damoiseau solved the remaining equations by replacing primarily the symbols c, g, μ, e, e', γ and $\frac{a}{a'}$ by their corresponding values. This solution was obtained by means of the method of successive approximations in a quite simple manner inspite of the large number of equations. In conclusion, Damoiseau solved equation (48) relative to v and obtained the expression of the longitude in terms of an explicit function of time.

We now compare the results obtained by Damoiseau with Hansen's theoretical predictions, the development of which requires a tremendous amount of work. This comparison can be illustrated by means of the following table which gives the number of coefficients in the expansion of the longitudes and the limits within which the differences between the values obtained by Damoiseau and the corresponding values obtained by Hansen vary:

Limits of the differences in the values of coefficients	Number of coefficients
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]

The eight largest differences are 3".33, 3".15, 1".82, 1".64, 1".25, 1".22, 1".21 and 1".20. These differences naturally depend on the use of different values for the constants. The possibility of these being some computational errors is not excluded. Finally, we have to note, that Hansen's calculations led to some values for the coefficients which involved errors of about tens of seconds.

We have thus seen that Laplace's method enables us to obtain the numerical values for the perturbations of the lunar motion in a relatively simple way. However, this method suffers from some essential drawbacks, which practically prevent us from applying this method to the construction of a complete lunar motion which can satisfy the more recent requirements of accuracy. Among these drawbacks are the following:

1- In order to obtain the value of the longitude within a given accuracy, it is necessary to calculate u with a considerably high accuracy. For example, in order to obtain second-order terms in the expansion of v, we had to calculate a part of the third-order terms in the expansion of u (see section 123). In calculating higher order terms of v, the situation will be much worse.

2- The solution of the system of equations, by which the unknown constants are obtained, rapidly becomes more difficult when the number of the unknowns increases.

3- The addition of new terms is quite difficult. At the same time as Damoiseau, who developed a numerical improvement to Laplace's method Plana and Carlini improved this method by means of an analytical approach. Plane reported his results in three large volumes where he obtained all the coefficients in theform of a power series of μ, e, e', γ and a/a' up to the fifth order inclusively. In some particular cases, he developed the expansion up to eight order terms. Inspite of this, he failed to obtain the accuracy obtained by Damoiseau in his numerical theory which required considerably less effort. The reason for this seems to be the slow convergence of the series developed in powers of μ. Laplace recognised this situation andfrom the very beginning preferred to construct the theory in a semi-numerical and a semi-algebraical manner. He immediately substituted μ by its numerical value and at the same time kept the other parameters e, e', γ and a/a' in their symbolic forms.

We pointed out in section 119 that Laplace used as a first approximation, an orbit obtained from the elliptic orbit by replacing v by cv in the expression of the latitude. Instead of this arbitrary way of obtaining the intermediary orbit of the moon, Guilden in 1885 suggested to keep, in the first approximation, some of the perturbing terms of the differential equations (1) and (2). In this manner, it was possible to obtain a more accurate intermediary orbit than that obtained by Laplace. This idea was further developed by Tisserand and Andoiyer(1).

Hill(2) suggested to separate the term proportional to the square of the radius vector and that proportional to the square of the distance from the moon to the ecliptical plane in the expression of the perturbation function which gave the perturbation caused by the sun. In other words, he suggested to put

$$U = \frac{1}{r} + \frac{1}{2} a (x^2 + y^2 + z^2) + \frac{1}{2} b z^2 + R',$$

where R' denotes the remaining part of the perturbation function. The first of the separated terms is proportional to r, and hence responsible for the rotation of the line of apses. The second term produces the force, which is responsible for the motion of the node. In the first approximation, we neglect R' and express the other terms in terms of u and s, so that

$$U = u(1 + s^2)^{-\frac{1}{2}} + \frac{1}{2} a u^{-2} + \frac{1}{2} (a + b) s^2 u^{-2}.$$

Substituting this expression into equations (1) and (2), we obtain

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(1) F. Tisserand, Traite de Mecanique celeste, 3, 118-140.

(2) G.W. Hill, On Intermediary Orbits in the Lunar Theory, Astr. Journal, 18, 1897, 81-87 (Works, 4, 1907, 136-149).

where

[illegible]

The first of these equation defines the unknown intermediary orbit. It can be integrated in a closed form by means of the elliptical functions if $s = 0$. Subsequently, the complete solution of the system can be easily performed by means of successive approximations, due to the smallness of s.[1]

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(1) The following works involve an application of the intermediary orbits in the theory of lunar motion:
A.M. Zdanov, The theory of intermediary orbits and its development for the purpose of investigating the lunar motion (Teorija Promezutocnyh orbit u prilozenie ee k issledovaniju dvizenija Luny) 1882;
A.V. Krasnov, Theory of solar inequalities in the lunar motion (Teorija solnecnyh neravenstv v dvizenii Luny) Kazan' 1894;
A.W. Krassnow, Zur Theorie der intermediaren Bahren des Mondes, Astr. Machr. 146, 1898.

CHAPTER XVIII

THEORY OF LUNAR MOTION. BASIS OF HILL'S METHOD

129. Introduction

Hill and Adams developed a method for obtaining the main inequalities of the lunar motion with arbitrarily high accuracy in a simple manner. As we have already pointed out in section 117, this method is mainly based on the old ideas developed by Euler. It enabled Brown to construct one of the most complete theories of lunar motion. The characteristic features of Hill's work is that he makes use of the rectangular rather than the polar coordinates. Hill pointed out that when rectangular coordinates are used, the differential equations of motion involve pure algebraic functions, while if the longitudes and latitudes are used, trigonometric functions will appear in these differential equations. In addition, in the case of unperturbed elliptical motion, the rectangular coordinates can easily be expressed in terms of explicit functions of time (see section 85), while the corresponding expressions of the true anomaly, given in section 82, and consequently those of the longitudes, are incomparably more complicated. One has the right to believe that also in the case of perturbed motion, the explicit expressions of the rectangular coordinates will be much simpler than the corresponding expressions of the polar coordinates.

Comparing the integration method of the differential equations of motion in rectangular coordinates with the method of variation of elements and with the method suggested by Delaunay, Hill again showed the advantage of the former method. In order to see this, let us assume that we wish to calculate the perturbations to a very high degree of accuracy. Then, we have to use the method of indefinite coefficients in order to obtain the scheme of the successive approximations. This method can equally be applied to differential equations of any order. However, when the number of unknowns and number of equations increase, the volume of the required work will be considerably increased.

On the basis of all the above arguments, Hill was convinced that it was more advantageous to integrate the differential equations of lunar motion by using rectangular coordinates. Once the rectangular coordinates are obtained, the calculation of the corresponding polar coordinates becomes quite simple.

Euler, in the second memoir, had already applied an elliptical rectangular coordinate system, rotating with a velocity equal to the mean velocity of the moon. Hill made use of a similar system, though rotating with a velocity equal to the mean velocity of the sun. Adams and subsequently Hill, systematically developed Euler's idea on the separate determination of inequalities of different powers relative to the parameters. For example, Hill first calculated the part of the motion of the perihelion which did not depend on the eccentricity of the solar orbit. Then, he calculated the part which was proportional to the first power of the eccentricity, and so on. Hill applied this idea even to the first approximation. Instead of starting with the elliptical orbit which results from the assumption that the mass m' of the sun is equal to zero, he assumed that, in the first apaproximation, the parallax of the sun can be set equal to zero. This assumption leads to an original version of the three-body problem, in which one of the three bodies goes to infinity and at the same time continues to influence the motion of the other two. By this way, one obtains a variational orbit which includes all the inequalities that depend on the angular distances of the moon and the sun. This orbit can be used as a trial intermediary orbit.

We should pay attention to one more feature of Hill's work. While all the previous authors used the ratio of the mean motions of the sun and the moon, i.e.

$$\mu = \frac{n'}{n}$$

as one of the parameters by which the expansions of the perturbations are developed, Hill preferred to expand the perturbations in powers of the ratio

$$m = \frac{n'}{n - n'},$$

since, in this case, the series converge more rapidly.

130. Equations of Motion

We take a rectangular heliocentric coordinate system, in which the xy-plane coincides with the plane of the ecliptic and which rotates about the z-axis with a constant angular velocity n'. The equations of motion of the moon in this coordinate system are given by (cf. section 38)

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} - n'^2x &= \frac{\partial V}{\partial x} \\ \frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} - n'^2y &= \frac{\partial V}{\partial y} \\ \frac{d^2z}{dt^2} \qquad\qquad\quad &= \frac{\partial V}{\partial z} \end{aligned}\right| \qquad (1)$$

where V is the force function. According to the results obtained in section 4, the function V is given by the following expression

$$V = k^2\frac{T+L}{r} + k^2\frac{T+L}{L}\frac{m'}{r'} + k^2\frac{T+L}{T}\frac{m'}{\Delta}$$

where T, L and m' denote as previously indicated, the masses of the earth, moon and sun, r and v' are the heliocentric distances of the moon and the sun, whereas Δ is the distance between them. As we have already seen in section 120 (annotation I1), the series-expansion of the last two terms yields.

$$V = k^2 \frac{T+L}{r} + \frac{k^2 m'}{r_1} \sum_{n=0}^{\infty} x_n \left(\frac{r}{r_1}\right)^n P_n(\cos H_1),$$

where r_1 is the distance from the sun to the centre of gravity of the earth and the moon, and H_1 is the angle between the radius vectors r and r_1.

Using the arguments given in annotation I of section 120, we assume that the motion of the sun relative to the centre of gravity of the earth-moon system is strictly elliptical. Hence, denoting the semi-major axis of the sun's orbit by a', we obtain the following relation

$$n'^2 a'^3 = k^2 m'.$$

where $T + L + m'$ has been replaced by m'. Consequently, noting that $x_2 = 1$,

$$V = k^2 \frac{T+L}{r} + n'^2 r^2 \left(\frac{a'}{r_1}\right)^3 \left[\frac{3}{2}\cos^2 H_1 - \frac{1}{2}\right] + \dots \tag{2}$$

The terms which we did not write are multiplied by

$$\frac{k^2 m'}{r_1^4} = n'^2 \left(\frac{a'}{r_1}\right)^3 \frac{1}{r_1}$$

They tend to zero, when a and m' increase to infinity in such a way that the ratio

$$\frac{k^2 m'}{a'^3} = n'^2.$$

remains finite. It is thus clear that, in order to obtain the inequalities of the moon that do not depend on the parallax of the sun, we must replace V in equation (1) by the function

$$V_1 = k^2 \frac{T+L}{r} + \frac{1}{2} n'^2 \left(\frac{a'}{r_1}\right)^3 [3r^2 \cos^2 H_1 - r^2].$$

Hill calculated the inequalities that depend on neither the parallax of the sun, nor on the eccentricity of the solar orbit. If $e' = 0$, then the sun moves uniformly with a velocity n' and we can therefore take the x-axis in such a way that it always passes by the sun.. In this case, we obtain

$$x' = a' \qquad y' = 0, \qquad z' = 0,$$

and, therefore,

$$r_1 = a', \qquad r \cos H_1 = \frac{xx' + yy' + zz'}{r_1} = x.$$

In this case, the force function may be taken as

$$V_2 = k^2 \frac{T + L}{r} + \frac{1}{2} n'^2 (3x^2 - r^2).$$

We finally introduce the parameter m by means of the following relation

$$m = \frac{n'}{n - n'},$$

where n' is the mean sidereal motion of the moon. We then write the function V_2 in the following final form

$$V_2 = k^2 \frac{T + L}{r} - \frac{1}{2} n'^2 (x^2 + y^2) + \frac{1}{2} m^2 (n - n')^2 (3x^2 - z^2). \qquad (1)$$

Furthermore, we put

$$V_1 = V_2 + \Omega,$$

where, as we can easily see,

$$\Omega = \frac{1}{2} m^2 (n - n')^2 \left[3 \frac{a'}{r_1^3} r^2 \cos^2 H_1 - 3x^2 + r^2 \left(1 - \frac{a'^3}{r_1^3}\right) \right]. \qquad (5)$$

Equation (1) for the force function $V = V_1$ can be written as

$$\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} = (n - n')^2 \left[-\frac{\kappa x}{r^3} + 3m^2 x + \frac{\partial \Omega}{\partial x} \right]$$

$$\frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} = (n - n')^2 \left[-\frac{\kappa y}{r^3} + \frac{\partial \Omega}{\partial y} \right]$$

$$\frac{d^2z}{dt^2} = (n - n')^2 \left[-\frac{\kappa z}{r^3} - m^2 z + \frac{\partial \Omega}{\partial z} \right],$$

where

$$\varkappa = \frac{k^2(T + L)}{(n - n')^2}.$$

Putting $\tau = (n - n')t$, we finally obtain the following equations of motion

$$\left.\begin{aligned}
\frac{d^2x}{d\tau^2} - 2m\frac{dy}{d\tau} + \frac{\varkappa x}{r^3} - 3m^2x &= \frac{\partial\Omega}{\partial x} \\
\frac{d^2y}{d\tau^2} + 2m\frac{dx}{d\tau} + \frac{\varkappa y}{r^3} \qquad &= \frac{\partial\Omega}{\partial y} \\
\frac{d^2z}{d\tau^2} \qquad + \frac{\varkappa z}{r^3} + m^2z &= \frac{\partial\Omega}{\partial z},
\end{aligned}\right\} \tag{6}$$

which correspond to the case in which the parallax of the sun is taken to be equal to zero. If the eccentricity of the solar orbit is also zero, then $\Omega = 0$. Since

$$n = \frac{2\pi}{27.32166\ldots}, \qquad n' = \frac{2\pi}{365.24220\ldots}$$

where t is expressed in seconds while τ is expressed in units, in which the period of the moon's sidereal rotation is equal to 2 .

The moon is at syzygies when $\tau = 0, \pi, 2\pi, \ldots$ and at quadratures when $\tau = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

131. The Hill Transformation

In order to simplify the application of the method of indefinite coefficients to equation (6), Hill introduced new variables. Putting

$$u = x + yi, \qquad s = x - yi,$$

then the first two of equations (6) may be replaced by

$$\left.\begin{aligned}
\frac{d^2u}{d\tau^2} + 2mi\frac{du}{d\tau} + \frac{\varkappa u}{r^3} - \frac{3}{2}m^2(u + s) &= 2\frac{\partial\Omega}{\partial s} \\
\frac{d^2s}{d\tau^2} - 2mi\frac{ds}{d\tau} + \frac{\varkappa s}{r^3} - \frac{3}{2}m^2(u + s) &= 2\frac{\partial\Omega}{\partial u}
\end{aligned}\right\} \tag{7}$$

Let us introduce the independent variable ζ by means of the following relation

$$\exp(t)$$

since

$$\frac{d}{dt} = \frac{d\zeta}{dt}\frac{d}{d\zeta} = \ldots \frac{d}{d\zeta},$$

then, using the operator

$$D = \frac{d}{d\zeta},$$

we reduce the rquations of motion into the following form

$$\left.\begin{aligned} &D^2u - 2mDu + \frac{3}{2}m^2(u + s) = -\frac{u}{r^3} + 2\frac{\partial\Omega}{\partial s},\\ &D^2s - 2mDs + \frac{3}{2}m^2(u + s) = -\frac{s}{r^3} + 2\frac{\partial\Omega}{\partial u},\\ &D^2z = -m^2z - \frac{z}{r^3} + \frac{\partial\Omega}{\partial z},\end{aligned}\right\} \qquad (8)$$

where $r^2 = us + z^2$.

In the following we need, apart from the previous equations, another relation analogous to the kinetic-energy integral. In order to deduce this relation, we multiply equations (6) by $2\frac{dx}{dt}$, $2\frac{dy}{dt}$ and $2\frac{dz}{dt}$ respectively, add and integrate the resulting equations. This procedure yields

$$\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2 + \left(\frac{dz}{d\zeta}\right)^2 = \frac{2}{r} + 3m^2x^2 - m^2z^2 + 2\int_0^{\zeta} d'\Omega - C,$$

where, we have put

$$d'\Omega = \left(\frac{\partial\Omega}{\partial x}\frac{dx}{d\zeta} + \frac{\partial\Omega}{\partial y}\frac{dy}{d\zeta} + \frac{\partial\Omega}{\partial z}\frac{dz}{d\zeta}\right)d\zeta,$$

On the left-hand side of this equation is the square of the relative velocity of the moon, which will be denoted by V. Inserting here our new variables, we obtain

$$Du\cdot Ds + (Dz)^2 + \frac{2\kappa}{r} + \frac{3}{4}\, m^2 (u+s)^2 - m^2 z^2 + 2\int_0 d\Omega = C. \qquad (9)$$

When $\Omega = 0$, this relation gives a first integral, known as the Jacobi integral (cf. section 38).

