V.M. KAMENKOVICH

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FUNDAMENTALS OF OCEAN DYNAMICS

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PREFACE TO 1973 EDITION

In essence, this book is an introduction to the theory of ocean currents. I have not made an attempt to cover the widest possible range of problems and to survey the very diverse and numerous studies undertaken in recent years. In contrast, I wanted to select and study thoroughly only a minimum amount of material which is required for an understanding of the features of the large-scale motions of the water of the oceans.

First of all, I have endeavoured to demonstrate the link between the basic concepts of the theory of ocean currents and the general laws of hydromechanics; therefore the book contains a brief exposition of the foundations of thermo-dynamics and hydro-dynamics of non-homogeneous fluids as well as necessary prerequisites of tensor analysis. In describing different hydrodynamic models of oceanic phenomena, special attention is given to a discussion of the physical premises of the theory, methods of mathematical analysis and comparison of theoretical results with observations. In principle, all derivations are given of the results considered in this book.

The contents of this book are clear from its chapter headings. For the sake of brevity, this book does not deal at all with the very important problems of numerical modelling of ocean currents; it also does not contain a historical survey of the development of the theory. Interested readers will find expositions of these aspects in published surveys and monographs by A.E. Gill [27], V.F. Kozlov [55], V.P. Kochergin [52], S. Manabe and K. Bryan [73], P.S. Lineikin [66, 67], G.I. Marchuk [74], A. Robinson [103–105], A.S. Sarkisyan [107], H. Stommel [115,117], A.I. Fel'zenbaum [18,19], N. Fofonoff [21], W.B. Stockmann [113,114] and V.V. Shuleikin [109].

The literature listed in this book in no way claims completeness. Literature references of a historical character are appended to each chapter; the text only contains references which are required for understanding the treatment.

This book is based on lectures read by me beginning from 1966 to students of the fourth year course at Moscow Physical-Technical Institute which specialized in oceanology.

I wish to thank all those who in one way or another assisted in the production of this book. Over a number of years, I had opportunities to discuss with my colleagues and friends a wide range of problems of ocean dynamics, and the results of such discussions found their reflections in this book. With special gratitude I wish to acknowledge conversations with W.B. Stokmann which had the greatest significance for me.

While working on this book, I often consulted A.S. Monin whose suggestions contributed essentially to the writing of this book. Observations by B.A. Kagan on the general lay-out of the book as well as on individual ques-

tions were rather valuable for me. Useful remarks and suggestions originated from V.A. Mitrofanov who read the complete manuscript, and likewise from G.M. Reznik and V.D. Larichev who read it in parts. I received great help in preparing the manuscript from L.I. Lavrishcheva, E.P. Belova and T.A. Yakusheva.

V.M. KAMENKOVICH

PREFACE TO ENGLISH EDITION 1977

For this edition, V.M. Kamenkovich revised completely Sections 8 of Chapter 3 and 7 of Chapter 5, wrote a new treatment on matching of asymptotic expansions (Appendix B), included additional material to Section 1 of Chapter III and made a large number of minor changes and additions. Cooperation over almost the largest possible distance on the surface of the Earth has been very close and successful.

R. RADOK

CONTENTS

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CHAPTER 1

STATICS. THERMODYNAMICS OF EQUILIBRIUM STATES

1.1 ENTROPY

The most important characteristic of a thermodynamic system is its entropy. More complete physical significance of entropy, as also of other thermodynamic parameters, is brought to light in statistical physics which deals with macroscopic properties of systems from the point of view of their microstructure.

From a microstructural point of view, a thermodynamic system consists of a huge number of particles (molecules, atoms, ions) the state of which is described by quantum mechanics. Each definite quantum state of a system of particles may be considered as a possible microstate of the thermodynamic system.

Consider an isolated system in a state of thermodynamic equilibrium. Without entering into the details of the description of quantum states of systems of particles, it may be asserted that to a given microstate of a thermodynamic system there correspond a finite number W of possible (compatible with given macroscopic conditions) microstates.

Statistical physics starts from the assumption that each possible microstate of a system i has a definite probability of its realization P_i. As a rule, this assumption is true for systems consisting of large numbers of particles. Obviously,

$$\sum_{1}^{W} P_i = 1 \,. \tag{1.1.1}$$

It is not difficult to define what must be understood by macroscopic parameters of a system, or functions of state of a system. For example, let ϵ_i be the internal energy of a system in the microstate i (the total mechanical energy of the particles constituting the system). Then the internal energy of the system ϵ as a function of state is given by

$$\epsilon = \sum_{1}^{W} P_i \epsilon_i = \langle \epsilon_i \rangle \,.$$

In other words, the quantity ϵ is the mathematical expectation of the stochastic quantities ϵ_i.

In essence, specification of the probabilities P_i completely characterizes that possible choice of microstates which corresponds to given macroscopic conditions. In order to have some "mean" measure of the molecular "disorder", one introduces the concept of entropy of a thermodynamic system η which is defined as

$$\eta = -k \sum_{1}^{W} P_i \ln P_i \,, \tag{1.1.2}$$

where k is Boltzmann's constant ($k = 1.38 \cdot 10^{-13}$ J/°C); the choice of the constant k as factor in (1.1.2) is only dictated by considerations of convenience.

In practice, entropy is more commonly denoted by S. However, in oceanographical literature, this symbol belongs by tradition to salinity; therefore entropy will be denoted here by η. It is interesting to note that the symbol η was, in fact, used by Gibbs [26].

In reality, when a system with certainty lies in a unique quantum state (ideal "order"), then $\eta = 0$. If the number of admissible microstates is fixed, then it is intuitively clear that the maximum "disorder" in the system will be under conditions of equal probability of all such microstates. This is in agreement with the measure of "disorder" introduced above, since the function $\eta(P_1, ..., P_W)$ attains a maximum under the condition (1.1.1) indeed when

$$P_1 = P_2 = ... = P_W = \frac{1}{W} \,;$$

then the magnitude of the maximum entropy η is

$$\eta = k \ln W \,. \tag{1.1.3.}$$

The definition of entropy, in accordance with (1.1.2), has been introduced for an arbitrary macrostate of a system. An attempt will now be made to single out equilibrium states which are the simplest among possible macrostates of a system; in fact, it will be assumed that in an equilibrium state a system achieves maximum molecular "disorder". Then the probabilities of all admissible microstates must be the same and the entropy of the equilibrium system will be determined by (1.1.3). It must be emphasized here that the definition of entropy, in accordance with (1.1.2) or (1.1.3), is not linked directly to such physical quantities as energy, interaction forces, etc.

In this book, electromagnetic processes (radiation, etc.) will not be considered. Therefore, in what follows, it is assumed that the macrostate of an equilibrium system of mass m is completely determined, if its internal energy ϵ, its volume V and the masses m_1, ..., m_n of the different substances constituting it (components of the system) are specified.

Sometimes the numbers of moles n_j or the numbers of particles (molecules, atoms, ions) N_j of the given components of a mixture are specified. Obviously, $n_j = m_j/M_j$, where M_j is the gram-molecular weight of the component j (m_j in grams). Also, $n_j/N_A = N_j$, where N_A is Avogadro's number equal to $6.02 \cdot 10^{23}$ mole^{-1}.

Having computed the number W of admissible microstates of a system in an equilibrium macrostate with parameters ϵ, V and m_j, one determines then the entropy of such a system as a single-valued function of the parameters ϵ, V and m_j from (1.1.3).

The concept of entropy of an equilibrium thermodynamic system permits to introduce into the consideration such parameters as the absolute temperature of the system T and the chemical potentials of the separate components of the system μ_j. By definition, one sets

$$\frac{1}{T} = \left(\frac{\partial \eta}{\partial \epsilon}\right)_{V, m_j}, \qquad \mu_j = -T\left(\frac{\partial \eta}{\partial m_j}\right)_{\epsilon, V, m_i}, \qquad j = 1, \ldots, n\,. \tag{1.1.4}$$

The subscripts V and m_j, for example, on the derivative $\partial\eta/\partial\epsilon$ indicate that during differentiation of the function η with respect to ϵ the parameters V and m_j remain unchanged. In this notation, it is immediately clear which are the independent variables. This notation will be employed whenever the independent variables are not especially specified.

It will be useful to apply this scheme to the simplest thermodynamic system, a mixture of two mono-atomic ideal gases. Consider a gas enclosed in a volume V which consists of N_1 atoms of the first kind and N_2 atoms of the second kind. The total number of atoms is then $N = N_1 + N_2$. Under normal conditions, the dynamics of systems of such atoms may be described by the laws of classical mechanics, assuming each atom to be a material point (mono-atomic gas) and neglecting the potential energy of interaction between atoms (ideal gas). Then the system will possess $3N$ degrees of freedom. Let q_1, q_2, q_3 be the Cartesian coordinates of the first atom and p_1, p_2, p_3 the associated impulses; q_4, q_5, q_6 and p_4, p_5, p_6 the Cartesian coordinates and associated impulses of the second atom, etc. At each instant of time, the state of the system of atoms is completely determined by specification of $6N$ numbers ($3N$ coordinates q_i and $3N$ impulses p_i). This is a microstate of the system which is conveniently represented as the point M with coordinates $(q_1, \ldots, q_{3N}; p_1, \ldots, p_{3N})$ in the $6N$-dimensional phase space of the system Γ.

Let the mass of the atoms of the first kind be equal to ν_1, that of the atoms of the second kind ν_2. Then, since $p_1 = \nu_1(\mathrm{d}q_1/\mathrm{d}t)$, $p_2 = \nu_1(\mathrm{d}q_2/\mathrm{d}t)$, etc., the Hamiltonian (total energy) of the system has the form

$$H = \sum_{1}^{3N_1} \frac{p_i^2}{2\nu_1} + \sum_{3N_1+1}^{3N} \frac{p_i^2}{2\nu_2} + U(q_i, V)\,. \tag{1.1.5}$$

In this expression, the potential energy $U(q_i, V)$ of the system has been introduced, in order to take into account the fact that the atoms of the gas may not leave the volume V. It is simplest to assume that for each atom $U = 0$ inside the volume V and $U = \infty$ outside this volume. Note that, by definition, the quantity H is the internal energy of the system in the microstate under consideration and that, by (1.1.5), one has

$$H(q_i, p_i, V) \geqslant 0 .$$

In correspondence with the general scheme, consider now the equilibrium state of an isolated system with parameters ϵ, V, m_j. How does one compute the number of possible microstates of the system W? Although the dynamics of the system of atoms are classical, quantum effects must be taken into account when computing W. It is a fact that, firstly, by strength of the uncertainty principle of the classical state contained in the volume h^{3N} of phase state (h is Planck's constant), they blend in a single quantum state and, secondly, that in quantum mechanics identical atoms are indistinguishable and $N_1!N_2!$ classical states likewise merge in a single quantum state. Thus, if the possible microstates of the system fill some finite volume Ω_Γ of phase space, then

$$W = \frac{\Omega_\Gamma}{h^{3N} N_1!N_2!} .$$

In the case under consideration, the possible microstates of the system lie on the surface $H(q_i, p_i, V) = \epsilon$ in the phase space Γ. However, it follows from the above that it will be convenient to assume for the computation of W that the energy of the system is not strictly constant but fluctuates within limits $(\epsilon, \epsilon + \delta\epsilon)$, where $\delta\epsilon$ is a small but finite quantity. It will be shown below that $\delta\epsilon$ does not affect the value of the entropy η.

The volume of phase space filled by the possible microstates of the system is now given by

$$\Omega_\Gamma = \int\limits_{\epsilon \leqslant H \leqslant \epsilon + \delta\epsilon} \mathrm{d}q_i \mathrm{d}p_i . \tag{1.1.6}$$

The integral over the coordinates in (1.1.6) is readily computed and seen to be equal to V^N. The integral over the impulses is reduced by a simple change of variables to the evaluation of the volume of a $3N$-dimensional sphere of given radius. Using a known formula for the volume $\Omega_n(R)$ of an n-dimensional sphere with radius R

$$\Omega_n(R) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)} R^n ,$$

one finds

$$W = \frac{\pi^{3N/2}(2\nu_1)^{3N_1/2}(2\nu_2)^{3N_2/2}\,\epsilon^{3N/2}V^N}{h^{3N}N_1!N_2!\Gamma\left(\dfrac{3N}{2}+1\right)}\left[\left(1+\frac{\delta\epsilon}{\epsilon}\right)^{3N/2}-1\right].$$

For large N, the expression for $\ln W$ effectively does not depend on the arbitrary parameter $\delta\epsilon$, since the ratio of the terms $\ln\,[(1+\delta\epsilon/\epsilon)^{3N/2}-1]$ and $\ln\,\epsilon^{3N/2}$ is of order $(\ln N)/N$. Therefore one obtains for the entropy of the system a completely defined expression which depends only on ϵ, V, N_1 and N_2. Using Stirling's formula

$$\ln N! = N\ln N - N + O(\ln N)\,,$$

and an analogous formula for $\ln[\Gamma(3N/2+1)]$, one arrives, finally, at

$$\eta = k\left\{\tfrac{3}{2}N\ln\frac{\epsilon}{\frac{3}{2}N} + N\ln\frac{V}{N} + N_1\ln\frac{a_1N}{N_1} + N_2\ln\frac{a_2N}{N_2}\right\},$$

$$a_i = \frac{(2\pi\nu_i)^{3/2}\,e^{5/2}}{h^3}\,,\qquad i=1,2\,. \tag{1.1.7}$$

Since it has been agreed to measure states of a mixture with different components by masses m_1 and m_2, one must introduce in (1.1.7) the substitutions $N_1 = m_1/\nu_1$ and $N_2 = m_2/\nu_2$.

Using (1.1.4), one obtains the formulae for the temperature T of a mixture of two ideal gases and the chemical potentials μ_i of the components of this mixture

$$T = \frac{\epsilon}{\frac{3}{2}kN}\,,\qquad \mu_i = -\frac{kT}{\nu_i}\left\{\ln\frac{V}{N} + \tfrac{3}{2}\ln\frac{\epsilon}{\frac{3}{2}N} + \ln\frac{a_iN}{N_i} - \tfrac{5}{2}\right\},\qquad i=1,2\,.$$

Note that the chemical potential of a component in a mixture depends on the mass of the other component and is an intensive quantity.

It will be recalled that the thermodynamic parameters of a system are subdivided into extensive and intensive parameters. Extensive parameters change by a factor M as the masses m_j of all components of the system change by a factor M; intensive parameters do not change for such a change of the masses m_j. In other words, extensive parameters are homogeneous functions of m_j of order one, while intensive parameters are homogeneous functions of m_j of order zero, and therefore depend only on the concentrations $c_j = m_j/m$.

It is seen from (1.1.7) that the entropy of a mixture of two ideal gases is extensive. This general property of entropy holds true for arbitrary systems [60].

1.2 EQUILIBRIUM PROCESSES

In the preceding section, it has been assumed for the computation of the entropy of an equilibrium thermodynamic system that the external conditions are unchanged. Now let the external conditions change in such a manner that the characteristic time of these changes is considerably larger than the relaxation time of the system (the characteristic time of transition to the equilibrium state). Then each state of a system may be assumed to be approximately an equilibrium state. Such processes are referred to as equilibrium or reversible processes.

For each equilibrium state, one has to determine the internal energy ϵ of the system. Therefore the first law of thermodynamics for equilibrium processes, one of the formulations of which asserts the existence for arbitrary systems of internal energy as a function of state, is fulfilled automatically.

A process is said to be an adiabatic equilibrium process when a change in its internal energy ϵ occurs only on account of work δA performed by the external forces acting on the system.

Here and below, symbols of the type δA are used to denote certain differential forms which, generally speaking, are not exact differentials of whatever functions of state under consideration.

Such processes must take place, on the one hand, sufficiently quickly, in order that heat- and mass-exchanges with the surrounding medium may be neglected, and, on the other hand, sufficiently slowly, so that the process may be assumed to be an equilibrium process.

In the case under consideration, the only external force is the pressure p (the average force exerted by the molecules on unit area of the surface Σ surrounding the volume V). The work performed by this force is $\delta A = -p\mathrm{d}V$ and, thus, for adiabatic processes

$$\mathrm{d}\epsilon = -p\mathrm{d}V\,. \tag{1.2.1}$$

How does the entropy of a system change during an adiabatic process? In order to answer this important question, consider again a mixture of two ideal gases and find an expression for the pressure p as a function of ϵ, V and m_j. During an arbitrary microstate of this system, one has the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_1^{3N} p_i q_i = \sum_1^{3N}\left(p_i\frac{\mathrm{d}q_i}{\mathrm{d}t} + q_i\frac{\mathrm{d}p_i}{\mathrm{d}t}\right) = \sum_1^{3N} q_i F_i + 2\left(\sum_1^{3N_1}\frac{p_i^2}{2\nu_1} + \sum_{3N_1+1}^{3N}\frac{p_i^2}{2\nu_2}\right), \tag{1.2.2}$$

where t is the time, $(F_1, F_2, F_3) = (\mathrm{d}p_1/\mathrm{d}t,\ \mathrm{d}p_2/\mathrm{d}t,\ \mathrm{d}p_3/\mathrm{d}t)$ the force acting at a given instant of time on the first atom, (F_4, F_5, F_6) the force acting on the second atom, etc. The mathematical expectation of the left-hand side of

(1.2.2) is zero, since in an equilibrium state the quantity $\langle \Sigma_1^{3N} p_i q_i \rangle$ obviously does not depend on time. Hence

$$-\langle \sum_1^{3N} F_i q_i \rangle = 2\epsilon \ .$$

Since one is dealing with an ideal gas, one must take into consideration only forces which the wall of the vessel exerts on the gas atoms. In other words, the forces acting on the atoms differ from zero only when the atoms hit the wall of the vessel; since the impact of an atom on a wall is postulated to be perfectly elastic, these forces act along the inward normal to the wall of the vessel. Since in an equilibrium state all microstates have the same probability, one finds

$$\langle \sum_1^{3N} F_i q_i \rangle = -p \int_\Sigma (\boldsymbol{r}, \boldsymbol{n})\, \mathrm{d}\Sigma \ ,$$

where the integral extends over the entire surface Σ of the vessel, $\boldsymbol{n}$ is the unit outward normal, $\boldsymbol{r}$ is the radius vector of points lying on the surface Σ. However, by the theorem of Gauss-Ostrogradskii,

$$\int_\Sigma (\boldsymbol{r}, \boldsymbol{n})\, \mathrm{d}\Sigma = \int_V \operatorname{div} \boldsymbol{r}\, \mathrm{d}V = 3V \ ,$$

and hence

$$p = \frac{2}{3} \frac{\epsilon}{V} \ . \tag{1.2.3}$$

Now it is already quite simple to show that during an adiabatic process the entropy of a mixture of two ideal gases does not change. In fact, by (1.1.7), (1.2.1) and (1.2.3),

$$\mathrm{d}\eta = kN\left(\frac{3}{2}\frac{\mathrm{d}\epsilon}{\epsilon} + \frac{\mathrm{d}V}{V}\right) = kN\left(-\frac{3}{2}\frac{p\mathrm{d}V}{\epsilon} + \frac{\mathrm{d}V}{V}\right) = 0 \ .$$

This important result is proved in courses of statistical physics for arbitrary systems (cf., for example, [60, § 11]). However, it follows directly from (1.2.1) that

$$\left(\frac{\partial \epsilon}{\partial V}\right)_{\eta, m_j} = -p \ .$$

One may now find an expression for $(\partial \eta / \partial V)_{\epsilon, m_j}$. Using a property of Jacobians, one finds

$$\left(\frac{\partial \eta}{\partial V}\right)_{\epsilon, m_j} = \frac{\partial(\eta, \epsilon)}{\partial(V, \epsilon)} = \frac{\partial(\eta, \epsilon)}{\partial(\eta, V)} \frac{\partial(\eta, V)}{\partial(V, \epsilon)} = -\left(\frac{\partial \epsilon}{\partial V}\right)_\eta \left(\frac{\partial \eta}{\partial \epsilon}\right)_V ,$$

and from the first formula (1.1.4)

$$\left(\frac{\partial \eta}{\partial V}\right)_{\epsilon, m_j} = \frac{p}{T}. \tag{1.2.4}$$

Now one has expressions for all partial derivatives of the entropy η with respect to ϵ, V and m_j. By (1.1.4) and (1.2.4),

$$\mathrm{d}\eta = \frac{1}{T}\mathrm{d}\epsilon + \frac{p}{T}\mathrm{d}V - \sum_{j=1}^{n} \frac{\mu_j}{T}\mathrm{d}m_j . \tag{1.2.5}$$

This formula, called Gibbs' relation, is one of the basic relations of thermodynamics.

The entropy of an isolated equilibrium thermodynamic system possesses the very important property of maximality. In order to formulate this property, imagine the system subdivided into m subsystems and denote by subscript k the parameters of the kth subsystem. Consider a possible perturbation of the parameters ϵ_k, V_k and m_k of a subsystem which are compatible with the condition of isolation of the system. In a perturbed state, this isolated system will have the entropy $\Sigma_{k=1}^{m} \eta_k(\epsilon_k, V_k, m_{kj})$. The property of maximality of entropy of the isolated equilibrium system implies that

$$\sum_{k=1}^{m} \eta_k(\epsilon_k, V_k, m_{kj}) \leqslant \eta(\epsilon, V, m_j) .$$

It is useful to verify this property for a mixture of two ideal gases. First of all, note that by strength of the isolated nature of the system the sum of the energies of the subsystems in their perturbed states as well as the sums of their volumes and of the numbers of atoms of each kind must be equal to the corresponding quantities in the original equilibrium state. Therefore, without reducing generality, one may subdivide each system with parameters 2ϵ, $2V$, $2N_1$, $2N_2$ into two subsystems with parameters $\epsilon \pm \delta\epsilon$, $V \pm \delta V$, $N_1 \pm \delta N_1$, $N_2 \pm \delta N_2$. Using the extensive character of entropy, the condition of maximality of entropy may be written in the form

$$\eta(\epsilon, V, N_1, N_2) - \tfrac{1}{2}\eta(\epsilon + \delta\epsilon, V + \delta V, N_1 + \delta N_1, N_2 + \delta N_2)$$
$$- \tfrac{1}{2}\eta(\epsilon - \delta\epsilon, V - \delta V, N_1 - \delta N_1, N_2 - \delta N_2) \geqslant 0 .$$

This is the condition of convexity of the function $\eta(\epsilon, V, N_1, N_2)$. It is readily proven by noting that, by (1.1.7), the expression for η consists of a sum of four functions of the type $x \ln (y/x)$ (there being no need to consider linear terms). Therefore it is sufficient to prove convexity of functions x ln

(y/x) along any straight line $y - y_0 = a(x - x_0)$ in the xy-plane. In fact,

$$\frac{d^2}{dx^2}\left[x \ln \frac{y_0 + a(x - x_0)}{x}\right]_{x=x_0} = -\frac{(y_0 - ax_0)^2}{x_0 y_0^2} < 0 .$$

For arbitrary isolated thermodynamic systems, the property of maximality of entropy is proved in courses on statistical physics (cf., for example, [60, § 7,8]).

Consider now an arbitrary equilibrium process for a closed system (such systems do not exchange mass with surrounding media; otherwise, a system is said to be open). Then $\delta\epsilon \neq \delta A$. By definition, the quantity δQ

$$\delta Q = d\epsilon - \delta A , \tag{1.2.6}$$

is called the amount of heat gained by the system.

Since all $dm_j = 0$ (chemical reactions not being taken into consideration) and $\delta A = -p\delta V$, it follows from Gibbs' relation (1.2.5) that

$$d\eta = \frac{\delta Q}{T} . \tag{1.2.7}$$

In essence, the truth of the second law of thermodynamics has been proved for equilibrium processes. In fact, from (1.2.7) follows the impossibility of a *perpetuum mobile* of the second kind. If there were given a cyclically operating engine, working by obtaining heat from some heat reservoir ($\delta Q > 0$), then over a complete cycle of such a machine $\oint(\delta Q/T) > 0$, which contradicts, by (1.2.7), the condition $\oint d\eta = 0$.

Thus, both laws of the thermodynamics of equilibrium processes follow from ordinary postulates of statistical physics.

Finally, it will be proved that the absolute temperature of a thermodynamic system, defined by the first relation (1.1.4), actually coincides with all properties of the absolute temperature, formulated in traditionally established thermodynamics which are not based on the ideas of statistical physics. Firstly, as it follows from (1.2.7), the function $1/T$ is an integrating factor for the differential form δQ; secondly, if two systems lie in thermal contact, then in an equilibrium state their temperatures are equal.

In order to prove the second assertion, use will be made of the maximum property of the entropy η of an isolated equilibrium system, consisting of two subsystems with energies ϵ_1 and ϵ_2. Since in the case of thermal contact, the volumes and constitutions of both subsystems do not change, one has

$$d\eta = \frac{\partial \eta_1}{\partial \epsilon_1}\delta\epsilon_1 + \frac{\partial \eta_2}{\partial \epsilon_2}\delta\epsilon_2 = 0 .$$

As $\delta\epsilon_1 + \delta\epsilon_2 = 0$, then, by (1.1.4), one has $T_1 = T_2$.

In conclusion, a definition will be given of the concept of the heat capacity of a system as the amount of heat δQ which must be added to a closed system in order to increase its temperature T by one centigrade. Since the quantity δQ is not a differential of any function of state and depends on the type of process, one may introduce heat capacities for constant pressure $C_p = \delta Q_p/\mathrm{d}T$ and for constant volume $C_v = \delta Q_v/\mathrm{d}T$, respectively. In order to express C_p and C_v in terms of derivatives of entropy, it is convenient to change over from the independent variables ϵ, V, m_j to the independent variables T, p, m_j and T, V, m_j, respectively. It then follows from (1.2.7) that

$$C_p = T\left(\frac{\partial \eta}{\partial T}\right)_{p,m_j}, \qquad C_v = T\left(\frac{\partial \eta}{\partial T}\right)_{V,m_j}. \tag{1.2.8}$$

1.3 THERMODYNAMIC POTENTIALS

By Gibbs' relation (1.2.5), the significance of the entropy of a system as a function of ϵ, V, m_j permits to determine all its basic thermodynamic parameters. In fact, by (1.1.2) and (1.2.4), one finds the temperature T, the chemical potentials μ_j and the pressure p as functions of ϵ, V, m_j. Further, eliminating from the expressions $T = T(\epsilon, V, m_j)$ and $p = p(\epsilon, V, m_j)$ the energy ϵ, one finds the equation of state of the system. For example, in the case of a mixture of ideal gases, one arrives without difficulties at the law

$$pV = kNT = R(n_1 + n_2)\,T\,,$$

where $R = kN_A = 8.314 \cdot 10^3$ J $\cdot$ mole$^{-1} \cdot {}^\circ$C^{-1} is referred to as the universal gas constant and n_1, n_2 are the numbers of moles of components of the mixture.

Further, it is readily verified that C_p and C_v, as also other thermodynamic parameters which will be defined in the following sections, may likewise be found from the known entropy $\eta = \eta(\epsilon, V, m_j)$. For this reason it is said that with respect to the independent variables ϵ, V, m_j the entropy of a system is a thermodynamic potential.

However, as has already been seen during the derivation of (1.2.8), the independent variables ϵ, V, m_j are not always themselves convenient. Sometimes it is, for example, convenient to use as independent variables η, V, m_j. Rewriting Gibbs' relation (1.2.5) in the form

$$\mathrm{d}\epsilon = T\mathrm{d}\eta - p\mathrm{d}V + \sum_1^n \mu_j \mathrm{d}m_j\,, \tag{1.3.1}$$

it is seen that in terms of the independent variables η, V, m_j the thermodynamic potential will be the internal energy of the system $\epsilon = \epsilon(\eta, V, m_j)$.

Using the invariance of the form of representing the first differential of a function, one obtains from (1.3.1) the relations

$$d(\epsilon + pV) = d\chi = Td\eta + Vdp + \sum_1^n \mu_j dm_j\ , \tag{1.3.2}$$

$$d(\epsilon - T\eta) = d\psi = -\eta dT - pdV + \sum_1^n \mu_j dm_j\ , \tag{1.3.3}$$

$$d(\epsilon + pV - T\eta) = d\zeta = -\eta dT + Vdp + \sum_1^n \mu_j dm_j\ . \tag{1.3.4}$$

The function $\chi = \epsilon + pV$ is called the enthalpy of the system (or the heat function); the function $\psi = \epsilon - T\eta$ is called the free energy of the system (or the Helmholtz function); the function $\zeta = \epsilon + pV - T\eta$ is called Gibbs' potential (or Gibbs' function) of the system.

Gibbs' notation [26] has been retained here. With regard to other notations and terminology, reference should be made to [3, pp. 19, 20].

It follows from (1.3.2)–(1.3.4) that the enthalpy χ is the thermodynamic potential with respect to the variable η, p, m_j, the free energy ψ with respect to T, V, m_j and Gibbs' potential ζ with respect to T, p, m_j. Obviously, the functions χ, ψ, ζ are extensive parameters of state of the system. Therefore, by Euler's formula for homogeneous functions of first order,

$$\zeta(T, p, m_j) = \sum_{k=1}^n \frac{\partial \zeta(T, p, m_j)}{\partial m_k} m_k\ .$$

Hence, by (1.3.4), one has Euler's identity

$$\epsilon + pV - T\eta = \sum_1^n \mu_k m_k\ . \tag{1.3.5}$$

Differentiating (1.3.5) and taking into account Gibbs' relation (1.2.5), one obtains

$$\eta dT - Vdp + \sum_1^n m_k d\mu_k = 0\ . \tag{1.3.6}$$

This formula is known as the Gibbs-Duhem relation.

If one relates an extensive parameter of a system to unit mass, then one may consider corresponding specific parameters. Thus, the specific volume $V/m = 1/\rho$ (ρ is the density of the system). Likewise, one may introduce the specific internal energy $\epsilon_m = \epsilon/m$, the specific entropy $\eta_m = \eta/m$, the specific enthalpy $\chi_m = \chi/m$, the specific free energy $\psi_m = \psi/m$ and the specific Gibbs' potential $\zeta_m = \zeta/m$. Obviously, all the specific parameters will be intensive quantities and not depend on m_j, but on the concentrations $c_j = m_j/m$. Further by Euler's identity (1.3.5) and Gibbs' relation (1.2.5), one obtains readily

$$Td\eta_m = d\epsilon_m + pd\left(\frac{1}{\rho}\right) - \sum_1^n \mu_k dc_k .$$

Since $\Sigma_1^n m_k = m$, then $\Sigma_1^n c_k = 1$. Therefore only $n - 1$ components may be taken as independent variables, characterizing the constitution of the mixture. Since $\Sigma_1^n dc_k = 0$, one has

$$Td\eta_m = d\epsilon_m + pd\left(\frac{1}{\rho}\right) - \sum_1^{n-1} (\mu_k - \mu_n)\, dc_k , \tag{1.3.7}$$

where already all $dc_1, \ldots, dc_{n-1}$ may be assumed to be independent. Note that one may interpret μ_n as chemical potential of any component.

The relations (1.3.2)–(1.3.4) and (1.3.6) may be rewritten in terms of specific quantities in the same manner.

Specific parameters have been introduced. However, by the same means, one could introduce a density $Z/V = \rho z_m$ (where Z is any extensive parameter and z_m its corresponding specific parameter). It is not difficult to write down Gibbs' relation in terms of densities. By (1.2.5), one has

$$Td(\rho\eta_m) = d(\rho\epsilon_m) - \sum_1^n \mu_k d\rho_k , \tag{1.3.8}$$

where $\rho_k = m_k/V$ is the density of the kth component of the mixture and $\Sigma_1^n \rho_k = \rho$. Note that the expression for the differential of the density of the entropy $\rho\eta_m$ has been obtained as a function of the independent variables $\rho\epsilon_m, \rho_1, \ldots, \rho_n$.

1.4 SEA WATER AS A TWO-COMPONENT SOLUTION

The principal substances, dissolved in sea water, are strong electrolytes and they are practically dissociated into ions. The basic components of the mixture are chloride ions (Cl^-), sodium ions (Na^+), sulphate ions (SO_4^{2-}), magnesium ions (Mg^{2+}), calcium ions (Ca^{2+}), potassium ions (K^+) and hydrocarbonate ions (HCO_3^-).

The concept of salinity will now be introduced. Consider a volume V and let $m_1, \ldots, m_{n-1}$ be the masses of the components of the admixture in solution (for example, $m_1 = \nu_1 N_1$, where ν_1 is the mass of the Cl^- ions, N_1, is the number of these ions), and m_n the mass of the pure water. Then salinity $s = \Sigma_1^{n-1} m_k/m$ (where m is the mass of the volume V). Defined in this manner,

TABLE 1.I

Ion composition of sea water (according to Sverdrup et al. [120, Chapter VI])

Ions	%	Ions	%
Na^+	30.61	Cl^-	55.04
Mg^{2+}	3.69	SO_4^{2-}	7.68
Ca^{2+}	1.16	HCO_3^-	0.41
K^+	1.10		

salinity turns out to be a non-dimensional quantity, usually expressed in parts per thousand and denoted by S. Clearly, $S = 1000s$.

The following important fact is experimental in origin: Far away from ocean shores, the composition of principal ions in sea water is constant

$$c_k = \frac{m_k}{m} = \lambda_k s\ , \qquad k = 1, \ldots, n-1\ , \qquad \sum_1^{n-1} \lambda_k = 1\ ,$$

where λ_k are constants (cf. Table 1.I). In other words, the salinity of sea water changes because of addition of pure water or its disappearance (precipitation, evaporation, formation and thawing of ice), but the composition of the salt of sea water remains unchanged.

As a consequence of the constancy of its salt composition, sea water may be considered as a two-component mixture: pure water and salt. Denote the concentration of pure water by c_w. Then $s + c_w = 1$. Selecting s as independent variable and denoting by μ_s the quantity $\Sigma\lambda_k\mu_k$ (the chemical potential of salt), Gibbs' relation for specific quantities (1.3.7) may be written in the form

$$T\mathrm{d}\eta_m = \mathrm{d}\epsilon_m + p\mathrm{d}\left(\frac{1}{\rho}\right) - \mu \mathrm{d}s\ , \tag{1.4.1}$$

where μ denotes the difference of the chemical potentials of salt μ_s and pure water μ_w, $\mu = \mu_s - \mu_w$.

The basic thermodynamic relations (1.3.1)—(1.3.6) for specific quantities will now be written down for sea water:

$$\mathrm{d}\epsilon_m = T\mathrm{d}\eta_m - p\mathrm{d}\left(\frac{1}{\rho}\right) + \mu \mathrm{d}s\ , \tag{1.4.2}$$

$$\mathrm{d}\chi_m = T\mathrm{d}\eta_m + \frac{1}{\rho}\mathrm{d}p + \mu \mathrm{d}s\ , \qquad \chi_m = \epsilon_m + \frac{p}{\rho}\ , \tag{1.4.3}$$

$$\mathrm{d}\psi_m = -\eta_m \mathrm{d}T - p\mathrm{d}\left(\frac{1}{\rho}\right) + \mu \mathrm{d}s\ , \qquad \psi_m = \epsilon_m - T\eta_m\ , \tag{1.4.4}$$

$$\mathrm{d}\zeta_m = -\eta_m \mathrm{d}T + \frac{1}{\rho}\,\mathrm{d}p + \mu \mathrm{d}s\,, \qquad \zeta_m = \epsilon_m + \frac{p}{\rho} - T\eta_m, \tag{1.4.5}$$

Euler's identity

$$\epsilon_m + \frac{p}{\rho} - T\eta_m = \mu s + \mu_w\,, \tag{1.4.6}$$

the Gibbs-Duhem relation

$$\eta_m \mathrm{d}T - \frac{1}{\rho}\,\mathrm{d}p + s\mathrm{d}\mu + \mathrm{d}\mu_w = 0\,. \tag{1.4.7}$$

Formulae (1.2.8) for the specific heats c_p and c_v assume the form

$$c_p = T\left(\frac{\partial \eta_m}{\partial T}\right)_{p,s}, \qquad c_v = T\left(\frac{\partial \eta_m}{\partial T}\right)_{\rho,s}. \tag{1.4.8}$$

1.5 ENTROPY, INTERNAL ENERGY AND CHEMICAL POTENTIAL OF SEA WATER

It has been shown in § 1.1 that starting from the statistical definition of entropy a simple analytic expression for the function $\eta(\epsilon, V, m_j)$ may be found for the case of a mixture of two ideal gases. However, it will be recalled that the simplicity of the calculations arose from the fact that for an ideal gas the effect of inter-atomic forces need not be taken into consideration. Unfortunately, because of the necessity of taking into account the forces of interaction between ions (especially Coulomb forces) by the methods of statistical physics, one cannot succeed in constructing a comprehensive analytic expression for the entropy of sea water. Known theories for weak solutions of strong electrolytes turn out to be valid for significantly weaker concentrations of ions of the admixtures [123].

It will be shown in this section that one may estimate the thermodynamic parameters (entropy, internal energy, etc.) on the basis of empirical data. For this purpose, a number of thermodynamic relations must be derived.

The measurement of temperature T, pressure p and salinity s is simplest. Therefore it is convenient to select as independent variables T, p and s. Clearly, for such a choice, Gibbs' potential ζ_m will play a basic role. By (1.4.5), one has

$$\left(\frac{\partial \zeta_m}{\partial T}\right)_{p,s} = -\eta_m\,, \qquad \left(\frac{\partial \zeta_m}{\partial p}\right)_{T,s} = \frac{1}{\rho}, \qquad \left(\frac{\partial \zeta_m}{\partial s}\right)_{T,p} = \mu\,. \tag{1.5.1}$$

Using the conditions $\partial^2\zeta_m/\partial T\partial p = \partial^2\zeta_m/\partial p\partial T$, etc., one readily finds (in terms of the variables T, p, s):

$$\frac{\partial \mu}{\partial T} = -\frac{\partial \eta_m}{\partial s}, \qquad \frac{\partial(1/\rho)}{\partial T} = -\frac{\partial \eta_m}{\partial p}, \qquad \frac{\partial(1/\rho)}{\partial s} = \frac{\partial \mu}{\partial \rho}. \tag{1.5.2}$$

Further, $\partial^2\zeta_m/\partial T^2 = -\partial\eta_m/\partial T = -c_p/T$.
Since $\partial^3\zeta_m/\partial p\partial T^2 = \partial^3\zeta_m/\partial T^2\partial p$ and $\partial^3\zeta_m/\partial s\partial T^2 = \partial^3\zeta_m/\partial T^2\partial s$, one has

$$-\frac{1}{T}\frac{\partial c_p}{\partial p} = \frac{\partial^2(1/\rho)}{\partial T^2}, \qquad -\frac{1}{T}\frac{\partial c_p}{\partial s} = \frac{\partial^2\mu}{\partial T^2}\,. \tag{1.5.3}$$

By (1.4.8) and (1.5.2),

$$\mathrm{d}\eta_m = \frac{c_p}{T}\,\mathrm{d}T - \frac{\partial(1/\rho)}{\partial T}\,\mathrm{d}p - \frac{\partial\mu}{\partial T}\,\mathrm{d}s\,. \tag{1.5.4}$$

At the present time sufficiently exact empirical formulae have been proposed for the equation of state of sea water [136,24,132] and the specific heat c_p at atmospheric pressure [8]. However, then, by the first formula (1.5.3), the specific heat c_p may be computed for any pressure, so that in the sequel the functions $\rho(T, p, s)$ and $c_p(T, p, s)$ may be assumed known. An idea about the dependence of ρ and c_p on T, p, s can be obtained from Figs. 1.1 and 1.2 and Table 1.II.

Further, by the third formula (1.5.2) and the second formula (1.5.3), the parameter μ may be computed exactly apart from a function $a(s)T + b(s)$, where $a(s)$ and $b(s)$ are arbitrary functions.

It will be shown below that $(\partial\mu/\partial s)_{T,p}$ may be computed as a function of s from data on the dependence of the compressibility of saturated vapour on

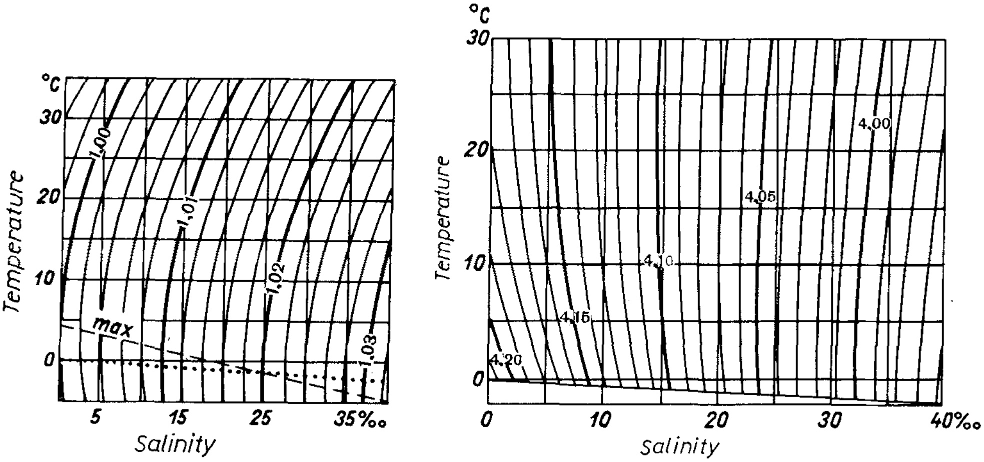


Fig. 1.1. Density ρ g/cm^3 as function of temperature and salinity at atmospheric pressure (Montgomery [80]). Dotted curve denotes freezing.

Fig. 1.2. Specific heat at constant pressure c_p J $\cdot$ g^{-1} $\cdot$ (°C)$^{-1}$ as function of temperature and salinity at atmospheric pressure (Fofonoff [24]).

TABLE 1.II

Dependence of ρ and c_p on pressure p (deviation from atmospheric pressure) for $T = 0°$C and $S = 35‰$ (Montgomery [80])

	Pressure p (dbar)					
	0	2000	4000	6000	8000	10 000
Density ρ g cm^{-3}	1.02813	1.03748	1.04640	1.05495	1.06315	1.07104
Difference $c_p(0) - c_p(p)$ $Jg^{-1}{}°C^{-1}$	0	0.0667	0.122	0.168	0.206	0.237

the salinity of sea water. Hence, if it is assumed that $(\partial\mu/\partial s)_{T,p}$ is known as a function of s (for two different values of the temperature and the same pressure), then arbitrariness in the determination of the parameter μ may be reduced to a linear function of T. Finally, by (1.5.4), the specific heat η_m may be computed as a function of T, p, s apart from a linear function of s.

How to estimate $(\partial\mu/\partial s)_{T,p}$? Consider for this purpose the equilibrium between sea water and its saturated vapour, enclosed in some constant volume. Since such a system may be assumed to be isolated, its entropy η must be a maximum. Taking into consideration (1.2.5), one has

$$0 = \delta\eta + \delta\eta_v = \left(\frac{1}{T} - \frac{1}{T_v}\right)\delta\epsilon + \left(\frac{p}{T} - \frac{p_v}{T_v}\right)\delta V - \left(\frac{\mu_w}{T} - \frac{\mu_v}{T_v}\right)\delta m_w$$

Quantities without subscripts relate to sea water, those with the subscript v to vapour. In this formula, μ_w is the chemical potential of the pure water in sea water, δm_w the change in the mass of pure water in the sea water (assuming that the ions of the salt do not evaporate form the solution), μ_v the specific Gibbs potential of pure vapour (which may be called the chemical potential of pure vapour.

Since $\delta\epsilon$, δV, δm_w are arbitrary, one finds

$$T = T_v\ , \qquad p = p_v\ , \tag{1.5.5}$$

$$\mu_w(T, p, s) = \mu_v(T, p)\ . \tag{1.5.6}$$

These equations give the dependence of the compressibility (pressure of saturated vapour p_v on T and s. Let T be constant and consider $p_v = p_v(s)$. Differentiating (1.5.6) with respect to s, one has

$$\frac{\partial\mu_w}{\partial p}\frac{\mathrm{d}p_v}{\mathrm{d}s} + \frac{\partial\mu_w}{\partial s} = \frac{\partial\mu_v}{\partial p}\frac{\mathrm{d}p_v}{\mathrm{d}s}\ . \tag{1.5.7}$$

In analogy with the derivation of (1.5.2), one finds readily from (1.3.4) that

$$\left(\frac{\partial \mu_w}{\partial p}\right)_{T, m_s, m_w} = \left(\frac{\partial V}{\partial m_w}\right)_{T, p, m_s} = V_w\,, \qquad \left(\frac{\partial \mu_v}{\partial p}\right)_T = V_v\,,$$

where V_w is the partial volume of pure water in sea water (in general, if $Z(T, p, m_j)$ is an extensive function of state of a system, then the intensive function of state $Z_j = \partial Z/\partial m_j$ is referred to as a partial parameter of the system), V_v the specific volume of vapour. Further, by the Gibbs-Duhem relation (1.4.7), one has

$$s\frac{\partial \mu}{\partial s}(T, p, s) = -\frac{\partial \mu_w}{\partial s}(T, p, s)\,.$$

Formula (1.5.7) may now be rewritten in the form

$$s\left(\frac{\partial \mu}{\partial s}\right)_{T,p} = (V_w - V_v)\frac{\mathrm{d}p_v}{\mathrm{d}s}\,.$$

Consider the relative lowering of the pressure of saturated vapour of sea water $r = (p_v^0 - p_v)/p_v^0$ (where p_v^0 is the pressure of saturated vapour over pure water). Since $V_w << V_v$, one has, finally,

$$s\left(\frac{\partial \mu}{\partial s}\right)_{T,p} = \frac{p_v V_v}{1-r}\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{R_v T}{1-r}\frac{\mathrm{d}r}{\partial s}, \tag{1.5.8}$$

where $R_v = R/M_v$, M_v is the gram-molecular weight of water vapour, R the universal gas constant, $R_v = 46.2 \cdot 10^5\ \mathrm{erg} \cdot \mathrm{g}^{-1} \cdot {}^\circ\mathrm{C}^{-1}$ (water vapour being assumed to be an ideal gas).

It is known that for normal conditions $r = 0.54s$ (cf. the survey of empirical data $\mathrm{d}r/\mathrm{d}s$ in Fofonoff [24]; it is shown there that $\partial\mu/\partial s$ may be estimated also from the known dependence of osmotic pressure on salinity, and likewise the boiling temperature or the freezing temperature of sea water). Then $s\partial\mu/\partial s \simeq 7.5 \cdot 10^8\ \mathrm{erg} \cdot \mathrm{g}^{-1}$ for $T = 27^\circ\mathrm{C}$.

An expression for the derivatives of the interval energy ϵ_m is also readily found. For example, one obtains from the second formula (1.4.5) in the system of variables T, p, s that

$$\frac{\partial \zeta_m}{\partial T} = \frac{\partial \epsilon_m}{\partial T} + p\frac{\partial(1/\rho)}{\partial T} - \eta_m - T\frac{\partial \eta_m}{\partial T}$$

and, since $\partial\zeta_m/\partial T = -\eta_m$ and $\partial\eta_m/\partial T = c_p/T$,

$$\frac{\partial \epsilon_m}{\partial T} = -p\frac{\partial(1/\rho)}{\partial T} + c_p\,. \tag{1.5.9}$$

Analogously,

$$\frac{\partial \epsilon_m}{\partial p} = -p\frac{\partial(1/\rho)}{\partial p} - T\frac{\partial(1/\rho)}{\partial T}, \tag{1.5.10}$$

$$\frac{\partial \epsilon_m}{\partial s} = -p\frac{\partial(1/\rho)}{\partial s} + \mu - T\frac{\partial \mu}{\partial T}. \tag{1.5.11}$$

Thus, from the same initial empirical data which permit the computation of the entropy of sea water $\eta(T, p, s)$, one may find also the specific internal energy of sea water $\epsilon(T, P, s)$ (it is readily seen that it will only be exact within a linear function of s). Clearly, it will then also be easy to find the specific Gibbs potential $\zeta_m(T, p, s) = \epsilon_m + p/\rho - T\eta_m$ exactly apart from a function of the form $a_1 + a_2T + a_3s + a_4Ts$ (where a_i are constants).

1.6 ADIABATIC TEMPERATURE GRADIENT AND COMPRESSIBILITY OF SEA WATER

In thermodynamics, one must often express parameters of state in different systems of independent variables. Such changes of variable lead to interesting relations. As an example, consider the heat capacity $c_v = T(\partial\eta_m/\partial T)_{\rho,s}$ and write down its expression in the system of variables T, p, s using for this purpose known properties of Jacobians. One has

$$c_v = T\left(\frac{\partial\eta_m}{\partial T}\right)_{\rho,s} = T\frac{\partial(\eta_m, 1/\rho)}{\partial(T, 1/\rho)} = T\frac{\partial(\eta_m, 1/\rho)}{\partial(T, p)}\frac{1}{\partial(T, 1/\rho)/\partial(T, p)} =$$

$$= T\left(\frac{\partial\eta_m}{\partial T}\frac{\partial(1/\rho)}{\partial p} - \frac{\partial(1/\rho)}{\partial T}\frac{\partial\eta_m}{\partial p}\right)\frac{1}{\partial(1/\rho)/\partial p}.$$

By (1.5.2) and (1.4.8), one finds

$$c_v = c_p + \frac{T[\partial(1/\rho)/\partial T]^2}{\partial(1/\rho)/\partial p}. \tag{1.6.1}$$

The formula (1.6.1) permits to determine c_v from c_p for a known equation of state of sea water. Since $\partial(1/\rho)/\partial p < 0$ (cf. § 1.7), one has $c_p > c_v$. The physical significance is as follows: If one adds to a system a definite quantity of heat, then for constant volume it heats up more than for constant pressure, since in the second case the system will perform work on account of part of the heat. Note that for sea water the ratio c_p/c_v is very close to unity.

Consider the change in the temperature of a system during an equilibrium adiabatic process. Since for such a process the entropy η_m and the

TABLE 1.III

Dependence of Γ on the pressure (difference from atmospheric pressure) for $T = 0°$C and $S = 35‰$ (according to Montgomery [80])

Pressure p (dbar)	0	2000	4000	6000	8000	10 000
Adiabatic temperature gradient Γ (°C/1000 dbar)	0.035	0.072	0.104	0.133	0.159	0.181

salinity s of the system remain unchanged, then this change is given by $(\partial T/\partial p)_{\eta_m, s}$. This quantity is denoted by Γ and referred to as the adiabatic temperature gradient of the system. In the system T, p, s, one finds for Γ the expression

$$\Gamma = \left(\frac{\partial T}{\partial p}\right)_{\eta_m, s} = \frac{\partial(T, \eta_m)}{\partial(p, \eta_m)} = \frac{\partial(T, \eta_m)}{\partial(T, p)} \frac{1}{\partial(p, \eta_m)/\partial(T, p)} = \frac{T}{c_p} \frac{\partial(1/\rho)}{\partial T} . \tag{1.6.2}$$

An idea of the numerical values of Γ can be gained from Table 1.III.

The concept of potential temperature θ of a system will now be introduced. This is the temperature which a system acquires during an equilibrium adiabatic transition from pressure p to atmospheric pressure p_a. The corresponding density for such a transition is termed potential density. By definition, one has

$$\theta(\eta_m, s) = T(\eta_m, p_a, s) = T(\eta_m, p, s) - \int_{p_a}^{p} \Gamma(\eta_m, p, s)\, \mathrm{d}p , \tag{1.6.3}$$

$$\rho_{\text{pot}}(\eta_m, s) = \rho(\eta_m, p_a, s) = \rho(\eta_m, p, s) - \int_{p_a}^{p} \left(\frac{\partial \rho}{\partial p}\right)_{\eta_m, s} \mathrm{d}p = \rho(\theta, s, p_a) . \tag{1.6.4}$$

The quantities θ and ρ_{pot} describe the effect of removal of pressure influence on the temperature and density of seawater.

Since the density of sea water differs little from the value 1 g · cm^{-3}, it is convenient to introduce the following quantities: $\sigma_{stp} = 10^3\ [\rho(T, s, p) - 1\ \text{g} \cdot \text{cm}^{-3}]$, $\sigma_t = \sigma_{stp}\ (T, s, p_a)$ and $\sigma_\theta = \sigma_{stp}\ (\theta, s, p_a)$. Table 1.IV permits to compare all these quantities by means of the example of a deep water station.

Finally, the following quantities will be introduced: the coefficient of thermal expansion (variables T, p, s)

TABLE 1.IV

Example of computations of θ, σ_{stp}, σ_t, σ_θ at a deep water station (according to Fofonoff [23])

z (m)	T (°C)	θ (°C)	S (‰)	σ_t	σ_θ	σ_{stp}	p (dbar)
0	7.47	7.47	32.47	25.39	25.39	25.39	0.0
50	7.17	7.16	32.50	25.45	25.45	25.68	50.3
100	7.02	7.01	33.26	26.07	26.07	26.53	100.6
200	6.41	6.39	33.94	26.68	26.69	27.61	201.4
500	5.03	4.99	34.13	27.00	27.01	29.34	504.1
1 000	3.19	3.12	34.39	27.40	27.41	32.10	1 009.6
1 500	2.36	2.26	34.52	27.58	27.59	34.62	1 516.5
2 000	1.93	1.79	34.61	27.69	27.70	37.05	2 024.6
2 500	1.82	1.65	34.64	27.72	27.73	39.37	2 533.8
3 000	1.66	1.44	34.66	27.75	27.76	41.68	3 044.2
3 500	1.58	1.31	34.67	27.76	27.78	43.95	3 555.7
4 000	1.59	1.26	34.67	27.76	27.78	46.18	4 068.3
4 500	1.64	1.25	34.67	27.76	27.78	48.38	4 582.0
5 500	1.78	1.26	34.67	27.75	27.78	52.71	5 612.6
6 500	1.92	1.25	34.67	27.74	27.78	56.96	6.647.3
7 500	2.08	1.24	34.67	27.72	27.79	61.13	7 686.2
8 500	2.23	1.22	34.67	27.71	27.79	65.23	8 729.2
10 000	2.48	1.16	34.67	27.69	27.79	71.24	10 301.1

$$\alpha = \rho\left[\frac{\partial(1/\rho)}{\partial T}\right]_{p,s}; \tag{1.6.5}$$

the coefficient of isothermal compression (variables T, p, s)

$$\kappa_T = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{T,s}; \tag{1.6.6}$$

the coefficient of adiabatic compression (variables η_m, p, s)

$$\kappa_\eta = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{\eta_m,s}. \tag{1.6.7}$$

The quantity κ_η is closely linked to the velocity of sound c in sea water which is given by

$$c = \sqrt{\left(\frac{\partial p}{\partial\rho}\right)_{\eta_m,s}}, \tag{1.6.8}$$

whence

$$c^2 = \frac{1}{\rho\kappa_\eta}.$$

TABLE 1.V

Dependence of speed of sound on pressure p (difference from atmospheric pressure) for $T = 0°$C and $S = 35‰$ (according to Montgomery [80])

Pressure p (dbar)	0	2000	4000	6000	8000	10 000
Speed of sound ($\mathrm{m \cdot sec^{-1}}$)	1448.6	1484.4	1519.7	1554.2	1587.7	1620.0

At the present time, the velocity of sound c in sea water as a function of T, p, s may be assumed to be known sufficiently well [131]. An idea regarding the dependence of the speed of sound c on T, p, s can be gained from Fig. 1.3 and Table 1.V.

The quantity κ_η will now be expressed in terms of the independent variables T, p, s:

$$\left(\frac{\partial \rho}{\partial p}\right)_{\eta_m, s} = \frac{\partial(\rho, \eta_m)}{\partial(p, \eta_m)} = \frac{\partial(\rho, \eta_m)}{\partial(T, p)} \frac{1}{\partial(p, \eta_m)/\partial(T, p)} =$$

$$-\left(\frac{\partial \rho}{\partial T}\frac{\partial \eta_m}{\partial p} - \frac{\partial \eta_m}{\partial T}\frac{\partial \rho}{\partial p}\right)\frac{1}{\partial \eta_m/\partial T} = \left\{-\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial T}\right)^2 + \frac{c_p}{T}\frac{\partial \rho}{\partial p}\right\}\frac{T}{c_p} = \Gamma\frac{\partial \rho}{\partial T} + \frac{\partial \rho}{\partial p}\ .$$

By (1.6.5) and (1.6.6), one has

$$\kappa_\eta = -\Gamma\alpha + \kappa_T \tag{1.6.9}$$

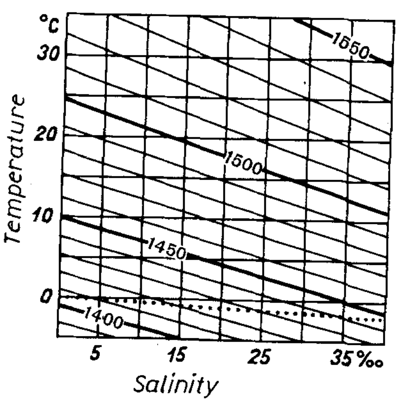


Fig. 1.3. Speed of sound c (m/sec), as function of temperature and salinity for atmospheric pressure (according to Montgomery [80]). Dotted curve refers to freezing temperature.

or, using (1.6.1) and (1.6.2),

$$\kappa_\eta = \frac{c_v}{c_p}\,\kappa_T\,. \tag{1.6.10}$$

Since $c_v < c_p$, then $\kappa_T > \kappa_\eta$. Formula (1.6.10) permits to compute c_p/c_v from the known speed of sound and the equation of state of sea water.