Equations (8) are not convenient for the application of the successive approximation method due to the prsence of terms involving r^{-3}. In order to obbtain a more convenient set of equations, we multiply equations (8) by s, u and 2z and add. We then obtain

$$s\,D^2u + u\,D^2s + 2z\,D^2z - 2m\,(u\,Ds - s\,Du) + \frac{3}{2}\, m^2 (u+s)^2 - 2m^2z^2 - \frac{2\kappa}{r} = 2\left(s\frac{\partial\Omega}{\partial s} + u\frac{\partial\Omega}{\partial u} + z\frac{\partial\Omega}{\partial z}\right).$$

Using equation (9), we eliminate the term that involves r^{-1} and obtain the first of the following two equations

$$\left.\begin{aligned} &D^2(\text{[illegible]}) - Du\,Ds - (D\text{[illegible]}) - 2m(u\,D\text{[illegible]}) \\ &+ \frac{9}{4}\, m^2 (u+s)^2 - 3m^2\text{[illegible]} = \text{[illegible]} - 2\int_0 \text{[illegible]} + 2\left(s\frac{\partial\Omega}{\partial s} + \text{[illegible]}\right) \\ &D(u\,Ds - s\,Du) - 2m\,D(us) + \frac{3}{2}\, m^2 (u^2 - s^2) = \text{[illegible]}\ s\frac{\partial\Omega}{\partial s} \text{[illegible]} \end{aligned}\right\} \quad (10)$$

The second equation has bee. obtained by multiplying equations (8) respectively by − s, + s and 0 and adding.

Considering now the simple case in which the motion of the moon is assumed to proceed in an ecliptical plane and the eccentricity of the solar orbit to be equal to zero; in this case, z = 0 an $\Omega = 0$, which reduces equations (10) and (9) to the following:

$$\left.\begin{aligned} D^2(us) - Du\,Ds - 2m(u\,Ds - s\,Du) + \frac{9}{4}m^2(u+s)^2 &= C \\ D(u\,Ds - s\,Du) - 2m\,D(us) + \frac{3}{2}m^2(u^2 - s^2) &= 0 \end{aligned}\right\} \qquad (11)$$

$$Du \cdot Ds + \frac{3}{4}m^2(u+s)^2 + 2\varkappa(us)^{-\frac{1}{2}} = C. \qquad (12)$$

It necessary to point out that equations (11) do not completely replace the initial equations

$$\left.\begin{aligned} D^2u + 2m\,Du + \frac{3}{2}m^2(u+s) - \varkappa u(us)^{-\frac{3}{2}} &= 0 \\ D^2s - 2m\,Ds + \frac{3}{2}m^2(u+s) - \varkappa s(us)^{-\frac{3}{2}} &= 0, \end{aligned}\right\} \qquad (13)$$

which can be obtained from equations (8) by putting z = 0 and Ω = 0. When the solution of equations (11) is already obtained, it is still necessary to substitute this solution into either equation (12) or equation (13) and find the relation between the constants x and C.

132. The variational Curve

The general solution of equations (11) or the equivalent equations.

$$\left.\begin{aligned} \frac{d^2x}{d\tau^2} - 2m\frac{dy}{d\tau} + \varkappa x r^{-3} - 3m^2x &= 0 \\ \frac{d^2y}{d\tau^2} + 2m\frac{dx}{d\tau} + \varkappa y r^{-3} &= 0, \end{aligned}\right\} \qquad (14)$$

which are obtained from equations (6) by putting z = 0 and Ω = 0, includes four arbitrary constants. We shall try to find these four constants in such a way, that the corresponding trajectory will be symmetric relative to the two coordinate axes. Since the trajectory is assumed to be free from singular points, we then require that it intersects each of the coordinate axes at right angles. Denoting by A the abscissa of the point of intersection with the x-axis, we then obtain

$$x = A, \quad y = 0, \quad \frac{dx}{d\tau} = 0 \quad \text{for} \quad \tau = \beta.$$

where β is an arbitrary constant. Combining these with the following equations

$$x = 0, \qquad \frac{dy}{d\tau} = 0,$$

which express the fact that the curve also intersects the y-axis at right angles (at moment τ, which can be excluded from these equations). We obtain four conditions which should, generally speaking, be satisfied by the solution of system (14). This solution will depend on the two arbitrary constants A and β. We can easily see that this solution is symmetric relative to the x-axis. In fact, equation (14) will not be altered if we replace $\beta + \tau$ by $\beta - \tau$, and at the same time replace y by $-$ y, keeping the value of x unaltered. Similarly, we can verify the symmetry of the solution relative to the y-axis.

In order to reduce the volume of the deduction, we put $\beta = 0$, which is equivalent to the time count from the moment of intersection of the x-axis. We choose the moment of intersection of the y-axis as the initial time and equal to $\frac{\pi}{2}$. This choice is defined by the rotation period under consideration. We search for the particular solution of equations (14) in the form

$$\left.\begin{aligned} x &= A_1 \cos \tau + A_3 \cos 3\tau + A_5 \cos 5\tau + \ldots \\ y &= A'_1 \sin \tau + A'_3 \sin 3\tau + A'_5 \sin 5\tau + \ldots \end{aligned}\right\} \tag{15}$$

or, by transforming to variabales u, s and ζ,

$$u = \sum_0^{\infty} \left\{ \frac{1}{2} (A_{2k+1} + A'_{2k+1}) \zeta^{2k+1} + \frac{1}{2} (A_{2k+1} - A'_{2k+1}) \zeta^{-2k-1} \right\}$$

$$s = \sum_0^{\infty} \left\{ \frac{1}{2} (A_{2k+1} - A'_{2k+1}) \zeta^{2k+1} + \frac{1}{2} (A_{2k+1} + A'_{2k+1}) \zeta^{-2k-1} \right\}.$$

Putting,

$$A_{2k+1} = a (a_{2k} + a_{-2k-2}), \quad A'_{2k+1} = a (a_{2k} - a_{-2k-2}),$$

we obtain

$$u = a \sum_{-\infty}^{+\infty} a_{2k} \zeta^{2k+1}, \quad s = a \sum_{-\infty}^{+\infty} a_{-2k} \zeta^{2k+1} \qquad (16)$$

The problem is thus reduced to a search for coefficients a_0, a_2, a_{-2}, a_4, a_{-4}, ... Since we have separated the common factor a, we can for example set $a_0 = 1$, after which the value of a can be defined. In order to simplify the substitution of expressioen (16) into equations (11), we initially evaluate the expressions involved in these equations. It is easy to see that

$$us = a^2 \sum_k a_{2k} \zeta^{2k+1} \sum_h a_{-2h} \zeta^{2h+1} =$$
$$= a^2 \sum_i \sum_k a_{2k} a_{2k-2i+2} \zeta^{2i},$$

where i = k + h + 1 runs all the values between $-\infty$ to $+\infty$. Similarly,

$$u^2 = a^2 \sum_i \sum_k a_{2k} a_{2i-2k-2} \zeta^{2i}, \qquad s^2 = a^2 \sum_i \sum_k a_{-2k} a_{2k-2i+2} \zeta^{2i}.$$

Since,

$$Du = a \sum_k (2k+1) a_{2k} \zeta^{2k+1}, \qquad Ds = a \sum_k (2k+1) a_{-2k} \zeta^{2k+1},$$

then,

$$Du\, Ds = a^2 \sum_i \sum_k (2k+1)(2i-2k-1) a_{2k} a_{2k-2i+2} \zeta^{2i}$$
$$u\,Ds - s\,Du = 2a^2 \sum_i \sum_k (i-2k-1) a_{2k} a_{2k-2i+2} \zeta^{2i}.$$

Finally,

$$D^2(us) = a^2 \sum_i \sum_k 4i^2 a_{2k} a_{2k-2i+2} \zeta^{2i}.$$

Substituting all of these expressions into equations (13) and equating the coefficients of ζ^{2i}, we obtain the following relations:

$$\left.\begin{aligned}&\sum_k \left[4i^2 + (2k+1)(2k-2i+1) + 4(2k-i+1)m + \frac{9}{2}m^2\right] a_{2k}\, a_{2k-2i} + \\ &\qquad + \frac{9}{4}m^2 \sum_k a_{2k}(a_{2i-2k-2} + a_{-2i-2k-2}) = 0 \\ &4i \sum_k (2k-i+1+m)\, a_{2k}\, a_{2k-2i} - \\ &\qquad - \frac{3}{2}m^2 \sum_k a_{2k}(a_{2i-2k-2} - a_{-2i-2k-2}) = 0.\end{aligned}\right\} \quad (17$$

For i = 0, the second of these equations becomes an identity, while the first equatic⁻ coŭld be replaced by

$$\sum_k \left[(2k+1)^2 + 4(2k+1)m + \frac{9}{2}m^2\right] a_{2k}^2 + \frac{9}{2}m^2 \sum_k a_{2k}\, a_{-2k-2} = a^2 C. \qquad (18)$$

In the next section, we shall see that the coefficients a_{2k}, which satisfy equation (17), will not only exist, but will also be relatively easy to obtair, for at least small values of the parameter m.

Since the corresponding series (16) or (17) are convergent, the existance of a particular solution for equation (14) having the required form, is thus proved by the above arguments.

The curve which is defined by equations (14) is usually called the variational curve. We shall now undertake to prove that this curve is indeed related to the inequality of the moon's motion, which we have called a vaiiation. We first of all observe that also in this case expressions (15) satisfy equations (14), in which τ is replaced by $\tau - \beta$ where β is an arbitrary constant. We denote the true longitude oi the moon at moment t by v. We represent the mean longitude of the moon at the same moment in the form of nt + ϵ . We consider the following expression:

$$r\cos(v - nt - \epsilon) \text{ and } r\sin(v - nt - \epsilon).$$

We have

[illegible]

where $n't + \epsilon'$ is the mean longitude of the sun, and

[illegible]

However, the x-axis is chosen to pass via the sun, then

[illegible]

and, therefore,

[illegible]

Replacing τ in equation (15) by $\tau - \beta$, we finally obtain

[illegible]

These two equations define the motion of the moon that corresponds to the particular solution of system (14) under consideration. In order to obtain the longitude of the moon from these equations, we can make use of the following equation

[illegible]

We rapidly see that the coefficients a_{2k} and a_{-2k} are 2k-order quantities relative to m. In particular,

[illegible]

Thus, Confining ourselves to third order terms, we obtain'

$$[\text{illegible}]\ m t_1 [\text{illegible}] \left(\frac{11}{[\text{illegible}]} m^2 + \frac{[\text{illegible}]}{6} m^{[\text{illegible}]} \ldots \right) \sin 2(\tau - [\text{illegible}]) + \ldots$$

Comparing this expansion with formulae (45) of section 727, we note that the terms which have been kept in the force function, reproduce the variation. This justifies the name which we have given to curve (15).

133. Calculation of the Coefficients

Let us now consider the solution of equations (17). First of all, we note that these equations can be rewritten in a much simpler manner by multiplying them by 2 and 3 respectively, adding, then constructing the sum and the difference of the resulting equations. This proceudre yields

$$\sum [8k^2 - 8(4i - 1)k + 20i^2 - 16i + 2 + [\text{illegible}] m + 9m^2]\, a_k a_{[\text{illegible}]} + 9m^2 \sum a_k a_{[\text{illegible}]} = 0$$

$$\sum [8k^2 - 8(2i + 1)k - 4i^2 + 8i + 2 + [\text{illegible}](i + 2)m - 9m^2]\, a_k a_{[\text{illegible}]} + 9m^2 \sum_k a_k a_{[\text{illegible}]} = 0. \qquad (19)$$

In order to obtain the terms that involve products $a_o\ a_{2i}$ and $a_o\ a_{-2i}$, it is necessary to take in the first sum of each of these equations, the terms that correspond to the values k = 0 and k = i. These ter[illegible] are

[illegible]

Because of this, we multiply the first equation by

[illegible]

and the second by

$$[illegible]$$

and add the resulting equations. We obtain anaeqation which does not contain a term involving the product $a_o\ a_{-2i}$. Putting

$$
\begin{aligned}
[i,k] &= [illegible] \\
[i] &= [illegible] \\
(i) &= [illegible]
\end{aligned}
\qquad [illegible]
$$

we finally obtain

$$\sum \{ [i, k]\, a\ [illegible]\ [i]\ a\ [illegible] \} = 0. \qquad [illegible]$$

It is easy to see that this single equation completely replaces system (17). In fact, equations (17) are equivalent to equations (19), and each of the latter equations is a consequence of the other. For example, if we replace k and i in the first of equations (19) by k - i and - i, we obtain the second.

Equations (21) have the most convenient form for the determination of a_{2i} since

$$[i, 0]\ [illegible]\ [i, i]\ [illegible]$$

while the quantities $[i]$ and (i) are second-order quantities relative to the parameter m, which we have agreed to consider as a small quantity. Let us now assume that the quantity a_{2k} is of the $|2k|$ - order relative to m. In this case, the sum

$$\sum (i)\, a\ \ a \qquad [illegible]$$

where $i > 0$, will consist of terms having orders at least four units larger than the order of equivalent terms in equation (21). The sum

$$\sum_k [i]\, a_k\, a_{i-k} \qquad (23)$$

in which $i < 0$ will have exactly the same property. We are not going to consider the value $i = 0$ either here or in equations (17).

Let us now consider the calculation of the coefficients a_{2k} in the first approximation. For this purpose, we write equations (21) for different values of i keeping each time only the terms having the least order. Taking into consideration the properties of the sums (22) and (23) which we have just mentioned, we obtain

$$\begin{aligned}
a_2 &= [1]\, a_0 a_0 \\
a_{-2} &= [-1]\, a_0 a_0 \\
a_4 &= [2]\,(a_0 a_2 + a_2 a_0) + [2,1]\, a_2 a_2 \\
a_{-4} &= [-2]\,(a_0 a_{-2} + a_{-2} a_0) + [-2,-1]\, a_{-2} a_{-2} \\
a_6 &= [3]\,(a_0 a_4 + a_2 a_2 + a_4 a_0) + [3,1]\, a_2 a_4 + [3,2]\, a_4 a_2 \\
a_{-6} &= [-3]\,(a_0 a_{-4} + a_{-2} a_{-2} + a_{-4} a_0) + [-3,-1]\, a_{-2} a_{-4} + [-3,-2]\, a_{-4} a_{-2} \\
a_8 &= [4]\,(a_0 a_6 + a_2 a_4 + a_4 a_2 + a_6 a_0) + [4,1]\, a_2 a_6 + \\
&\qquad + [4,2]\, a_4 a_4 + [4,3]\, a_6 a_2 \\
a_{-8} &= [-4]\,(a_0 a_{-6} + a_{-2} a_{-4} + a_{-4} a_{-2} + a_{-6} a_0) + [-4,-1]\, a_{-2} a_{-6} + \\
&\qquad + [-4,-2]\, a_{-4} a_{-4} + [-4,-3]\, a_{-6} a_{-2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

where we keep in these equations the coefficient $a_0 = 1$ in order to clearly show the scheme construction of the successive terms. The solution of this system of recurring equations is not difficult. It yields for the coefficient a_{2k}, a quantity having the order $|2k|$.

In order to obtain more accurate values, we have to repeat the calculation keeping not only the terms that have the least order but also the terms which have next to least order. To the first approximation, we obtain a_{2k} with an error of the $(|2k| + 4)$ - order, and in the second approximation an error of the $|2k| + 8$ order, and so on. This illustrates that successive approximations converge sufficiently rapidly for small values of the parameter m.

In order to show how simple these calculations are, we write the equation which can be used to calculate the coefficient a_2 within an error of the 14th order. This equation is

$$a_2 = [1](1 + 2a_{[illegible]}a_{[illegible]} + 2a_{[illegible]}a_1) + (1)(a^2_{[illegible]} + 2a_{[illegible]}a_{[illegible]} + 2a_{[illegible]}a_{[illegible]}) +$$
$$+ [1,-2]a_{[illegible]}a_{[illegible]} + [1,-1]a_{[illegible]}a_{[illegible]} + [1,2]a_1a_3 + [1,3]a_{[illegible]}a_4.$$

Before calculating the quantities $[i, k]$, $[i]$ and (i), we can simplify equations (20). Indeed, it is not difficult to see that

$$\left.\begin{aligned} [i,k] + [-i,-k] &= -\frac{2k}{i} + \frac{8k(i-k)}{2(4i^2-1)-4m+m^2} \\ [i,k] - [-i,-k] &= \frac{8k(k-i)}{i}\,\frac{1+m}{2(4i^2-1)-4m+m^2} \end{aligned}\right\} \qquad (21)$$

Similarly,

$$\left.\begin{aligned} [i] + (-i) &= -\frac{3}{2i}\,\frac{3i^2+1+2m}{2(4i^2-1)-4m+m^2}\,m^2 \\ [i] - (-i) &= \frac{27}{8i^2}m^2 - \frac{3}{2i^3}\,\frac{16i^4 \; [illegible] \; 5-(3i+11)m}{2(4i^2-1)-4m+m^2}\,m \end{aligned}\right\} \qquad (24')$$

These formulae are more convenient than equations (20).