1.7 THERMODYNAMIC INEQUALITIES

In the preceding sections, the property of maximality of entropy has been used to determine conditions of equilibrium of an isolated system (cf. the proof of the equality of the temperatures of two systems lying in thermal contact in § 1.2 and likewise the conditions of equilibrium of the system of sea water and its saturated vapour in § 1.5). However, it is known that for an isolated system the extreme value of entropy is a maximum. Therefore one has not only the condition $\delta\eta = 0$ but also the condition $\delta^2\eta \leqslant 0$ (naturally, for definite limitations on the possible variations $\delta\epsilon$, δV, δm_j).

The condition $\delta^2\eta \leqslant 0$ for admissible variations $\delta\epsilon$, δV, δm_j permits to obtain a number of important thermodynamic inequalities. In order to derive these, subdivide an equilibrium system into two subsystems with identical mass m, volume V and internal energy ϵ and vary the parameters of each subsystem. Restricting consideration to a two-component mixture (sea water) and varying only the mass of the salt, one finds

$$\begin{aligned}
0 &\geqslant \eta(\epsilon + \delta\epsilon', V + \delta V, m_s + \delta m_s) + \eta(\epsilon + \delta\epsilon'', V - \delta V, m_s - \delta m_s) \\
&\quad - \eta(2\epsilon, 2V, 2m_s) = \frac{\partial\eta}{\partial\epsilon}\,\delta\epsilon' + \frac{\partial\eta}{\partial\epsilon}\,\delta\epsilon'' + \frac{1}{2}\frac{\partial^2\eta}{\partial\epsilon^2}\,(\delta\epsilon')^2 + \frac{1}{2}\frac{\partial^2\eta}{\partial\epsilon^2}\,(\delta\epsilon'')^2 \\
&\quad + \frac{\partial^2\eta}{\partial\epsilon\partial V}\,\delta\epsilon'\delta V + \frac{\partial^2\eta}{\partial\epsilon\partial m_s}\,\delta\epsilon'\delta m_s - \frac{\partial^2\eta}{\partial\epsilon\partial V}\,\delta\epsilon''\delta V - \frac{\partial^2\eta}{\partial\epsilon\partial m_s}\,\delta\epsilon''\delta m_s' \\
&\quad + 2\frac{\partial^2\eta}{\partial V\partial m_s}\,\delta V\delta m_s + \frac{\partial^2\eta}{\partial V^2}(\delta V)^2 + \frac{\partial^2\eta}{\partial m_s^2}\,(\delta m_s)^2 + \ldots
\end{aligned} \tag{1.7.1}$$

The conditions for the variations $\delta\epsilon'$ and $\delta\epsilon''$ will now be written down. Assume that in the perturbed state the subsystems may have macroscopic velocities $\delta\boldsymbol{v}_1$ and $\delta\boldsymbol{v}_2$ (the question of possible macroscopic motions of an equilibrium system is treated in detail in the following section).

Then it is obvious that, by strength of the laws of conservation of the total impulse, the moment of total impulse and the energy of the system, one may write down the linking conditions

$$(m + \delta m_s)\,\delta\boldsymbol{v}_1 + (m - \delta m_s)\,\delta\boldsymbol{v}_2 = 0\,,$$

$$(m + \delta m_s)\, \boldsymbol{r} \times \delta \boldsymbol{v}_1 + (m - \delta m_s)\, \boldsymbol{r} \times \delta \boldsymbol{v}_2 = 0\,,$$

$$(m + \delta m_s)\, \frac{\delta v_1^2}{2} + (m - \delta m_s)\, \frac{\delta v_2^2}{2} + \delta\epsilon' + \delta\epsilon'' = 0\,,$$

where $\boldsymbol{r}$ is the radius vector relative to some point 0. It follows from the first two relations, accurately to second order of magnitude, that

$$\delta \boldsymbol{v}_1 = -\delta \boldsymbol{v}_2 = \delta \boldsymbol{v}$$

Expressing from the third relation $\delta\epsilon''$ in terms of $\delta\epsilon'$ and $\delta \boldsymbol{v}$ and substituting into (1.7.1), one arrives at a quadratic form with respect to now already arbitrary variations $\delta \boldsymbol{v}$, $\delta\epsilon'$, δV, δm_s. As this form is negative definite, one derives now inequalities for the principal diagonal minors $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ of its matrix:

$$\Delta_1 < 0, \qquad \Delta_2 > 0, \qquad \Delta_3 < 0, \qquad \Delta_4 > 0\,. \tag{1.7.2}$$

The first of the inequalities (1.7.2) leads directly to an important deduction relating to the positiveness of the absolute temperature of the system. By (1.2.5), (1.2.8) and known properties of Jacobians, one has

$$\frac{\Delta_2}{\Delta_1} = \frac{\partial(\partial\eta/\partial\epsilon)}{\partial\epsilon} = \frac{\partial(1/T)}{\partial\epsilon} = -\frac{1}{T^2}\frac{\partial T}{\partial\epsilon} = -\frac{1}{T^2}\,\frac{1}{T(\partial\eta/\partial T)_{V,m_s}} = -\frac{1}{T^2 C_v}\,,$$

$$\frac{\Delta_3}{\Delta_1} = \frac{\partial(\partial\eta/\partial\epsilon, \partial\eta/\partial V)}{\partial(\epsilon, V)} = \frac{\partial(1/T, p/T)}{\partial(\epsilon, V)} = -\frac{1}{T^3}\frac{\partial(T,p)}{\partial(\epsilon, V)} = -\frac{1}{T^3}\frac{\partial(T,p)}{\partial(T, V)}$$

$$\times \frac{1}{\partial(\epsilon, V)\partial(T, V)} = \frac{1}{T}\left(\frac{\partial p}{\partial V}\right)_T \frac{\Delta_2}{\Delta_1}\,,$$

$$\frac{\Delta_4}{\Delta_1} = \frac{\partial(\partial\eta/\partial\epsilon, \partial\eta/\partial V, \partial\eta/\partial m_s)}{\partial(\epsilon, V, m_s)} = \frac{\partial(1/T, p/T, -\mu_s/T)}{\partial(\epsilon, V, m_s)} =$$

$$= \frac{1}{T^4}\frac{\partial(T,p,\mu_s)}{\partial(\epsilon, V, m_s)} = \frac{1}{T^4}\frac{\partial(T,p,\mu_s)}{\partial(T,p,m_s)}\,\frac{1}{\partial(\epsilon, V, m_s)/\partial(T,p,m_s)} =$$

$$= -\frac{1}{T}\left(\frac{\partial\mu_s}{\partial m_s}\right)_{T,p}\frac{\Delta_3}{\Delta_1}\,.$$

Finally, the inequalities (1.7.2) may now be given the forms

$$T > 0, \qquad C_v > 0, \qquad (\partial p/\partial V)_{T, m_s, m_w} < 0, \qquad (\partial\mu_s/\partial m_s)_{T,p,m_w} > 0\,. \tag{1.7.3}$$

Since for T and p = const.

$$\left(\frac{\partial\mu_s}{\partial m_s}\right)_{m_w} = \frac{\partial\mu_s}{\partial s}\left(\frac{\partial s}{\partial m_s}\right)_{m_w} = \frac{\partial\mu_s}{\partial s}\frac{m_w}{m^2}\,,$$

one finds, rewriting the Gibbs-Duhem relation (1.4.7) in the form $(1-s)\mathrm{d}\mu = \mathrm{d}\mu_s$, from the last inequality (1.7.3)

$$\left(\frac{\partial \mu}{\partial s}\right)_{T,p} > 0 \,. \tag{1.7.4}$$

Strictly speaking, one must not exclude from the inequalities (1.7.2) the equality signs, since the quadratic form (1.7.1) must, in general, be only non-positive. A thermodynamic state for which one of the inequalities (1.7.2) would become an equality is said in thermodynamics to be critical and is subjected to special study. In the following it will be assumed that states of sea water under consideration are not critical.

In this context, it is interesting to note that during simultaneous variations in ϵ, V, m_s, m_w the corresponding quadratic form in $\delta \boldsymbol{v}$, $\delta\epsilon'$, δV, δm_s, δm_w will already no longer be negative definite, but only non-positive. In fact, side by side with the minors (1.7.2) one would have to consider also the determinant Δ_5

$$\Delta_5 = \frac{\partial(\partial\eta/\partial\epsilon,\ \partial\eta/\partial V,\ \partial\eta/\partial m_s,\ \partial\eta/\partial m_w)}{\partial(\epsilon,\ V,\ m_s,\ m_w)}\,\Delta_1 = \frac{1}{T^2}\frac{\partial(\mu_s,\ \mu_w)}{\partial(m_s,\ m_w)}\,\Delta_3 \,.$$

However, this determinant vanishes identically, since by the Gibbs-Duhem relation (1.3.6) one has for two-component mixtures

$$m_s \mathrm{d}\mu_s + m_w \mathrm{d}\mu_w = 0$$

with T, p = const., and therefore

$$\frac{\partial(\mu_s,\ \mu_w)}{\partial(m_s,\ m_w)} = 0 \,.$$

Consider now the second and third inequality (1.7.3). Since during differentiation m_s and m_w remain constant, these inequalities are equivalent to the inequalities

$$c_v > 0, \qquad (\partial\rho/\partial p)_{T,s} > 0 \,. \tag{1.7.5}$$

However, then, by (1.6.1), one has $c_p > c_v$ and $c_p/c_v > 1$ (a fact which has already been used in § 1.6). Further, it follows from (1.6.6) and (1.6.10) that $\kappa_\eta > 0$ and the speed of sound, by (1.6.8), may be determined for any medium. It is interesting to note that, by (1.5.8), one has $\mathrm{d}p_v/\mathrm{d}s < 0$: The pressure of saturated vapour drops as the salinity of sea water is increased.

1.8 CONDITIONS OF EQUILIBRIUM OF SEA WATER

For the definition of entropy in § 1.1 it has been assumed that the system is at rest (in a macroscopic sense) and that no external forces act on it.

Consider now the problem of equilibrium of a finite volume V of fluid located in a field of stationary conservative forces with specific potential $U(M)$ [where M is a point of the medium]; in addition, assume that separate parts of the fluid may undergo macroscopic motion with velocity $\boldsymbol{v}(M)$. For definition of the entropy of such a system, we decompose the volume V into separate particles, which are somewhat small and such that within their limits the fields U and $\boldsymbol{v}$ may be assumed to be homogeneous, but at the same time sufficiently large so that the statistical concept of entropy may make sense for them (here and below, a particle is conceived in a macroscopic sense). Then the entropy of each particle will depend only on its internal energy, volume and composition; the presence of the fields U and $\boldsymbol{v}$ will not affect the magnitude of the entropy of separate particles.

It is simpler to demonstrate this fact by the example of a mixture of two ideal gases. In the case of fields U and $\boldsymbol{v}$ which are homogeneous within the bounds of the entire volume of the mixture, the phase volume Ω_Γ must be computed from the formula [cf. (1.1.6)]

$$\Omega_\Gamma = \int\limits_{\epsilon+U\leqslant H+U\leqslant \epsilon+\delta\epsilon+U} \mathrm{d}q_i \mathrm{d}p_i \,,$$

where H is given by (1.1.5) in which p_1^2 must be replaced by $(p_1 + \nu_1 v_1)^2$, p_2^2 by $(p_2 + \nu_2 v_2)^2$, etc. Clearly, the same magnitude is obtained for Ω_Γ as from (1.1.6).

Thus, by definition, the entropy of a finite volume V of the liquid will be

$$\eta = \int\limits_V \rho(M)\, \eta_m(M)\, \mathrm{d}V \,, \tag{1.8.1}$$

where the specific entropy η_m is a function of the specific internal energy ϵ_m, the density of the medium ρ and the concentration of the admixture s. Analogous formulae may be written down for the internal energy of a finite volume V and likewise for other extensive thermodynamic parameters.

The conditions of thermodynamic equilibrium will now be derived. Assume that the system under consideration is isolated; then its entropy must be a maximum and simultaneously the following laws of conservation must be fulfilled.

(1) The total impulse of the system must be constant

$$\int\limits_V \rho \boldsymbol{v} \mathrm{d}V = \text{constant} \,; \tag{1.8.2}$$

(2) The total moment of momentum of the system must be constant

$$\int\limits_V \boldsymbol{r} \times \rho \boldsymbol{v} \mathrm{d}V = \text{constant}, \tag{1.8.3}$$

where r is the radius vector of the point.

(3) The total energy of the system (sum of kinetic energy of macroscopic motions with velocity $\boldsymbol{v}$, potential energy U and internal energy ϵ_m) must be constant

$$\int_V \rho\left[\frac{\boldsymbol{v}^2}{2} + \epsilon_m + U\right]\mathrm{d}V = \text{constant} \tag{1.8.4}$$

(4) The masses of each component of the system must be constant

$$\int_V \rho_s \mathrm{d}V = \text{constant}, \qquad \int_V \rho_w \mathrm{d}V = \text{constant}\,. \tag{1.8.5}$$

The relations (1.8.2)—(1.8.5), in essence, yield an exact formulation of what must be understood by the condition of isolation of a system.

At the end of § 1.3 [(1.3.8)], it has been seen that the density of the entropy $\rho\eta_m$ may be assumed to be a function of $\rho\epsilon_m$, ρ_s, ρ_w. However, the entropy of a finite volume V, by the constraint (1.8.4), will, in general, depend not only on the fields $\rho\epsilon_m$, ρ_s, ρ_w, but also on the field $\rho\boldsymbol{v}$ (although the entropy of separate particles also does not depend on the field $\rho\boldsymbol{v}$; since the field U is stationary, it has not been included in the list of functions on which η depends; for example, for the gravity field, $U = -gz$ with g the gravitational acceleration and z the downward vertical coordinate). Thus, the determination of the conditions of equilibrium of a system has been reduced to an analysis of the extrema of the functional $\eta(\rho\epsilon_m, \rho_s, \rho_w, \rho\boldsymbol{v})$ for the constraints (1.8.2)—(1.8.5), imposed on the possible functions $\rho\epsilon_m$, ρ_s, ρ_w, $\rho\boldsymbol{v}$.

The method of Lagrange multipliers will be employed to determine the extremum η. Introduce the auxiliary function G

$$G = \int_V \rho\eta_m \mathrm{d}V + \lambda_s \int_V \rho_s \mathrm{d}V + \lambda_w \int_V \rho_w \mathrm{d}V + \lambda \int_V \rho\left(\frac{\boldsymbol{v}^2}{2} + \epsilon_m + U\right)\mathrm{d}V$$

$$+ \boldsymbol{a} \int_V \rho\boldsymbol{v}\mathrm{d}V + \boldsymbol{b} \int_V \boldsymbol{r} \times \rho\boldsymbol{v}\mathrm{d}V\,, \tag{1.8.6}$$

where λ_s, λ_w, λ, $\boldsymbol{a}$, $\boldsymbol{b}$ are constant numbers and vectors. Recall that $\rho = \rho_s + \rho_w$ and that for variations in $\rho\epsilon_m$, ρ_s, ρ_w, $\rho\boldsymbol{v}$ the volume of the system V remains unchanged [the region of integration in (1.8.6)].

Constructing the expression for the first variation of the functional G and using (1.3.8), one has

$$\delta G = \int_V \left\{\left(\frac{1}{T} + \lambda\right] \delta(\rho\epsilon_m) + \left[-\frac{\mu_s}{T} + \lambda U + \lambda_s - \lambda\frac{\boldsymbol{v}^2}{2}\right]\delta\rho_s\right.$$

$$+\left[-\frac{\mu_w}{T}+\lambda U+\lambda_w-\lambda\frac{\boldsymbol{v}^2}{2}\right]\delta\rho_w+[\lambda\boldsymbol{v}+\boldsymbol{a}+\boldsymbol{b}\times\boldsymbol{r}]\,\delta(\rho\boldsymbol{v})\Bigg\}\,\mathrm{d}V\,.$$

Setting equal to zero the coefficients of the variations $\delta(\rho\epsilon_m)$, $\delta\rho_s$, $\delta\rho_w$, $\delta(\rho\boldsymbol{v})$ yields now

$$T=-\frac{1}{\lambda}, \tag{1.8.7}$$

$$\mu_s=-U-\frac{\lambda_s}{\lambda}+\frac{\boldsymbol{v}^2}{2}, \tag{1.8.8}$$

$$\mu_w=-U-\frac{\lambda_w}{\lambda}+\frac{\boldsymbol{v}^2}{2}, \tag{1.8.9}$$

$$\boldsymbol{v}=-\frac{\boldsymbol{a}}{\lambda}-\frac{1}{\lambda}(\boldsymbol{b}\times\boldsymbol{r})\,. \tag{1.8.10}$$

These are the necessary conditions of thermodynamic equilibrium of a finite volume of sea water. Thus, in an equilibrium state:

(1) The temperature T is constant throughout the entire volume of fluid.

(2) The chemical potentials μ_s and μ_w differ only by a constant amount.

(3) The volume of fluid may move only like a rigid body with velocities of translation $-\boldsymbol{a}/\lambda$ and of rotation $-\boldsymbol{b}/\lambda$.

Introduce the strain rate tensor $e_{\alpha\beta}$. Consider for this purpose the Cartesian coordinate system x^α. It is easily shown that the distribution of velocities at the point M' near the point M is given by

$$v_\alpha(M')=v_\alpha(M)+\tfrac{1}{2}[(\operatorname{rot}\boldsymbol{v}\times\delta\boldsymbol{r})_\alpha+e_{\alpha\beta}\delta r^\beta]\,, \tag{1.8.11}$$

where

$$e_{\alpha\beta}=\frac{\partial v_\alpha}{\partial x^\beta}+\frac{\partial v_\beta}{\partial x^\alpha}\,,\qquad \delta\boldsymbol{r}=OM'-OM\,.$$

The arrangement of the indices in (1.8.11) is convenient for the following exposition, although in an orthogonal Cartesian coordinate system it is, of course, indifferent where the indices are placed — up or down (without violating the summation convention).

It is natural to define the strain rate tensor $e_{\alpha\beta}$ in the general tensorial form as

$$e_{\alpha\beta}=\nabla_\alpha v_\beta+\nabla_\beta v_\alpha\,. \tag{1.8.12}$$

Then the relation (1.8.11) may be assumed to hold true in any coordinate system (inasmuch as it is a coordinate system in which it applies, cf. § A.3).

By (1.8.11), the strain rate tensor exactly describes the difference between the motion of the fluid particles and the fluid's motion as a rigid body. Therefore it is obvious that the equilibrium condition (1.8.10) must be equivalent to the condition $e_{\alpha\beta} \equiv 0$ (a fact which is also readily established formally).

Subtracting (1.8.9) from (1.8.8), the equilibrium condition for sea water can be written in the form

$$T \equiv \text{constant}, \qquad \mu \equiv \text{constant}, \qquad e_{\alpha\beta} \equiv 0\,. \tag{1.8.13}$$

This formulation will be employed extensively in what follows.

Finally, the equations of hydrostatics will be derived from the general equilibrium conditions (1.8.7)–(1.8.10). Assuming the system to be at rest ($v = 0$), one has, by the Gibbs-Duhem relations (1.4.7), the first two conditions (1.8.13) and (1.8.9).

$$\frac{1}{\rho}\nabla_\alpha p = \nabla_\alpha \mu_w = -\nabla_\alpha U\,.$$

Since $\nabla_\alpha U = -X_\alpha$ (where X_α is the body force), then

$$\frac{1}{\rho}\nabla_\alpha p = X_\alpha\,. \tag{1.8.14}$$

It is seen that in the presence of mass forces X the pressure p cannot be a constant quantity. However, then also the salinity cannot be constant. In fact, since T, μ = constant, then

$$\frac{\partial \mu}{\partial s}\nabla_\alpha s + \frac{\partial \mu}{\partial p}\nabla_\alpha p = 0\,.$$

The equilibrium vertical salinity gradient in a gravitational force field is readily estimated from this formula. Since $s\partial\mu/\partial s \simeq 7.5 \cdot 10^8$ erg $\cdot$ g^{-1} $|\partial\mu/\partial p| \simeq 1$ erg $\cdot$ g^{-1} (dyne $\cdot$ cm^2)$^{-1}$, then $\partial s/\partial z \simeq 1.3 \cdot 10^{-4} s$ (m^{-1}), which differs essentially from what is observed in the oceans.

1.9 CONDITION FOR THE ABSENCE OF CONVECTION. VÄISÄLÄ FREQUENCY

Assume that the temperature of the sea water T, the salinity s and the density ρ, and also all other thermodynamic parameters depend only on the vertical coordinate z (increasing downwards). Such a fluid is said to be stratified. Since the temperature of the fluid is not constant, then it cannot find itself in a state of thermodynamic equilibrium (cf. § 1.8). However, let it be assumed that the fluid is in a state of mechanical equilibrium and study the condition of stability of such an equilibrium (the condition of absence of convection).

Let a particle of the fluid, located at the level z, move adiabatically to the nearby level $z + \xi$. Assume that at each instant of time the thermodynamic state of the particle may be assumed to be an equilibrium state. The density of the particle at the level $z + \xi$ is $\rho(z) + (\mathrm{d}\rho/\mathrm{d}z)_{\eta_m,s} \cdot \xi$, and the density of the surrounding medium at this level $\rho(z) + (\mathrm{d}\rho/\mathrm{d}z)\xi$. However, then the Archimedian force acting per unit volume of particle is $g[(\mathrm{d}\rho/\mathrm{d}z)_{\eta_m,s} - (\mathrm{d}\rho/\mathrm{d}z)]\xi$. Clearly, this force will tend to return the particle to its former level z only if

$$\frac{\mathrm{d}\rho}{\mathrm{d}z} > \left(\frac{\mathrm{d}\rho}{\partial z}\right)_{\eta_m,s} . \tag{1.9.1}$$

This is the well known condition of stability of the equilibrium of a stratified fluid (the condition of absence of convection). If the fluid were incompressible, then the stability condition would be simply $\mathrm{d}\rho/\mathrm{d}z > 0$; taking account of the adiabatic compression of the particle (and connected with it of its increase in density) has led to the formulation (1.9.1).

If the condition (1.9.1) is fulfilled, then one may introduce the frequency N defined by

$$N^2 = \frac{g}{\rho}\left[\frac{\mathrm{d}\rho}{\mathrm{d}z} - \left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\eta_m,s}\right]. \tag{1.9.2}$$

It follows from the preceding work that the fluid particle displaced from equilibrium will perform small oscillations about its equilibrium position with this frequency N. This frequency, an important parameter of a stratified medium, is referred to as Väisälä frequency (this name not being used universally).

Formula (1.9.2) may be written differently. By (1.8.14) and (1.6.8), one has

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}z}\right)_{\eta_m,s} = \left(\frac{\partial\rho}{\partial p}\right)_{\eta_m,s} \frac{\mathrm{d}p}{\mathrm{d}z} = \frac{g\rho}{c^2} ,$$

and hence

$$N^2 = \frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z} - \frac{g^2}{c^2} . \tag{1.9.3}$$

In the upper layers of the sea (outside a homogeneous layer), the first term on the right-hand side of (1.9.3) significantly exceeds the second term, and the stability condition may be given the approximate form $\mathrm{d}\rho/\mathrm{d}z > 0$.

Assuming ρ to be a function of T, p, s and using (1.6.9), one finds

$$N^2 = \frac{g}{\rho}\left[\left(\frac{\partial\rho}{\partial T}\right)_{p,s}\left(\frac{\mathrm{d}T}{\mathrm{d}z} - g\rho\Gamma\right) + \left(\frac{\partial\rho}{\partial s}\right)_{T,p}\frac{\mathrm{d}s}{\mathrm{d}z}\right]. \tag{1.9.4}$$

Since for sea water $g\rho\Gamma \sim 10^{-4\circ}\mathrm{C/m}$ (a change of temperature of 0.01°C for every 100 m depth), then in the upper layers of the sea for a not large salinity gradient the stability condition reduces effectively to $dT/dz < 0$ [recall that for a temperature, characteristic for the upper layers of the sea, $(\partial\rho/\partial T)_{p,s} < 0$].

Finally, assuming again ρ to be a function of T, p, s, one has, from (1.9.3) and (1.6.10),

$$N^2 = \frac{g}{\rho}\left(\frac{d\rho}{dz}\right)_p + \frac{g^2}{c^2}\left(\frac{c_p}{c_v} - 1\right). \tag{1.9.5}$$

This formula is convenient for obtaining an estimate of N within the limits of homogeneous layers, where the temperature and salinity vary little with depth. In contrast, in those layers of the sea where dT/dz and ds/dz are large, the second term in (1.9.5) may be omitted (since $c_p \simeq c_v$). Therefore, assuming that $(d\rho/dz)_p \simeq 10^{-3}\ (d\sigma_t/dz)$, one obtains an approximate formula for the evaluation of N in those layers where T and s vary essentially with depth:

$$N^2 = 10^{-3}\frac{g}{\rho}\frac{d\sigma_t}{dz}. \tag{1.9.6}$$

In conclusion, it should be pointed out that in the ocean the smallest value of N is of the order of $10^{-3} \div 10^{-4}$ sec^{-1} (corresponding to periods of 1.7—17 hours), while the largest value of N, usually attained in the seasonal thermocline, is of an order of 10^{-2} sec^{-1} (corresponding to a period of ~10 minutes).

COMMENT ON CHAPTER 1

The material of this chapter is classical. Therefore it is difficult here to give references. General courses on thermodynamics and statistical physics have been used which were written by Landau and Lifshits [60], Kubo [56], Morse [82], Khinchin [50] and Haase [29], and likewise the survey paper by Fofonoff [24]. The derivation of the formula (1.2.3) has been taken from Kubo [56]. For the derivation of N, the work of Väisälä [124] has been followed (confer also Eckart [13]).

CHAPTER 2

DYNAMICS, THERMODYNAMICS OF IRREVERSIBLE PROCESSES

2.1 THERMODYNAMIC PARAMETERS IN A NON-EQUILIBRIUM STATE

Hitherto, only states of thermodynamic equilibrium have been considered when the internal state of a system is characterized completely by such parameters as, for example, ϵ, V, m_s and m_w. As it has been seen, for an equilibrium state, one may introduce entropy η as a function of ϵ, V, m_s and m_w. Further, changes of the function of state have been studied for transition from one equilibrium state to another (equilibrium processes) and conditions of thermodynamic equilibrium of a finite fluid volume have been derived.

Next, non-equilibrium processes of transition in a fluid medium will be studied. In other words, a system will be investigated which is not in a state of thermodynamic equilibrium. Is it possible to use for this purpose the results of the thermodynamics of equilibrium states?

Assume that the characteristic relaxation time of the system (time of transition to an equilibrium state) decreases as the dimensions of the system decrease. Therefore subdivide this system into a set of somewhat small particles (containing, however, a large number of molecules) in order that the relaxation time of each particle will be significantly shorter than the characteristic time scale of the process under consideration. Then it may be assumed in approximation that at any instant of time any particle finds itself in a state of thermodynamic equilibrium, and for each particle the entropy may be determined as an *equilibrium* function of its internal energy, volume and composition. After this, temperature, pressure, chemical potentials, etc., may be determined by ordinary means. In this manner, Gibbs' relation proves to be valid for each particle and, consequently, also all formulae of equilibrium thermodynamics.

In the sequel, entropy of a non-equilibrium system will be understood to be the sum of the entropies of all equilibrium particles into which the original system has been decomposed. By strength of the extensiveness of the entropy of an equilibrium system, further decomposition of equilibrium particles into small parts does not affect the magnitude of the entropy of a non-equilibrium system. Thus, the entropy of a finite volume V of a fluid volume is defined by

$$\eta(t) = \int_V \rho(M, t)\, \eta_m(M, t)\, \mathrm{d}V\,, \tag{2.1.1}$$

where $\eta_m(M, t)$ is the specific entropy; it is important that $\eta_m = \eta_m(\epsilon_m, \rho, s)$. In an analogous manner, also other extensive thermodynamic parameters for a finite volume of a continuous medium may be defined (compare the reasoning in § 1.8 for the definition of entropy of a finite volume of a fluid; it was explained there why specific entropy η_m does not depend on the fields of macroscopic velocities and external forces).

Thus, in the case of a liquid medium, one may speak of fields of temperature $T(M, t)$, pressure $p(M, t)$, specific entropy $\eta_m(M, t)$, specific internal energy $\epsilon_m(M, t)$, etc., and assume Gibbs' relation (1.4.1) for specific quantities to be true at each point of the medium.

It should be emphasized that, in contrast to the specific entropy η_m, the entropy η of a finite volume V of a fluid medium depends, of course, not only on the internal energy of this volume, the magnitude of the volume and its composition, but also on a number of other parameters (naturally, this statement does not relate to the case when this volume is in an equilibrium state).

The approximation introduced is normally referred to as approximation of local thermodynamic equilibrium. A definition of entropy of a non-equilibrium system and an estimate of the accuracy of the approximation introduced presents a very difficult problem and will not be considered here (cf., for example, [135]). However, it is clear intuitively that for systems with not very large gradients in the basic parameters the approximation above must be true. In what follows, consideration will be restricted to just such systems.

The basic physical laws for continuous fluid media will now be formulated.

2.2 EQUATIONS OF CONSERVATION OF MASS

Assume that each component of a mixture may be considered as a continuous medium with its own velocity field. Let ρ_s, ρ_w, $\boldsymbol{v}_s$ and $\boldsymbol{v}_w$, respectively, be the densities and velocities of the salt component and pure water. Then one may postulate the conservation laws

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_s} \rho_s \mathrm{d}V = 0\,, \tag{2.2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_w} \rho_w \mathrm{d}V = 0\,. \tag{2.2.2}$$

Here and below, it is assumed that integration is extended over the individual volume, i.e. the moving, deforming volume consisting of one and the same particles of the medium [in (2.2.1), the particles of salt, in (2.2.2), those of pure water]. Note that in writing down (2.2.1) and (2.2.2), chemical reactions between the components of the mixture have not been taken into account.

Using (A.7.15) and the arbitrariness of an individual volume, one arrives at the differential form of the equations of conservation of mass:

$$\frac{\partial \rho_s}{\partial t} + \operatorname{div}(\rho_s \boldsymbol{v}_s) = 0\,, \tag{2.2.3}$$

$$\frac{\partial \rho_w}{\partial t} + \operatorname{div}(\rho_w \boldsymbol{v}_w) = 0\,. \tag{2.2.4}$$

Obviously, equations (2.2.3) and (2.2.4) follow from (2.2.1) and (2.2.2) only if all the functions entering into (2.2.3) and (2.2.4) are sufficiently smooth. Throughout this book, it will be assumed that this condition is fulfilled.

Next, consider a more convenient characteristic. Introduce the velocity field $\boldsymbol{v}$ as the velocity of the centre of inertia of a particle of sea water:

$$\boldsymbol{v} = \frac{\rho_s \boldsymbol{v}_s + \rho_w \boldsymbol{v}_w}{\rho_s + \rho_w} \tag{2.2.5}$$

Since the density of sea water is $\rho = \rho_s + \rho_w$, addition of equations (2.2.3) and (2.2.4) yields the equation of conservation of mass of sea water

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0\,. \tag{2.2.6}$$

Equations (2.2.3) and (2.2.4) may be written in the form,

$$\frac{\partial \rho_s}{\partial t} = -\operatorname{div}(\rho_s \boldsymbol{v} + \boldsymbol{I}_s)\,, \tag{2.2.7}$$

$$\frac{\partial \rho_w}{\partial t} = -\operatorname{div}(\rho_w \boldsymbol{v} + \boldsymbol{I}_w)\,, \tag{2.2.8}$$

where

$$\boldsymbol{I}_s = \rho_s(\boldsymbol{v}_s - \boldsymbol{v})\,, \tag{2.2.9}$$

$$\boldsymbol{I}_w = \rho_w(\boldsymbol{v}_w - \boldsymbol{v})\,. \tag{2.2.10}$$

The vectors $\rho\boldsymbol{v}$, $\rho\boldsymbol{v}_s + \boldsymbol{I}_s$, $\rho_w\boldsymbol{v} + \boldsymbol{I}_w$ are normally referred to as vectors of density of mass fluxes of sea water, salt and pure water, respectively. If $\boldsymbol{n}$ is the external normal to the surface Σ, then the expressions $(\rho\boldsymbol{v}, \boldsymbol{n})$, $(\rho_s\boldsymbol{v} + \boldsymbol{I}_s, \boldsymbol{n})$ and $(\rho_w\boldsymbol{v} + \boldsymbol{I}_w, \boldsymbol{n})$ represent the corresponding masses passing in unit time through unit area of the fixed surface Σ, bounding a volume V. The vectors $\rho\boldsymbol{v}$, $\rho_s\boldsymbol{v}$ and $\rho_w\boldsymbol{v}$ characterize *advective* transport of mass (caused by macroscopic motion of sea water particles with velocity $\boldsymbol{v}$), the vectors $\boldsymbol{I}_s$ and $\boldsymbol{I}_w$ diffusive transport of mass (not linked to macroscopic motion). In fact, integrating (2.2.7) and (2.2.8) over the volume V, one finds

$$\frac{d}{dt}\int_V \rho_s dV = -\oint_\Sigma (\boldsymbol{I}_s, \boldsymbol{n})\, d\Sigma\,, \tag{2.2.11}$$

$$\frac{d}{dt}\int_V \rho_w dV = -\oint_\Sigma (\boldsymbol{I}_w, \boldsymbol{n})\, d\Sigma\,, \tag{2.2.12}$$

where the individual volume V consists of one and the same particles of sea water. It is seen that the vectors $\boldsymbol{I}_s$ and $\boldsymbol{I}_w$ characterize mass transport through the moving surface Σ bounding the volume V.

Equation (2.2.7) and (2.2.8) are normally referred to as diffusion equations for salt and sea water, respectively. Using (2.2.6), the definition of total derivative (cf. § A.7) and the fact that $\rho_s = \rho s$, equations (2.2.6) and (2.2.7) may be rewritten in the alternate forms

$$\frac{d\rho}{dt} + \rho \operatorname{div} \boldsymbol{v} = 0\,, \tag{2.2.13}$$

$$\rho\frac{ds}{dt} = -\operatorname{div} \boldsymbol{I}_s\,. \tag{2.2.14}$$

which often prove useful.

Adding (2.2.9) and (2.2.10), one obtains, by (2.2.5),

$$\boldsymbol{I}_s + \boldsymbol{I}_w = 0\,. \tag{2.2.15}$$

Thus, two laws of mass conservation have been postulated: (2.2.1) and (2.2.2) or, in differential form, (2.2.3) and (2.2.4). These equations have been written in terms of parameters of the medium such as ρ_s, ρ_w, $\boldsymbol{v}_s$ and $\boldsymbol{v}_w$. If one selects as parameters of the medium the quantities $\rho_s = \rho s$, ρ, $\boldsymbol{v}$ and $\boldsymbol{I}_s$, then it is convenient to employ as equations of mass conservation the equations of conservation of mass of sea water (2.2.6) and the diffusion equation for salt (2.2.7) [or in the form (2.2.13) and (2.2.14)].

In conclusion, consider mass exchange through the free ocean surface: evaporation, precipitation, formation and thawing of ice. The total effect of these processes may be described by specification of the flux of pure water b in unit time per unit area. Then one has that at the ocean surface $F(\boldsymbol{r}, t) = 0$

$$\rho_w(\boldsymbol{v}_w - \boldsymbol{v}_F, \boldsymbol{n}) = b, \qquad \rho_s(\boldsymbol{v}_s - \boldsymbol{v}_F, \boldsymbol{n}) = 0\,, \tag{2.2.16}$$

where $\boldsymbol{v}_F = d\boldsymbol{r}/dt$ is the velocity of the motion of points of the surface F, $\boldsymbol{n} = \nabla F/|\nabla F|$ is the normal to this surface, and it has been assumed in writing down (2.2.16) that $b > 0$, if ∇F is directed into the ocean and $b < 0$ if ∇F points out of the ocean. Differentiating the equation $F(\boldsymbol{r}, t) = 0$ with respect to time, one finds

$$(\boldsymbol{v}_F, \boldsymbol{n}) = \frac{1}{|\nabla F|}\frac{\partial F}{\partial t}.$$

Introducing instead of $\boldsymbol{v}_s$ and $\boldsymbol{v}_w$ the velocity $\boldsymbol{v}$ and diffusive flux $\boldsymbol{I}_s$, equations (2.2.16) may be rewritten in the form

$$\frac{\partial F}{\partial t} + (v, \nabla F) = \frac{b}{\rho}|\nabla F|, \qquad (\boldsymbol{I}_s, \nabla F) = -bs|\nabla F|. \tag{2.2.17}$$

This is the final form of the boundary conditions at the free ocean surface.

2.3 EQUATIONS OF MOTION

As a rule, it may be assumed that the Earth has the shape of a sphere and rotates with constant angular velocity Ω: $\Omega = 7.29 \cdot 10^{-5}\ \mathrm{sec}^{-1}$. It is natural to consider motion of sea water from the point of view of an earthbound observer. However, then one is forced to operate with a non-inertial reference system and to take in the equations of motion inertia forces into account, i.e., centripetal and Coriolis forces.

External forces acting on an individual volume V of a continuous medium may be subdivided into mass (volume) and surface forces. The most important mass force is that of gravity; it is equal to the Earth's attraction and the centripetal force. It will be assumed that the specific value of the gravity force (or gravitational acceleration) is a constant vector directed along the Earth's radius towards its centre. The specific value of the Coriolis force is known to be given by $2\boldsymbol{v} \times \boldsymbol{\Omega}$. In many problems, importance also attaches to the tide-generating forces of Moon and Sun. In what follows, the resultant of all mass forces (excluding the Coriolis force) per unit mass will be denoted, as in Chapter 1, by $\boldsymbol{X}$, and it will be assumed that it has the potential U so that $X_\alpha = -\nabla_\alpha U$.

External surface forces with which a surrounding medium acts on an area $\mathrm{d}\Sigma$ are conveniently described by $\boldsymbol{p}(\boldsymbol{n})\mathrm{d}\Sigma$ (where $\boldsymbol{n}$ is the external normal to $\mathrm{d}\Sigma$; it determines the orientation of this element in space). Analogously, the vector $\boldsymbol{p}(\boldsymbol{n})$ may also characterize internal surface forces at the point M. For this purpose, one needs only switch mentally the medium, located on the side of the normal $\boldsymbol{n}$ to the area $\mathrm{d}\Sigma$, and replace its effect on $\mathrm{d}\Sigma$ by the vector $\boldsymbol{p}(\boldsymbol{n})$. By Newton's third law, $\boldsymbol{p}(\boldsymbol{n}) = -\boldsymbol{p}(-\boldsymbol{n})$.

Thus, consider an arbitrary finite volume V of a continuous medium and let Σ be the surface bounding V. Postulate Newton's second law for this volume in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \boldsymbol{v}\mathrm{dV} = \int_V \rho(\boldsymbol{X} + 2\boldsymbol{v} \times \boldsymbol{\Omega})\ \mathrm{d}V + \oint_\Sigma \boldsymbol{p}(\boldsymbol{n})\ \mathrm{d}\Sigma\ . \tag{2.3.1}$$

As in the preceding section, let the motion V be individualized. Thus, the change in unit time of the total impulse of V (the impulse of unit volume, by definition, is equal to $\rho \boldsymbol{v}$) equals the resultant of all mass as well as surface forces, external with respect to this volume.

Apply (2.3.1) to an infinitesimal tetrahedron with vertex at M and edges along the coordinate axes [assuming (2.3.1) to be applicable to any volume]. Using a Cartesian orthogonal coordinate system with base vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, and letting the volume of the tetrahedron vanish, one obtains

$$\boldsymbol{p}(\boldsymbol{n}) = \boldsymbol{p}(\boldsymbol{e}_1)\, n' + \boldsymbol{p}(\boldsymbol{e}_2)\, n^2 + \boldsymbol{p}(\boldsymbol{e}_3)\, n^3 \ ,$$

where $n = n^i e_i$.

Thus, in a Cartesian orthogonal coordinate system, the vector $\boldsymbol{p}(\boldsymbol{n})$ depends linearly and homogeneously on the vector $\boldsymbol{n}$. Obviously, such characteristic dependence is conserved in any coordinate system at the point M. Thus,

$$p_\alpha(\boldsymbol{n}) = p_{\alpha\beta} n^\beta \ . \tag{2.3.2}$$

According to the tensor criterion (cf. § A.3), $p_{\alpha\beta}$ is a tensor; as a rule, it is called the stress tensor at the point M. Directing $\boldsymbol{n}$ along the coordinate lines, the physical significance of the individual tensor components $p_{\alpha\beta}$ is readily explained.

It is customary to present $p_{\alpha\beta}$ in the form

$$p_{\alpha\beta} = -p m_{\alpha\beta} + \sigma_{\alpha\beta} \ , \tag{2.3.3}$$

where $\sigma_{\alpha\beta}$ is the viscous stress tensor, p the pressure and $m_{\alpha\beta}$ the metric tensor.

Newton's second law will now be written down in differential form. Ordinarily, one employs (A.7.15) for each component of the vector $\rho \boldsymbol{v}$ and utilizes the arbitrariness of the volume V to obtain first this law in a Cartesian orthogonal coordinate system; then it is written in general tensorial form [cf. the derivation of (1.8.11)]. One has

$$\frac{\partial}{\partial t}(\rho v^\alpha) = -\nabla_\beta \Pi^{\alpha\beta} + \rho X^\alpha + 2\rho \epsilon^{\alpha\beta\gamma} v_\beta \Omega_\gamma \ , \tag{2.3.4}$$

where

$$\Pi^{\alpha\beta} = \rho v^\alpha v^\beta + p m^{\alpha\beta} - \sigma^{\alpha\beta} \ . \tag{2.3.5}$$

The tensor $\Pi^{\alpha\beta}$ is called the tensor of density of the momentum flux. Newton's law in the form (2.3.4) is normally referred to as momentum flux equation.

It follows from (2.3.4) that generation of momentum (per unit volume in unit time) is linked to action of mass forces X and Coriolis forces. If these were absent, then change in the momentum $\int_V \rho \boldsymbol{v} \mathrm{d}V$ of the volume V in unit time would be equal to the flux $\oint_\Sigma \Pi^{\alpha\beta} n_\alpha \mathrm{d}\Sigma$ through the surface Σ and, under the assumption that there exists a shell such that $\oint_\Sigma \Pi^{\alpha\beta} n_\alpha \mathrm{d}\Sigma = 0$, the momentum $\int_V \rho \boldsymbol{v} \mathrm{d}V$ would not change in time. Therefore, in cases when the change per unit time of density of some scalar (or vector) may be presented in the form of divergence of some vector (or tensor), one speaks of the equation of conservation of this scalar (or vector). For example, equation (2.2.7) is the equation of conservation of salt; in the absence of mass and Coriolis forces, equation (2.3.4) is the equation of conservation of momentum, etc.

Using the mass conservation equation (2.2.13) and the concept of the total derivative of a vector (cf. § A.7) equation (2.3.4) may be rewritten in the form

$$\rho \frac{\mathrm{d}v_\alpha}{\mathrm{d}t} = -\nabla_\alpha p + \rho X_\alpha + \nabla_\beta \sigma^\beta_\alpha + 2\rho \epsilon_{\alpha\beta\gamma} v^\beta \Omega^\gamma \,. \tag{2.3.6}$$

It is generally referred to as equation of motion.

2.4 EQUATIONS OF ANGULAR MOMENTUM

Let O be an arbitrary point and $\boldsymbol{r} = \boldsymbol{OM}$ the corresponding vector of the point M. Then one may postulate independently of (2.3.1) for an arbitrary individualized volume V the law

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \boldsymbol{r} \times \rho \boldsymbol{v} \mathrm{dV} = \int_V \{\boldsymbol{r} \times \rho(\boldsymbol{X} + 2\boldsymbol{v} \times \boldsymbol{\Omega})\} \,\mathrm{d}V + \oint_\Sigma \boldsymbol{r} \times \boldsymbol{p}(\boldsymbol{n}) \,\mathrm{d}\Sigma \,, \tag{2.4.1}$$

or, in words, the change in unit time of the total moment of the momentum of the volume V about the point O equals the resultant moment (about the same point O) of all forces which are external with respect to this volume.

In some models of continuous media, one must take into account in the writing down of (2.4.1) internal moments of momentum and internal couples (cf. for more details [108, Chapter III, § 3]).

Using (A.7.15) and the arbitrariness of V, one has

$$\frac{\partial}{\partial t} (\epsilon_{\alpha\beta\gamma} x^\beta \rho v^\gamma) + \nabla_\delta (\epsilon_{\alpha\beta\gamma} x^\beta \rho v^\gamma v^\delta) = \epsilon_{\alpha\beta\gamma} x^\beta \rho (X^\gamma + 2\epsilon^{\gamma\omega\chi} v_\omega \Omega_\chi) +$$
$$+ \nabla_\delta (\epsilon_{\alpha\beta\gamma} x^\beta p^{\gamma\delta}) \,. \tag{2.4.2}$$

Multiply (2.3.4) vectorially from the left by $\boldsymbol{r}$ and subtract the result from (2.4.2), to obtain after some simple transformations

$$\epsilon_{\alpha\beta\gamma} p^{\beta\gamma} = 0 \,, \tag{2.4.3}$$

which establishes the symmetric nature of the stress tensor $p^{\beta\gamma}$.

Equation (2.4.3) is equivalent to (2.4.2). In what follows, it will be assumed that the tensor $p^{\alpha\beta}$, by (2.4.3), is symmetric, and therefore (2.4.2) will no longer be considered to be independent of (2.3.4).

2.5 EQUATION OF CONSERVATION OF ENERGY

A start will be made with a study of an equilibrium system which is open, i.e., exchanges mass with a surrounding medium. The equality $d\epsilon = \delta A$, by definition, is true only for adiabatic processes (cf. § 1.2); recall that δA is the work of all external forces acting on the system. In the general case, one has $\epsilon \neq \delta A$. For an open system, the quantity of heat δQ received is represented conveniently by the formula

$$\delta Q = d\epsilon - \delta A - \sum_1^n \chi_j \delta_e m_j , \tag{2.5.1}$$

where $\chi_j = (\partial \chi / \partial m_j)_{T,p,m_i}$ is the partial enthalpy of component j of the mixture, and $\delta_e m_j$ is the increase in the mass m_j of the component due to mass exchange with the surrounding medium. Equation (2.5.1) may be called equation of conservation of energy for an open equilibrium system.

Definition (2.5.1) is not the only possible definition of δQ for an open system (cf. Landau and Lifshits [59, § 57] and Eckart [14]). In fact, the very quantity δQ is very conditional. Thus, for a closed system, it is defined as that part of the change in the internal energy of a system which is not linked to work done by external forces on the system. In the case of an open system, it is convenient to especially separate out the change in energy caused by mass exchange with the surrounding medium and not to include it in δQ. The definition (2.5.1) has been constructed in just this manner. In fact, during equilibrium increase in mass of a system for constant T and p, one has

$$\delta Q = d\epsilon + p dV - \sum_1^n \chi_j \delta_e m_j = d\chi - \sum_1^n \chi_j dm_j = 0 .$$

In analogy with equation (2.5.1) of energy conservation for an equilibrium system, let it be postulated that for a finite volume V of sea water

$$- \oint_\Sigma q(\boldsymbol{n})\, d\Sigma = \frac{d}{dt} \int_V \rho \left(\frac{v^2}{2} + \epsilon_m \right) dV - \int_V (\rho \boldsymbol{X}, \boldsymbol{v})\, dV - \oint_\Sigma [\boldsymbol{p}(\boldsymbol{n}), \boldsymbol{v}]\, d\Sigma$$
$$+ \oint_\Sigma \{\chi_s(\boldsymbol{I}_s, \boldsymbol{n}) + \chi_w(\boldsymbol{I}_w, \boldsymbol{n})\}\, d\Sigma , \tag{2.5.2}$$

where χ_s is the partial enthalpy of salt and χ_w that of pure water.

The significance of the individual terms in (2.5.2) will now be explained. On the left-hand side of the equation, one has an expression for the amount of heat obtained by the system from outside media during unit time; since heat enters only through the surface, this quantity must be represented by a surface integral (if the shell is adiabatic, there will be no heat flux) with some scalar density $q(\boldsymbol{n})$ which depends on the orientation of $\mathrm{d}\Sigma$ (the negative sign is due to the fact that $\boldsymbol{n}$ is the outward normal). The first term on the right-hand side describes the rate of change of the energy of the system (where it is natural to understand by the energy of the system during irreversible processes the sum of its kinetic and internal energies). The following two terms on the right-hand side describe the work done by external forces (mass and surface) on the system in unit time (the work done by the Coriolis force equals zero). Finally, the last term on the right-hand side of (2.5.2) represents the mass exchange with the surrounding medium, since the mass fluxes through $\mathrm{d}\Sigma$ of salt and pure water are $-(\boldsymbol{I}_s, \boldsymbol{n})$ and $-(\boldsymbol{I}_w, \boldsymbol{n})$, respectively [cf. (2.2.11) and (2.2.12)].

Applying (2.5.2) to an infinitesimal tetrahedron at the point M and letting the volume of the tetrahedron vanish, one finds that the scalar $q(\boldsymbol{n})$ depends linearly on the normal $\boldsymbol{n}$ [cf. the derivation of (2.3.2)]. Therefore one may introduce at each point of a medium a vector $\boldsymbol{q}$ such that

$$q(\boldsymbol{n}) = (\boldsymbol{q}, \boldsymbol{n}). \tag{2.5.3}$$

This vector is called the heat flux density vector.

In the ordinary manner, one obtains from (2.5.2) the differential form of the energy equation:

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{v^2}{2} + \epsilon_m\right)\right\} = -\nabla_\alpha\left\{\rho\left(\frac{v^2}{2} + \epsilon_m\right) v^\alpha + q^\alpha - p^{\alpha\beta}v_\beta + \chi_s I_s^\alpha + \chi_w I_w^\alpha\right\} + \rho(\boldsymbol{X}, \boldsymbol{v}) . \tag{2.5.4}$$

This equation will now be transformed. Firstly, by (2.2.15), one has $\chi_s \boldsymbol{I}_s + \chi_w \boldsymbol{I}_w = (\chi_s - \chi_w)\boldsymbol{I}_s$, and further $\chi_s - \chi_w = (\partial \chi_m/\partial s)_{T,p}$ which follows from the identity

$$\mathrm{d}\left(\frac{\chi}{m}\right) = \chi_s \mathrm{d}\left(\frac{m_s}{m}\right) + \chi_w \mathrm{d}\left(\frac{m_w}{m}\right) + (\chi_s m_s + \chi_w m_w - \chi)\frac{\mathrm{d}m}{m^2},$$

true for T, p = constant. Secondly, since the body force has the potential U, one has

$$-\rho(\boldsymbol{X}, \boldsymbol{v}) = \rho\nabla_\alpha U v^\alpha = \frac{\partial}{\partial t}(\rho U) + \nabla_\alpha(\rho U v^\alpha) - \rho\frac{\partial U}{\partial t} . \tag{2.5.5}$$

Therefore (2.5.4) may be rewritten in the form of the energy transport equation

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{v^2}{2} + \epsilon_m + U\right)\right\} = -\nabla_\alpha\left\{\rho\left(\frac{v^2}{2} + U + \epsilon_m\right) v^\alpha + q^\alpha - p^{\alpha\beta}v_\beta + \left(\frac{\partial \chi_m}{\partial s}\right)_{T,p} I_s^\alpha\right\} + \rho\frac{\partial U}{\partial t} . \tag{2.5.6}$$

It is natural to call the quantity $E_m = \frac{1}{2}v^2 + \epsilon_m + U$ the total specific energy of a particle (sum of kinetic, internal and potential energies). If $\partial U/\partial t = 0$, as this is the case for the gravitational force, then (2.5.6) leads to

$$\frac{\partial}{\partial t}(\rho E_m) = -\mathrm{div}\,\boldsymbol{I}_E\,, \tag{2.5.7}$$

the equation of conservation of energy, where

$$I_E^\alpha = \rho\left(\frac{v^2}{2} + \epsilon_m + U\right) v^\alpha + q^\alpha - p^{\alpha\beta}v_\beta + \left(\frac{\partial \chi_m}{\partial s}\right)_{T,\,p} I_s^\alpha, \tag{2.5.8}$$

The vector $\boldsymbol{I}_E$ is called the vector of the density of energy flux. Obviously, the vector $\rho(\epsilon_m + U + \boldsymbol{v}^2/2)v^\alpha$ describes the advective energy flux (caused by macroscopic motion with velocity $\boldsymbol{v}$). The remaining terms in the expression for I_E are linked to influx of heat, work done by surface forces and mass exchange. It follows from (2.5.7) that, if the volume V be surrounded by a shell such that $\oint_\Sigma(I_E, \boldsymbol{n})\mathrm{d}\Sigma = 0$, then the total energy of such a volume will not change in time.

Note that (as has been seen in § 1.5) specific internal energy ϵ_m may be defined from experimental data only exactly apart from a function $as + b$, where a and b are constants. It is readily shown, using (2.5.6) and (2.2.14), that the definition of heat influx into a volume V, according to (2.5.2), does not depend on the function $as + b$.

Thus, the basic equations of conservation of mass of sea water, diffusion of salt, motion and energy conservation have been written down. However, this system of six equations contains at least eighteen unknown functions. These are temperature T, pressure p, salinity s, the velocity vector $\boldsymbol{v}$, as well as the vectors $\boldsymbol{I}_s$ and $\boldsymbol{q}$ and the symmetric tensor $\sigma_{\alpha\beta}$. Using thermodynamic relations, one may express in terms of T, p and s the density (equation of state of sea water), the specific internal energy ϵ_m, the specific entropy η_m and other thermodynamic parameters. In future, the functions T, p, s and $\boldsymbol{v}$ will be referred to as basic parameters of the medium.

The parameters $\boldsymbol{I}_s$, $\boldsymbol{q}$ and $\sigma_{\alpha\beta}$ characterize non-equilibrium processes in the medium (diffusion, heat conduction, internal friction) and therefore may not be evaluated on the basis of formulae of equilibrium thermodynamics. As a rule, they are referred to as thermodynamic fluxes.

The remaining part of this chapter is devoted to establishment of relations between the thermodynamic fluxes $\boldsymbol{I}_s$, $\boldsymbol{q}$ and $\sigma_{\alpha\beta}$ and the basic parameters of the medium T, p, s and $\boldsymbol{v}$. Analysis of changes in the system's entropy plays here a resolving role.

2.6 EQUATIONS FOR MECHANICAL AND INTERNAL ENERGY

Consider certain simple consequences of the basic equations. Multiply (2.3.6) scalarly by $\boldsymbol{v}$; since the Coriolis force does not perform any work, it follows that

$$\rho\frac{\partial}{\partial t}\left(\frac{v^2}{2}\right) + \rho v^\alpha \nabla_\alpha\left(\frac{v^2}{2}\right) = \rho X_\alpha v^\alpha + v_\alpha \nabla_\beta p^{\alpha\beta} .$$

Integrating this relation over the volume V and using the symmetry of the tensor $p^{\alpha\beta}$, one finds after some elementary transformations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\frac{v^2}{2}\mathrm{d}V = \oint_\Sigma p^{\alpha\beta} n_\alpha v_\beta \mathrm{d}\Sigma + \int_V \rho X_\alpha v^\alpha \mathrm{d}V - \tfrac{1}{2}\int_V p^{\alpha\beta} e_{\alpha\beta}\mathrm{d}V , \tag{2.6.1}$$

where $e_{\alpha\beta}$ is the strain rate tensor (cf. § 1.8).

The first two terms on the right-hand side of (2.6.1) represent work done (in unit time) by the external (mass and surface) forces. Since a change in the system's kinetic energy is equal to the work done by external forces acting on the system, it is clear that the last term on the right-hand side of (2.6.1) expresses the work done by the internal forces of the given system (in unit time).

Since the body force X has a potential U, one has from (2.5.5)

$$\frac{\partial}{\partial t}\left\{\rho\left(\frac{v^2}{2} + U\right)\right\} = -\nabla_\alpha\left\{\rho\left(\frac{v^2}{2} + U\right) v^\alpha - p^{\alpha\beta} v_\beta\right\} - \tfrac{1}{2}p^{\alpha\beta} e_{\alpha\beta} + \rho\frac{\partial U}{\partial t} . \tag{2.6.2}$$

It is natural to call the quantity $U + v^2/2$ the specific mechanical energy and the vector $\rho(v^2/2 + U)v^\alpha + p^{\alpha\beta}v_\beta$ the vector of the density of mechanical energy flux. The expression $-\frac{1}{2}p^{\alpha\beta}e_{\alpha\beta} + \rho\partial U/\partial t$ yields the amount of mechanical energy generated in unit volume in unit time. The equation (2.6.2) is called the equation of mechanical energy transfer.

Subtract now term by term equation (2.6.2) from (2.5.6):

$$\frac{\partial}{\partial t}(\rho\epsilon_m) = -\nabla_\alpha\left\{\rho\epsilon_m v^\alpha + q^\alpha + \left(\frac{\partial\chi_m}{\partial s}\right)_{T,p} I_s^\alpha\right\} + \tfrac{1}{2}p^{\alpha\beta}e_{\alpha\beta} . \tag{2.6.3}$$

This is the equation of internal energy transfer. By (2.6.3), the vector of the density of internal energy flux is equal to $\rho\epsilon_m v^\alpha + q^\alpha + (\partial\chi_m/\partial s)_{T,p} I_s^\alpha$, and the amount of internal energy generated in unit time in unit volume equals $\frac{1}{2}p^{\alpha\beta}e_{\alpha\beta}$.