Let us consider the case when it is required to calculate the variational curve for only one given value of m. In this case, it is easier to immediately calculate the numerical values of the coefficients. Hill adopted that

[illegible]

for which case be obtained

$$m = \frac{n'}{n - n'} = [illegible]$$

At the start, he calculated all the necessary values for the quantities

$\left[i,k\right]$, $\left[i\right]$ and (i), by means of the above equations for this value of m. He then proceeded to compute the values of the coefficients a_2, a_{-2},... by means of successive approximations. The computation of the first two coefficients is done in the following way

1st approx.,	2-order term	+	0.00151 58491 71593	−	0.00869 58084 99634
2nd approx.,	6-order term	−	0.00000 01416 98831	+	0.00000 00615 51932
3rd approx.,	10-order term	+	0.00000 00000 06801	−	0.00000 00000 13838
	$a_2 =$	+	0.00151 57074 79563	$a_{-2} =$ −	0.00869 57469 61540

Hill's final results were given in thefollowing form:

r cos(v − nt − ε) = a[1 − 0.00718 [illegible] 81977 cos 2(τ − ξ)
+ 0.00000 [illegible] 17061 cos 4(τ − ξ)
+ 0.00000 [illegible] 9202[illegible] cos 6(τ − ξ)
+ [illegible] cos 8(τ − ξ)
+ [illegible] 01171 cos 10(τ − ξ)
+ [illegible] cos 12(τ − ξ)]

r sin(v − nt − ε) = a[+ 0.0102[illegible] [illegible]1511 [illegible] sin 2(τ − ξ)
+ [illegible] 5711[illegible] sin 4(τ − ξ)
+ 0.00000 00275 [illegible] sin 6(τ − ξ)
+ [illegible] sin 8(τ − ξ)
+ [illegible] 01012 sin 10(τ − ξ)
+ [illegible] sin 12(τ − ξ)]

Giving this result in a supplement to the translation of Euler's book "New Theory of Lunar Mo on" cited above in section 117, Academician A.N. Krylov observed that a $\approx 10^{-14}$ produces a correction to the distance between the centres of gravity of the earth and moon, approximately equal to 4 microns. This unusual and practically useless accuracy, shows the tremendous power of Hill's method, which enables us to obtain this accuracy through a relatively small amount of work.

134. General Expressions for the Coefficients

The equations, deduced in the previous section, can be not only used for obtaining the numerical values of the coefficients a_{2i}, but also for deriving general expressions for these coefficients as functions of m. Formulae (24) and (24') show that the factors $\left[i,k\right]$, $\left[i\right]$ and (i) are rational functions of m with denominators of the form

$$2(i^2-1)-4m+m^2 \tag{25}$$

It is thus clear that each of the unknown coefficients can easily be represented in the result of the successive approximations by a double series in the form

[illegible]

where each of the quantities M_0, M_1, ..., N_1, ... is a double term in the form

[illegible]

having rational coefficients. This series only converges for the values of m which are smaller than the least, by a modulus of the roots of the denominator (25), i.e. for $m < \sqrt{6}$.

Expanding each term of series (26) in powers of m, Hill obtained

[illegible]

The difficult problem of the convergence of the power-series obtained here was studied by A.M. Ljapunov in his excellent book "On the series suggested by Hill forrepresenting the lunar motion" (O rjadah predlozennyh Hillom dlja predstavlenija dvizenija Luny), in which he proved that these series converge for $m < \frac{1}{7}$. Since, for the moon, $m = 0.0808 \ldots \approx \frac{1}{12}$, we can then consider that the application of Hill's method to this case is justified. The exact limits of convergence of Hill's series are still unknown.

If we confine ourselves to second-order terms relative to m, then

[illegible]

and, hence, equations (15) yield

[illegible]

Taking into account that

[illegible]

we can easily reduce the equation of the variational curve, within the accuracy required, into the form

[illegible]

In this manner, the variational curve has the form of a circle for $m = 0$. When m is increased, this curve will have a form similar to an ellipse with a centre at the origin of the coordinates and a semimajor axis equal to $(1-m^2)/(1 + m^2)$. We note that it follows from the last two equations that

[illegible]

We finally consider the calculation of the common factor a given by equation (16). For this purpose we take any one of the non-homogeneous equations relative to u and s. We take the first of equations (13), which may be written as

[illegible]

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or

$$a \sum \left| (2i + 1)^2 + 2m(2i + 1) + \frac{9}{2} m^2 \right| a_{2i} \ldots [illegible]$$
$$+ \frac{3}{2} m^2 a \sum a_{2i} \ldots [illegible]$$

because

$$a \sum [illegible] \qquad a \sum a_{2i} [illegible] \qquad a \sum a [illegible]$$

when $\zeta = 1$, this equation becomes

$$a \sum \left| (2i + 1 + m)^2 + 2m^2 \right| a_{2i} \ldots a \left(\sum a_{2i} \right) [illegible]$$

Since

$$[illegible] (l' + L)(1 + m) [illegible] \qquad [illegible] (l' + L)(1 + m) [illegible]$$

then

$$[illegible] \left(\sum a_{2i} \right) \left| \sum \right| (2i + 1 + m)^2 + 2m^2 \left| a_{2i} \right. [illegible]$$

Denoting by a the semi-major axis corresponding to the means motion n of the unperturbed motion, the third law of Kepler gives

$$n^2 a^3 = [illegible] (l' + L)$$

Comparing this equation with the previous ones and using the values of the coefficients a_{2k} obtained above, we easily obtain

$$a = a \left(1 - \frac{1}{6} m^2 [illegible] \frac{1}{3} m^3 + [illegible] m^4 - \frac{67}{288} m^5 - [illegible] m^6 \right.$$
$$\left. [illegible] m^7 [illegible] m^8 [illegible] m^9 \ldots \right).$$

We thus conclude that the form of the variational curve is completely defined by the value of m. In order to find the dimensions of this curve, characterized by the quantity a, we have also to know the value of n.

In the following, we shall also need the relation (*) between the quantities x and a. This relation may be given, to within terms of the order of magnitude of m^2 inclusively, by the following equation

[illegible]

135. Orbits Infinitely Close to the Variational Curve

In the previous section, we studied in detail the variational curves which appear as a particular solution of equations (14) having the form (15). It is interesting to know to what extent this solution is applicable from the point of view of closeness to the actual lunar orbit. If m = 0, equations (14) are reduced to the well known equations of the two-body problem. They describe an elliptical motion. On the other hand, equations (15) are reduced to

[illegible]

i.e., represent a circular motion. Hence, for small values of m we can regard the motion described by the variational curve as a motion along a circular orbit, deformed by the attraction of the sun. The actual orbit of the moon looks more like an ellipse than a circle, hence, we cannot confine ourselves to the study of solution (15) for equations (14). We have to consider more general solutions for these equations.

We call the general solution for equations (14) which involves four arbitrary constants, a variational orbit. We shall consider the difficult problem of determining the variational orbit and start by studying the particular case of calculating orbits, infinitely close to a variational orbit, i.e. orbits that correspond to the elliptical orbits of the two-body problem, the eccentricities of which are so small that their squares may be neglected. We write equations (14) in the following manner

$$\frac{d^2x}{dt^2} - 2m\frac{dy}{dt} = \frac{\partial F}{\partial x}, \quad \frac{d^2y}{dt^2} + 2m\frac{dx}{dt} = \frac{\partial F}{\partial y} \qquad (27)$$

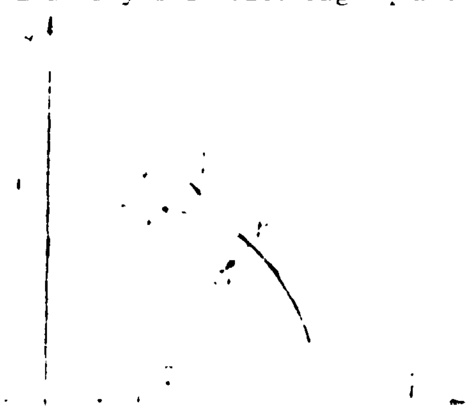
where

$$F = \frac{\mu}{r} + \frac{3}{2} m^2 x^2$$

We denote by x and y the coordinates of an arbitrary point on the variational curve (15), and by x + δx and y + δy the coordinates of a corresponding point P' on a close curve. By "corresponding point", we mean a point related to the same moment τ. Considering that the increments δx and δy are infinitesimal quantities, their squares may be neglected.

Substituting the coordinates of points P and P' into equations (27) and substracting the resulting equations term by term, we obtain for the determination of the increments δx and δy the following equations:

$$\left.\begin{aligned} \frac{d^2}{dt^2}\delta x - 2m\frac{d}{dt}\delta y &= \frac{\partial}{\partial x}\delta F \\ \frac{d^2}{dt^2}\delta y + 2m\frac{d}{dt}\delta x &= \frac{\partial}{\partial y}\delta F, \end{aligned}\right\} \qquad (28)$$



where

$$\delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial y}\delta y$$

Fig. 15

In our case, z = 0 and Ω = 0 and hence equation (9) leads to the following form of the Jacobi integral

$$V^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2F - C \qquad (29)$$

We confine ourselves to the consideration of only those adjacent orbits, for which the constant C has the same value as the initial variational curve. We use the Jacobi integral in the same manner as we have just used equation (27). We thus obtain [illegible] (30)

Denoting by δT and δN, the tangential and normal displacement of point P' relative to point P, and by ψ, the angle formed by the tangent to the variational curve with the x-axis (see figure 15), we obtain

[illegible]

which enables us to obtain from equations (28) and (29) the equations required for the determination of δN and δT. We start by transforming equation (29). First of all, we have

[illegible]

and since

[illegible]

we replace equation (30) by

[illegible]

On the other hand, we obtain from equation (33)

[illegible]

then, because of equation (27),

[illegible]

or, using again equation (32) and noting that

[illegible]

we finally obtain

$$[illegible] \qquad (36)$$

Once again we consider the Jacobi integral (29), which yields

$$[illegible]$$

However,

$$[illegible]$$

from which it follows that

$$[illegible] \qquad (38)$$

We write equation (34) in the following manner

$$[illegible]$$

Eliminating the partial derivatives of the function F by means of equations (36) and (38), we obtain

$$[illegible]$$

Having transformed the Jacobi integral and derived the auxiliary relations, we consider equations (28). Multiplying these equations by $-\sin\psi$ and $+\cos\psi$ respectively and adding, we obtain

$$[illegible]$$

Using equations (32), we transform the quantities found inside the square brackets into

$$[illegible]$$

Similarly,

$$-\frac{d\delta x}{d\tau}\sin\psi+\frac{d\delta y}{d\tau}\cos\psi=\frac{d\delta N}{d\tau}+\frac{d\psi}{d\tau}\delta T.$$

Differentiating this equation, and using again equations (32), we obtain

$$-\frac{d^2\delta x}{d\tau^2}\sin\psi+\frac{d^2\delta y}{d\tau^2}\cos\psi=\frac{d^2\delta N}{d\tau^2}+2\frac{d\psi}{d\tau}\frac{d\delta T}{d\tau}-\left(\frac{d\psi}{d\tau}\right)^2\delta N+\frac{d^2\psi}{d\tau^2}\delta T.$$

This enables us to represent equation (40) in the following manner

$$\left\{\frac{d^2}{d\tau^2}-\left(\frac{d\psi}{d\tau}\right)^2-2m\frac{d\psi}{d\tau}\right\}\delta N+\frac{d^2\psi}{d\tau^2}\delta T+2\left(\frac{d\psi}{d\tau}+m\right)\frac{d\delta T}{d\tau}-\frac{\partial\delta F}{\partial N}=0. \quad (41)$$

We now have to transform the last term. First of all, we have

$$\frac{\partial\delta F}{\partial N}=\delta\frac{\partial F}{\partial N}=\frac{\partial^2 F}{\partial N^2}\delta N+\frac{\partial^2 F}{\partial N\,\partial T}\delta T. \quad (42)$$

Differentiating by τ equation

$$\frac{\partial F}{\partial N}=\cos\psi\frac{\partial F}{\partial y}-\sin\psi\frac{\partial F}{\partial x}$$

we obtain

$$\frac{d}{d\tau}\frac{\partial F}{\partial N}=-\frac{d\psi}{d\tau}\frac{\partial F}{\partial T}+\cos\psi\frac{d}{d\tau}\frac{\partial F}{\partial y}-\sin\psi\frac{d}{d\tau}\frac{\partial F}{\partial x}=$$
$$=-\frac{d\psi}{d\tau}\frac{\partial F}{\partial T}+V\left[\left(-\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)\cos\psi\sin\psi+\frac{\partial^2 F}{\partial x\,\partial y}(\cos^2\psi-\sin^2\psi)\right],$$

or, noting that it follows from equation (35) that

$$\frac{\partial^2 F}{\partial N\,\partial T}=\left(-\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)\cos\psi\sin\psi+\frac{\partial^2 F}{\partial x\,\partial y}(\cos^2\psi-\sin^2\psi),$$

we finally obtain

$$\frac{d}{d\tau}\frac{\partial F}{\partial N}=-\frac{d\psi}{d\tau}\frac{\partial F}{\partial T}+V\frac{\partial^2 T}{\partial N\,\partial T}.$$

Using this relation to exclude the second derivative of F from equation (42), we obtain

$$\frac{\partial \delta F}{\partial N} = \frac{\partial^2 F}{\partial N^2}\delta N + \frac{\delta T}{V}\left(\frac{d}{d\tau}\frac{\partial F}{\partial N} + \frac{d\psi}{d\tau}\frac{\partial F}{\partial T}\right)$$
$$= \frac{\partial^2 F}{\partial N^2}\delta N + \frac{\delta T}{V}\left[V\frac{d^2\psi}{d\tau^2} + \frac{dV}{d\tau}\left(\frac{d\psi}{d\tau}+2m\right) + \frac{d\psi}{d\tau}\frac{\partial F}{\partial T}\right].$$

where $\frac{\partial F}{\partial N}$ is replaced by expression (36). Finally, using equation (38), we obtain

$$\frac{\partial \delta F}{\partial N} = \frac{\partial^2 F}{\partial N^2}\delta N + \frac{d^2\psi}{d\tau^2}\delta T + 2\left(\frac{d\psi}{d\tau}+m\right)\frac{dV}{d\tau}\frac{\delta T}{V}.$$

Having on hand this final expression for the last term of equation (41) we can rewrite this equation in the following manner

$$\left\{\frac{d^2}{d\tau^2} - \left(\frac{d\psi}{d\tau}\right)^2 - 2m\frac{d\psi}{d\tau} - \frac{\partial^2 F}{\partial N^2}\right\}\delta N + 2\left(\frac{d\psi}{d\tau}+m\right)\left(\frac{d\delta T}{d\tau} - \frac{dV}{d\tau}\frac{\delta T}{V}\right) = 0$$

or, using equation (39),

$$\frac{d^2\delta N}{d\tau^2} + \Theta\delta N = 0, \tag{43}$$

where

$$\Theta = 3\left(\frac{d\psi}{d\tau}+m\right)^2 + m^2 - \frac{\partial^2 F}{\partial N^2}. \tag{44}$$

In order to calculate the function Θ, we make use of the following formulae

$$\frac{\partial^2 F}{\partial N^2} = \sin^2\psi\frac{\partial^2 F}{\partial x^2} - 2\sin\psi\cos\psi\frac{\partial^2 F}{\partial x\,\partial y} + \cos^2\psi\frac{\partial^2 F}{\partial y^2},$$
$$\frac{d\psi}{d\tau} = V^{-2}\left(\frac{d^2y}{d\tau^2}\frac{dx}{d\tau} - \frac{d^2x}{d\tau^2}\frac{dy}{d\tau}\right),$$

in combination with equation (33).

After finding the increment δN from equation (43), we can obtain increment δT by using equation (39).

This method of obtaining infinitely close solutions can be applied to all equations having the form (27), in which the force function F does not explicitly depend on time. As we have already seen, the problem will be reduced to the solution of the principal equation (43).

In our particular case, the force function has the following form:

$$F = \kappa (x^2 + y^2)^{-\frac{1}{2}} + \frac{3}{2} m^2 x^2,$$

where x and y are defined by the series (15). Since the coordinates x and y either do not change or only change their signs when the variable τ is changed into $-\tau$ or $\tau + \pi$, then the force function F is an even periodic function of τ, having a period equal to π. It is easy to see that the second partial derivative $\frac{\partial^2 F}{\partial N^2}$ as well as the derivative $\frac{d\psi}{d\tau}$ will also have this property. Accordingly, the function Θ can be expanded into a series of the type

$$\Theta = q_0 + 2q_1 \cos 2\tau + 2q_2 \cos 4\tau + \ldots$$

The corresponding equation (43) is called Hill's equations.

It can be proved that the function Θ can be expanded in a series, developed by positive integral powers of the quantities m, $m^2 \zeta$, and $m^2 \zeta^{-2}$, from which it follows that the expansion of the coefficient q_k in powers of m starts from a term having the order 2k. We shall not give here the proof of this property of the coefficients q_k. The expansion of these coefficients is conveniently carried out if the coordinates x and y are replaced by the variables s and u. The result of the expansion is the following[1]:

---

(1) G.W. Hill, Literal Expansion for the Motion of the Moon's Perigee, Annals of Mathematics, 9, 1894, 31-41 (Works, IV, 41-50).

$$q^2 = 1 + 2m - \frac{1}{2} m^2 + \frac{255}{32} m^4 + 19 m^5 + \frac{80}{3} m^6 +$$

$$+ \frac{533}{2.3^2} m^7 + \frac{11\,230\,225}{2^{11}.3^3} m^8 + \frac{1\,576\,037}{2^7.3^4} m^9 +$$

$$+ \frac{49\,359\,583}{2^9.3^5} m^{10} + \frac{720\,508\,007}{2^8.3^6.5} m^{11} + \ldots$$

$$q_1 = - \frac{15}{2} m^2 - \frac{57}{4} m^3 - 11 m^4 - \frac{23}{2.3} m^5 - \frac{68\,803}{2^4.3^2} m^6 -$$

$$- \frac{1\,792\,417}{2^{10}.3^3} m^7 - \frac{7\,172\,183}{2^7.3^4.5} m^8 - \frac{596\,404\,499}{2^9.3^5.5^2} m^9 -$$

$$- \frac{2\,641\,291\,011\,773}{2^{17}.3^6.5^3} m^{10} + \ldots$$

$$q_2 = + \frac{111}{16} m^4 + \frac{1397}{2^6} m^5 + \frac{8807}{2^4.3.5} m^6 + \frac{319\,003}{2^5.3^2.5^2} m^7 +$$

$$+ \frac{252\,382\,507}{2^{10}.3^3.5^3} m^8 + \ldots$$

$$q_3 = - \frac{11\,669}{2^9} m^6 + \ldots$$

In the following, we give the numerical values of these coefficients for $m = 0.08084\ 89338\ 08212$ used by Hill:

$$\begin{aligned}
\theta = {} & 1.15884\ 39395\ 96583 \\
& - 0.11408\ 80374\ 93807 \cos 2\tau \\
& + 0.00076\ 64759\ 95109 \cos 4\tau \\
& - 0.00001\ 83465\ 77790 \cos 6\tau \\
& + 0.00000\ 01088\ 95009 \cos 8\tau \\
& - 0.00000\ 00020\ 98671 \cos 10\tau \\
& + 0.00000\ 00000\ 12103 \cos 12\tau \\
& - 0.00000\ 00000\ 00211 \cos 14\tau .
\end{aligned}$$

However, the numerical method enables us to obtain the coefficients of this series with the same accuracy but more simply than the algebraic method based on the above-mentioned expansions in powers of m.

In conclusion, we note that equation (39) devoted to the determination of δT can be rewritten to within the second powers of m in the following manner.

$$\frac{d\delta T}{d\tau} + \frac{7}{2} m^2 \sin 2\tau \, \delta T - 2\left(1 + m - \frac{5}{4} m^2 \cos 2\tau\right) \delta N = 0. \qquad (45)$$

136. Some properties of Hill's Equation

In the previous section, we have reduced the problem of finding orbits infinitely close to the variational orbit, to the solution of Hill's equation

$$\frac{d^2 x}{d\tau^2} + \Theta x = 0, \qquad (46)$$

in which the coefficient

$$\Theta = q_0^2 + 2q_1 \cos 2\tau + 2q_2 \cos 4\tau + 2q_3 \cos 6\tau + \ldots$$

is a periodic function of period π. We shall first of all consider some properties of this equation, which are particular cases of the properties of all the linear differential equations with periodic coefficients.

Let us denote by $f(\tau)$ and $\varphi(\tau)$ two of the particular solutions of equations (46), which satisfy the following initial conditions

$$f(0) = 1, \quad f'(0) = 0; \qquad \varphi(0) = 0, \quad \varphi'(0) = 1$$

Since these solutions from a fundamental system, then any arbitrary solution, $F(\tau)$, of equation (46) may be represented by

$$F(\tau) = Af(\tau) + B\varphi(\tau),$$

where A and B are constants. Equations (46) does not change when τ is replaced by $\tau + \pi$. Hence, the functions $f(\tau + \pi)$ and $\varphi(\tau + \pi)$ are also solutions of this equation. Consequently there exists such constant numbers as α, β, γ and δ_1, thus

$$\left.\begin{aligned} f(\tau+\pi) &= \alpha f(\tau)+\beta\varphi(\tau) \\ \varphi(\tau+\pi) &= \gamma f(\tau)+\delta\varphi(\tau). \end{aligned}\right\} \tag{47}$$

We prove that equation (46) has a solution which satisfies the following condition

$$F(\tau+\pi) = \nu F(\tau), \tag{48}$$

where ν is a constant. This condition gives

$$A(\alpha f+\beta\varphi)+B(\gamma f+\delta\varphi) = \nu(Af+B\varphi),$$

where the argument τ is dropped. The functions f and φ form a fundamental system and hence it follows from the previous condition that

$$A(\alpha-\nu)+B\gamma = 0, \qquad A\beta+B(\delta-\nu) = 0.$$

Since A and B cannot simultaneously be equal to zero, then

$$\begin{vmatrix} \alpha-\nu & \gamma \\ \beta & \delta-\nu \end{vmatrix} = 0,$$

or

$$\nu^2-(\alpha+\delta)\nu+(\alpha\delta-\beta\gamma) = 0. \tag{49}$$

Each of the roots of this equation gives a solution F(τ) which satisfies relation (48). In this way, the determination of the factor which will have a fundamental value in our future discussions, is reduced to the search for the substitution (47) that the fundamental system f, φ is subject to when the argument τ is increased by a period of π. If the functions f and φ satisfy the above-mentioned initial conditions, then equation (49) may be simplified. Indeed equation

$$\frac{d^2 f}{d\tau^2} + \Theta f = 0, \qquad \frac{d^2 \varphi}{d\tau^2} + \Theta \varphi = 0$$

gives

$$f\frac{d^2\varphi}{d\tau^2} - \varphi\frac{d^2 f}{d\tau^2} = 0,$$

Integrating and making use of the initial conditions, we obtain

$$f(\tau)\varphi'(\tau) - \varphi(\tau)f'(\tau) = 1.$$

Putting $\tau = \pi$ in this equation and noting that when $\tau = 0$, equations (47) give

$$f(\pi) = \alpha, \qquad \varphi(\pi) = \gamma, \qquad f'(\pi) = \beta, \qquad \varphi'(\pi) = \delta,$$

we obtain

$$\alpha\delta - \beta\gamma = 1.$$

In the case under consideration, equation (49) will then have the following form

$$\nu^2 - (\alpha + \delta)\nu + 1 = 0,$$

so that its roots may be denoted by ν and $1/\nu$. Hence

$$\nu + \frac{1}{\nu} = \alpha + \delta.$$

When $\tau = 0$ and $\tau = -\pi$, equation (48) yields

$$F(\pi) = \nu F(0), \qquad F(-\pi) = \frac{1}{\nu}F(0),$$

from which it follows that

$$\nu + \frac{1}{\nu} = \frac{F(\pi) + F(-\pi)}{F(0)}.$$

On the other hand, it is easy to see that $f(\tau)$ is an even function of τ, while $\varphi(\tau)$ is an odd function. Therefore,

$$F(\pi) = Af(\pi) + B\varphi(\pi)$$
$$F(-\pi) = Af(\pi) - B\varphi(\pi).$$

Moreover, since $F(0) = A$, then we finally obtain

$$\nu + \frac{1}{\nu} = 2f(\pi).$$

This form of equation (49) shows that, for small values of the parameter m, the roots ν and ν^{-1} are complex conjugate numbers having a modulus equal to unity. Indeed, the latter equation yields

$$\nu = f(\pi) \pm \sqrt{[f(\pi)]^2 - 1}.$$

On the other hand, using the approximate values of q^2, q_1, q_2, ..., given at the end of the previous section and neglecting terms of the order of m^2, we obtain

$$\frac{d^2x}{d\tau^2} + (1 + 2m)x = 0,$$

from which it follows that, in the first approximation

$$f(\tau) = \cos(1 + m)\tau,$$

and hence $|f(\pi)| < 1$. Thus, if we put

$$\nu = \exp(ic\pi),$$

then c would be a real number differing slightly from unity, at least for small values of m.

Considering the function $\exp(ic\tau)$. Evidently,

$$\exp[ic(\tau + \pi)] = \nu\exp(ic\tau).$$

i.e. this function satisfies the same relation (48), satisfied by $F(\tau)$. Hence, it follows that the ratio

$$\frac{F(\tau)}{\exp(ic\tau)} = \Phi(\tau)$$

does not change when π is added to the argument.

Finally, noting that equation (46) does not change when $+\tau$ is replaced by $-\tau$, we may conclude that equation (46) has two solutions of the form

$$\Phi(\tau)\cdot\exp(ic\tau) \ \& \ \Phi(-\tau)\exp(-ic\tau),$$

where $\Phi(\tau)$ is a function of the period π. As it can be easily seen, these solutions form a fundamental system.

Introducing, as in the previous section, the following independent variable

$$\zeta = \exp(i\tau),$$

and putting $q_o = q^2$ and $q_{-k} = q_k$ we can write the function

$$\Theta = q^2 + 2q_1 \cos 2\tau + \ . \ . \ . = \sum_{-\infty}^{+\infty} q_k \cos 2k\tau$$

in the following manner

$$\Theta = \sum q_k \zeta^{2k}.$$

In the following, we assume that the coefficients q_k are such that the series $\sum |q_k|$ converges. Since the function $\Phi(\tau)$ has the same period π, it can then be expanded in a series of a similar form. Hence, putting

$$\Phi(\tau) = \sum b_k \zeta^{2k},$$

we obtain

$$x(\tau) = \Phi(\tau)\exp(ic\tau) = \sum_{-\infty}^{+\infty} b_k \tau^{2k+c}. \tag{51}$$

Thus, Hill's equations have a general solution of the form (51). If this solution is found, for which it is necessary to compute the coefficients c and b_k, the general solution may be given in the form

$$C_1 x(\tau) + C_2 x(-\tau),$$

where C_1 and C_2 are arbitrary constants.

137. <u>Application of the method of indefinite coefficients</u>

In order to find the constants c and b_k, we substitute expression (51) into equation (46), and, taking into account equation (50), we obtain

$$-\sum b_k (2k+c)^2 \tau^{2k+c} + \sum \sum q_h b_k \tau^{2k+2h+c} = 0.$$

Equating the coefficient τ^{2k+c} to zero, we obtain the following system of equations:

$$[q_0 - (2k+c)^2] b_k + \sum q_{k-i} b_i = 0. \qquad (i \neq k) \tag{52}$$

This is an infinite system of linear equations with infinite number of unknowns b_k. This did not prevent Hill in applying theorems only proved for the case of finite systems of linear equations to this system. The results he obtained were strictly justified by Poincare, who developed for this purpose, a theory of infinite determinants.

Let us consider the simple case in which $q_1 = q_2 = \ldots = 0$. Since $q_0 = q^2$ and $q_{-k} = q_k$, then equation (52) will in this case have the following form:

$$[q^2 - (2k + c)^2]\, b_k = 0.$$

We are only interested in the solutions of the type (51) for which all the coefficients b_k are equal to zero. Hence, we will have

$$c = -2n \pm q, \qquad b_k = 0 \text{ для } k \neq n;$$

and consequently obtain two such solutions

$$x = b_n e^{iqt}, \qquad x = b_n e^{-iqt}$$

of equation (46) which in the present case reads

$$\frac{d^2x}{dt^2} + q^2 x = 0.$$

We shall now consider the general case in which the coefficients q_1, q_2, ... are not equal to zero. We shall confine ourselves to the case when case when all of these coefficients are small. The corresponding values of c will not be equal to $-2n \pm q$, where n is an integer, but will be slightly different from these quantities. Hence, we may consider that neither of the expressions

$$q^2 - (2k + c)^2$$

will be equal to zero. Accordingly, we rewrite equations (52) as follows

$$b_k + \sum \frac{q_{k-i}}{q^2 - (2k + c)^2}\, b_i = 0, \qquad (i \neq k)$$

or, in the unfolded form:

$$\ldots + \frac{q_2}{q^2 - (2k + c)^2}\, b_{k-2} + \frac{q_1}{q^2 - (2k + c)^2}\, b_{k-1} + b_k +$$
$$+ \frac{q_1}{q^2 - (2k + c)^2}\, b_{k+1} + \ldots = 0. \qquad (53)$$

It is easy to see that the determinant of this system belongs to the class of normal infinite determinants. These are the determinants of the type

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 + a_{-1,-1} & a_{-1,0} & a_{-1,1} \cdots \\ \cdots & a_{0,-1} & 1 + a_{0,0} & a_{0,1} \cdots \\ \cdots & a_{1,-1} & a_{1,0} & 1 + a_{1,1} \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

for which the double series $\sum_{i,j} |a_{ij}|$ is convergent. In fact, in the case under consideration,

$$a_{ii} = 0, \qquad a_{ik} = \frac{q_k}{q^2 - (2k + c)^2}$$

and hence

$$\sum_{i,k} |a_{ik}| = 2 \sum_i |q_i| \sum_k \left| \frac{1}{q^2 - (2k + c)^2} \right|,$$

where both of the series standing on the right-hand sides are convergent.

In the following, we shall only consider the bounded system for the solution of equations (53). This is the system of solutions which satisfies the following condition

$$|b_k| < A,$$

where A is some constant. It is well known that in the case of bounded systems of solutions, an infinite system of linear equations for which the determinant composed by coefficients is normal, will have the same properties as that of finite systems. In particular, one may conclude that when the determinant composed by the coefficients vanishes, the system (53) will have a solution only in the case in which all of the coefficients are equal to zero. We denote this determinant by $\Delta(c)$.

Thus, the problem of finding solutions of the type (53) of Hill's equation can be divided into two parts. Firstly, it is required to find the roots of the following equation:

$$\Delta(c) = 0; \tag{54}$$

and, secondly, to solve equations (53) for the resulting values of c. We start by the first part and try to find the solution of equation (54). The existance of the roots of this equation follows from the arguments given in the preceding section. We initially consider the function Δ (z) of the complex variable z. By this function we mean the value of the determinant, the k^{th} row of which

$$\ldots, a_{k,k-2}, \quad a_{k,k-1}, \quad 1+a_{k,k}, \quad a_{k,k+1}, \ldots$$

consists of terms respectively equal to

$$\ldots, \frac{q_2}{q^2-(2k+z)^2}, \quad \frac{q_1}{q^2-(2k+z)^2}, \quad 1, \quad \frac{q_1}{q^2-(2k+z)^2}, \ldots \tag{55}$$

where k = ..., -2, -1, 0, + 1, + 2, ... It follows from the previous arguments that such a determinant is normal for all values of z, except

$$z = \pm q - 2k. \tag{56}$$

Hence, the function Δ (z) is holomorphic for all the points z, except the points defined by equation (56). It is easy to see that these latter points are first-order poles of this function. In fact, each of points (56) is a pole of the first order for all the points of the k^{th} row, except for the single term that is equal to unity, and is a regular point for all of the other terms. When we work out the normal determinant, we obtain a convergent series, in which each term has one of the k^{th} row terms as a multiplying factor. It is thus clear that the points (56) can only be poles of an order not higher than the first for the function $\Delta(z)$. Therefore, $\Delta(z)$ is a meromorphic function. It is easy to see that this function is an even function such, that

$$\Delta(-z) = \Delta(z). \qquad (57)$$

Indeed, if z is replaced by -z and if the columns are at the same time replaced by the rows, then the determinant $\Delta(z)$ will not be charged. Similarly,

$$\Delta(z+2) = \Delta(z), \qquad (58)$$

since if z is replaced by z + 2, each column and each row will change place with the next column and the next row. Accordingly, the function $\Delta(z)$ is a periodic function with a period equal to 2.

It follows from equations (54), (57) and (58) that all points $z = \pm C - 2k$, where k is an arbitrary integer, are nodes of the function $\Delta(z)$.

We point out another property of the function $\Delta(z)$. Putting $z = x \pm yi$ and letting y tend to $\pm\infty$, all terms (55) of the k^{th} row tend to zero, except the term $1 + a_{kk} = 1$.

Consequently,

$$\lim_{y \to \pm\infty} \Delta(x + yi) = 1.$$

We shall now prove that this latter property completely defines the function $\Delta(z)$. We consider the following meromorphic function

$$\frac{\cos \pi z - \cos \pi c}{\cos \pi z - \cos \pi j},$$

which has first-order nodes at points (59) and first-order poles at points (56). This is an even function having a period equal to 2. If we put $z = x + yi$ and let y tend to $\pm\infty$, then this function tends to unity in analogy with function $\Delta(z)$. Hence, we conclude that the ratio of the functions under consideration, i.e.,

$$F(z) = \Delta(z)\,\frac{\cos \pi z - \cos \pi q}{\cos \pi z - \cos \pi c},$$

is a regular function of period 2, which tends to unity when y tends to $\pm\infty$. However, this function must be equal to a constant because it remains finite for all the points of the complex z-plane. Making $y \longrightarrow \pm\infty$, we obtain that F (z) = 1, and hence we finally obtain

$$\Delta(z) = \frac{\cos \pi z - \cos \pi c}{\cos \pi z - \cos \pi q}.$$

When z = 0, this equation yields

$$\sin^2 \frac{\pi c}{2} = \Delta(0)\,\sin^2 \frac{\pi q}{2} \qquad (60)$$

In this manner, we have reduced the problem of solving equation (54) to the problem of finding the determinant $\Delta(0)$, which is a relatively easy problem.

Putting z = ½, or z = 1, or z = q, we obtain different forms for the equation that determines c. However these forms are not as convenient as equation (60).