It is seen that the quantity $p^{\alpha\beta}e_{\alpha\beta}/2$ enters with opposite signs into (2.6.2) and (2.6.3). Hence it is clear that it describes the interconversion of mechanical and internal energies.

Using (2.3.3), one has

$$\tfrac{1}{2}p^{\alpha\beta}e_{\alpha\beta} = -p \operatorname{div} v + \tfrac{1}{2}\sigma^{\alpha\beta}e_{\alpha\beta} .$$

It will be shown below that always $\sigma^{\alpha\beta}e_{\alpha\beta} > 0$. It is natural to call this term dissipation of mechanical energy (in unit time in unit volume). Note that for an incompressible fluid div $v = 0$ and mechanical energy always goes over (dissipates) into internal energy. For a compressible fluid, generally speaking, reverse conversion is possible.

Using the equation of mass conservation (2.2.13), equation (2.6.3) may be rewritten in the form

$$\rho\frac{\mathrm{d}\epsilon_m}{\mathrm{d}t} = -\nabla_\alpha\left[q^\alpha + \left(\frac{\partial\chi_m}{\partial s}\right)_{T,\,p} I_s^\alpha\right] + \tfrac{1}{2}p^{\alpha\beta}e_{\alpha\beta}\,. \tag{2.6.4}$$

2.7 EQUATION OF ENTROPY TRANSFER

In § 2.1, the basic proposition on the validity of Gibbs' relation for specific entropy η_m (approximation of local thermodynamic equilibrium) has been discussed. In studying changes in the parameters of particles in unit time, one finds

$$T\frac{\mathrm{d}\eta_m}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon_m}{\mathrm{d}t} + p\frac{\mathrm{d}\left(\frac{1}{\rho}\right)}{\mathrm{d}t} - \mu\frac{\mathrm{d}s}{\mathrm{d}t}.$$

Using (2.2.13), (2.2.14) and (2.6.4), express the rate of change of the entropy η_m in terms of the thermodynamic fluxes $\boldsymbol{I}_s$, $\boldsymbol{q}$ and $\sigma_{\alpha\beta}$. After simple transformations, one obtains

$$\rho\frac{\mathrm{d}\eta_m}{\mathrm{d}t} = -\frac{1}{T}\mathrm{div}\left\{\boldsymbol{q} + \left(\frac{\partial\chi_m}{\partial s}\right)_{T,\,p}\boldsymbol{I}_s\right\} + \frac{1}{2T}\sigma^{\alpha\beta}e_{\alpha\beta} + \frac{\mu}{T}\,\mathrm{div}\,\boldsymbol{I}_s\,. \tag{2.7.1}$$

This equation will be called the equation of entropy evolution.

Proceed now to formulation of the second law of thermodynamics. Consider an individual volume V of sea water as a thermodynamic system; the change of the entropy η of such a system (in unit time) is obtained by integration of (2.7.1) over the volume V. One has

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\eta_m\mathrm{d}V = \int_V\left\{-\frac{1}{T}\,\mathrm{div}\left[\boldsymbol{q} + \left(\frac{\partial\chi_m}{\partial s}\right)_{T,\,p}\boldsymbol{I}_s\right] + \frac{1}{2T}\sigma^{\alpha\beta}e_{\alpha\beta} + \frac{\mu}{T}\,\mathrm{div}\,\boldsymbol{I}_s\right\}\mathrm{d}V\,. \tag{2.7.2}$$

Entropy changes of any system may be presented in the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{\delta_e\eta}{\mathrm{d}t} + \frac{\delta_i\eta}{\mathrm{d}t},$$

where $\delta_e\eta/\mathrm{d}t$ is the entropy change due to heat and mass exchange with the surrounding medium and $\delta_i\eta/\mathrm{d}t$ is the entropy change caused by non-equilibrium of the processes under consideration. By the second law of thermodynamics, one has for irreversible processes

$$\frac{\delta_i \eta}{dt} > 0 .$$

The second law of thermodynamics leads to important results relating to transfer processes in fluids.

In order to separate the expression for $\delta_i \eta/dt$ (or, what is the same thing, $\delta_e \eta/dt$) from the general expression for $d\eta/dt$, the following procedure will be adopted. Assume first that at a given instant of time the system lies in an equilibrium state. Then, on the basis of the equilibrium conditions [cf. (1.8.13)], one has $T \equiv$ constant, $\mu \equiv$ constant, $e_{\alpha\beta} \equiv 0$, and, since $d\eta/dt = \delta_e \eta/dt$, it follows from (2.7.2) that

$$\frac{\delta_e \eta}{dt} = -\int_V \operatorname{div} \boldsymbol{I}_\eta dV = -\oint_\Sigma (\boldsymbol{I}_\eta, \boldsymbol{n})\, d\Sigma , \tag{2.7.3}$$

where

$$\boldsymbol{I}_\eta = \frac{\boldsymbol{q}}{T} + \frac{1}{T}\left\{\left(\frac{\partial \chi_m}{\partial s}\right)_{T,p} - \mu\right\}\boldsymbol{I}_s .$$

Proceeding to the case of non-equilibrium systems, note that by its very definition the quantity $\delta_e \eta/dt$ is not linked directly to the non-equilibrium nature of the state under consideration. Therefore let the validity of (2.7.3) be postulated also for the general case. Then one finds from (2.7.2)

$$\frac{\delta_i \eta}{dt} = \int_V \vartheta_\eta dV ,$$

$$\vartheta_\eta = q^\alpha \nabla_\alpha \left(\frac{1}{T}\right) + I_s^\alpha \left\{\left(\frac{\partial \chi_m}{\partial s}\right)_{T,p} \nabla_\alpha \left(\frac{1}{T}\right) - \nabla_\alpha \left(\frac{\mu}{T}\right)\right\} + \frac{1}{2T}\sigma^{\alpha\beta} e_{\alpha\beta} .$$

The vector $\rho \eta_m \boldsymbol{v} + \boldsymbol{I}_\eta$ is called the entropy density flux vector. In this context, it is useful to compare the formula for $\boldsymbol{I}_\eta$ with the formula $d\eta = \delta Q/T$ [cf. (1.2.7)], which gives the change in entropy of a closed equilibrium system when heat δQ is introduced. The expression $-\oint_\Sigma (\boldsymbol{I}_\eta, \boldsymbol{n}) d\Sigma$ gives the total inflow of entropy (in unit time) through the moving boundary Σ of the volume V, caused by heat and mass exchange with the surrounding medium; the sign of the flux may be arbitrary. The quantity ϑ_η is called entropy production: It is the amount of entropy which arises in unit time in unit volume of fluid due to the non-equilibrium nature of the processes taking place in the medium. By the second law of thermodynamics, one has

$$\vartheta_\eta > 0 . \tag{2.7.4}$$

The expressions for $\boldsymbol{I}_\eta$ and ϑ_η may be rewritten in the form

$$\boldsymbol{I}_\eta = \frac{\boldsymbol{q}}{T} + \left(\frac{\partial \eta_m}{\partial s}\right)_{T,p} \boldsymbol{I}_s , \tag{2.7.5}$$

$$\vartheta_\eta = q^\alpha \nabla_\alpha\left(\frac{1}{T}\right) + I_s^\alpha\left[-\frac{1}{T}(\nabla_\alpha\mu)_T\right] + \sigma^{\alpha\beta}\frac{1}{2T}e_{\alpha\beta}\ , \tag{2.7.6}$$

if one employs the thermodynamic identity

$$T\left(\frac{\partial\eta_m}{\partial s}\right)_{T,\,p} = \left(\frac{\partial\chi_m}{\partial s}\right)_{T,\,p} - \mu\ ,$$

which is readily obtained from Gibbs' relation (1.4.1). The subscript T in $(\nabla_\alpha\mu)_T$ denotes that during the computation of the gradient of $\mu(T, p, s)$ the temperature is assumed to be constant.

Employing the formulae derived, equation (2.7.1) will, finally, be written in the form

$$\frac{\partial}{\partial t}(\rho\eta_m) = -\mathrm{div}(\rho\eta_m \boldsymbol{v} + \boldsymbol{I}_\eta) + \vartheta_\eta\ . \tag{2.7.7}$$

This equation for the entropy is usually referred to as equation of entropy transfer.

2.8 THE BASIC PROPOSITIONS OF THE THERMODYNAMICS OF IRREVERSIBLE PROCESSES

For the ensuing work, it will be useful to introduce the concept of thermodynamic forces

$$\nabla_\alpha\left(\frac{1}{T}\right), \qquad -\frac{1}{T}(\nabla_\alpha\mu)_T\ , \qquad \frac{1}{2T}e_{\alpha\beta}\ ,$$

defining them as corresponding multipliers of the thermodynamic fluxes q^α, I_s^α and $\sigma^{\alpha\beta}$ in (2.7.6).

In accordance with the conditions (1.8.13) of thermodynamic equilibrium, thermodynamic forces vanish in equilibrium states. It is natural to assume that thermodynamic forces are such additional parameters which must be introduced as characteristics of a non-equilibrium state of a finite volume of fluid as a thermodynamic system. In other words, postulate that thermodynamic fluxes are functions of thermodynamic forces. Assuming the gradients of the basic parameters of a medium T, p, s and $\boldsymbol{v}$ to be not large, restrict consideration to linear approximations

$$\begin{aligned}
q^\alpha &= A_{11}^{\alpha\beta}\nabla_\beta\left(\frac{1}{T}\right) + A_{12}^{\alpha\beta}\left[-\frac{1}{T}(\nabla_\beta\mu)_T\right] + A_{13}^{\alpha\beta\gamma}\left(\frac{1}{2T}e_{\beta\gamma}\right)\\
I_s^\alpha &= A_{21}^{\alpha\beta}\nabla_\beta\left(\frac{1}{T}\right) + A_{22}^{\alpha\beta}\left[-\frac{1}{T}(\nabla_\beta\mu)_T\right] + A_{23}^{\alpha\beta\gamma}\left(\frac{1}{2T}e_{\beta\gamma}\right)\ ,\\
\sigma^{\alpha\beta} &= A_{31}^{\alpha\beta\gamma}\nabla_\gamma\left(\frac{1}{T}\right) + A_{32}^{\alpha\beta\gamma}\left[-\frac{1}{T}(\nabla_\gamma\mu)_T\right] + A_{33}^{\alpha\beta\gamma\chi}\left(\frac{1}{2T}e_{\gamma\chi}\right)\ .
\end{aligned} \tag{2.8.1}$$

By the tensor criterion, $A_{11}^{\alpha\beta}$, $A_{12}^{\alpha\beta}$, ..., being functions of T, p, and s, will be tensors. The relations (2.8.1) are called phenomenological laws. They do not contain terms of zero order, since in a state of thermodynamic equilibrium all fluxes vanish. The components of the tensors $A_{11}^{\alpha\beta}$ ... are called phenomenological coefficients.

The laws (2.8.1) form the foundation of the thermodynamics of irreversible processes. For isotropic media, they simplify significantly, since for such media all tensors of phenomenological coefficients must be isotropic (cf. § A.5). However, an isotropic tensor of third order can only be zero, and one arrives at the conclusion that fluxes of heat and salt $\boldsymbol{q}$ and $\boldsymbol{I}_s$ do not depend on the thermodynamic forces $e_{\alpha\beta}/2T$, and that momentum fluxes $\sigma_{\alpha\beta}$ caused by viscous stresses do not depend on the thermodynamic forces

$$\nabla_\alpha\left(\frac{1}{T}\right), \qquad -\frac{1}{T}(\nabla_\alpha\mu)_T$$

This fact is known as Curie's theorem.

Furthermore, an isotropic second-order tensor depends only on one scalar a and has the form $am^{\alpha\beta}$. The isotropic tensor $A_{33}^{\alpha\beta\gamma\varkappa}$, by strength of the symmetry of the tensors $\sigma^{\alpha\beta}$ and $e_{\alpha\beta}$, may be assumed to be symmetric in the superscripts α, β and γ, κ, separately. Such a tensor depends on two scalars a_1 and a_2 and has the form (cf. § A.5)

$$A_{33}^{\alpha\beta\gamma\varkappa} = a_1 m^{\alpha\beta} m^{\gamma\varkappa} + a_2(m^{\alpha\gamma} m^{\beta\varkappa} + m^{\alpha\varkappa} m^{\beta\gamma}) \ .$$

Thus, one may write, finally, for isotropic media formulae (2.8.1) in the form

$$\begin{aligned} q_\alpha &= a\nabla_\alpha\left(\frac{1}{T}\right) + b\left[-\frac{1}{T}(\nabla_\alpha\mu)_T\right] , \\ I_{s(\alpha)} &= b'\nabla_\alpha\left(\frac{1}{T}\right) + c\left[-\frac{1}{T}(\nabla_\alpha\mu)_T\right] , \\ \sigma_{\alpha\beta} &= de^{x}_{\cdot x}\, m_{\alpha\beta} + 2fe_{\alpha\beta} \ , \end{aligned} \qquad (2.8.2)$$

where a, b, b', d and f are scalar functions of T, p and s.

Finally, by Onsager's principle, proved in statistical physics (cf., for example, [60, § 122]), the matrix of phenomenological coefficients must be symmetric and $b' = b$. Thus, processes of heat conduction, diffusion and internal friction in isotropic fluids are described with the aid of five phenomenological coefficients: a, b, c, d and f which are functions of T, p and s. In the following sections, the laws (2.8.2) will be studied in greater detail.

2.9 THE RELATIONSHIP BETWEEN THE VISCOUS STRESS TENSOR AND THE STRAIN RATE TENSOR

Present the tensor $e_{\alpha\beta}$ as $e_{\alpha\beta} = 2(S_{\alpha\beta} + V_{\alpha\beta})$, where

$$S_{\alpha\beta} = \tfrac{1}{2}(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha) - \tfrac{1}{3}(\operatorname{div} \boldsymbol{v})\, m_{\alpha\beta}\,, \qquad V_{\alpha\beta} = \tfrac{1}{3}(\operatorname{div} \boldsymbol{v})\, m_{\alpha\beta}\,.$$

The tensor $S_{\alpha\beta}$ characterizes pure shear deformation, the tensor $V_{\alpha\beta}$ deformation in pure extension or compression; note that $S^{\alpha}_{.\alpha} = 0$.

Substituting the expression $\frac{1}{2}e_{\alpha\beta} = S_{\alpha\beta} + V_{\alpha\beta}$ into the last formula (2.8.2), it becomes

$$\sigma_{\alpha\beta} = 2\nu_1 \rho S_{\alpha\beta} + 3\nu_2 \rho V_{\alpha\beta}\,. \tag{2.9.1}$$

The coefficients ν_1 and ν_2 are called first and second coefficients of viscosity, respectively; they have been introduced in place of the coefficients d and f. The first coefficient of viscosity ν_1 is called, as a rule, simply viscosity coefficient. For an incompressible fluid, $V_{\alpha\beta} = 0$ and $S_{\alpha\beta} = e_{\alpha\beta}/2$, and the second coefficient of viscosity does not play any role.

Compute now the entropy increase in unit time in unit volume caused only by the process of internal friction. By (2.7.6), one has

$$\vartheta_\eta = \frac{2\nu_1\rho}{T} S_{\alpha\beta}S^{\alpha\beta} + \frac{3\nu_2\rho}{T} V_{\alpha\beta}V^{\alpha\beta}\,,$$

since $S_{\alpha\beta}V^{\alpha\beta} = \frac{1}{2}e^{\alpha}_{.\alpha}S^{\beta}_{.\beta}$. As far as, by the second law of thermodynamics, $\vartheta_\eta > 0$, one arrives at the result that the first and second coefficients of viscosity are postive.

2.10 THE RELATIONSHIP BETWEEN FLUXES OF HEAT AND SALT AND TEMPERATURE, PRESSURE AND SALINITY GRADIENTS

Consider the first two formulae (2.8.2) and introduce instead of a, b and c more customary coefficients. Recall that $b' = b$. Start with the vector $\boldsymbol{q}$. Expressing $-(1/T)\ (\nabla_\alpha \mu)_T$, by strength of (2.8.2), in terms of $I_{(s)\alpha}$ and $\nabla_\alpha(1/T)$, one obtains

$$q_\alpha = a\nabla_\alpha\left(\frac{1}{T}\right) + \frac{b}{c}I_{(s)\alpha} - \frac{b^2}{c}\nabla_\alpha\left(\frac{1}{T}\right) = -\kappa\nabla_\alpha T + \frac{b}{c}I_{(s)\alpha}\,.$$

The new coefficient $\kappa = (1/T^2)[a - (b^2/c)]$ has been introduced. It characterizes processes of heat transfer in the absence of diffusion ($\boldsymbol{I}_s = 0$) and is called thermal conductivity.

Next, find an expression for the entropy increase caused only by processes of heat conduction and diffusion in unit time in unit volume. By (2.7.6), one has

$$\vartheta_\eta = \frac{\kappa}{T^2}\,(\nabla T,\ \nabla T) + \frac{1}{c}(\boldsymbol{I}_s, \boldsymbol{I}_s)$$

Since $\vartheta_\eta > 0$, by the second law of thermodynamics, one finds that $\kappa > 0$, $c > 0$.

Rewrite the second formula (2.8.2) in the form

$$I_s = -\rho D\left(\nabla s + \frac{k_T}{T}\nabla T + \frac{k_p}{p}\nabla p\right), \tag{2.10.1}$$

where $D = (c/\rho T)(\partial\mu/\partial s)_{T,p}$; the coefficient D is called diffusion coefficient and characterizes diffusive transfer in the presence of a salinity gradient only. Since, by (1.7.4), one has $(\partial\mu/\partial s)_{T,p} > 0$, one finds $D > 0$; the coefficient $k_T = b/[c(\partial\mu/\partial s)_{T,p}]$, referred to as thermo-diffusion ratio, characterizes diffusive transfer in the presence of a temperature gradient only (thermo-diffusion). The coefficient $(\rho k_p D)/p$, by analogy, may be called baro-diffusion coefficient; it is interesting to note that $k_p = p[\partial(1/\rho)/\partial s]_{T,p}\ (\partial\mu/\partial s)_{T,p}$ and that it does not depend on the phenomenological coefficients a, b and c.

Introducing the coefficient k_T, write the expression for q finally in the form

$$q = -\kappa\nabla T + k_T\left(\frac{\partial\mu}{\partial s}\right)_{T,p} I_s\,. \tag{2.10.2}$$

Thus, instead of a, b and c, new phenomenological coefficients κ, D and k_T have been introduced which have clear physical significance. Note that k_T and k_p may have arbitrary signs.

TABLE 2.I

Phenomenological coefficients for atmospheric pressure (according to Montgomery [80])

	Pure water		Sea water	
	0° C	20° C	0° C	20° C
Coefficient of dynamic viscosity $\nu\rho$ $g \cdot cm^{-1} \cdot sec^{-1}$	$1.787 \cdot 10^{-2}$	$1.002 \cdot 10^{-2}$	$1.877 \cdot 10^{-2}$	$1.075 \cdot 10^{-2}$
Coefficient of heat conductivity κ Watt $\cdot cm^{-1} \cdot {}^{\circ}C^{-1}$	$5.66 \cdot 10^{-3}$	$5.99 \cdot 10^{-3}$	$5.63 \cdot 10^{-3}$	$5.96 \cdot 10^{-3}$
Coefficient of kinematic viscosity ν $cm^2 \cdot sec^{-1}$	$1.787 \cdot 10^{-2}$	$1.004 \cdot 10^{-2}$	$1.826 \cdot 10^{-2}$	$1.049 \cdot 10^{-2}$
Coefficient of thermal conductivity $\kappa/c_p\rho$ $cm^2 \cdot sec^{-1}$	$1.34 \cdot 10^{-3}$	$1.43 \cdot 10^{-3}$	$1.39 \cdot 10^{-3}$	$1.49 \cdot 10^{-3}$
Coefficient of diffusion D $cm^2 \cdot sec^{-1}$ (NaCl)	$0.74 \cdot 10^{-5}$	$1.41 \cdot 10^{-5}$	$0.68 \cdot 10^{-5}$	$1.29 \cdot 10^{-5}$
Prandtl number $\nu/(\kappa/c_p\rho)$	13.3	7.0	13.1	7.0

In the further treatment, phenomenological coefficients will be assumed to be known functions of T, p and s. Generally speaking, they may be determined (or linked to other parameters) on the basis of the microscopic theory of transfer processes in fluids or measured empirically.

Table 2.I gives an idea of the magnitude of the phenomenological coefficients. Apparently, the second coefficient of viscosity ν_2 has the same order of magnitude as ν_1 [59, § 78]. The thermo-diffusion ratio k_T for sea water is unknown; De Groot and Masur [9, Chapter XI, § 7] state that for liquid mixtures the quantity k_T/Ts has an order of $10^{-3} \div 10^{-5}\,^{\circ}C^{-1}$.

Thus, if one replaces in the equations of mass conservation (2.2.13), salt diffusion (2.2.14), motion (2.3.6) and energy transfer (2.5.6) the expressions for $\boldsymbol{q}$, $\boldsymbol{I}_s$ and $\sigma_{\alpha\beta}$ by gradients of T, p, s and $\boldsymbol{v}$, in accordance with (2.9.1), (2.10.1) and (2.10.2), one actually obtains a closed system of equations for the determination of the basic parameters of the medium T, p, s and $\boldsymbol{v}$. Note that it is at times convenient to select in place of the equation of energy transfer (2.5.6) the equation of internal energy transfer (2.6.3) or the equation of entropy evolution (2.7.1).

COMMENT ON CHAPTER 2

This chapter pursues basically the normal treatment of the thermodynamics of irreversible processes in a continuous medium. The following monographs have been used: De Groot [10], De Groot and Masur [9], Landau and Lifshits [59], Levich et al. [61], Sedov [108] and Haase [29].

CHAPTER 3

WAVE MOTION IN THE OCEAN

3.1. BASIC EQUATIONS

As usual, during an analysis of wave motions, dissipative processes (friction, heat conduction, diffusion) will be neglected. The starting point will be the equations of motion (2.3.6), of conservation of mass of sea water (2.3.13), of diffusion of salt (2.2.14) and of evolution of entropy (2.7.1). In the case under consideration, the number of these equations may be reduced. In fact, since $\mathrm{d}s/\mathrm{d}t = 0$ and $\mathrm{d}\eta_m/\mathrm{d}t = 0$, by (2.2.14) and (2.7.1), one obtains

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \left(\frac{\partial \rho}{\partial p}\right)_{\eta_m, s} \frac{\mathrm{d}p}{\mathrm{d}t} . \qquad (3.1.1)$$

The quantity $(\partial p/\partial \rho)_{\eta_m,\, s}$ has the meaning of the square of the local velocity of sound in the medium; if it is assumed that it is a known function of the pressure and density of the medium, then the three equations of motion (without friction), the equation of conservation of mass of sea water and equation (3.1.1) contain only the five unknown functions $v(u, v, w), p, \rho$. In other words, a closed system of equations has been obtained.

In the sequel, wave motions in the ocean will be considered as small oscillations of a layer of liquid of constant depth H in a gravity field.

Since the force of gravity singles out in space the vertical direction, it is convenient to go to a spherical system of coordinates λ, φ, z (where λ is the longitude, $0 < \lambda < 2\pi$, φ is the latitude, $-\pi/2 < \varphi < \pi/2$, and z is reckoned upward from the undisturbed surface of the ocean $-H \leqslant z \leqslant 0$). Since $|z| << a$ (where a is Earth's radius), one may replace, in writing down the basic equations in a spherical coordinate system [cf. (A.9.4—14)] the Lamé coefficients $h_\lambda, h_\varphi, h_z$ by the approximate expressions

$$h_\lambda = a \cos \varphi, \qquad h_\varphi = a, \qquad h_z = 1 . \qquad (3.1.2)$$

Apparently, this approximation involves an insignificant error; however, it is important that it does not violate the tensorial nature of the individual terms of the equations, and therefore, if a certain equation has the form of a law of conservation (cf. § 2.3), this property is maintained in the approximate version.

It is interesting to note that in Euclidean space there do not exist curvilinear orthogonal systems q_1, q_2, q_3, the Lamé coefficients of which would be equal to $a \cos q_2, a, 1$. This follows directly from (A.8.7) and (A.8.8): The curvature tensor in this coordinate system is not equal to zero.

A start will now be made with free oscillations. Assume that the basic state of rest is characterized by density $\rho_0(z)$, velocity of sound $c(z)$ and pressure $p_0(z) = p_a - \int_0^z g\rho_0(z)\mathrm{d}z$. Linearizing the initial equations with respect to such a basic state, one obtains

$$\frac{\partial u}{\partial t} - 2\Omega v \sin\varphi = -\frac{1}{\rho_0}\frac{\partial p'}{a\cos\varphi\partial\lambda}, \tag{3.1.3}$$

$$\frac{\partial v}{\partial t} + 2\Omega u \sin\varphi = -\frac{1}{\rho_0}\frac{\partial p'}{a\partial\varphi}, \tag{3.1.4}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\rho'}{\rho_0}, \tag{3.1.5}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\rho_0}{\mathrm{d}z} + \rho_0 \operatorname{div} \boldsymbol{v} = 0, \tag{3.1.6}$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = \frac{1}{c^2}\left(\frac{\partial p'}{\partial t} - g\rho_0 w\right), \tag{3.1.7}$$

where u, v, w are the zonal, meridional, vertical components, respectively, of the velocity $\boldsymbol{v}$, p' and ρ' are the deviations of pressure and density from their undisturbed values $p_0(z)$ and $\rho_0(z)$.

The Coriolis force in equations (3.1.3)—(3.1.5) has been presented in the so called "traditional" approximation; its complete expression is $2\Omega \times \boldsymbol{v}$ $(-2\Omega v \sin\varphi + 2\Omega w \cos\varphi,\ 2\Omega u \sin\varphi,\ -2\Omega u \cos\varphi)$. The accuracy of this approximation is not always clear. As a rule, it is based on the smallness of vertical velocities compared with horizontal ones, but this condition is only true for long waves and, besides, at the equator ($\varphi = 0$) the neglected terms $2\Omega w \cos\varphi$, $-2\Omega u \cos\varphi$ may turn out to be significant. However, the problem would be considerably more complicated, if the complete expressions for the Coriolis forces were retained (since it would preclude the ensuing employment of the method of separation of variables).

A discussion of the "traditional" approximation is given in [13, §§ 37, 38, 53; 87; 95; 96; 126].

Boundary conditions will be formulated next. At the free surface of the ocean $z = \zeta(\lambda, \varphi, t)$ (where ζ is the sea level), there must be fulfilled the dynamic condition of continuity of pressure $p_0 + p' = p_a$ (ideal fluid) and the first condition (2.2.17) which is usually called kinematic condition (under the assumption that evaporation, precipitation, thawing and formation of ice are absent). Linearization of the dynamic condition for small ζ yields

$$p_a = p_0(\zeta) + p'(\zeta) = p_0(0) + \zeta\frac{\mathrm{d}p_0}{\mathrm{d}z}(0) + p'(0) + \ldots = p_a - g\rho_0(0)\,\zeta + p'(0) + \ldots$$

Further, since $(\boldsymbol{v}, \boldsymbol{n}) \simeq \partial\zeta/\partial t$, $\boldsymbol{n} \simeq (0, 0, 1)$ and ζ is small, the boundary conditions at the free ocean surface may be written in the form

$$p' = g\rho_0\zeta, \qquad w = \frac{\partial \zeta}{\partial t} \qquad \text{for} \qquad z = 0\,. \tag{3.1.8}$$

Obviously, the boundary condition at the sea floor is

$$w = 0 \qquad \text{for} \qquad z = -H\,. \tag{3.1.9}$$

Next, the energy equation for wave motions will be derived. Initially, it will be useful to treat this problem under the assumptions that dissipative processes are absent and the oscillations of the fluid layer are finite. In this chapter, the basic equations are the equations of motion, of conservation of mass, of diffusion of salt and of evolution of entropy; therefore the equation of conservation of energy must be a consequence of these equations. In fact, it follows from Gibbs' relation (1.4.2), since $\mathrm{d}s/\mathrm{d}t = 0$ and $\mathrm{d}\eta_m/\mathrm{d}t = 0$, that $\rho(\mathrm{d}\epsilon_m/\mathrm{d}t) = -p\mathrm{div}\,\boldsymbol{v}$; using equation (2.6.2) for the mechanical energy, one obtains

$$\frac{\partial}{\partial t}(K + E) = -\mathrm{div}[(K + E + p)\,\boldsymbol{v}]\,, \tag{3.1.10}$$

where $K = \rho(v^2/2)$ is the kinetic energy per unit volume and $E = \rho\epsilon + \rho g z$ is the sum of the internal and potential energies per unit volume.

For the free motions under consideration, the total energy of the fluid remains constant. Integrating equation (3.1.10) over the entire volume of fluid and assuming that the fluid covers the entire sphere (if there is a coast, the normal velocity vanishes on it) and $p|_\zeta = 0$, one finds

$$\int_{-H}^{\zeta}\int_{\Sigma}(K + E)\,\mathrm{d}z\mathrm{d}\Sigma = \text{constant}\,. \tag{3.1.10'}$$

where Σ is the surface of the entire sphere (or, if there are shores, of the ocean).

Is it possible to give an analogue of the equation of conservation of energy for equations (3.1.3)—(3.1.7) describing small oscillations of a fluid? Although the answer to this question turns out to be positive, it is far from being trivial, since equations (3.1.3)—(3.1.7) are "distorted images" of the initial non-linear equations, and existence of a positive definite function (with respect to $\boldsymbol{v}$, p' and ρ') which remains constant during motion of the fluid is not obvious beforehand.

The equation of conservation of energy for wave motions has the form

$$\frac{\partial}{\partial t}\left\{\rho_0\frac{v^2}{2} + \frac{p'^2}{2\rho_0 c^2} + \frac{g^2}{2\rho_0 N^2}\left(\rho' - \frac{1}{c^2}p'\right)^2\right\} = -\mathrm{div}(p'\boldsymbol{v})\,, \tag{3.1.11}$$

where $N^2 = -\left(\dfrac{g}{\rho_0}\dfrac{\mathrm{d}\rho_0}{\mathrm{d}z} + \dfrac{g^2}{c^2}\right)$

is the square of the Väisälä frequency (cf. § 1.9).

The truth of this equation is most simply established by direct verification. For this purpose, one must rewrite first equations (3.1.6) and (3.1.7) in the form

$$\frac{\partial p'}{\partial t} = g\rho_0 w - c^2\rho_0 \operatorname{div} \boldsymbol{v} \,, \tag{3.1.12}$$

$$\frac{\partial}{\partial t}\left(\rho' - \frac{1}{c^2} p'\right) = \frac{\rho_0 N^2}{g} w \,. \tag{3.1.13}$$

and, differentiating with respect to t the contents of the curly bracket on the left-hand side of (3.1.11), employ (3.1.3)—(3.1.5), (3.1.12) and (3.1.13).

Integrating equation (3.1.11), as in the case of the derivation of (3.1.10′), over the entire volume of the liquid, one arrives, by (3.1.8), at the equation

$$\int_{-H}^{0} \int_{\Sigma} \left\{\frac{\rho_0}{2} \boldsymbol{v}^2 + \frac{1}{2\rho_0 c^2} p'^2 + \frac{g^2}{2\rho_0 N^2}\left(\rho' - \frac{p'}{c^2}\right)^2\right\} \mathrm{d}z\mathrm{d}\Sigma + \int_{\Sigma} \frac{g\rho_0}{2} \zeta^2 \mathrm{d}\Sigma = \text{constant} \tag{3.1.14}$$

which yields a simple proof of the stability of the equilibrium state of stratified fluid under the condition $N^2 > 0$ (cf. § 1.9).

It is not difficult to prove that (3.1.14) and the derivation relating to stability are true also in the case when $N^2 = 0$ in separate layers or throughout (in these layers one must retain in the integrand of the volume integral only the first two terms).

The quantity

$$\mathcal{E} = \frac{1}{2\rho_0 c^2} p'^2 + \frac{g^2}{2\rho_0 N^2}\left(\rho' - \frac{1}{c^2} p'\right)^2 , \tag{3.1.15}$$

by the significance of the energy equation (3.1.11), must be somehow linked to the sum of the internal and potential energies of the fluid E. In order to explain this connection, expand the characteristics under consideration in the form of series of the type

$$E = E_0 + E_1 + E_2 + \ldots \,, \qquad K = K_2 + \ldots \,, \tag{3.1.16}$$

and analogously expand p, ρ, v, η_m, etc., where the subscript indicates the order of magnitude of the term under consideration. Substituting these series into the original equations and equating to zero terms of different orders, one derives equations of zero, first and second orders. For example, equations (3.1.3)—(3.1.6) will be the equations of motion and conservation of mass of first order (in terms of the new notation used up to the end of § 3.1, one must replace in all equations (3.1.3)—(3.1.15) $\boldsymbol{v}$ by $\boldsymbol{v}_1$, p' by p_1, ρ'

by ρ_1, etc.). Furthermore, write down in place of equations (3.1.7) the equations of salt diffusion and entropy evolution of first order

$$\frac{\partial s_1}{\partial t} + w_1 \frac{\mathrm{d}s_0}{\mathrm{d}z} = 0 , \tag{3.1.17}$$

$$\frac{\partial \eta_1}{\partial t} + w_1 \frac{\mathrm{d}\eta_0}{\mathrm{d}z} = 0 , \tag{3.1.18}$$

and derive from the second-order equations only the equation of conservation of mass

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div}(\rho_0 \boldsymbol{v}_2 + \rho_1 \boldsymbol{v}_1) = 0 . \tag{3.1.19}$$

In an analogous manner, one obtains from (3.1.10)

$$\frac{\partial E_1}{\partial t} = 0 , \qquad \frac{\partial E_1}{\partial t} = -\operatorname{div}[(E_0 + p_0)\, \boldsymbol{v}_1] ,$$

$$\frac{\partial}{\partial t}(K_2 + E_2) = -\operatorname{div}[(E_0 + p_0)\boldsymbol{v}_2 + (E_1 + p_1)\boldsymbol{v}_1] . \tag{3.1.20}$$

Obviously, these energy relations are consequences of the equations of motion, conservation of mass, salt diffusion and entropy evolution of zero, first and second orders.

The third equation (3.1.20) is of special interest. Integrating it over the entire fluid volume, as for the derivation of equation (3.1.14), one finds

$$\frac{\partial}{\partial t}[\int_{-H}^{0} \int_{\Sigma} (K_2 + E_2)\, \mathrm{d}z\mathrm{d}\Sigma] + \int_{\Sigma} \frac{g\rho_0}{2} \zeta^2 \mathrm{d}\Sigma = -\int_{\Sigma} [(E_0 + p_0)\, w_2 + E_1 w_1]_{z=0}\, \mathrm{d}\Sigma . \tag{3.1.21}$$

For the transformation of this equation, write E in the form

$$E = \rho\epsilon_m + g\rho z = \rho\chi_m - p + g\rho z ,$$

using Gibbs' relation (1.4.3), to find

$$E_0 + p_0 = \rho_0\chi_0 + \rho_0 g z, \qquad E_1 = \rho_0 T_0 \eta_1 + \rho_0 \mu_0 s_1 + \rho_1 \chi_0 + \rho_1 g z .$$

Recall that all quantities with zero subscript depend on z only. Using now (3.1.17), (3.1.18) and (3.1.19), equation (3.1.21) is readily rewritten in the form

$$\int_{-H} \int_{\Sigma} (K_2 + E_2)\, \mathrm{d}z\mathrm{d}\Sigma + \int_{\Sigma} \frac{g\rho_0}{2}\zeta^2 \mathrm{d}\Sigma - \int_{-H} \int_{\Sigma} \chi_0 \rho_2 \mathrm{d}z\mathrm{d}\Sigma -$$

$$- \int_{\Sigma} \left(\frac{\rho_0 \mu_0}{2\mathrm{d}s_0/\mathrm{d}z}\, s_1^2 + \frac{\rho_0 T_0}{2\mathrm{d}\eta_0/\mathrm{d}z}\, \eta_1^2 \right) \mathrm{d}\Sigma = \text{constant} .$$

Subtracting from this relation (3.1.14) and taking into consideration that $K_2 = \rho_0(v_1^2/2)$, one finds

$$\int_{-H}^{0}\int_{\Sigma}(E_2 - \mathcal{E})\,\mathrm{d}z\mathrm{d}\Sigma - \int_{-H}^{0}\int_{\Sigma}\chi_0\rho_2\mathrm{d}z\mathrm{d}\Sigma - \int_{\Sigma}\left(\frac{\rho_0\mu_0}{2\mathrm{d}s_0/\mathrm{d}z}\,s_1^2 + \frac{\rho_0 T_0}{2\mathrm{d}\eta_0/\mathrm{d}z}\,\eta_1^2\right)\mathrm{d}\Sigma$$

$$= \text{constant}\,. \tag{3.1.22}$$

A discussion of these relations will be started with the case of a fluid with a rigid lid at $z = 0$; in that case, the last term on the left-hand side of (3.1.14) drops out, and so do the last two terms on the left-hand side of (3.1.22). Since first-order motions are under consideration, the magnitude of the kinetic energy of such motion is of second order. Therefore it is obvious that, generally speaking, changes in the integral $\int_{-H}^{0}\int_{\Sigma}K_2\,\mathrm{d}z\mathrm{d}\Sigma$ may be connected only to changes in $\int_{-H}^{0}\int_{\Sigma}E_2\mathrm{d}z\mathrm{d}\Sigma$. However, it is more important for an analysis of the system of equations of first order that changes in $\int_{-H}^{0}\int_{\Sigma}K_2\mathrm{d}z\mathrm{d}\Sigma$ are linked, by (3.1.14), to changes in $\int_{-H}^{0}\int_{\Sigma}\mathcal{E}\mathrm{d}z\mathrm{d}\Sigma$, since only characteristics of first order enter into the expression for $\mathcal{E}$, while, as is easily shown, also second-order characteristics enter into the expression for E_2. Therefore, since by (3.1.22), the difference $\int_{-H}^{0}\int_{\Sigma}(E_2 - \mathcal{E})\mathrm{d}z\mathrm{d}\Sigma$ remains unchanged with time, it is convenient to imagine that during small oscillations of a fluid not all the energy $\int_{-H}^{0}\int_{\Sigma}E_2\mathrm{d}z\mathrm{d}\Sigma$, but only its part $\int_{-H}^{0}\int_{\Sigma}\mathcal{E}\mathrm{d}z\mathrm{d}\Sigma$ may convert into kinetic energy $\int_{-H}^{0}\int_{\Sigma}K_2\mathrm{d}z\mathrm{d}\Sigma$.

For a liquid with a free surface, the considerations above remain valid; however, one must take in (3.1.14) and (3.1.22) all terms into account. The quantity $\int_{-H}^{0}\int_{\Sigma}(E_2 - \mathcal{E})\mathrm{d}z\mathrm{d}\Sigma$ will now change in the course of time, but these changes will not be reflected directly in changes of the kinetic energy $\int_{-H}^{0}\int_{\Sigma}K_2\mathrm{d}z\mathrm{d}\Sigma$.

In analogy with a concept introduced by Lorenz [71] for a definite class of atmospheric motions, the quantity $\mathcal{E}$ may be called *available potential energy* (per unit volume) for wave motions of small amplitude in a stably stratified fluid. Generally speaking, the quantity $\mathcal{E}$ contains internal as well as potential energy contributions; however, by strength of the small compressibility of sea water, the contribution of the internal energy is not large, and therefore, for the sake of brevity, the quantity $\mathcal{E}$ will be referred to as *available potential energy*.

3.2. SEPARATION OF VARIABLES

In what follows, consideration will be given to wave motions in an unbounded ocean which are periodic in time. For this purpose, solutions of the system of equations (3.1.3)—(3.1.7) for the entire sphere will be studied for the boundary conditions (3.1.8) and (3.1.9) in the form

$$(u, v, w, p', \rho') = \mathrm{Re}\{(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}, \tilde{\rho}) \exp(-i\sigma t)\}, \tag{3.2.1}$$

where the complex amplitudes of the oscillations $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}$ and $\tilde{\rho}$ are functions of λ, φ, z, and σ is the frequency of the oscillations. Substituting (3.2.1) into (3.1.3)–(3.1.7) and eliminating $\tilde{\rho}$ from (3.1.5) and (3.1.6) with the aid of (3.1.7), one obtains

$$-i\sigma\tilde{u} - 2\Omega\tilde{v}\sin\varphi = -\frac{1}{\rho_0}\frac{\partial\tilde{p}}{a\cos\varphi\partial\lambda}, \tag{3.2.2}$$

$$-i\sigma\tilde{v} + 2\Omega\tilde{u}\sin\varphi = -\frac{1}{\rho_0}\frac{\partial\tilde{p}}{a\partial\varphi}, \tag{3.2.3}$$

$$(\sigma^2 - N^2)\,\tilde{w} + \frac{i\sigma}{\rho_0}\left(\frac{\partial\tilde{p}}{\partial z} + \frac{g}{c^2}\tilde{p}\right) = 0, \tag{3.2.4}$$

$$\rho_0\,\mathrm{div}_h(\tilde{u}, \tilde{v}) + \rho_0\frac{\partial\tilde{w}}{\partial z} - \frac{i\sigma}{c^2}\tilde{p} - \frac{g\rho_0}{c^2}\tilde{w} = 0. \tag{3.2.5}$$

Only stable (or, in separate layers, neutral) stratified fluids will be considered ($N^2 \geqslant 0$). Possible frequencies σ must then be real numbers; otherwise all functions v, p' and ρ' will be in the course of time either exponentially increasing or decreasing, which contradicts, by (3.1.13), the conditions of constancy of total energy.

The structure of (3.2.2)–(3.2.5) permits to seek solutions of the problem in the form

$$(\tilde{u}, \tilde{v}) = \frac{1}{\rho_0(z)}P(z)[U(\lambda, \varphi),\ V(\lambda, \varphi)], \qquad \tilde{w} = i\sigma W(z)\,\Pi(\lambda, \varphi),$$

$$\tilde{p} = P(z)\,\Pi(\lambda, \varphi). \tag{3.2.6}$$

Substituting (3.2.6) into (3.2.5), this equation may be written in the form

$$\frac{\mathrm{div}_h(U, V)}{i\sigma\Pi} = \frac{\frac{1}{c^2}P + \frac{g\rho_0}{c^2}W - \rho_0\frac{\mathrm{d}W}{\mathrm{d}z}}{P} = \epsilon, \tag{3.2.7}$$

where ϵ is the separation constant of the variables (with dimensionality $\mathrm{sec}^2/\mathrm{cm}^2$).

Final substitution of (3.2.6) into (3.2.2)–(3.2.5), taking into account (3.2.7), yields for U, V and Π Laplace's tidal equations

$$-i\sigma U - 2\Omega V\sin\varphi = -\frac{\partial\Pi}{a\cos\varphi\partial\lambda}, \tag{3.2.8}$$

$$-i\sigma V + 2\Omega U\sin\varphi = -\frac{\partial\Pi}{a\partial\varphi}, \tag{3.2.9}$$

$$-i\sigma\epsilon\Pi + \mathrm{div}_h(U,\ V) = 0\ , \tag{3.2.10}$$

and for P and W the equations

$$\frac{\mathrm{d}P}{\mathrm{d}z} + \frac{g}{c^2}P + (\sigma^2 - N^2)\,\rho_0 W = 0\ , \tag{3.2.11}$$

$$\frac{\mathrm{d}W}{\mathrm{d}z} - \frac{g}{c^2}W + \left(\epsilon - \frac{1}{c^2}\right)\frac{1}{\rho_0}P = 0\ . \tag{3.2.12}$$

Equations (3.2.8)—(3.2.10) were considered a long time ago by Laplace in connection with tidal theory. It may be shown that they describe low frequency ($\sigma << N$) oscillations of a homogeneous ocean of constant depth $H = 1/g\epsilon$ (cf. § 3.9).

The boundary conditions for P and W will now be rewritten. Eliminating ζ, one has

$$W = 0 \qquad \text{for} \qquad z = -H \qquad \text{and} \qquad P + g\rho_0 W = 0 \qquad \text{for} \qquad z = 0\ . \tag{3.2.13}$$

The problem of free oscillations in an ocean may now be posed in the following manner: Find all possible pairs of characteristic numbers ϵ and σ for which Problem H (the system of equations (3.2.8)—(3.2.10) for conditions of boundedness) and Problem V [the system of equations (3.2.11) and (3.2.12) for Conditions (3.2.12)] have non-trivial solutions. In the $(\epsilon,\ \sigma)$-plane, such pairs of numbers for each problem form a discrete set of curves called eigenvalue curves of the corresponding problems. Points of intersection of eigenvalue curves of Problem H and Problem V yield the frequencies of the possible free oscillations of the ocean. Thus, one may also obtain the dependence of σ on, in a corresponding manner determined, wave numbers with respect to λ, φ and z (dispersion relations).

Note that the forms of System (3.2.11) and (3.2.12) and Conditions (3.2.13) do not change for a study by the method stated of problems of free oscillations of a liquid layer on a non-rotating sphere, and likewise on a rotating or non-rotating plane. In other words, the effects of the Earth's rotation and spherical shape appear explicitly only during an analysis of Problem H. At the same time, effects of stratification, compressibility, gravity forces and boundary conditions at the upper surface of the liquid layer appear explicitly only during a study of Problem V. Such a *division* of effects turns out to be very useful for a study of the general problem.

3.3. ANALYSIS OF THE SIMPLEST CASES

When equilibrium is disturbed in a stably stratified compressible ocean, there appear various restoring forces which give rise to a range of wave motions. The following factors are basic: gravity force, stratification and compressibility of sea water, rotation and spherical shape of Earth. All these factors combine; however, at first, it is useful to study by means of simple models each of these effects separately.

The general approach of § 3.2 will be pursued. Consider Problem V for an incompressible and inhomogeneous ($N = N_0$ = constant) fluid. If $N = N_0$ = constant and $c = \infty$, then $\rho_0(z) = \rho_0(0)\exp[-(N_0^2/g)z]$. However, it may be assumed with a very small error (cf. § 3.4) that ρ_0 = constant, and then System (3.2.11) and (3.2.12) reduces readily to the single equation with constant coefficients

$$\frac{d^2 W}{dz^2} + \epsilon(N_0^2 - \sigma^2)\, W = 0\,.$$

Let σ be fixed and look first for a positive eigenvalue ϵ. If $\sigma^2 > N_0^2$, then $W = \sinh\{\sqrt{\epsilon(\sigma^2 - N_0^2)}\ \ (z + H)\}$, $P = -(\rho_0/\epsilon)\sqrt{\epsilon(\sigma^2 - N_0^2)} \cdot \cosh\{\sqrt{\epsilon(\sigma^2 - N_0^2)}\ (z + H)\}$, and the eigenvalues ϵ will be roots of the equation

$$\tanh\sqrt{\epsilon(\sigma^2 - N_0^2)}\, H = \frac{\sqrt{\sigma^2 - N_0^2}}{g\sqrt{\epsilon}}\,. \tag{3.3.1}$$

In the case when $\sigma^2 < N_0^2$, $W = \sin\{\sqrt{\epsilon(N_0^2 - \sigma^2)}(z+H)\}$, $P = -(\rho_0/\epsilon)\sqrt{\epsilon(N_0^2 - \sigma^2)}\ \cos\{\sqrt{\epsilon(N_0^2 - \sigma^2)}\ (z + H)\}$, and the eigenvalues ϵ will be roots of the equation

$$\tan\sqrt{\epsilon(N_0^2 - \sigma^2)}\, H = \frac{\sqrt{N_0^2 - \sigma^2}}{g\sqrt{\epsilon}}\,. \tag{3.3.2}$$

If $\epsilon < 0$, then one has for the determination of ϵ the equations:

$$\text{for } \sigma^2 < N_0^2, \qquad \tanh\sqrt{-\epsilon(N_0^2 - \sigma^2)}\, H = -\frac{\sqrt{N_0^2 - \sigma^2}}{g\sqrt{-\epsilon}}\,; \tag{3.3.3}$$

$$\text{for } \sigma^2 > N_0^2, \qquad \tan\sqrt{-\epsilon(\sigma^2 - N_0^2)}\, H = -\frac{\sqrt{\sigma^2 - N_0^2}}{g\sqrt{-\epsilon}}\,. \tag{3.3.4}$$

Patterns for graphical solution of equations (3.3.1) through (3.3.4) are shown in Fig. 3.1. It is seen that equation (3.3.1) has a single root ϵ_0, equation (3.3.3) has no roots and equations (3.3.2) and (3.3.4) have countable sets of roots $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ and $\epsilon_{-1}, \epsilon_{-2} \ldots$, respectively (where the subscripts have been chosen to differ for all these roots).

The behaviour of the eigenvalue curves $\epsilon_i(\sigma)$ is also readily assessed. Thus, it follows from Fig. 3.1a that $\epsilon_0 \to \infty$ as $\sigma \to \infty$. However, then, by (3.3.1), $g^2\ \epsilon_0 \sim \sigma^2$. Further, as $\sigma \to 0$, one has approximately $\epsilon_0(0) \simeq 1/gH$; by (3.3.2), the relative error of this approximation is small and of order $N_0^2 H/g$ (for the ocean $\sim 10^{-2}$). To this degree of accuracy, one finds easily ϵ_i, replacing equations (3.3.2) and (3.3.4), respectivelv, by

$$\tan H\sqrt{\epsilon(N_0^2 - \sigma^2)} = 0 \qquad \text{and} \qquad \tan H\sqrt{-\epsilon(\sigma^2 - N_0^2)} = 0$$

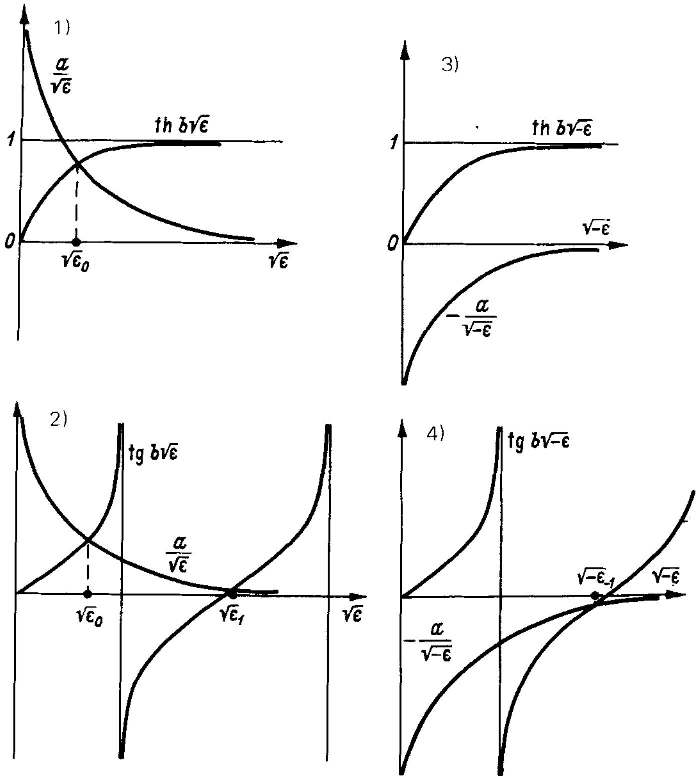


Fig. 3.1. Graphical solution of equations (3.3.1)–(3.3.4).
(1) $\epsilon > 0$, $\sigma^2 > N_0^2$, $a = \sqrt{\sigma^2 - N_0^2}/g$, $b = H\sqrt{\sigma^2 - N_0^2}$;
(2) $\epsilon > 0$, $\sigma^2 < N_0^2$, $a = \sqrt{N_0^2 - \sigma^2}/g$, $b = H\sqrt{N_0^2 - \sigma^2}$;
(3) $\epsilon < 0$, $\sigma^2 < N_0^2$, $a = \sqrt{N_0^2 - \sigma^2}/g$, $b = H\sqrt{N_0^2 - \sigma^2}$;
(4) $\epsilon < 0$, $\sigma^2 > N_0^2$, $a = \sqrt{\sigma^2 - N_0^2}/g$, $b = H\sqrt{\sigma^2 - N_0^2}$.

(cf. Figs. 3.1b and d). Thus,

$$\epsilon_n = \frac{(n\pi)^2}{H^2(N_0^2 - \sigma^2)}, \qquad n = 1, 2, 3, \ldots, \tag{3.3.5}$$

$$\epsilon_n = -\frac{(n\pi)^2}{H^2(\sigma^2 - N_0^2)}, \qquad n = -1, -2, -3, \ldots. \tag{3.3.6}$$

Figure 3.2 presents the final description of the behaviour of the eigenvalue curves $\epsilon_i(\sigma)$.

Next, consider Problem H for the simplest case of a plane rotating layer. The form of equations (3.2.8)–(3.2.10) must, of course, change:

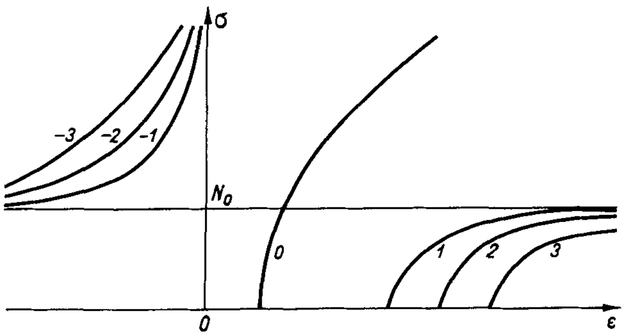


Fig. 3.2. Pattern of eigenvalue curves of Problem V ($c = \infty$, ρ_0 = constant, N_0 = constant). The curves are numbered with the corresponding order n. The pattern is symmetric about the axis $\sigma = 0$.

$a \cos \varphi \, \partial\lambda$ is replaced by ∂x, $a\partial\varphi$ by ∂y, $2\Omega \sin \varphi$ by 2Ω, $\mathrm{div}_h(U, V)$ by $\partial U/\partial x + \partial V/\partial y$.

Seek the solution of Problem H in the form

$$(U, V, \Pi) = (U_0, V_0, \Pi_0) \exp i(kx + ly) ,$$

where U_0, V_0 and Π_0 are constants, k and l are (dimensional) wave numbers along the x- and y-axes. Substituting these expressions into (3.2.8)—(3.2.10), one obtains a homogeneous algebraic system of equations for U_0, V_0 and Π_0. Setting the determinant of this system equal to zero, one finds

$$\sigma^2 = (2\Omega)^2 + \frac{k^2 + l^2}{\epsilon} . \tag{3.3.7}$$

Figure 3.3. shows the pattern of the eigenvalue curves (3.3.7). It is convenient to refer to the curves in the half-planes $\epsilon > 0$ and $\epsilon < 0$ as curves of first and third order, respectively.

The results obtained permit now a study of the free oscillations of a plane rotating layer of incompressible and non-homogeneous (N = constant) fluid in a gravity field. Consider for this purpose the (ϵ, σ)-plane. The solutions of (3.1.3)—(3.1.7), corresponding to points of intersection of the characteristic curve $\epsilon_0(\sigma)$ of Problem V with first type eigencurves of Problem H are called gravitational surface waves; their frequencies satisfy $\sigma^2 > 4\Omega^2$. It is readily seen that replacement of the second boundary conditions (3.2.13) by the rigid cover condition $W(0) = 0$ converts $\epsilon_0(\sigma)$ into $\epsilon = 0$, without intersection with the first type eigenvalue curves of Problem H. Hence the waves under consideration are caused entirely by the effect of the free surface in the gravity force field (it is not difficult to show that the inhomogeneity of the water exerts almost no effect). The physical reason for the occurrence of surface waves is obvious; when the free surface is deflected, there appear pressure gradients which play the role of restoring forces. The dispersion relation for surface waves for $\sigma > N_0$ is obtained by elimination of ϵ from (3.3.1) and (3.3.7):

$$\tanh k_h H \sqrt{\frac{\sigma^2 - N_0^2}{\sigma^2 - 4\Omega^2}} = \frac{\sqrt{(\sigma^2 - N_0^2)(\sigma^2 - 4\Omega^2)}}{g k_h}, \qquad k_h = \sqrt{k^2 + l^2}\,. \quad (3.3.8)$$

In this way one finds immediately the limiting formulae for short ($k_h H >> 1$) and long ($k_h H << 1$) waves:

$$\sigma^2 = g k_h \quad \text{for} \quad k_h H >> 1, \qquad \sigma^2 - 4\Omega^2 = gHk_h^2 \quad \text{for} \quad k_h H << 1 \quad (3.3.9)$$

The first formula (3.3.9) has been written down using the fact that for short waves in the ocean $|\sigma| >> N_0$ and $|\sigma| >> 2\Omega$. The second formula (3.3.9) is true also for $|\sigma| < N_0$.