138. Calculation of the Determinant $\Delta(0)$

Introducing the following notation

$$\beta_k = \frac{1}{q^2 - 4k^2},$$

we write the determinant $\Delta(0)$ as

$$\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \beta_{k-1} q_1 & \beta_{k-1} q_2 & \cdots \\ \cdots & \beta_k q_1 & 1 & \beta_k q_1 & \cdots \\ \cdots & \beta_{k+1} q_2 & \beta_{k+1} q_1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (61),$$

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At the end of the previous section, we pointed out that the coefficient q_j is a quantity of the $2j$ order relative to m. This situation enables us to work out the determinant Δ (0) into a rapidly convergent series. We carry out this expansion on the basis of the following property of the determinant (61):

If $A q_\alpha q_\beta \ldots q_\gamma$ is one of the terms of the expansion of the determinant (61), then the sum of indices

$$\alpha + \beta + \ldots + \gamma$$

is always an even number.

In order to prove this, we replace the quantity q_i in all the terms of expression (61) by the quantity $q_j z^j$ and show that the determinant Δ (0,z) obtained thisway is an even function of z. Indeed, the determinant Δ (0, − z) is obtained from the determinant Δ (0,z) if the signs of all the terms of the rows and of the columns are alternately changed. Since the number of rows is equal to the number of columns in all the finite determinants, the limit of which is (61), then this change in signs of the terms will not change the determinant. It therefore immediately follows that the expansion of the determinant (61) will consist of terms, each of which has an order relative to m that is divisible by 4. Dropping the 12-order terms, we obtain

$$\Delta(0) = 1 + Aq_1^2 + Bq_1^4 + Cq_1^2 q_2 + Dq_2^2.$$

where the sum of indices are only 0, 2 and 4. It is easy to see that

$$Aq_1^2 = \sum_k \begin{vmatrix} 0, & \beta_{k+1} q_1 \\ \beta_k q_1, & 0 \end{vmatrix} = -q_1^2 \sum_k \beta_{k+1} \beta_k$$

$$Bq_1^4 = \sum_k \sum_l \begin{vmatrix} 0, & \beta_{k+1} q_1 \\ \beta_k q_1, & 0 \end{vmatrix} \begin{vmatrix} 0, & \beta_{l+1} q_1 \\ \beta_l q_1, & 0 \end{vmatrix} \quad q_1^4 \sum_k \sum_l \beta_{l+1} \beta_k \beta_{l-1} \beta_l$$

In the latter equation, k cannot be equal to i, i-1 and i + 1. Hence

$$0 = \sum \beta_k^2 \ldots - 2 \sum \beta_{k-1} \beta_i^2 \beta_k \ldots$$

The coefficients A, B, C, D, ... are easily expressed in terms of q. For example

$$A = -\sum_k \beta_{k-1} \beta_k = -\sum \frac{1}{16\left[\frac{1}{4}q^2 - (k-1)^2\right]\left[\frac{1}{4}q^2 - k^2\right]} =$$

$$= -\sum_k \frac{1}{16q(q+1)}\left(\frac{1}{\frac{1}{2}q + k} + \frac{1}{\frac{1}{2}q - k + 1}\right) - \sum_k \frac{1}{16q(q-1)}\left(\frac{1}{\frac{1}{2}q - k} - \frac{1}{\frac{1}{2}q + k - 1}\right) =$$

$$\sum_k \frac{1}{4q(1-q^2)} \frac{1}{\frac{1}{2}q + k} = \frac{1}{4q(1-q^2)}\left(\frac{1}{\frac{1}{2}q} + q \sum_1 \frac{1}{\frac{1}{4}q^2 - k^2}\right) =$$

$$= \frac{\pi}{4q(1-q^2)} \operatorname{ctg} \frac{\pi q}{2}.$$

Hill calculated all the terms of the expansion of $\Delta(0)$ having an order relative to m less than 16. The results which he obtained are

$$\Delta(0) = 1 + \frac{\pi}{4q} \operatorname{ctg} \frac{\pi q}{2} \left| \frac{q_1^2}{1-q^2} + \frac{q_2^2}{4-q^2} + \frac{q_3^2}{9-q^2} \right|$$

$$+ \frac{\pi \operatorname{ctg} \frac{\pi q}{2}}{32q(1-q^2)^2} \left| \frac{\pi \operatorname{ctg} \pi q}{q} - \frac{1}{q^2} + \frac{2}{1-q^2} + \frac{9}{2(4-q^2)} \right| q_1^4$$

$$+ \frac{3\pi \operatorname{ctg} \frac{\pi q}{2}}{8q(1-q^2)(1-q^2)} q_1^2 q_2 +$$

$$+ \frac{\pi \operatorname{ctg} \frac{\pi q}{2}}{128q(1-q^2)^2} \left| \left(-\frac{1}{q^4} + \frac{2}{1-q^2} + \frac{9}{2(4-q^2)}\right) \frac{\pi \operatorname{ctg} \pi q}{q} - \frac{25}{8q^2} - \right.$$

$$\left. - \frac{1}{q^4} + \frac{2}{1-q^2} - \frac{4}{(1-q^2)^2} - \frac{9}{8(4-q^2)} + \frac{9}{(4-q^2)^2} + \frac{4}{9-q^2} - \frac{\pi^2}{3q^2} \right| q_1^6 +$$

$$+ \frac{3\pi \operatorname{ctg} \frac{\pi q}{2}}{32q(1-q^2)^2(4-q^2)} \left| \frac{\pi \operatorname{ctg} \pi q}{q} - \frac{1}{q^2} + \frac{2}{1-q^2} + \right.$$

$$\left. + \frac{2}{4-q^2} + \frac{20}{3(9-q^2)} \right| q_1^4 q_2 +$$

$$+ \frac{\pi \operatorname{ctg} \frac{\pi q}{2}}{16q(1-q^2)(4-q^2)} \left| \frac{\pi \operatorname{ctg} \pi q}{q} - \frac{1}{q^2} + \frac{2}{1-q^2} + \frac{2}{4-q^2} + \right.$$

$$\left. + \frac{10}{9-q^2} \right| q_1^2 q_2^2 + \frac{(7-3q^2)\pi \operatorname{ctg} \frac{\pi q}{2}}{4q(1-q^2)(4-q^2)(9-q^2)} q_1 q_2 q_3 +$$

$$+ \frac{5\pi \operatorname{ctg} \frac{\pi q}{2}}{16q(1-q^2)(4-q^2)(9-q^2)} q_1^2 q_3.$$

Substituting into these equations the values of q, q_1, q_2, ... that correspond to the adopted value of m (section 135), Hill obtained

The zero-order term	1.00000 00080 00000 0
The 4-order term	0.00180 46110 93422 7
The sum of 8-order terms	0.00080 01808 63109 9
The sum of 12-order terms	0.00080 09000 64478 6
	1.00180 47920 24011 2

Judging by the law of decrease of terms of the different orders, we may conclude that the first thirteen decimals are correct.

Considering again equation (60), we find

$$c = 1.07158\ 32774\ 160.$$

where we expect that the error in $\Delta(0)$ is transferred to c increasing it 2.8 times. The fact that we have found a real value for c is of great importance. This finding suggests that the variational curve is a stable solution of equations (14). Indeed, by consulting sections 135 and 136, we find that the deviations δN and δT from the motion, represented by the variational curve, will be such that if the initial deviation from the motion along the variational curve is small, the deviation will remain small during any further motion only if the value of c is real.

If we replace q^2, q_1, ... by their expressions in terms of the parameter m (section 135), we then obtain $\Delta(0)$ as well as c in terms of explicit functions of m. In the book, quoted at the end of section 135, Hill obtained by means of the successive approximations method

$$c = 1 + m - \frac{3}{4} m^2 - \frac{201}{32} m^3 - \frac{2367}{2^7} m^4 - \frac{111749}{2^{11}} m^5 -$$
$$- \frac{4095991}{2^{13}.3} m^6 - \frac{332532037}{2^{16}.3^2} m^7 -$$
$$- \frac{15106211789}{2^{18}.3^3} m^8 - \frac{5975332916861}{2^{23}.3^4} m^9 -$$
$$- \frac{1547804933375567}{2^{26}.3^5.5} m^{10} -$$
$$- \frac{818293211830767367}{2^{29}.3^6.5^2} m^{11} - \ldots$$

139. <u>Calculation of the Coefficients</u>

Once the value of the fundamental constant c is obtained, the solution of equations (52) relative to the coefficients b_k is quite simple. In order to obtain the numerical values of these coefficients, it is recommended to replace the infinite system (52), written in the form

$$b_k + \sum_i \frac{q_{k-i}}{q^2 - (2k + c)^2} b_i = 0, \qquad (i \neq k) \quad (63)$$

by a finite system, obtained by neglecting all the negligibly small coefficients. We are assuming that the unknowns b_k are bounded, the magnitude of each term of this equation will then depend on the absolute value $|q^2 - (2k + c)^2|$ of the denominator and on the absolute value $|k-i|$ of the index of q_{k-i}, and will rapidly decrease when these quantities increase. Consequently, this proves that the unknown coefficients will rapidly tend to zero when the quantity $|k|$ is increased. The easiest manner to obtain the general expressions of b_k is the following: We replace the zero row

$$\ldots, \frac{q_{-2}}{q^2 - z^2}, \frac{q_{-1}}{q^2 - z^2}, 1, \frac{q_1}{q^2 - z^2}, \frac{q_2}{q^2 - z^2}, \ldots$$

of the determinant $\Delta(z)$, composed by the terms (55), by the following indefinite quantities

$$\ldots, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \ldots$$

The determinant D(z) obtained by this replacement will be convergent if the condition $|x_i| < A$, where A is independent of i, is satisfied. Expanding D(z) by the elements of the zero row, we find

$$D(z) = \ldots + x_{-1}B_{-1}(z) + x_0 B_0(z) + x_1 B_1(z) + \ldots,$$

where $B_k(z)$ is a meromorphic function of z, having the same poles (56) that the function $\Delta(z)$ has, with theexception of points $z = \pm q$. It is easy to see that

$$b_k = -B_k(c).$$

Indeed, if we replace the quantity x_i in the determinant D(c) by

$$\frac{q_{k-i}}{q^2 - (2k+c)^2}$$

where $i \neq k_1$ and the quantity x_k by unity, we then obtain $D(c) = 0$ since the determinant will have two equal rows in the case of $k \neq 0$, and will tend to $\Delta(c)$ in the case $k = 0$ and hence D(c) will disappear. However, this replacement makes D(c) identical to the left-hand side of the $k^{\underline{th}}$ row of equations (63). This proves our conclusion.

It is possible to show that the coefficient b_k will have a multiplying factor of $m^{|2k|-1}$. However, we shall not consider this here.

140. <u>The most important inequalities of the lunar motion</u>

In order to elucidate the relation between Hill's and Laplace's theories, we calculate the leading terms in the expansions of the quantities under consideration in powers of m. Confining ourselves to an order of terms not higher than the third in equations (52), and taking into account the values of q^2, q_1, q_2, ... obtained in section 135, we find

$$[q^2 - c^2]\, b_0 + q_1 b_1 + q_1 b_{-1} = 0,$$
$$[q^2 - (c+2)^2]\, b_1 + q_1 b_0 = 0, \quad [q^2 - (c-2)^2]\, b_{-1} + q_1 b_0 + q_1 b_{-2} = 0$$
$$[q^2 - (c+4)^2]\, b_2 + q_1 b_1 = 0, \quad [q^2 - (c-4)^2]\, b_{-2} + q_1 b_{-1} = 0.$$

$$q^2 - c^2 = \frac{225}{16} m^4 + \frac{3645}{64} m^5 + \ldots, \quad q_1 = -\frac{15}{2} m^2 - \frac{57}{4} m^3 - 11 m^4 + \ldots$$
$$q^2 - (c+2)^2 = -8 - 4m + 3m^2 + \ldots, \quad q^2 - (c-2)^2 = 4m - 3m^2 - \ldots$$
$$q^2 - (c+4)^2 = -24 - 8m + \ldots, \quad q^2 - (c-4)^2 = -8 + 8m - 6m^2 - \ldots.$$

Neglecting the terms having an order of magnitude of m^3, we obtain from these equations that

$$b_1 = -\frac{15}{16} m\, b_0, \quad b_{-1} = \left(\frac{15}{8} m + \frac{159}{32} m\right) b_0, \quad b_2 = b_{-2} = 0$$

The corresponding formula (51) gives the general solution of the equation that defines δN in the following form

$$\delta N = -\frac{15}{16} m^2 (C_1 \zeta^{2+c} + C_2 \zeta^{-2-c}) + (C_1 \zeta^{c} + C_2 \zeta^{-c}) +$$
$$+ \left(\frac{15}{8} m + \frac{159}{32} m^2\right)(C_1 \zeta^{-2+c} + C_2 \zeta^{2-c}) + \ldots ,$$

where C_1 and C_2 are arbitrary constants. Putting

$$C_1 + C_2 = A \cos \omega, \quad i(C_1 - C_2) = -A \sin \omega,$$

where A and ω are new arbitrary constants, we obtain

$$A^{-1} \delta N = -\frac{15}{16} m \cos [(c+2)\tau + \omega] + \cos (c\tau + \omega) +$$
$$+ \left(\frac{15}{8} m + \frac{159}{32} m^2\right) \cos [(c-2)\tau + \omega].$$

In order to simplify the following deductions, we also neglect the second-order terms. Then, equation (45) becomes

$$\frac{d}{d\tau} \delta T = 2(1 + m)\, \delta N,$$

The previous expression for δN gives

$$\delta N = A\cos(c\tau+\omega) + \frac{15}{8}Am\cos[(c-2)\tau+\omega].$$

Hence,

$$\delta T = 2A\sin(c\tau+\omega) - \frac{15}{4}Am\sin[(c-2)\tau+\omega] + B,$$

where B is a new constant, since C = 1 + m and C-2 = - (1-m) within the degree of accuracy desired. On theother hand, we have seen in section 134 that the equation of the variational curve is given by

$$x = a\cos\tau\left[1 - m^2 - \frac{3}{4}m^2\sin^2\tau\right], \quad y = a\sin\tau\left[1 + m^2 + \frac{3}{4}m^2\cos^2\tau\right].$$

to within terms of the order of m^2. Hence equation (33), that defines' the angle between the tangent to the variational curve and the x-axis, yields

$$\psi = \frac{\pi}{2} + \tau.$$

within an error of the order of m^2. Consequently,

$$\delta x = -\delta T\sin\tau - \delta N\cos\tau, \quad \delta y = \delta T\cos\tau - \delta N\sin\tau.$$

Substituting here the values of δ N and δ T just found, and equating the resulting δ x and δ y to the coordinates of the points of the variational curve, we obtain

$$\left.\begin{aligned}
x &= a\cos\tau - B\sin\tau - am^2\left(1 + \frac{3}{4}\sin^2\tau\right)\cos\tau - \\
&\quad - A\Big\{\frac{15}{8}m\cos[(c-2)\tau+\omega]\cos\tau + \cos(c\tau+\omega)\cos\tau - \\
&\quad - \frac{15}{4}m\sin[(c-2)\tau+\omega]\sin\omega - 2\sin(c\tau+\omega)\sin\tau\Big\} \\
y &= a\sin\tau + B\cos\tau + am^2\left(1 + \frac{3}{4}\cos^2\tau\right)\sin\tau - \\
&\quad - A\Big\{\frac{15}{8}m\cos[(c-2)\tau+\omega]\sin\tau + \cos(c\tau+\omega)\sin\tau + \\
&\quad + \frac{15}{4}m\sin[(c-2)\tau+\omega]\cos\tau + 2\sin(c\tau+\omega)\cos\tau\Big\}
\end{aligned}\right\} \quad (61)$$

The coefficients A and B should be regarded as infinitesimal quantities of the first order. Hence, it is possible to put

$$a\cos\tau - B\sin\tau = a\cos(\tau + \delta\tau_0), \quad a\sin\tau + B\cos\tau = a\sin(\tau + \delta\tau_0),$$

where $\delta\tau_0$ is an infinitesimal constant equivalent to B. Since the origin of counting τ is unknown, we can drop $\delta\tau_0$. This will change the previous values of x and y by only second-order quantities relative to A and $\delta\tau_0$.

We shall first consider the case of m = 0. In this case the perturbation caused by the sun is absent. The differential equations (14) defining the variational orbit are reduced to the equations of the two-body problem. Hence, when m = 0, equation (64) must describe an ellipitical motion in which only first powers of the eccentricity are included. Howerver, when m = 0, these equations yield

$$x = a\cos\tau - \frac{3}{2}A\cos\omega + \frac{1}{2}A\cos(2\tau + \omega)$$
$$y = a\sin\tau - \frac{3}{2}A\sin\omega + \frac{1}{2}A\sin(2\tau + \omega).$$

Rotating the coordinate axis by an angle of w, the new coordinates

$$x' = x\cos\omega - y\sin\omega, \quad y' = x\sin\omega + y\cos\omega$$

will be equal to

$$x' = -\frac{3}{2}A + a\cos(\tau + \omega) + \frac{1}{2}A\cos 2(\tau + \omega)$$
$$y' = a\sin(\tau + \omega) + \frac{1}{2}A\sin 2(\tau + \omega).$$

comparing these expressions with the formulae of section 89 that define the coordinates in the elliptical motion, namely

$$\xi = a(\cos E - e) = a\left(-\frac{3}{2}e + \cos M + \frac{e}{2}\cos 2M\right),$$
$$\eta = a\sqrt{1 - e^2}\sin E = a\left(\sin M + \frac{e}{2}\sin 2M\right),$$

and confining ourselves to first powers of the eccentricity, we obtain

$$a = a, \quad A = ae, \quad \tau + \omega \ [illegible]$$

Let us now consider again the motion of the moon in the general case when $m \neq 0$. Denoting as previously the radius vector and the longitude of the moon by r and v, and the longitude of the sun by $v' = n't + \epsilon'$ and by putting $B = 0$ and $A = \underline{a}e$, we obtain

$$\left.\begin{aligned}
\frac{r}{a}\cos(v-v') &= \cos\tau - am^2\left(1+\frac{3}{4}\sin^2\tau\right)\cos\tau - \\
&-\frac{15}{8}me\cos[(c-2)\tau+\omega]\cos\tau - e\cos(c\tau+\omega)\cos\tau + \\
&+\frac{15}{4}me\sin[(c-2)\tau+\omega]\sin\tau - 2e\sin(c\tau+\omega)\sin\tau \\
\frac{r}{a}\sin(v-v') &= \sin\tau + am^2\left(1+\frac{3}{4}\cos^2\tau\right)\sin\tau - \\
&-\frac{15}{8}me\cos[(c-2)\tau+\omega]\sin\tau - e\cos(c\tau+\omega)\sin\tau - \\
&-\frac{15}{4}me\sin[(c-2)\tau+\omega]\cos\tau + 2e\sin(c\tau+\omega)\cos\tau.
\end{aligned}\right\} \quad (65)$$

Squaring these equations and adding, we obtain after evident manipulations

$$r^2 = a^2\left\{1 - 2e\cos(c\tau+\omega) - 2m^2\cos 2\tau - \frac{15}{4}me\cos[(c-2)\tau+\omega]\right\};$$

from which, taking into account that (Sec. 134)

$$a = a\left(1 - \frac{1}{6}m^2 + \ldots\right),$$

we obtain

$$r = a\left\{1 - \frac{1}{6}m^2 - e\cos(c\tau+\omega) - \frac{15}{8}me\cos[(c-2)\tau+\omega] - m^2\cos 2\tau\right\}. \quad (66)$$

Dividing each of equations (65) term by term by the obtained value of r, we obtain

$$\cos(v-v') = \cos\tau\left[1-\frac{11}{4}m^2\sin^2\tau\right]+$$
$$+\sin\tau\left\{\frac{15}{4}me\sin[(c-2)\tau+\omega]-2e\sin(c\tau+\omega)\right\}$$
$$\sin(v-v') = \sin\tau\left[1+\frac{11}{4}m^2\cos^2\tau\right]-$$
$$-\cos\tau\left\{\frac{15}{4}me\sin[(c-2)\tau+\omega]-2e\sin(c\tau+\omega)\right\},$$

from which it follows that

$$\sin(v-v'-\tau) = \frac{11}{8}m^2\sin 2\tau-\frac{15}{4}me\sin[(c-2)\tau+\omega]+2e\sin(c\tau+\omega),$$

or, within the same accuracy, i.e., to within small quantities of the second order relative to m and of first order relative to e,

$$v-v'-\tau = 2e\sin(c\tau+\omega)-\frac{15}{4}me\sin[(c-2)\tau+\omega]+\frac{11}{8}m^2\sin 2\tau. \qquad (67)$$

Evidently, the periodic terms in the expansions (66) and (67) represent the elliptical inequalities of the moon. Hence, the argument of these terms is nothing else but the mean anomaly. Keeping the notations of section 126, the mean anomaly is equal to nt $-\Pi$. Consequently,

$$c\tau+\omega = nt-\Pi.$$

Differentiating this equation gives the motion of the perihelion:

$$\frac{d\Pi}{dt} = n-c(n-n') = n\left(1-\frac{c}{1+m}\right)$$

Substituting here the value of c obtained in section 136, we obtain for that part of the perihelions motion, that does not depend on the eccentricities of the moon and the sun, the following equation

$$\frac{1}{n}\frac{d\Pi}{dt} = \frac{3}{2^2}m^2+\frac{177}{2^5}m^3+\frac{1659}{2^7}m^4+\frac{85\,205}{2^{11}}m^5+\frac{3\,073\,531}{2^{13}.3}m^6+$$
$$+\frac{258\,767\,293}{2^{16}.3^2}m^7+\frac{12\,001\,004\,273}{2^{18}.3^3}m^8+\frac{4\,823\,236\,506\,653}{2^{23}.3^4}m^9+\ldots$$

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This series converges so slowly, that it is recommended to use the numerical method given in section 138 for the actual calculation of the perihelion's motion. By this method, Hill found

$$\frac{1}{n}\frac{d\Pi}{dt} = 0.00857\,25730\,04864.$$

On comparing the results obtained here with those obtained by the methods developed by Laplace, de Pontecoulant and others, it is interesting to note that

$$m = \frac{n'}{n - n'} = \frac{\mu}{1-\mu}, \quad c = \frac{n}{n-n'}, \; c = (1+m)c.$$

Expressing the motion of the perihelion in terms of μ, we obtain

$$\frac{1}{n}\frac{d\Pi}{dt} = 1 - c = \frac{3}{4}\mu^2 + \frac{225}{32}\mu^3 + \frac{4071}{2^7}\mu^4 + \frac{265\,493}{2^{11}}\mu^5 + \frac{12\,822\,631}{2^{13}.3}\mu^6 + \ldots$$

The coefficients of these series are considerably larger than the corresponding coefficients of the series developed by powers of m. Accordingly, it is more useful to use parameter m in the lunar theory than parameter μ. However, in the case when a particularly high degree of accuracy is desired, it is also not recommended to use the expansion in powers of m. It is simpler and more direct to apply the numerical method as we have already pointed out in section 138.

Let us again consider equations (66) and (67). Since the mean longitudes of the moon and the sun are nt + ϵ and n't + ϵ' respectively, then

$$v' + \tau = nt + \epsilon, \quad \tau = (n - n')t - \beta,$$

where $\beta = \epsilon' - \epsilon$ and τ is the angular distance between the mean position of the moon and the sun. Putting

$$c\tau + \omega = \frac{cn}{n-n'}\tau + \omega = cnt - \pi,$$

where

$$\pi = \frac{cn\beta}{n - n'} - \varpi.$$

we represent equations (67) and (66) in the following way

$$\begin{aligned} v = nt + \varepsilon + 2e \sin (cnt - \pi) + \frac{11}{8} m^2 \sin [2(n - n')t - 2\beta] + \\ + \frac{15}{4} me \sin [(2n - 2n' - cn)t - 2\beta + \pi] \end{aligned}$$

$$\begin{aligned} r = a \Big\{ 1 - \frac{1}{6} m^2 - e \cos (cnt - \pi) - m^2 \cos [2(n - n')t - 2\beta] - \\ - \frac{15}{8} me \cos [(2n - 2n' - cn)t - 2\beta + \pi] \Big\}. \end{aligned}$$

We compare these expressions with those obtained in Laplace's theory particularly with formula (45) obtained in section 127. We find that Hill's theory leads to the principal terms of the equation of the centre, the variation and the evection. The advantage of this theory over Laplace's consists, first of all, in that Hill's theory allows us to calculate these inequalaities as well as the motion of the perihelion in a relatively simple way and with the desired arbitrarily high degree of accuracy for the parameter μ .

141. Inequalities depending on the eccentricity of the lunar orbit

We have studied in detail the method developed by Hill to calculate the inequalities that depend on the first power of the eccentricity of the lunar orbit. We shall now consider the inequalities that have higher powers of eccentricity as multiplying factors. This problem is equivalent to calculating the general solution of equations (14) or (13). We first consider the solution, orbits infinitely close to the variational curve, which has been found in sections 135-139. It follows from equations (31) that

$$\delta N = -\delta x \sin\varphi + \delta y \cos\varphi = \frac{1}{2}\, i(\delta s \cdot e^{i\varphi} - \delta u \cdot e^{-i\varphi})$$

$$\delta T = \delta x \cos\varphi + \delta y \sin\varphi = \frac{1}{2}\,(\delta s \cdot e^{i\varphi} + \delta u \cdot e^{-i\varphi}),$$

if we again put u = x + yi and s = x - yi. On the other hand, equations (33) yield

$$Ve^{i\varphi} = i\,Du, \quad Ve^{-i\varphi} = -i\,Ds,$$

from which it follows that

$$\delta u = \frac{Du}{V}(\delta T - \delta N), \quad \delta s = \frac{Ds}{V}(\delta T + \delta N). \qquad (68)$$

In order to simplify the forthcoming deductions, we rewrite the general expression of δ N, given by equation (51), in the following manner

$$\delta N = \zeta^{ic} \sum b_{\alpha} \zeta^{2\alpha},$$

where

$$\zeta = \exp i(\tau - \tau_1),$$

and τ_1 is an arbitrary constant.

Equation (39), which can given the form

$$V \frac{d}{d\tau}\left(\frac{\delta T}{V}\right) = 2\left(\frac{d\varphi}{d\tau} + m\right)\delta N,$$

or,

$$iV \frac{d}{d\tau}\left(\frac{\delta T}{V}\right) = \left(\frac{D^2 u}{Du} - \frac{D^2 s}{Ds} + m\right)\delta N,$$

enables us to conclude that δ T has the same form as δ N although it is necessary to take into consideration that V is not an even function of

Considering equation (68), we see that δ u and δs are even functions of ζ , expandable in the following series

$$\delta u = \zeta_1^{c} \zeta \sum b_{2k} \zeta^{2k}, \quad \delta s = \zeta_1^{-c} \zeta^{-1} \sum b_{2k} \zeta^{-2k}$$

that have constant coefficients. Indeed, the sum $\delta u + \delta s = 2 \delta x$ should have a real quantity, while the difference $\delta u - \delta s$ should be an imaginary one. In this way, taking equation (16) into consideration, we can represent the solution of equations (13), that differs slightly from the variational curve, in the following manner

$$u = a \zeta \sum_k \sum_p A_{2k+pc} \zeta^{2k} \zeta_1^{pc}, \quad s = a \zeta^{-1} \sum_k \sum_p A_{-2k-pc} \zeta^{2k} \zeta_1^{pc}, \tag{69}$$

where k runs over all the integral values from $-\infty$ to $+\infty$, while p takes only the three values -1, 0 and +1. In particular, if p = 0 then $A_{2k} = a_{2k}$.

Since the general solution differs from the variational curve by finite quantities, we can then consider that the solution just given is obtained by the leading terms of a more general expansion, representing the general solution of equations (13). Following Brown, we search for the general solution in the form of the same series (69) but under the condition that p runs over all the integral values from $-\infty$ to $+\infty$. In doing this, we assume that the coefficients A_{2k+pc} are quantities of the order $|p|$ relative to some small parameter e. It is interesting to note, that Brown obtained expressions (69) for u and s by using de Pontecoulant's theory.

In order to show that there exists a solution of the type (69) for the system (13), it is necessary to be convinced in the possibility of finding such coefficients A_{2k+pc}, for which the series (69) formally satisfies conditions (13). We substitute series (69) into equations (11), which follow from equations (13). Since, on one hand,

$$D(\zeta^{2k+1}\zeta_1^{pc}) = (2k+1+pc)\zeta^{2k+1}\zeta_1^{pc},$$

and on the other hand,

$$D(\zeta^{2k+1+pc}) = (2k+1+pc)\zeta^{2k+1+pc},$$

then the equations resulting from the substitution and the equation of coefficients of equal powers of ζ, do not change if we put $\zeta_1 = \zeta$ in the series (69). It is only necessary to remember that, in the final result, the quantities $\zeta^{2k+1+pc}$ are to be replaced by

$$\exp i[(2k+1)\tau + pc(\tau - \tau_1)].$$

In this way, instead of substituting in equation (11) the series given by equations (69), we substitute series

$$u = a\sum_k\sum_p A_{2k+1+pc}\zeta^{2k+1+pc}, \quad s = a\sum_k\sum_p A_{-2k-1-pc}\zeta^{-2k-1-pc}.$$

We shall not repeat here the calculations carried out in section 137 but directly give the result of this substitution, which leads us to the following equations.

$$\sum_k\sum_p a_{2k+pc}\{[2i+qc,\ 2k+pc]A_{2i+1-2k+qc-pc} + [2i+qc]A_{2i-2k+1+qc-pc} + (2i+qc)A_{2i-2k-1+qc-pc}\} = 0. \qquad (70)$$

These equations correspond to equations (21). Starting with the above values of c and a_{2k}, we can find the solution of these equations by means of the successive approximations method. In this way, the part of the motion of the perihelion, that depends on the eccentricity of the lunar orbit, can be obtained. However, we are not going to go through the details of these calculations since they principally involve nothing new.

142. <u>Inequalities depending on the slope of the lunar orbit</u>

We have always assumed that the moon moves in an ecliptical plane and for this reason we have substituted $z = 0$ into the equations of motion, derived in section 130. We shall now investigate the variations introduced to the lunar motion on discarding this assujmption and taking into account the slope of the lunar orbit.

Neglecting as before the eccentricity of the solar orbit and consequently putting $\Omega = 0$, we obtain from equations (8) and (10) the following equations:

$$\left.\begin{aligned} D^2(us + z^2) - Du \cdot Ds - (Dz)^2 - 2m(uDs - sDu) + \\ + \frac{9}{4} m^2(u + s)^2 - 3m^2z^2 = 0 \\ D(uDs - sDu) - 2mD(us) + \frac{3}{2} m^2(u^2 - s^2) = 0 \end{aligned}\right\} \quad (71)$$

$$D^2z - m^2z - \varkappa z r^{-3} = 0. \quad (72)$$

One of the arbitrary constants involved in the general solution of equation (72) should be chosen such, that when this constant disappears the coordinate z also disappears. Denoting this constant by γ, we assume that z has γ as a multiplying factor, where γ is a small quantity of the order of the slope of the lunar orbit.

If we neglect quantities of the order of magnitude of γ^2, then equations (71) will not depend on z and will give the solutions that we, have already studied in detail in the previous section. Substituting the values of u and s, obtained this way, into equation (72), we are able to find z with an error having an order of magnitude of γ^2. Substituting this value of z into equation (71) we obtain the terms in the expansions of s and u that have the same order of magnitude as γ^2, and so on. By means of this alternate application of equations (71) and (72), we can obtain terms having an arbitrarily high order of magnitude relative to γ

in the expressions of all three coordinates u, s and z. Evidently, u and s will involve even powers of γ while z will involve odd powers.

We shall now consider in greater detail the calculation of the first-order terms of the coordinate z, assuming that the eccentricity of the lunar orbit can be set equal to zero. The corresponding solution of equations (71) is the variational curve

$$u = a\sum a_{2k}\zeta^{2k+1}, \quad s = a\sum a_{-2k-2}\zeta^{2k+1}.$$

We substitute these values of u and s into equation (72). Since

$$m^2 + \varkappa r^{-3} = m^2 + \varkappa(u^2 + s^2)^{-\frac{3}{2}}$$

is an even function of ζ, and does not change when ζ is replaced by ζ^{-1}, then

$$m^2 + \varkappa r^{-3} = 2\sum_k M_k\zeta^{2k}, \tag{73}$$

where $M_{-k} = M_k$. It is easy to see that M_k is a quantity of the same order of magnitude as $m^{|2k|}$, from which it follows that equation (72) can be reduced to the form

$$D^2 z - z\, 2\sum M_k\zeta^{2k} = 0 \tag{74}$$

or

$$\frac{d^2 z}{d\tau^2} + (2M_0 + 4M_1\cos 2\tau + 4M_2\cos 4\tau + \ldots)\, z = 0,$$

i.e. becomes identical to Hill's equation, studied in detail in sections 136-139.

Thus, the general solution of equation (74) is

$$z = C_1\zeta^{g}\sum\beta_k\zeta^{2k} + C_2\zeta^{-g}\sum\beta_k\zeta^{-2k}, \tag{75}$$

where C_1 and C_2 are arbitrary constants while the characteristic exponents g and -g are the roots of the equation

$$\sin^2 \frac{\pi g}{2} = \Delta_1 (0) \sin^2 \frac{\pi \sqrt{2M_0}}{2}, \tag{76}$$

in which the determinant Δ (0) has been obtained from the determinant Δ (0) by means of replacing q^2 by $2M_o$ and q_k by $2M_k$. Finally, the coefficients β_k are defined by the equations

$$\beta_k (g + 2k)^2 - 2 \Sigma M_{k-i} \beta_i = 0, \tag{77}$$

which correspond to equations (52).