Solutions of System (3.1.3)–(3.1.7) corresponding to points of intersection of eigenvalue curves $\epsilon_i(\sigma)$ of Problem V with characteristic curves of Type 1 of Problem H are called internal gravitational waves. Obviously, eigenvalue curves of Problem H and V intersect only when $2\Omega < N_0$ (cf. Figs. 3.2 and 3.3); hence the frequencies of internal waves lie in the range $2\Omega < |\sigma| < N_0$. In essence, these waves are not linked to the presence of a free surface; it is easily seen that the approximate formula (3.3.5) for the characteristic curves $\epsilon_i(\sigma)$ is equivalent to a replacement of the second condition (3.2.13) by the condition of a rigid lid $W(0) = 0$. The physical reason for the generation of internal waves is the effect of Archimedes forces which play in a stably stratified fluid the role of restoring forces. The dispersion relation for internal waves is obtained by elimination of ϵ from (3.3.5) and (3.3.7)

$$\sigma^2 = \frac{4\Omega^2 m^2 + N_0^2 k_h^2}{k_h^2 + m^2}, \quad (3.3.10)$$

where

$$m^2 = \frac{(n\pi)^2}{H^2}, \qquad n = 1, 2, 3, \ldots$$

If the medium were unbounded, then (3.3.10) would describe the dispersion relation for plane internal waves of the form $\exp[i(kx + ly + mz - \sigma t)]$. Therefore it may be said that m^2 has the meaning of the square of a vertical wave number. In the case of no rotation

$$\sigma^2 = N_0^2 \frac{k_h^2}{k_h^2 + m^2} = N_0^2 \sin^2\theta\,, \quad (3.3.11)$$

and θ is the angle formed by the wave number vector (k, l, m) with the vertical.

Assume now that the liquid is homogeneous ($N_0 = 0$); then the eigenvalue curves $\epsilon_i(\sigma)$, $i = -1, -2, \ldots$, of Problem V will necessarily intersect the third type eigenvalue curves of Problem H (for $N_0 \neq 0$, this is only possible when $2\Omega > N_0$). The corresponding solutions of equation (3.1.3)—(3.1.7) are referred to as gyroscopic waves.

As a rule, these waves are called inertial waves. The terminology proposed by Tolstoy [122, p. 217] is used here; it is based on the similarity of the particle motions in such waves (in an unbounded medium) and the motions of a gyroscope, since one understands by inertial also oscillations with frequencies equal or near to $2\Omega \sin \varphi$ (independently of the nature of these vibrations).

The physical reason for the existence of these waves is the Earth's rotation. In fact, one obtains from (3.1.3) and (3.1.4), for example, that $\partial^2 u/\partial t^2 + (2\Omega)^2 u = (-1/\rho_0)(\partial^2 p'/\partial x \partial t + 2\Omega \partial p'/\partial y)$, whence it is clear that the Coriolis force may play the role of restoring force. As internal gravitational waves, gyroscopic waves are practically not distorted when the free surface is replaced by a rigid lid (an approximation to (3.3.6) has been derived essentially for $W(0) = 0$). The dispersion relation for gyroscopic waves is obtained by elimination of ϵ from (3.3.6) and (3.3.7). For $N_0 = 0$, one has

$$\sigma^2 = (2\Omega)^2 \frac{m^2}{k_h^2 + m^2} = (2\Omega)^2 \cos^2 \vartheta \,, \tag{3.3.12}$$

where

$$m^2 = (n\pi)^2/H^2, \qquad n = -1, -2, \ldots$$

has the significance of the square of a vertical wave number and ϑ is the angle between the wave vector (k, l, m) and the axis of rotation (z-axis). Clearly, one has always $\sigma^2 < 4\Omega^2$.

A special case are oscillations of a plane rotating layer with frequency $\sigma^2 = \pm 2\Omega$. This case must be studied on the basis of initial equations of the type (3.1.3)—(3.1.7) without separation of variables. It is easily seen that $(u, v) \sim \mathrm{Re}(1, \pm i) \exp(\pm i 2\Omega t)$, $w \equiv 0, p' \equiv 0$, $\rho' \equiv 0$, so that the oscillations will proceed without participation of pressure gradients (pure inertial oscillations).

Thus, in the problem considered here, one may find surface gravitational waves and internal gravitational waves (if $2\Omega < N_0$) or gyroscopic waves (if $2\Omega > N_0$).

Introduce now a new factor, namely, Earth's spherical shape, and consider free oscillations of a non-rotating ($\Omega = 0$) spherical layer of incompressible ($c = \infty$) and inhomogeneous (ρ_0 = constant, $N = N_0$ = constant) fluid. Problem V does not change under these conditions, but Problem H must be studied all over again. Since the coefficients of the System (3.2.8)—(3.2.10) depend only on φ, solutions of Problem H will be sought in the form

$$(U,\ V,\ \Pi) = (\tilde{U},\ \tilde{V},\ \tilde{\Pi}) \exp ik\lambda\ ,$$

where $\tilde{U}$, $\tilde{V}$ and $\tilde{\Pi}$ depend on φ, and $k = 0, 1, 2, \ldots$ is the longitudinal (non-dimensional) wave number. It is convenient to change over to another dependent variable

$$\mu = \sin\varphi, \qquad -1 \leqslant \mu \leqslant 1\ .$$

Eliminating $\tilde{U}$ and $\tilde{V}$, one finds for $\tilde{\Pi}$ the equation

$$L\tilde{\Pi} + a^2\sigma^2\epsilon\tilde{\Pi} = 0\ ,$$

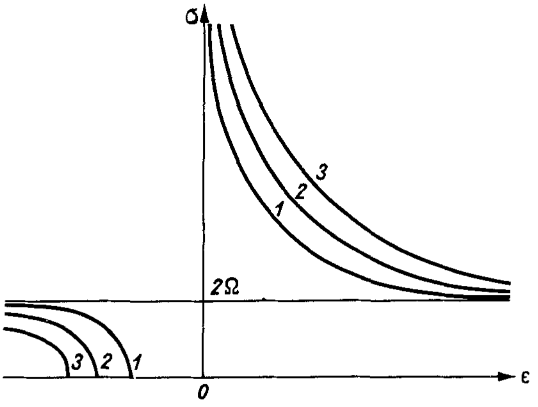
where

$$L = \frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1 - \mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}\right] - \frac{k^2}{1 - \mu^2}\ . \tag{3.3.13}$$

Solutions of this equation which are limited to the interval $[-1, 1]$ exist only for $a^2\sigma^2\epsilon = n(n+1)$, $n = k, k+1, \ldots$; these are associate Legendre functions of the first kind $P_n^k(\mu)$ (cf. [62, pp. 327—335]). Thus, the eigenfunctions and eigenvalue curves of Problem H in the case of a non-rotating spherical layer have the form

$$\Pi = P_n^k(\mu) \exp ik\lambda, \qquad \sigma^2 = \frac{n(n+1)}{a^2\epsilon}, \qquad n = k, k+1, \ldots; \qquad k = 0, 1, \ldots \tag{3.3.14}$$

The number $n - k$ gives the number of zeros of Π along the meridian: It is



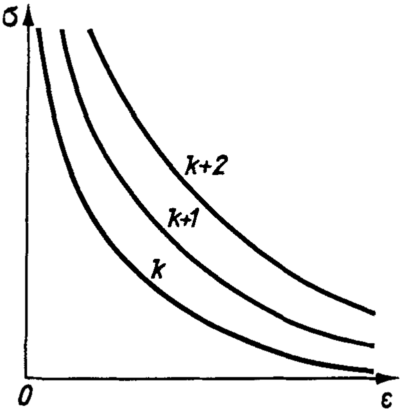


Fig. 3.3. Pattern of eigenvalue curves of Problem H for a rotating plane layer. Numbers indicate definite values of $k^2 + l^2$. The pattern is symmetric about $\sigma = 0$.

Fig. 3.4. Pattern of eigenvalue curves of Problem H for non-rotating spherical layer for fixed k. Each curve is numbered. The pattern is symmetric about $\sigma = 0$ axis.

natural to call it the latitudinal wave number (non-dimensional). The pattern of eigenvalue curves (3.3.14) is shown in Fig. 3.4. Since $\Omega = 0$, only first type eigenvalue curves are possible (compare with Fig. 3.3).

Eliminating ϵ from (3.3.1) and (3.3.14), one obtains the dispersion relation for gravitational surface waves in a spherical layer for $|\sigma| > N_0$

$$\tanh\left\{\frac{\sqrt{n(n+1)}}{a} H \frac{\sqrt{\sigma^2 - N_0^2}}{\sigma}\right\} = \frac{\sigma\sqrt{\sigma^2 - N_0^2}}{g\dfrac{\sqrt{n(n+1)}}{a}}, \qquad n = k, k+1, \ldots \tag{3.3.15}$$

The limiting cases of this formula for short and long waves are readily found:

$$\sigma^2 = g\frac{\sqrt{n(n+1)}}{a} \quad \text{for} \quad \frac{\sqrt{n(n+1)}}{a} H >> 1\,, \tag{3.3.16}$$

$$\sigma^2 = gH\frac{n(n+1)}{a^2} \quad \text{for} \quad \frac{\sqrt{n(n+1)}}{a} H << 1, \qquad n = k, k+1, \ldots$$

In writing down the first of these formulae, as also for the first formula (3.3.9), it has been assumed that $|\sigma| >> N_0$; the second formula is also true for $|\sigma| < N_0$.

Next, the dispersion relation for internal gravitational waves in a spherical layer will be written down. Eliminating ϵ from (3.3.5) and (3.3.15), one has

$$\sigma^2 = N_0^2 \frac{\dfrac{n(n+1)}{a^2}}{m^2 + \dfrac{n(n+1)}{a^2}}\,, \qquad m^2 = \frac{(n'\pi)^2}{H^2}\,, \qquad n' = 1, 2, \ldots, \tag{3.3.17}$$

$n = k, k + 1, \ldots$

In formulae (3.3.15)—(3.3.17), the symbols k, $n - k$, m denote longitudinal, latitudinal and vertical wave numbers, respectively (the first two of which are non-dimensional). It is useful to compare formulae (3.3.15)—(3.3.17) with (3.3.8)—(3.3.10).

The case of a rotating spherical layer is very complex and will be studied below. It is natural to assume that gravitational (surface and internal) and gyroscopic waves will exist also in this case. However, one may already immediately demonstrate the existence of a new, very important class of waves usually referred to as Rossby waves.

Consider the eigenvalue curve $\epsilon_0(\sigma)$ of Problem V. It has been seen that $\epsilon_0 \simeq 1/gH$ for $\sigma = 0$, and consequently, for small σ, the curve $\epsilon_0(\sigma)$ is located close to the $\epsilon = 0$ axis. Does Problem H possess such eigenvalue curves which intersect the $\epsilon = 0$ axis for small values of σ? In the cases of a plane rotating layer and a non-rotating spherical layer, such eigenvalue curves do not exist, by (3.3.7) and (3.3.14). Proceeding to the general case, set $\epsilon = 0$ in (3.2.10). Then, introducing a stream function ψ

$$U = -\frac{\partial \psi}{a \partial \varphi}, \qquad V = \frac{\partial \psi}{a \cos \varphi \partial \lambda}$$

and eliminating the function Π from (3.2.8) and (3.2.9) one finds

$$(-i\sigma)\, \Delta \psi + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} = 0\,, \tag{3.3.18}$$

where Δ is the Laplace operator on a sphere with radius a. If one seeks solutions of this equation in the form $\psi = \tilde{\psi}(\sin \varphi) e^{ik\lambda}$, $k = 1, 2, \ldots$, one finds readily

$$\tilde{\psi}(\sin \varphi) = P_n^k(\sin \varphi), \qquad \sigma = -\frac{2\Omega k}{n(n+1)}, \qquad n = k, k+1, \ldots \tag{3.3.19}$$

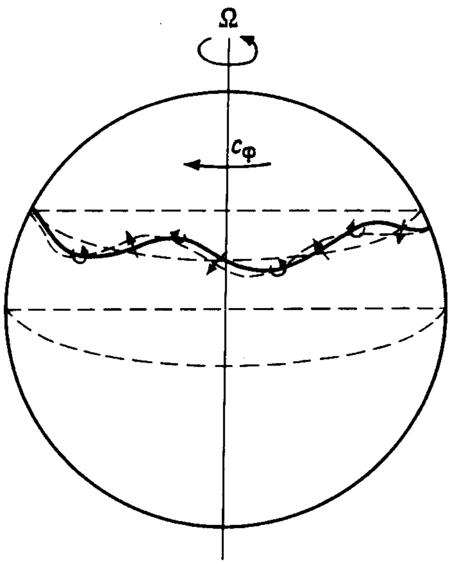
Formula (3.3.19) gives a positive answer to the question posed above. Thus, in the case of a rotating spherical layer, there arise a new class of eigenvalue curves of Problem H for which $\sigma_n(0) = 2\Omega k/n(n+1)$, $n = k, k+1$; they will be called, in future, eigenvalue curves of second type of Problem H. It is clear that the eigenvalue curves of second type of Problem H must intersect the eigenvalue curve $\epsilon_0(\sigma)$ of Problem V, which runs close to the $\epsilon = 0$ axis for small σ. The corresponding solutions of the system of equations (3.1.3)–(3.1.7) will be referred to as barotropic Rossby waves. In essence, these waves were known already a long time ago (cf., for example, [75,32]); however, Rossby [112] first showed the significance of these waves in geophysics. A detailed historical survey of these aspects is given in [99].

It is clear from the above that barotropic Rossby waves are described approximately by (3.3.18) (the error of the approximation will be estimated at the end of § 3.8, and likewise in § 3.7). It is not difficult to show that for such waves $P(z) \simeq$ constant, $W \simeq 0$, i.e., that the vertical structure of Rossby waves practically does not depend on the stratification of the sea water (whence they obtain the term barotropic). In order to explain the physical mechanism of the generation of barotropic Rossby waves, rewrite (3.3.18) in the form

$$\frac{\partial \omega_z}{\partial t} + \frac{\mathrm{d}f}{a\mathrm{d}\varphi} v = 0\,, \tag{3.3.20}$$

where $-i\sigma$ has been replaced by the operator $\partial/\partial t$ and also the Coriolis parameter $f = 2\Omega \sin \Psi$ and the vertical component of the vorticity vector $\omega_z = \Delta\psi$ have been introduced. Equation (3.3.20) is nothing else but the (linearized) equation of vorticity for such fluid motions, when u and v do not depend on z and $\omega = 0$ (cf. § 4.1).

Introduce now the displacement y of particles of the fluid along the meridian y: $\partial y/\partial t = v$. It follows then from (3.3.20) that the vorticity ω_z and



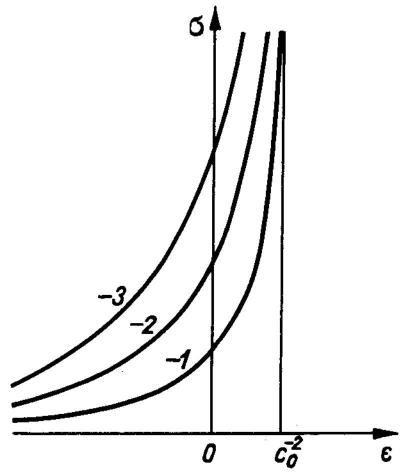


Fig. 3.5. Towards explaining the physical mechanism of generation of barotropic Rossby waves [47]. c_ϕ = phase velocity.

Fig. 3.6. Pattern of eigenvalue curves of Problem *V*. Orders of curves are indicated. The pattern is symmetric about the $\sigma = 0$ axis.

the displacement y of a particle have opposite phases. Therefore during motion of a particle towards north (south) from some mean position, the vorticity ω_z of the particle drops (grows). A vorticity field thus perturbed gives rise to a velocity field (Fig. 3.5) for which each particle executes an oscillation about its mean position while the wave form moves to the west. Thus, in the case of a rotating spherical layer ($\mathrm{d}f/\mathrm{d}\varphi \neq 0$), there exists a distinct wave-type mechanism which has no analogue in the case of a rotating plane layer or of a non-rotating sphere.

Note that, by (3.3.19), barotropic Rossby waves have low frequencies and that the zonal components of their phase velocities are always negative, i.e., they may propagate only in a westerly direction.

In conclusion of this section, the effect of compressibility will be briefly investigated. For this purpose, consider a homogeneous (ρ_0 = constant, $N_0 = 0$), compressible ($c = c_0$ = constant) fluid in the absence of gravity forces ($g = 0$). Then the system (3.2.11) and (3.2.12) reduces to the single equation

$$\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} + \sigma^2\left(\frac{1}{c_0^2} - \epsilon\right) W = 0$$

with the conditions $W(-H) = 0$ and $(\mathrm{d}W/\mathrm{d}z)_{z=0} = 0$. It is readily seen that

$$W = \sin\left\{\sigma\sqrt{\frac{1}{c_0^2} - \epsilon}(z + H)\right\}, \qquad \sigma^2\left(\frac{1}{c_0^2} - \epsilon\right) = \left(\frac{\pi}{2H}\right)^2 (2|n| - 1)^2,$$

$$n = -1, -2, \ldots \tag{3.3.21}$$

Figure 3.6 presents schematically the eigenvalue curves (3.3.21). Comparison of Figs. 3.2. and 3.6 shows that the eigenvalue curves of Fig. 3.6 display the distorting effect of compressibility (displacement of vertical asymptote). The disappearance in Fig. 3.6 of eigenvalue curves with numbers $n = 0, 1, \ldots$ is completely clear, since there is no gravity field.

Solutions of System (3.1.3)–(3.1.7) corresponding to intersections of eigenvalue curves $\epsilon_{-1}(\sigma)$, $\epsilon_{-2}(\sigma)$, ... of Problem V with eigenvalue curves of the first type of Problem H (cf. Figs. 3.3 and 3.4) are referred to as acoustic waves. They are high-frequency waves which exist due to a medium's elasticity (obviously, these waves disappear for $c_0 = \infty$). The dispersion relation for acoustic waves on a non-rotating plane is

$$\frac{\sigma^2}{c_0^2} = k^2 + l^2 + m^2,$$

where $m^2 = (\pi/2H)^2 (2|n|-1)^2$ $(n = -1, -2, \ldots)$ has the significance of the square of a vertical wave number. It is not difficult to write down the analogous formula also for a non-rotating spherical layer (rotation exerts insignificant influence on acoustic waves).

3.4. THE EIGENVALUE CURVES FOR PROBLEM V

Consider now Problem V for the general case of a compressible stratified ocean. Starting from results of § 3.3, one might expect existence of eigenvalue curves of the type $\sigma_0(\epsilon)$, $\sigma_1(\epsilon)$, ..., $\sigma_{-1}(\epsilon)$, $\sigma_{-2}(\epsilon)$, ... (cf. Figs. 3.2 and 3.6), although, generally speaking, new types of eigenvalue curves are also possible (the ensuing analysis shows that new types of such curves do not exist).

A beginning will be made with a study of the disposition of the eigenvalue curves of Problem V in the (ϵ, σ^2)-plane by derivation of integral relations.

In fact, it is convenient to study the (ϵ, σ^2)-plane (and not the (ϵ, σ)-plane) and to assume that $-\infty < \sigma^2 < \infty$, although in the problem of free oscillations of a stably stratified fluid only $\sigma^2 > 0$ has physical significance.

Replace equation (3.2.11) by a complex conjugate equation and, multiplying it by W, add it to (3.2.12) after it has been multiplied by the function conjugate complex to P. Integrating the relation obtained from $-H$ to 0 and using the conjugate complex form of (3.2.13), one finds

$$\int_{-H}^{0} (N^2 - \sigma^2)\, \rho_0 |W|^2\, dz + g\rho_0(0)|W(0)|^2 = \int_{-H}^{0} \left(\epsilon - \frac{1}{c^2}\right) \frac{1}{\rho_0} |P|^2\, dz\ .$$

Since it has already been proved that all possible frequencies σ are real, it follows that the eigenvalues ϵ are real. Therefore also the eigenfunctions P and W may be assumed to be real. Likewise, it is obvious that for $\sigma^2 < N^2_{\min}$ the eigenvalue curves of Problem V cannot lie in a region where $\epsilon < 1/c^2_{\max}$.

If $F(z)$ is some function on the segment $(-H, 0)$, then $F_{\min}$ (or $F_{\max}$) denotes the minimum (or maximum) of the function on $(-H, 0)$.

Introduce instead of P the new function Φ

$$\Phi = P + g\rho_0 W$$

and rewrite System (3.2.11) and (3.2.12) in the form

$$\Phi' + g\epsilon\Phi + (\sigma^2 - g^2\varepsilon)\ \rho_0 W = 0\ , \tag{3.4.1}$$

$$W' - g\epsilon W + \left(\epsilon - \frac{1}{c^2}\right) \frac{1}{\rho_0} \Phi = 0\ . \tag{3.4.2}$$

where here and throughout this section dashes denote differentiation with respect to z.

Multiply (3.4.1) by W, and (3.4.2) by Φ and add the results. After integration of the relation obtained from $-H$ to 0, one obtains, by the boundary conditions (3.2.13),

$$(g^2\epsilon - \sigma^2) \int_{-H}^{0} \rho_0 W^2 dz = \int_{-H}^{0} \left(\epsilon - \frac{1}{c^2}\right) \frac{1}{\rho_0}\, \Phi^2 dz\ .$$

It follows from this identity that for $g^2\epsilon > \sigma^2$ the eigenvalue curves of Problem V cannot lie in the region $\epsilon < 1/c^2_{\max}$, and for $g^2\epsilon < \sigma^2$ in the region $\epsilon > 1/c^2_{\min}$.

Finally, the monotonic nature of the eigenvalue curves will be proved. For this purpose consider an arbitrary point (ϵ, σ^2) one some eigenvalue curve and its corresponding eigenfunctions P and W. Move along this eigenvalue curve by $(d\epsilon, d\sigma^2)$; then also the eigenfunctions P and W will change by δP and δW. Taking variations of (3.2.11), (3.2.12) and (3.2.13), one finds

$$(\delta P)' + \frac{g}{c^2}\delta P + (\sigma^2 - N^2)\, \rho_0 \delta W + \rho_0 W d\sigma^2 = 0\ , \tag{3.4.3}$$

$$(\delta W)' - \frac{g}{c^2}\, \delta W + \left(\epsilon - \frac{1}{c^2}\right) \frac{1}{\rho_0} \delta P + \frac{1}{\rho_0} P d\epsilon = 0\ , \tag{3.4.4}$$

$$\delta W = 0 \quad \text{for} \quad z = -H, \qquad \delta P + g\rho_0 \delta W = 0 \quad \text{for} \quad z = 0 \,. \tag{3.4.5}$$

Subtract from equation (3.2.11), multiplied by δW, equation (3.4.3), multiplied by W, from equation (3.4.4), multiplied by P, equation (3.2.12), multiplied by δP, and add the results. Integrate the relation obtained from $-H$ to 0. Since $[P\delta W - W\delta P]_{-H}^{0} = 0$, on the basis of the boundary conditions (3.2.13) and (3.4.5), one finds

$$\left(\int_{-H}^{0} \rho_0 W^2 \mathrm{d}z\right) \mathrm{d}\sigma^2 = \left(\int_{-H}^{0} \frac{1}{\rho_0} P^2 \mathrm{d}z\right) \mathrm{d}\epsilon \,.$$

Hence follows the monotonic nature of the eigenvalue curves in the (ϵ, σ^2)-plane. It should be noted here that there do not exist for the problem under consideration eigensolutions of the form $P \equiv 0$, $W \not\equiv 0$ or $P \not\equiv 0$, $W \equiv 0$. Therefore an ensuing transition from the analysis of a system of equations for P and W to a study of separate equations for P or W does not lead to a loss of eigensolutions.

It is convenient to introduce new functions

$$\hat{p} = P \exp\left(\int_{-H}^{z} \frac{g}{c^2} \mathrm{d}z\right), \qquad \hat{w} = W \exp\left(-\int_{-H}^{z} \frac{g}{c^2} \mathrm{d}z\right),$$

$$\hat{\rho}_0 = \rho_0 \exp\left(2 \int_{-H}^{z} \frac{g}{c^2} \mathrm{d}z\right) \tag{3.4.6}$$

and to write System (3.2.11) and (3.2.12) in the simpler form

$$\hat{p}' + (\sigma^2 - N^2)\,\hat{\rho}_0 \hat{w} = 0 \,, \tag{3.4.7}$$

$$\hat{w}' + \left(\epsilon - \frac{1}{c^2}\right) \frac{1}{\hat{\rho}_0} \hat{p} = 0 \,. \tag{3.4.8}$$

Consider the following cases:

(1) $\sigma^2 < N^2_{\min}$

Recall that in this region eigenvalues may only occur for $\epsilon > 1/c^2_{\max}$. Eliminating from (3.4.7) and (3.4.8) the function $\hat{w}$, one obtains for $\hat{p}$ the equation

$$(K\hat{p}')' - G\hat{p} = 0 \,, \tag{3.4.9}$$

where

$$K = \frac{1}{(N^2 - \sigma^2)\,\hat{\rho}_0} > 0, \qquad G = \left(\frac{1}{c^2} - \epsilon\right) \frac{1}{\hat{\rho}_0} \,. \tag{3.4.10}$$

Rewrite (3.2.13) in the form

$$\hat{p}' = 0 \quad \text{for} \quad z = -H, \qquad \hat{p} + \frac{g}{N^2 - \sigma^2}\hat{p}' = 0 \quad \text{for} \quad z = 0. \tag{3.4.11}$$

First, the existence of eigenvalues for the problem (3.4.9) and (3.4.11) will be proved. Let $\hat{p}(z)$ be the solution of the Cauchy problem for equations (3.4.9): $\hat{p}(-H) = 1, \hat{p}'(-H) = 0$. Using Sturm's comparison theorems [39, Chapter X; 30, Chapter XI], it is not difficult to obtain qualitatively graphs of the quantity $K(0)\hat{p}'(0)/\hat{p}(0)$ as a function of ϵ (for fixed σ^2).

In fact, fix σ^2 and increase ϵ from $-\infty$ to ∞. During this process, the function K does not change, but the function G decreases (for all z) and, according to the first comparison theorem, in the interval $-H \leqslant z \leqslant 0$ the number of zeros of $\hat{p}(z)$ may only increase, where each new zero appears first at the point $z = 0$ and the moves in the direction of the point $z = -H$ (zeros cannot pass through the point $z = -H$). Let ϵ_1, ϵ_2, ... be values of ϵ for which $\hat{p}(0) = 0$. There will be an infinite number of such values of ϵ, since $G \to \infty$ for $\epsilon \to \infty$ for all z. Then, by the second comparison theorem, the quantity $K(0)\hat{p}'(0)/\hat{p}(0)$ will monotonically decrease with ϵ in the intervals $(-\infty, \bar{\epsilon}_1)$, $(\bar{\epsilon}_1, \bar{\epsilon}_2)$, ...

Further, replacing K and G in (3.4.9) by $K_{\min}$ and $G_{\min}$:

$$K_{\min} = \frac{1}{(N_{\max}^2 - \sigma^2)\,\hat{\rho}_{0\,\max}}, \qquad G_{\min} = \left(\frac{1}{c_{\max}^2} - \epsilon\right)\frac{1}{\hat{\rho}_{0\,\min}}, \tag{3.4.12}$$

and using the second comparison theorem, it is not difficult to show that $K(0)\hat{p}'(0)/\hat{p}(0) \to +\infty$ for $\epsilon \to -\infty$.

Schematic graphs of $K(0)\hat{p}'(0)/\hat{p}(0)$ as function of ϵ are given in Fig. 3.7. By (3.4.11), intersections of this graph with the straight line $K(0)\hat{p}'(0)/\hat{p}(0) = -1/g\rho_0(0)$ give the eigenvalues of the problem under consideration.

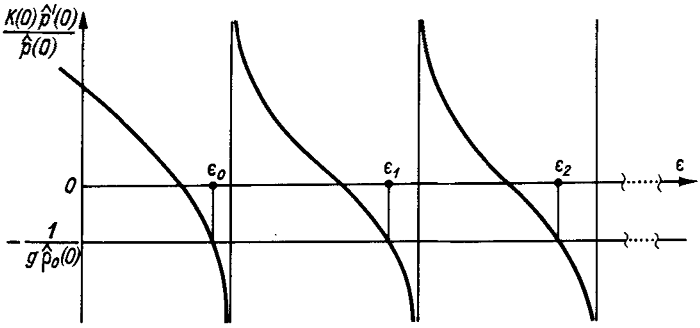


Fig. 3.7. Schematic graph of $K(0)\hat{p}'(0)/\hat{p}(0)$ as function of ϵ for fixed $\sigma^2 < N_{\min}^2$. The eigenvalues ϵ_0, ϵ_1, ϵ_2, ... are indicated.