At the end of section 134, we have seen that the following relations hold

$$\begin{gathered} \kappa a^{-3} = 1 + 2m + \frac{3}{2} m^2 + \dots \\ r = a (1 - m^2 \cos 2\tau + \dots). \end{gathered}$$

to withing terms of the order of magnitude of m^3. Consequently,

$$m^2 + \kappa r^{-3} = 1 + 2m + \frac{5}{2} m^2 + 3m^2 \cos 2\tau + \dots$$

Therefore, within the degree of accuracy desired,

$$2M_0 = 1 + 2m + \frac{5}{2} m^2, \quad 2M_1 = 2M_{-1} = \frac{3}{2} m^2,$$

while all the other coefficients M_k are equal to zero. Since the coefficients $\beta \pm 2$, $\beta \pm 3$, ... are not less than a third order, then equations (77) yield

$$\begin{aligned} &\left[1 + 2m + \frac{5}{2} m^2 - g^2\right] \beta_0 + \frac{3}{2} m^2 \beta_{-1} + \frac{3}{2} m^2 \beta_1 = 0 \\ &\left[1 + 2m + \frac{5}{2} m^2 - (g - 2)^2\right] \beta_{-1} + \frac{3}{2} m^2 \beta_0 = 0 \\ &\left[1 + 2m + \frac{5}{2} m^2 - (g + 2)^2\right] \beta_1 + \frac{3}{2} m^2 \beta_0 = 0 \end{aligned} \tag{78}$$

We can approximately obtain the value of g from these equations. Indeed the first equation shows that the following relation holds

$$g^2 = 1 + 2m + \frac{5}{2}m^2 + \dots,$$

to an error of the order of not less than m^3, from which it follows that

$$g = 1 + m + \frac{3}{4}m^2 + \dots$$

Substituting this value for g into the second of equations (78), we see that the coefficient β_{-1} only involves a first order terms, so that

$$\beta_{-1} = -\frac{3}{8}m\beta_0.$$

Therefore, it is of no consequence to keep terms of the second order in the expression of β_1. Hence we can take $\beta_1 = 0$. Thus, to an error of the order of magnitude of m^3, equation (75) yields

$$z = \beta_0\left\{\cos(g\tau + \varepsilon_1) - \frac{3}{8}m\cos[(g-2)\tau + \varepsilon_1]\right\}, \qquad (79)$$

and making use of the arbitrary nature of β_0, we can put

$$C_1 = \frac{1}{2}\exp(i\varepsilon_1), \quad C_2 = \frac{1}{2}\exp(-i\varepsilon_1).$$

The first term in formula (79) evidently corresponds to the unperturbed motion. The second term is called the evection of the latitutde. Denoting by i the slope of the lunar orbit, and by Ω the longitude of the ascending node, and using the arguments in section 119, we write

$$z = \gamma r\sin(v - \Omega),$$

where γ = tg i. Since,

$$r = a(1 - m^2\cos 2\tau), \quad v = nt + \varepsilon + \frac{11}{8}m^2\sin 2\tau + \dots$$

then, neglecting terms of order of magnitude of m^2, we obtain

$$\vartheta_0 = g(n - n')t + \varepsilon - \Omega + 90°.$$

Let us take the derivatives with respect to t on both sides of the last equation. Then

$$g(n - n') = n - \frac{d\Omega}{dt},$$

from which it follows that

$$\frac{d\Omega}{dt} = n\left(1 - \frac{g}{1 + m}\right).$$

Since, in Laplace's theory, we have put

$$\frac{d\Omega}{dt} = n(1 - g),$$

then

$$g = \frac{g}{1 + m}.$$

Equation (76) enables us to obtain thevalue of g and consequently that of g, with an arbitrary degree of accuracy. The leading terms in the expansion of g in powers of m are

$$g = 1 + \frac{3}{4} m^2 + \frac{177}{32} m^3 + \frac{1659}{2^7} m^4 + \frac{85\,205}{2^{11}} m^5 + \frac{3\,073\,531}{2^{13} \cdot 3} m^6 +$$
$$+ \frac{258\,767\,293}{2^{16} \cdot 3^2} m^7 + \frac{12\,001\,004\,273}{2^{18} \cdot 3^2} m^8 + \frac{4\,823\,236\,506\,653}{2^{23} \cdot 3^4} m^9 + \ldots$$

If a very high degree of accuracy is required, then it is recommended not to expand the coefficients g and β_k in powers of m, but to directly calculate their values numerically. In this case, it is better to obtain the expansion of the function (73) by calculating separate values for r.
Using the abovementioned expansion of the function r/a cos v and r/a sin v,

Hill obtained the following specific values for the function [illegible]

$\tau = 0°$	1.1[illegible] 57017 234[illegible]
15	1.19318 [illegible]
30	1.18[illegible] 76716
45	1.1[illegible]25 [illegible]
60	1.1[illegible]876 77987 2[illegible]687
75	1.14978 67679 [illegible]5764
90	1.14[illegible]

Using the conventional formulae of harmonic analysis, we obtain from the previous value

$$
\begin{aligned}
m^2 + \kappa r^{-3} = {} & 1.17804\ 45712\ 77166 \\
& + 0.0[illegible]\ [illegible]24\ 97860 \cos 2\tau \\
& + 0.0002[illegible]\ 15[illegible]\ [illegible]0012 \cos 4\tau \\
& + 0.00000\ 21[illegible]\ [illegible] \cos 6\tau \\
& + 0.00[illegible]\ 0022[illegible]\ [illegible]5851 \cos 8\tau \\
& + 0.00000\ [illegible]\ [illegible]750 \cos 10\tau \\
& + 0.00000\ [illegible]\ [illegible]1908 \cos 12\tau \\
& + 0.00000\ 00000\ 00017 \cos 14\tau
\end{aligned}
$$

Adam[1] was the first to obtain the value of the quantity g, that characterises the translational motion of the node of the lunar orbit but means of this method.

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(1) J.C. Adams, On the Motion of the Moon's node in the case when the Orbits of the Sun and Moon are supposed to have no eccentricities etc., Monthly Notices R.A.S., 38, 1877, 43-49. Coll. Works, 181-188.

Table I

The coefficients $C_k^{n,m}$ of the expansion in powers of the eccentricity (sec.8?)

n	m	$k=0$				$k=1$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
[illegible]	0	$+1$	$\frac{3}{2}$	$+\frac{15}{8}$	$+\frac{35}{16}$	$+3$	$\frac{27}{8}$	$+\frac{261}{64}$	$+\frac{11399}{3072}$
2	0	-1	$+\frac{1}{2}$	$\frac{3}{8}$	$+\frac{5}{16}$	$+2$	$\frac{3}{4}$	$\frac{65}{96}$	$+\frac{2675}{4608}$
1	0	$+1$	0	0	0	$+1$	$-\frac{1}{8}$	$\frac{1}{192}$	$\frac{1}{9216}$
-1	0	$+1$	$\frac{1}{2}$	0	0	-1	$+\frac{3}{8}$	$-\frac{5}{192}$	$+\frac{7}{9216}$
$+2$	0	$+1$	$+\frac{3}{2}$	0	0	2	$+\frac{1}{4}$	$\frac{1}{96}$	$+\frac{1}{4608}$
$+3$	0	$+1$	3	$+\frac{3}{8}$	0	-3	$-\frac{9}{8}$	$+\frac{15}{64}$	$-\frac{55}{3072}$
-2	2	0	0	0	0	1	$+\frac{1}{3}$	$+\frac{1}{381}$	$+\frac{77}{11520}$
-1	2	0	$\frac{1}{4}$	$\frac{1}{8}$	$+\frac{5}{64}$	$-\frac{3}{2}$	$+\frac{13}{12}$	$+\frac{103}{768}$	$+\frac{31}{210}$
0	2	0	$+\frac{3}{4}$	$\frac{1}{8}$	$+\frac{3}{64}$	2	$+\frac{11}{6}$	$-\frac{[illegible]}{192}$	$+\frac{751}{11520}$
$+1$	2	0	$-\frac{3}{2}$	0	0	$-\frac{5}{2}$	$+\frac{25}{12}$	$\frac{187}{768}$	$\frac{29}{2880}$
$+2$	2	0	$+\frac{5}{2}$	0	0	-3	$+\frac{4}{3}$	$\frac{37}{384}$	$\frac{11}{3840}$
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
2	1	0	0	0	0	$+1$	$\frac{3}{8}$	$\frac{5}{192}$	$-\frac{7}{9216}$
1	1	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{16}$	$-\frac{5}{128}$	1	$-\frac{9}{8}$	$-\frac{5}{64}$	$\frac{889}{9216}$
0	1	-1	0	0	0	$+1$	$-\frac{9}{8}$	$+\frac{25}{192}$	$\frac{49}{9216}$
-1	1	$-\frac{3}{2}$	0	0	0	$+1$	$-\frac{3}{8}$	$+\frac{5}{192}$	$\frac{7}{9216}$
2	1	-2	$\frac{1}{2}$	0	0	$+1$	$+\frac{9}{8}$	$\frac{25}{64}$	$\frac{245}{9216}$

Table I - (continued)

n	m	$k = 2$: e^0	e^2	e^4	e^6	$k = 3$: e	e^3	e^5	e^7
3	0	0	$+\frac{9}{2}$	$+\frac{7}{2}$	$+\frac{111}{32}$	0	$+\frac{53}{8}$	$+\frac{393}{128}$	$\frac{21753}{5120}$
2	0	0	$+\frac{5}{2}$	$+\frac{1}{3}$	$+\frac{21}{32}$	0	$\frac{13}{4}$	$-\frac{25}{64}$	$+\frac{393}{512}$
1	0	0	$+1$	$-\frac{1}{3}$	$+\frac{1}{24}$	0	$\frac{9}{8}$	$\frac{81}{128}$	$+\frac{729}{5120}$
+1	0	0	$\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{16}$	0	$\frac{3}{8}$	$+\frac{45}{128}$	$\frac{567}{5120}$
+2	0	0	$-\frac{1}{2}$	$+\frac{1}{6}$	$-\frac{1}{48}$	0	$\frac{1}{4}$	$+\frac{9}{64}$	$-\frac{81}{560}$
+3	0	0	0	$-\frac{1}{2}$	$+\frac{3}{16}$	0	$+\frac{1}{8}$	$-\frac{45}{128}$	$+\frac{189}{1024}$
−2	2	+1	$-\frac{7}{2}$	$+\frac{85}{48}$	$\frac{319}{1440}$	+3	$\frac{69}{8}$	$+\frac{3663}{640}$	$-\frac{3597}{2560}$
−1	2	+1	−4	$+\frac{47}{16}$	$-\frac{13}{36}$	$+\frac{5}{2}$	$-\frac{131}{16}$	$+\frac{8861}{1280}$	$-\frac{9921}{5120}$
0	2	+1	−4	$+\frac{167}{18}$	$-\frac{503}{720}$	+2	$\frac{27}{4}$	$+\frac{2079}{320}$	$\frac{1427}{640}$
+1	2	+1	$-\frac{7}{2}$	$+\frac{71}{24}$	$-\frac{551}{720}$	$+\frac{3}{2}$	$\frac{75}{16}$	$+\frac{5751}{1280}$	$-\frac{8829}{5120}$
+2	2	+1	$\frac{5}{2}$	$+\frac{11}{8}$	$-\frac{179}{720}$	+1	$-\frac{19}{8}$	$+\frac{1053}{640}$	$\frac{243}{512}$
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
−2	1	+2	$\frac{4}{3}$	$+\frac{1}{4}$	$-\frac{1}{15}$	0	$\frac{27}{8}$	$-\frac{405}{128}$	$+\frac{5103}{5120}$
−1	1	$+\frac{3}{2}$	$-\frac{11}{6}$	$+\frac{9}{32}$	$-\frac{9}{80}$	0	$\frac{17}{8}$	$-\frac{385}{128}$	$+\frac{5201}{5120}$
0	1	+1	$-\frac{4}{3}$	$+\frac{3}{8}$	$-\frac{2}{45}$	0	$\frac{9}{8}$	$-\frac{225}{128}$	$+\frac{3969}{5120}$
+1	1	$+\frac{1}{2}$	$-\frac{1}{3}$	$+\frac{1}{16}$	$-\frac{1}{180}$	0	$+\frac{3}{8}$	$-\frac{45}{128}$	$+\frac{567}{5120}$
+2	1	0	$+\frac{2}{3}$	$-\frac{3}{8}$	$\frac{1}{15}$	0	$\frac{1}{8}$	$\frac{75}{128}$	$\frac{141}{1024}$

Table I - (continued)

n	m	$k = 4$			$k = 5$		
		e^4	e^5	e^6	e^5	e^6	e^7
3	0	0	$\frac{77}{8}$	$+\frac{124}{60}$	0	$+\frac{1773}{128}$	$\frac{4967}{3072}$
2	0	0	$\frac{103}{24}$	$\frac{129}{80}$	0	$+\frac{1097}{192}$	$-\frac{16\,521}{4608}$
−1	0	0	$+\frac{4}{3}$	$\frac{16}{15}$	0	$+\frac{625}{384}$	$\frac{15\,625}{9216}$
−1	0	0	$-\frac{1}{3}$	$+\frac{2}{5}$	0	$-\frac{125}{384}$	$-\frac{4375}{9216}$
+2	0	0	$-\frac{1}{6}$	$+\frac{2}{15}$	0	$\frac{25}{192}$	$-\frac{625}{4608}$
3	0	0	$+\frac{1}{8}$	$-\frac{3}{10}$	0	$+\frac{15}{128}$	$-\frac{875}{3072}$
−2	2	$+\frac{13}{2}$	$-\frac{55}{3}$	$+\frac{10\,723}{720}$	$+\frac{295}{24}$	$\frac{13\,745}{384}$	$+\frac{1\,102\,775}{32\,256}$
1	2	$\frac{19}{4}$	$-\frac{121}{8}$	$+\frac{10\,597}{720}$	$+\frac{389}{48}$	$\frac{20\,267}{768}$	$+\frac{626\,681}{21\,504}$
	2	$\frac{13}{4}$	$\frac{259}{24}$	$+\frac{8401}{720}$	$+\frac{59}{12}$	$-\frac{3221}{192}$	$\frac{163\,363}{8064}$
1	2	$+2$	$\frac{19}{3}$	$+\frac{62}{9}$	$+\frac{125}{18}$	$-\frac{6625}{768}$	$+\frac{679\,375}{64\,512}$
2	2	1	$-\frac{5}{2}$	$+\frac{94}{45}$	$+\frac{25}{24}$	$\frac{1075}{384}$	$\frac{29\,375}{10\,752}$
		e^3	e^5	e^7	e^4	e^6	—
−2	1	$\frac{16}{3}$	$-\frac{32}{5}$	$+\frac{128}{45}$	$+\frac{3125}{384}$	$\frac{169\,375}{9216}$	—
1	1	$+\frac{71}{24}$	$\frac{387}{80}$	$\frac{387}{160}$	$+\frac{523}{128}$	$\frac{70\,273}{9216}$	—
0	1	$+\frac{4}{3}$	$\frac{12}{5}$	$+\frac{64}{45}$	$+\frac{625}{384}$	$-\frac{30\,625}{9216}$	—
1	1	$\frac{1}{3}$	$\frac{2}{5}$	$+\frac{8}{45}$	$+\frac{125}{384}$	$-\frac{4375}{9216}$	—
2	1	$\frac{1}{6}$	$\frac{3}{5}$	$-\frac{8}{15}$	$-\frac{25}{128}$	$+\frac{6125}{9216}$	—

Table I- (continued)

n	m	$k = 6$		$k = 7$		$k = 8$	$k = 9$
		e^4	e^6	e^5	e^7	e^6	e
3	0	0	$+\frac{3167}{160}$	0	$+\frac{432\,091}{15\,360}$	0	0
−2	0	0	$+\frac{1223}{160}$	0	$+\frac{47\,273}{4608}$	0	0
1	0	0	$+\frac{81}{40}$	0	$\frac{117\,649}{46\,080}$	0	0
1	0	0	$-\frac{27}{80}$	0	$-\frac{16\,807}{46\,080}$	0	0
+2	0	0	$-\frac{9}{80}$	0	$-\frac{2401}{23\,040}$	0	0
+3	0	0	$+\frac{9}{80}$	0	$+\frac{343}{3072}$	0	0
2	2	$+\frac{345}{16}$	$-\frac{10\,569}{160}$	$+\frac{69\,251}{1920}$	$-\frac{5\,391\,109}{46\,080}$	$+\frac{42\,037}{720}$	$+\frac{3\,306\,[illegible]}{35\,840}$
1	2	$+\frac{209}{16}$	$-\frac{887}{20}$	$+\frac{8\,077}{3840}$	$-\frac{2\,228\,929}{30\,720}$	$+\frac{17\,807}{576}$	$+\frac{3\,313\,213}{71\,680}$
0	2	$+\frac{115}{16}$	$-\frac{2049}{80}$	$+\frac{98[illegible]3}{960}$	$-\frac{889\,303}{23\,040}$	$+\frac{42\,037}{2880}$	$+\frac{367\,439}{17\,920}$
1	2	$+\frac{27}{8}$	$-\frac{189}{16}$	$+\frac{16\,807}{3840}$	$\frac{1\,495\,823}{92\,160}$	$+\frac{256}{45}$	$+\frac{531\,441}{71\,680}$
2	2	$+\frac{9}{[illegible]}$	$\frac{261}{80}$	$\frac{2401}{1920}$	$-\frac{12\,005}{3072}$	$\frac{64}{45}$	$+\frac{59\,049}{35\,840}$
		e^3	e^5	e^4	—	e^5	—
2	1	$\frac{243}{20}$	$-\frac{729}{35}$	$+\frac{823\,543}{46\,080}$	—	$\frac{8192}{315}$	
1	1	$+\frac{899}{160}$	$-\frac{6617}{560}$	$+\frac{355\,[illegible]}{46\,080}$		$+\frac{47\,259}{4480}$	
0	1	$\frac{81}{40}$	$-\frac{162}{35}$	$+\frac{117\,649}{46\,080}$	—	$\frac{1[illegible]21}{315}$	
+1	1	$\frac{27}{80}$	$-\frac{81}{140}$	$+\frac{16\,807}{46\,080}$	—	$\frac{128}{315}$	
+2	1	$-\frac{9}{40}$	$+\frac{27}{35}$	$-\frac{2401}{9216}$	...	$\frac{32}{105}$	

Table II

The coefficients $S_k^{n,m}$ of the expansion in powers of the eccentricity (sec. 82)

n	m	$k=1$				$k=2$			
		e^0	e^2	e^4	e^6	e	e^3	e^5	e^7
2	1	-1	$\frac{5}{8}$	$\frac{11}{192}$	$\frac{457}{9216}$	$+2$	$\frac{5}{3}$	$\frac{1}{6}$	$\frac{1}{15}$
1	1	$+1$	$\frac{7}{8}$	$+\frac{3}{64}$	$\frac{199}{9216}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{35}{[illegible]}$	$\frac{13}{210}$
0	1	$+1$	$\frac{7}{8}$	$\frac{17}{192}$	$\frac{271}{9216}$	$+1$	$\frac{7}{6}$	$\frac{1}{3}$	$-\frac{19}{360}$
+1	1	$+1$	$\frac{5}{8}$	$-\frac{11}{192}$	$-\frac{457}{9216}$	$+\frac{1}{2}$	$\frac{5}{12}$	$+\frac{1}{24}$	$-\frac{1}{45}$
+2	1	$+1$	$\frac{1}{8}$	$-\frac{25}{64}$	$-\frac{613}{9216}$	0	$+\frac{1}{3}$	$\frac{7}{24}$	$-\frac{3}{80}$
+3	1	$+1$	$-\frac{5}{8}$	$-\frac{151}{192}$	$-\frac{1387}{9216}$	$\frac{1}{2}$	$+\frac{5}{6}$	$-\frac{17}{48}$	$+\frac{47}{2440}$
−2	3	0	$\frac{7}{8}$	$\frac{85}{384}$	$-\frac{237}{5120}$	2	$+\frac{9}{2}$	$-\frac{103}{60}$	$\frac{2}{15}$
1	3	0	$+\frac{13}{8}$	$-\frac{329}{384}$	$\frac{2009}{15360}$	$-\frac{5}{2}$	$+\frac{13}{2}$	$\frac{1709}{480}$	$+\frac{1}{40}$
0	3	0	$-\frac{21}{8}$	$-\frac{243}{128}$	$-\frac{339}{5120}$	-3	$\frac{33}{4}$	$-\frac{111}{[illegible]}$	$-\frac{13}{15}$
+1	3	0	$-\frac{31}{8}$	$-\frac{1213}{384}$	$+\frac{143}{1024}$	$-\frac{7}{[illegible]}$	$+\frac{75}{8}$	$-\frac{821}{120}$	$\frac{22}{15}$
+2	3	0	$+\frac{43}{8}$	$-\frac{1637}{384}$	$+\frac{3397}{15360}$	-4	$\frac{19}{2}$	$-\frac{791}{120}$	$-\frac{37}{24}$
		e	e^3	e^5	e^7	e^0	e^2	e^4	e^6
−2	2	-1	$+\frac{5}{12}$	$+\frac{9}{128}$	$+\frac{1321}{23040}$	$+1$	$-\frac{[illegible]}{2}$	$+\frac{89}{48}$	$-\frac{211}{2440}$
1	2	$\frac{[illegible]}{2}$	$\frac{25}{24}$	$+\frac{15}{256}$	$+\frac{971}{15360}$	1	-4	$\frac{47}{16}$	$\frac{29}{72}$
0	2	2	$\frac{5}{3}$	$\frac{5}{64}$	$\frac{151}{2880}$	$+1$	4	$+\frac{163}{48}$	$\frac{527}{72}$
+1	2	$\frac{5}{2}$	$-\frac{43}{24}$	$\frac{35}{256}$	$+\frac{943}{4608}$	1	$-\frac{7}{2}$	$\frac{35}{12}$	$\frac{131}{180}$
+2	2	3	$\frac{23}{12}$	$\frac{19}{128}$	$\frac{471}{2560}$		$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{73}{360}$

Table II- (continued)

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l	m	$k=3$: e	e^2	e^3	e^4	$k=4$: e	e^2	e^3	e^4
−2	1	0	27/8	159/128	1[illegible]7/5120	0	16/3	104/15	26/9
1	1	0	17/8	467/128	5489/5120	0	21/24	−1129/240	1177/480
0	1	0	9/8	207/128	+3081/5120	0	−4/3	35/15	121/90
−1	1	0	3/8	51/128	543/5120	0	−1/3	13/30	+13/72
+2	1	0	1/8	+53/128	253/1024	0	1/6	−29/60	107/210
+3	1	0	3/8	+81/128	168[illegible]/5120	0	7/24	131/240	203/576
2	3	+1	17/[illegible]	951/64	38 947/5120	+1	−21	241/6	5309/210
1	3	+1	9	+1143/64	11 689/1024	7/2	179/8	2009/48	−103 781/3360
0	3	+1	−9	+1215/64	71 813/5120	+3	51/2	155/4	26 843/810
−1	3	+1	17/2	+1143/64	72 131/5120	+5/2	63/4	757/24	15 387/560
−2	3	+1	15/2	+927/64	56 599/5120	+2	23/2	65/3	31 013/1680
		e	e^2	e^3	e^4	e^2	e^3	e^4	
2	2	+3	69/8	+3717/640	213/160	13/2	55/3	1073/720	
1	2	5/2	131/16	+8879/1280	[illegible]/2560	19/4	121/8	10 613/720	
0	2	+2	27/4	2061/320	[illegible]/11 520	13/4	259/24	+8509/720	—
−1	2	3/2	75/16	5709/1280	1353/2560	2	19/3	247/16	—
−2	2	+1	13/8	1087/640	59/128	+1	1/2	163/360	—

Table II- (continued)

n	m	k = 5			k = 6		
		e^3	e^4	e^5	e^4	e^5	e^6
2	1	0	$+\frac{3125}{384}$	$\frac{115\,625}{9216}$	0	$+\frac{899}{160}$	$\frac{817}{70}$
−1	1	0	$+\frac{523}{128}$	$-\frac{69\,023}{9216}$	0	$+\frac{243}{20}$	$-\frac{1215}{56}$
0	1	0	$+\frac{625}{384}$	$\frac{29\,375}{9216}$	0	$+\frac{81}{40}$	$\frac{2511}{560}$
+1	1	0	$+\frac{125}{384}$	$\frac{4625}{9216}$	0	$+\frac{27}{80}$	$-\frac{135}{224}$
+2	1	0	$-\frac{25}{128}$	$+\frac{5275}{9216}$	0	$\frac{9}{40}$	$+\frac{387}{560}$
+3	1	0	$\frac{95}{384}$	$+\frac{4765}{9216}$	0	$-\frac{9}{40}$	$+\frac{117}{224}$
−2	3	$+\frac{85}{8}$	$\frac{1355}{24}$	$\frac{98\,525}{1024}$	$\frac{47}{2}$	$-\frac{239}{2}$	$-\frac{33\,951}{160}$
−1	3	$+\frac{67}{8}$	$\frac{2279}{48}$	$+\frac{274\,315}{3072}$	$+17$	$-\frac{2049}{32}$	$+\frac{57\,213}{320}$
0	3	$+\frac{51}{8}$	$-\frac{593}{16}$	$+\frac{75\,643}{1024}$	$+\frac{47}{4}$	$\frac{525}{8}$	$+\frac{43\,011}{320}$
+1	3	$+\frac{37}{8}$	$-\frac{635}{24}$	$+\frac{54\,765}{1024}$	$-\frac{61}{8}$	$\frac{677}{16}$	$+\frac{56\,487}{640}$
+2	3	$+\frac{25}{8}$	$\frac{50}{3}$	$\frac{98\,875}{3072}$	$+\frac{9}{2}$	$-\frac{169}{8}$	$+\frac{3807}{80}$
		e^5	e^6	e^7	e^4	e^5	
−2	2	$+\frac{295}{24}$	$-\frac{13\,745}{384}$	$+\frac{276\,475}{8064}$	$+\frac{345}{16}$	$-\frac{10569}{160}$	—
−1	2	$+\frac{389}{48}$	$-\frac{20\,267}{768}$	$+\frac{101\,551}{3584}$	$+\frac{209}{16}$	$-\frac{887}{20}$	—
0	2	$+\frac{59}{12}$	$-\frac{3221}{192}$	$+\frac{326\,101}{16\,128}$	$+\frac{115}{16}$	$-\frac{2049}{80}$	—
+1	2	$+\frac{125}{48}$	$-\frac{6625}{768}$	$+\frac{338\,875}{32\,256}$	$+\frac{27}{8}$	$-\frac{189}{16}$	
+2	2	$+\frac{25}{24}$	$-\frac{1075}{384}$	$+\frac{925}{336}$	$+\frac{9}{8}$	$-\frac{261}{80}$	—

Table II- (continued)

n	m	$k = 7$		$k = 8$		$k = 9$	$k = 10$
		c^6	c^7	c	c^2	c^6	c^1
2	1	0	$\frac{823543}{46080}$	0	$\frac{8192}{315}$	0	0
1	1	0	$+\frac{355081}{46080}$	0	$+\frac{17249}{4480}$	0	0
0	1	0	$+\frac{117649}{46080}$	0	$+\frac{1024}{315}$	0	0
+1	1	0	$+\frac{16807}{46080}$	0	$+\frac{128}{315}$	0	0
+2	1	0	$\frac{2401}{9216}$	0	$-\frac{32}{105}$	0	0
+3	1	0	$-\frac{9917}{46080}$	0	$-\frac{68}{315}$	0	0
−2	3	$+\frac{17969}{384}$	$-\frac{301973}{1280}$	$+\frac{2611}{30}$	$-\frac{39893}{90}$	$+\frac{790053}{5120}$	$+\frac{532315}{2016}$
−1	3	$+\frac{12015}{384}$	$\frac{32419}{192}$	$\frac{26371}{480}$	$\frac{1710983}{5760}$	$+\frac{471527}{5120}$	$+\frac{[illegible]}{4032}$
0	3	$+\frac{2567}{128}$	$-\frac{35563}{320}$	$+\frac{2011}{80}$	$-\frac{87599}{480}$	$+\frac{263351}{5120}$	$+\frac{[illegible]}{1344}$
+1	3	$+\frac{4753}{384}$	$-\frac{84169}{1280}$	$+\frac{8551}{480}$	$\frac{288221}{2880}$	$+\frac{26809}{1024}$	$+\frac{305593}{[illegible]}$
+2	3	$+\frac{2101}{384}$	$-\frac{127253}{3840}$	$+\frac{128}{15}$	$\frac{416}{9}$	$+\frac{59[illegible]49}{5120}$	$+\frac{15625}{1008}$
		c^1	c^7	c^8		c^7	
2	2	$+\frac{69251}{1920}$	$-\frac{5794109}{46080}$	$+\frac{42047}{720}$		$+\frac{3306951}{[illegible]}$	
−1	2	$+\frac{78077}{3840}$	$\frac{2228929}{30720}$	$+\frac{17807}{576}$	−	$+\frac{3313213}{[illegible]}$	−
0	2	$+\frac{9803}{960}$	$-\frac{889203}{23040}$	$+\frac{42037}{2880}$	−	$+\frac{367439}{17920}$	
+1	2	$+\frac{16807}{3840}$	$-\frac{1497823}{92160}$	$+\frac{256}{45}$		$+\frac{531441}{71680}$	
+2	2	$+\frac{2101}{1920}$	$-\frac{12005}{3072}$	$+\frac{64}{45}$	−	$+\frac{59019}{35840}$	

Table III

The Enke function and its logarithm (section 96)

q	f				$\lg f$				q
	$q>0$		$q<0$		$q>0$		$q<0$		
0.000	3.0000	75	3.0000	75	0.47712	108	0.47712	109	0.000
0.001	2.9925	74	3.0075	76	0.47604	109	0.47821	109	0.001
0.002	2.9851	74	3.0151	76	0.47495	108	0.47930	109	0.002
0.003	2.9777	74	3.0227	76	0.47387	107	0.48039	109	0.003
0.004	2.9703	74	3.0303	76	0.47280	108	0.48148	110	0.004
0.005	2.9629	73	3.0379	77	0.47172	107	0.48258	110	0.005
0.006	2.9556	73	3.0456	78	0.47065	107	0.48368	110	0.006
0.007	2.9483	72	3.0534	77	0.46958	107	0.48478	110	0.007
0.008	2.9411	72	3.0611	78	0.46851	107	0.48588	111	0.008
0.009	2.9339	72	3.0689	79	0.46744	106	0.48699	111	0.009
0.010	2.9267	71	3.0768	79	0.46638	106	0.48810	111	0.010
0.011	2.9196	71	3.0847	79	0.46532	106	0.48921	111	0.011
0.012	2.9125	71	3.0926	79	0.46426	106	0.49032	112	0.012
0.013	2.9054	71	3.1005	80	0.46320	105	0.49144	112	0.013
0.014	2.8983	70	3.1085	81	0.46215	106	0.49256	112	0.014
0.015	2.8913	70	3.1166	81	0.46109	105	0.49368	112	0.015
0.016	2.8843	69	3.1247	81	0.46004	104	0.49480	113	0.016
0.017	2.8774	69	3.1328	81	0.45900	105	0.49593	113	0.017
0.018	2.8705	69	3.1409	82	0.45795	104	0.49706	113	0.018
0.019	2.8636	69	3.1491	82	0.45691	105	0.49819	113	0.019
0.020	2.8567	68	3.1573	83	0.45586	104	0.49932	114	0.020
0.021	2.8499	68	3.1656	83	0.45482	104	0.50046	113	0.021
0.022	2.8431	68	3.1739	84	0.45378	104	0.50159	115	0.022
0.023	2.8363	67	3.1823	84	0.45275	103	0.50271	114	0.023
0.024	2.8296	67	3.1907	84	0.45172	103	0.50385	115	0.024
0.025	2.8229	67	3.1991	85	0.45069	103	0.50503	115	0.025
0.026	2.8162	67	3.2076	85	0.44966	103	0.50618	115	0.026
0.027	2.8095	66	3.2161	85	0.44863	102	0.50733	115	0.027
0.028	2.8029	66	3.2246	86	0.44761	102	0.50848	116	0.028
0.029	2.7963	66	3.2332	87	0.44659	102	0.50964	116	0.029
0.030	2.7897		3.2419		0.44557		0.51080		0.030

Table IV

Different constants used in computing the perturbations

		Value	logarithm
The radius of the arc in degrees		57°.2957 7951	1.758 1226 824
The radius of the arc in minutes		3 437'.746771	3.536 2738 828
The radius of the arc in seconds		206 264".806247	5.314 4251 332
The gaussian constant	k	0.017 20209895	8.235 5814 414 − 10
	k''	3548".18761	3.550 0066
	k^2	0.000 295 9122	6.471 1629 − 10

Intervals ω:		10^d	20^d	40^d	80^d
Planet	$\frac{1}{m}$	The upper line gives lg $(\omega^2 k^2 m)$ The lower line gives, lg $(10^7 \omega^4 k^4 m)$			
Mercury	6 000 000	7.7719−10 8.6939−10	8.0729 −10 9.2951 −10	8.3739 −10 9.8972 −10	8.6749 −10 0.4993
Venus	408 000	8.9394−10 9.6605−10	9.2404−10 0.4626	9.5414 −10 1.0646	9.8424 −10 1.6667
Earth + Moon	329 390	9.0323 −10 9.9534 −10	9.3333 −10 0.5555	9.6344 −10 1.1576	9.9354 −10 1.7596
Mars	3 093 500	8.0596 −10 8.9807 −10	8.3606−10 9.5828 −10	8.6616 −10 0.1848	8.9626 −10 0.7869
Jupiter	1047.355	1.529 013 2.451 069	1.830 943 3.053 129	2.131 973 3.655 189	2.433 003 4.257 249
Saturn	3501.6	1.60571 1.92600	1.90677 2.52896	1.65780 3.13102	1.90883 3.733[illegible]8
Uranus	22 869	0.1908 1.1119	0.4918 1.7140	0.7928 2.3160	1.0938 2.9181
Neptune	19 700	0.2556 1.1767	0.5566 1.7788	0.8576 2.3808	1.1586 2.9829
Pluto	350 000	– –	–	–	9.9070−10 1.733
	$\omega^2 k^2$ =	0.029 5912	0.118 365	0.473 460	1.893 838
	lg $(\omega^2 k^2)$ =	8.471 163 −10	9.073 223 −10	9.675 283 −10	0.277 343