For $\epsilon > 1/c^2_{\max}$ in the interval $(-H, 0)$ one can always find a subinterval where G is negative. However, then, for $\sigma^2 \to -\infty$, the function K decreases (for all z), and, by the first comparison theorem, the solution $p(z)$ will oscillate over this subinterval and hence over the entire interval $(-H, 0)$. In other words, for any fixed $\epsilon > 1/c^2_{\max}$, the problem under consideration must have an infinite number of eigenvalues σ^2. Since the eigenvalue curves cannot intersect (cf. Fig. 3.7), then for $\sigma^2 \to -\infty$ all eigenvalue curves tend to the vertical asymptote $\epsilon = 1/c^2_{\max}$.

Using Sturm's comparison theorems, one may obtain simple estimates of the eigenvalues $\epsilon_0, \epsilon_1, \ldots$. For this purpose, replace in equation (3.4.9) the functions F and G by their minimum values $K_{\min}$ and $G_{\min}$ and consider the equation with constant coefficients obtained for the boundary conditions $\hat{p}'(-H) = 0, \hat{p}'(-H) = 0$. For the changed equation (3.4.9), one may likewise introduce the solution of the Cauchy problem: $\hat{p}(-H) = 1, \hat{p}'(-H) = 0$ and, constructing the graph of $K_{\min}\hat{p}'(0)/\hat{p}(0)$, compare it with the graph of $K(0)\hat{p}'(0)/\hat{p}(0)$ already constructed for the same equation (3.4.9). It is not difficult to show then that for the same numbers the eigenvalues of the initial problem will be larger than the eigenvalues of the changed problem Since the last can be found immediately, known lower estimates of the eigenvalues $\epsilon_0, \epsilon_1, \ldots$ have been obtained. Analogously, replacing in equation (3.4.9) the functions K and G by their maximum values:

$$K_{\max} = \frac{1}{(N^2_{\min} - \sigma^2)\,\hat{\rho}_{0\,\min}}, \qquad G_{\max} = \left(\frac{1}{c^2_{\min}} - \epsilon\right)\frac{1}{\hat{\rho}_{0\,\max}}, \tag{3.4.13}$$

one obtains upper bounds for the eigenvalues $\epsilon_0, \epsilon_1, \ldots$. Thus, one has, finally,

$$\frac{(n\pi)^2\hat{\rho}_{0\,\min}}{(N^2_{\max} - \sigma^2)\,H^2\hat{\rho}_{0\,\max}} + \frac{1}{c^2_{\max}} < \epsilon_n(\sigma^2) < \frac{1}{c^2_{\min}} + \frac{[(n+1)\,\pi]^2\,\hat{\rho}_{0\,\max}}{(N^2_{\min} - \sigma^2)\,H^2\hat{\rho}_{0\,\min}},$$
$$n = 0, 1, \ldots \tag{3.4.14}$$

Generally speaking, the upper bounds (3.4.14) are true only for $\epsilon > 1/c^2_{\min}$, since for $\epsilon < 1/c^2_{\min}$ formula (3.4.13) for $G_{\max}$ is not valid. However, since it has been proved that the eigenvalue curves are monotonic, these estimates are also true for $\epsilon < 1/c^2_{\min}$.

The estimates (3.4.14) are rather coarse (especially for ϵ_0); however, all the same, they yield a definite idea on the distributions of the eigenvalue curves $\epsilon_1(\sigma^2)$, $\epsilon_2(\sigma^2)$, ... (for $\epsilon_0(\sigma^2)$ another estimate will be derived, cf. (3.4.19)).

It is not difficult to show, reverting to System (3.4.1) and (3.4.2), that the eigenvalue curves of the problem in hand may not intersect the straight line $\sigma^2 = g^2\epsilon$. Therefore, since the curves $\epsilon_n(\sigma^2)$ $(n = 0, 1, 2, \ldots)$ are monotonic, they may not have vertical asymptotes for $\epsilon > 1/c^2_{\max}$ and must without fail decrease in the region $\epsilon > 1/c^2_{\min}$.

(2) $\epsilon > 1/c^2_{\min}$

Eliminating from (3.4.7) and (3.4.8) the function $\hat{p}$, write the equation for $\hat{w}$ in the form

$$(K\hat{w}')' - G\hat{w} = 0\,, \tag{3.4.15}$$

where

$$K = \frac{\hat{\rho}_0}{\epsilon - \dfrac{1}{c^2}} > 0, \qquad G = (\sigma^2 - N^2)\,\hat{\rho}_0\,. \tag{3.4.16}$$

The boundary conditions are

$$\hat{w} = 0 \quad \text{for} \quad z = -H, \qquad \hat{w}' - \left(\epsilon - \frac{1}{c^2}\right) g\hat{w} = 0 \quad \text{for} \quad z = 0\,. \tag{3.4.17}$$

Let $w(z)$ be the solution of the Cauchy problem for equation (3.4.15): $\hat{w}(-H) = 0$, $\hat{w}'(-H) = 1$. Since, as σ^2 decreases from ∞ to $-\infty$ and when ϵ is fixed, the function G decreases (for all z) and the function K does not change, then, using the comparison theorems, in the same manner as in the preceding case, it is not difficult to establish the graph of $K(0)\hat{w}'(0)/\hat{w}(0)$ as a function of σ^2 (for fixed ϵ) (cf. Fig. 3.8). The intersection of this graph with the straight line $K(0)\hat{w}'(0)/\hat{w}(0) = g\hat{\rho}_0(0)$ yields, by (3.4.17), the eigenvalues $\sigma_0^2, \sigma_1^2, \ldots$.

As in the preceding case, one may obtain upper and lower bounds for the eigenvalues σ_1^2, σ_2^2, ... (replacing conditions (3.4.17) by the conditions $\hat{w}(-H) = 0$, $\hat{w}(0) = 0$ and K, G by $K_{\min}$, $G_{\min}$ and $K_{\max}$, $G_{\max}$, respectively):

$$-\frac{[(n+1)\,\pi]^2\,\hat{\rho}_{0\,\max}}{\left(\epsilon - \dfrac{1}{c^2_{\min}}\right) H^2 \hat{\rho}_{0\,\min}} + N^2_{\min} < \sigma_n^2(\epsilon) < N^2_{\max} - \frac{(n\pi)^2\,\hat{\rho}_{0\,\min}}{\left(\epsilon - \dfrac{1}{c^2_{\max}}\right) H^2 \hat{\rho}_{0\,\max}}\,,$$

$$n = 1, 2, \ldots \tag{3.4.18}$$

Note that the estimated curves in (3.4.14) and (3.4.18) for $n = 1, 2, \ldots$ in the region $\sigma^2 < N^2_{\min}$, $\epsilon > 1/c^2_{\mathrm{m}}$ are identical.

Thus, for all $\epsilon > 1/c^2_{\min}$, by (3.4.18), one has $\sigma_n^2(\epsilon) < N^2_{\max}$ $n = 1, 2, \ldots$. Since for $\sigma^2 < N^2_{\max}$ in the interval $(-H, 0)$ one can always find a subinterval where G will be negative, and K decreases for $\epsilon \to \infty$ (for all z), then it is easily shown that the eigenvalue curves $\sigma_n^2(\epsilon)$ ($n = 1, 2, \ldots$) tend for large ϵ to the straight line $\sigma^2 = N^2_{\max}$ as an asymptote.

The eigenvalue curve $\sigma_0^2(\epsilon)$ requires separate consideration. Without proof (cf. [55] for a proof) one has first the estimate

$$\frac{1}{gH} \leqslant \epsilon_0(0) < \frac{1}{c_{\min}^2} + \frac{1}{gH}\frac{\hat{\rho}_{0\,\max}}{\hat{\rho}_0(0)} . \tag{3.4.19}$$

Setting $N_0 = 0$ in (3.3.2) (incompressible homogeneous fluid), one has that $\epsilon_0(0) = 1/gH$. In the general case, the estimate (3.4.19) gives a very narrow range for $\epsilon_0(0)$.

In order to avoid a misunderstanding, it will be recalled that (3.3.2) for $N_0 \neq 0$ is approximate; hence the fact that, by this formula, $\epsilon_0(N_0) = 1/gH$ does not contradict inequality (3.4.19).

Further, since the curve $\sigma_0^2(\epsilon)$ has been seen to lie below the straight line $\sigma^2 = g^2\epsilon$, one has $\sigma_0^2(\epsilon) < g^2\epsilon$ for all ϵ. It will be proved now that for $\epsilon \to \infty$ the eigenvalue curve $\sigma_0^2(\epsilon)$ approaches to the straight line $\sigma^2 = g^2\epsilon$ asymptotically. For this purpose, replace in (3.4.15) the quantities K and G by K_1 and G_1

$$K_1 = \frac{\hat{\rho}_0(z)}{\epsilon - \dfrac{1}{c_{\min}^2}}, \qquad G_1 = \sigma^2\hat{\rho}_0(z) ,$$

and the boundary conditions (3.4.17) by

$$\hat{w} = 0 \quad \text{for} \quad z = -H, \qquad \hat{w}' - (\epsilon - 1/c_{\min}^2)\, g\hat{w} = 0 \quad \text{for} \quad z = 0 . \tag{3.4.20}$$

It is readily shown that the problem changed in this manner has only one eigenvalue σ_*^2. In order to determine it, let $\hat{w}(z) = \hat{\rho}_0^{-1/2}\psi(z)$. Then one finds

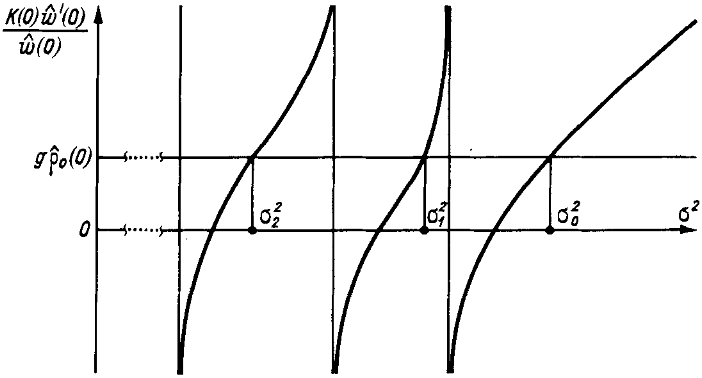


Fig. 3.8. Schematic graph of $K(0)\hat{w}'(0)/\hat{w}(0)$ as function of σ^2 for fixed $\epsilon > 1/c_{\min}^2$. The eigenvalues σ_0^2, σ_1^2, σ_2^2, ... are indicated.

for ψ the equation

$$\psi'' - \left\{\left(\epsilon - \frac{1}{c_{\min}^2}\right)\sigma^2 + \frac{1}{\hat{\rho}_0^{1/2}}\left(\hat{\rho}_0^{1/2}\right)''\right\}\psi = 0\,. \tag{3.4.21}$$

It is easily verified that as $\epsilon \to \infty$ the eigenvalue of the problem (3.4.21) and (3.4.20) (condition (3.4.20) must be rewritten in terms of ψ) tends asymptotically to $g^2\epsilon$. Since $\sigma_0^2 > \sigma_*^2$ for all ϵ, then also the curve $\sigma_0^2(\epsilon)$ approaches for $\epsilon \to \infty$asymptotically to the straight line $\sigma^2 = g^2\epsilon$.

(3) $\epsilon > 1/c_{\max}^2$

In this region, there may only occur eigenvalues for $\sigma^2 > N_{\min}^2$. Write down for $\hat{w}$ an equation of the form (3.4.15) with the functions K and G given by

$$K = \frac{\hat{\rho}_0}{\dfrac{1}{c^2} - \epsilon} > 0, \qquad G = (N^2 - \sigma^2)\,\hat{\rho}_0\,,$$

which differ only in sign from the corresponding formulae (3.4.16). Consider again the solution of the Cauchy problem for (3.4.15): $\hat{w}(-H) = 0$, $\hat{w}'(-H) = 1$. The graph of the quantity $K(0)\hat{w}'(0)/\hat{w}(0)$ as function of σ^2 (for fixed ϵ) is shown in Fig. 3.9 which also shows the eigenvalues σ_{-1}^2, σ_{-2}^2, ... (to discriminate between these eigenvalues and σ_1^2, σ_2^2 ... negative indices are introduced, c.f. (3.3.6)).

It is readily proved that all eigenvalue curves $\sigma_n^2(\epsilon)$ ($n = -1, -2, \ldots$) tend for $\epsilon \to \infty$ to the straight line $\sigma^2 = N_{\min}^2$ as asymptote. Only lower bounds for these eigenvalues will be obtained. For this purpose, one must consider equation (3.4.15) for the boundary conditions $\hat{w}(-H) = 0$, $\hat{w}'(0) = 0$, replacing in them K, G by $K_{\min}$, $G_{\min}$. One finds

$$\sigma_n^2(\epsilon) > N_{\min}^2 + \frac{(|n| - 1/2)^2\,\pi^2\hat{\rho}_{0\,\min}}{\left(\dfrac{1}{c_{\min}^2} - \epsilon\right)H^2\hat{\rho}_{0\,\max}}\,, \qquad n = -1, -2, \ldots \tag{3.4.22}$$

Since the curves $\sigma_n^2(\epsilon)$ ($n = -1, -2, \ldots$) do not intersect the straight line $\sigma^2 = g^2\epsilon$, one finds that, because they are monotonic, they cannot have horizontal asymptotes for $\epsilon \to \infty$ and therefore must lie in the region $\sigma^2 > N_{\max}^2$.

(4) $\sigma^2 > N_{\max}^2$

Write the equation for $\hat{p}$ in the form (3.4.9), determining the functions K, G by

$$K = \frac{1}{(\sigma^2 - N^2)\,\hat{\rho}_0} > 0, \qquad G = \left(\epsilon - \frac{1}{c^2}\right)\frac{1}{\hat{\rho}_0}\,,$$

which differ from (3.4.10) only in sign.

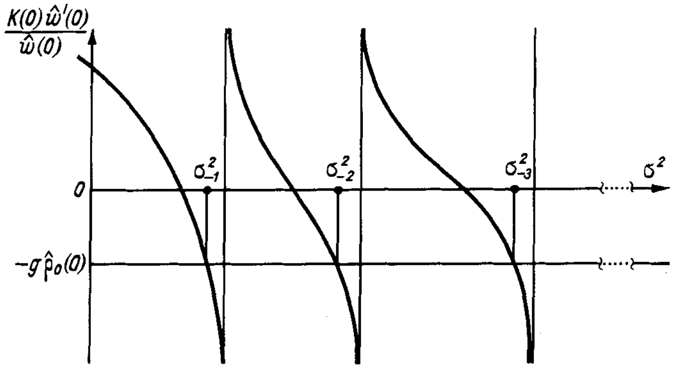


Fig. 3.9. Schematic graph of the quantity $K(0)\hat{w}'(0)/\hat{w}(0)$ as function of σ^2 for fixed $\epsilon < 1/c^2_{\max}$. The eigenvalues σ^2_{-1}, σ^2_{-2}, ... are indicated.

Consider the solution of the Cauchy problem for (3.4.9): $\hat{p}(-H) = 1$, $\hat{p}'(-H) = 0$. The graph of the quantity $K(0)\hat{p}'(0)/\hat{p}(0)$ as function of ϵ (for fixed σ^2) is shown in Fig. 3.10, which also indicates the eigenvalues ϵ_0, ϵ_{-1}, ϵ_{-2}, ... For a derivation of upper bounds for the eigenvalues ϵ_{-1}, ϵ_{-2}, ... consider equation (3.4.9) for the boundary conditions $\hat{p}'(-H) = 0, \hat{p}(0) = 0$, replacing K, G by $K_{\min}$, $G_{\min}$. One finds

$$\epsilon_n(\sigma^2) < \frac{1}{c^2_{\min}} - \frac{(|n| - 1/2)^2\, \pi^2 \hat{\rho}_{0\,\min}}{(\sigma^2 - N^2_{\min})\, H^2 \hat{\rho}_{0\,\max}}, \qquad n = -1, -2, \ldots \tag{3.4.23}$$

The estimated curves in (3.4.22) and (3.4.23) for $n = -1, -2, \ldots$ in the region $\sigma^2 > N^2_{\max}$ and $\epsilon < 1/c^2_{\max}$ are identical.

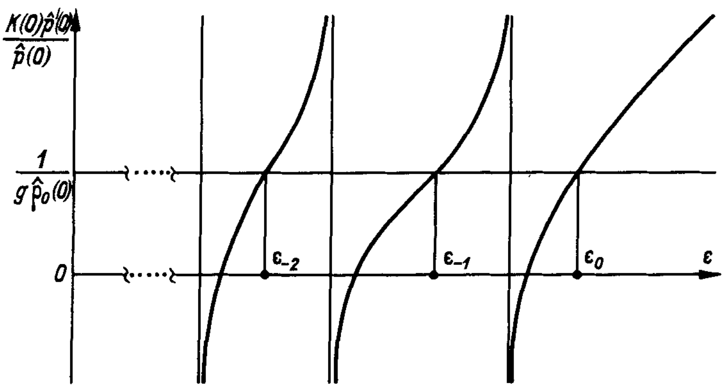


Fig. 3.10. Schematic graph of the quantity $K(0)\hat{p}'(0)/\hat{p}(0)$ as function of ϵ for fixed $\sigma^2 > N^2_{\max}$. The eigenvalues ϵ_0, ϵ_{-1}, ϵ_{-2}, ... are indicated.

TABLE 3.I

Number of zeros of $P_n(z)$ inside the interval $[-H, 0]$

Region	Index	Number of zeros
$\sigma^2 < N^2_{\min}$	$n \geqslant 0$	n
$\sigma^2 > N^2_{\max}$	$n \leqslant 0$	$\lvert n \rvert$

Thus, one has for all σ^2, by (3.4.23), that $\epsilon_n(\sigma^2) < 1/c^2_{\min}$ ($n = -1, -2, ...$). It is easily proved that the straight line $\epsilon = 1/c^2_{\min}$ is a vertical asymptote for all eigenvalue curves $\epsilon_n(\sigma^2)$ ($n = -1, -2, ...$).

(5) $1/c^2_{\max} < \epsilon < 1/c^2_{\min}, N^2_{\min} < \sigma^2 < N^2_{\max}$

Since the eigenvalue curves, in general, are monotonic, there cannot be closed curves in the region under consideration, and consequently there can pass through this region only eigenvalue curves which have already been studied. Naturally, it will be proposed that the eigenvalue curves may not have singular points and cannot break up at any finite point of the (ϵ, σ^2)-plane.

In summarizing the results obtained, it is easy to display schematically the eigenvalue curves of Problem V in the (ϵ, σ^2)-plane (Fig. 3.11). It will be useful to compare Fig. 3.11 and Figs. 3.2 and 3.6 keeping in mind that Figs. 3.2 and 3.6 are graphed on a plane ϵ, σ. In fact, there do not arise new types of eigenvalue curves (compared with Figs. 3.2 and 3.6) in the general case.

Let $\hat{p}_n(z)$ and $\hat{w}_n(z)$ be eigenfunctions which correspond to the eigenvalue curves with index $n (n = 0, \pm 1, \pm 2, ...)$. Then, applying Sturm's theorems to the problem under consideration (cf. Figs. 3.7—3.10), one may point out the number of zeros of $\hat{p}_n(z)$ and $\hat{w}_n(z)$ (i.e., also of $P_n(z)$ and $W_n(z)$, cf. (3.4.6)) inside the interval $[-H, 0]$ (Tables 3.I and 3.II). In the general case when $1/c^2_{\max} < \epsilon < 1/c^2_{\min}$, it is impossible to say anything on the number of zeros of $\hat{w}_n(z)$; apparently, the same is true for the zeros of $\hat{p}_n(z)$, when $N^2_{\min} < \sigma^2 < N^2_{\max}$.

The results of this section permit to obtain rather easily estimates of changes of the parameters $c(z)$ and $N(z)$ for different types of waves in the general case.

TABLE 3.II

Number of zeros of $W_n(z)$ inside the interval $[-H, 0]$

Region	Index	Number of zeros
$\epsilon < 1/c^2_{\max}$	$n < 0$	$\lvert n \rvert - 1$
$\epsilon > 1/c^2_{\min}$	$n \geqslant 0$	n

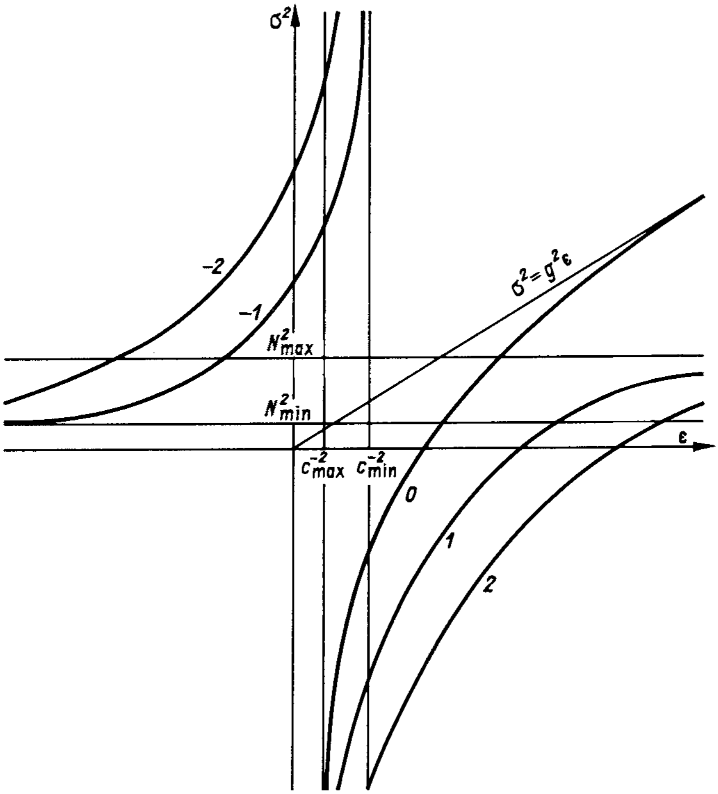


Fig. 3.11. Schematic presentation of eigenvalue curves of Problem V in (ϵ, σ^2)-plane (general case). Numerals indicate corresponding values of n.

3.5 THE EIGENVALUE CURVES OF PROBLEM H.

In the general case of a rotating spherical layer of fluid, one may expect, on the basis of the results of § 3.3, that all three types of eigenvalue curves of Problem H exist: The first type, an analogue of the curves shown in Figs. 3.3 and 3.4 for $\epsilon > 0$; the second type, curves with the property of boundedness of $\sigma_n(\epsilon)$ as $\epsilon \to 0$ (this type of curves are, in fact, characteristic for a rotating spherical layer; these curves are missing in Figs. 3.3 and 3.4); the third type, an analogue of the curves shown in Fig. 3.3. for $\epsilon < 0$.

Thus, proceed to the analysis of Problem H. It will be convenient to seek the solution of this problem in the form

$$(U, V, \Pi) = \left(\frac{U_*}{\cos\varphi}, \frac{V_*}{i\cos\varphi}, 2\Omega a \Pi_*\right) \exp ik\lambda\,, \tag{3.5.1}$$

where the functions U_*, V_*, Π_* depend only on φ, and k is an integer which is not equal to zero (the case $k = 0$ is discussed at the end of this section) and has the significance of a (non-dimensional) longitudinal wave number. Substituting (3.5.1) into (3.2.8)–(3.2.10) and changing over to the variable $\mu = \sin\varphi$, one finds

$$\sigma_* U_* - \mu V_* - k\Pi_* = 0\,, \tag{3.5.2}$$

$$-\sigma_* V_* + \mu U_* + D\Pi_* = 0\,, \tag{3.5.3}$$

$$kU_* - DV_* - \epsilon_*\sigma_*(1-\mu^2)\,\Pi_* = 0\,, \tag{3.5.4}$$

where

$$D = (1-\mu^2)\frac{\mathrm{d}}{\mathrm{d}\mu}, \qquad \sigma_* = \frac{\sigma}{2\Omega}, \qquad \epsilon_* = 4\Omega^2 a^2 \epsilon\,. \tag{3.5.5}$$

Use equation (3.5.2) to eliminate U_* from equations (3.5.3) and (3.5.4) and find

$$\left(D + \frac{\mu k}{\sigma_*}\right)\Pi_* = -\left(\frac{\mu^2}{\sigma_*} - \sigma_*\right) V_*\,, \tag{3.5.6}$$

$$\left(D - \frac{\mu k}{\sigma_*}\right) V_* = \left[\frac{k^2}{\sigma_*} - \epsilon_*\sigma_*(1-\mu^2)\right]\Pi_*\,. \tag{3.5.7}$$

Thus, the problem has been reduced to finding solutions bounded in the interval $[-1, +1]$ for the two ordinary differential equations (3.5.6) and (3.5.7). It is not difficult to establish (cf., for example, [110, Chapter 5]) that there can exist only one-parameter families of solutions of this system which enter the neighbourhoods of the North and South Poles like $|\mu - 1|^{k/2}$ and $-|\mu + 1|^{k/2}$, respectively (there cannot be singular points inside the interval $[-1, 1]$).

Replace equation (3.5.6) by its conjugate complex equation, multiply it by V_* and add the result to equation (3.5.7) after having multiplied it by the function, conjugate complex to Π_*. Multiply the result by $(1-\mu^2)^{-1}$ and integrate it with respect to μ from -1 to 1. On the basis of the behaviour stated of (Π_*, V_*), one has for $\mu = \pm 1$ that $[\Pi_*, V_*]_{-1}^{1} = 0$; one obtains, finally,

$$\epsilon_*\sigma_*^2 \int_{-1}^{1} |\Pi_*|^2\,\mathrm{d}\mu = \int_{-1}^{1} \frac{\sigma_*^2 - \mu^2}{1-\mu^2}\,|V_*|^2\,\mathrm{d}\mu + \int_{-1}^{1} \frac{k^2}{1-\mu^2}\,|\Pi_*|^2\,\mathrm{d}\mu\,.$$

Since σ_* is real, it follows that the eigenvalues ϵ_* of Problem H are real (recall that it has already been proved for Problem V that the ϵ are real). Therefore the functions Π_* and V_* may also be assumed to be real. Besides, it follows from the identity derived that for $\epsilon_* < 0$ the eigenvalue curves of Problem H cannot lie outside the strip $|\sigma_*| < 1$.

It is easily shown [cf. (3.2.1), (3.2.6), (3.5.1) and (3.5.6), (3.5.7)] that, if

$$\mathrm{Re}\left\{\left[\frac{1}{\rho_0}P\frac{U_*}{\cos\varphi}, \frac{1}{\rho_0}P\frac{V_*}{i\cos\varphi}, 2\Omega ai\sigma W\Pi_*, 2\Omega aP\Pi_*\right]\exp[i(k\lambda-\sigma t)]\right\}$$

is a solution of Problem (3.1.3)—(3.1.9), then also

$$\mathrm{Re}\left\{\left[\frac{1}{\rho_0}P\frac{U_*}{\cos\varphi}, -\frac{1}{\rho_0}P\frac{V_*}{i\cos\varphi}, -2\Omega ai\sigma W\Pi_*, 2\Omega aP\Pi_*\right]\exp[-i(k\lambda-\sigma t)]\right\}$$

is a solution of this problem (ρ' has not been written down). Since both expressions yield one and the same solution, one may, without reducing generality, consider only positive k. Thus one has $k = 1, 2, \ldots$. Recall that $\sigma > 0$ will then correspond to waves which travel eastwards and $\sigma < 0$ to waves which travel westwards.

It is very difficult to find the eigenvalue curves $\sigma_*(\epsilon_*)$ of the system of equations (3.5.6) and (3.5.7) for $k = 1, 2, \ldots$. However, by studying the simpler problem of the asymptotic behaviour of these eigenvalue curves for small and large ϵ_*, one may construct a qualitative picture of the distribution of these curves in the (ϵ, σ)-plane.

In what follows, separate equations for V_* or Π_* are required [note that system (3.5.6), (3.5.7) does not have solutions of the type $V_* \equiv 0, \Pi_* \not\equiv 0$ or $V_* \not\equiv 0, \Pi_* \equiv 0$]. After some simple manipulations, one obtains

$$\left\{L-\frac{k}{\sigma_*}-\frac{2\epsilon_*\sigma_*^2\mu}{k^2-\epsilon_*\sigma_*^2(1-\mu^2)}\left(D-\frac{k\mu}{\sigma_*}\right)+\epsilon_*(\sigma_*^2-\mu^2)\right\}V_* = 0\,, \tag{3.5.8}$$

$$\left\{L+\frac{k}{\sigma_*}+\frac{2\mu}{\sigma_*^2-\mu^2}\left(D+\frac{k\mu}{\sigma_*}\right)+\epsilon_*(\sigma_*^2-\mu^2)\right\}\Pi_* = 0\,, \tag{3.5.9}$$

where the operators L and D are given by (3.3.13) and (3.5.5). Note that although the coefficients in equations (3.5.8) and (3.5.9) have singularities inside $[-1, 1]$, the functions V_* and Π_* have already been shown not to have any singularities inside this interval.

Thus, fix k and consider the following cases.

(1) $\epsilon_* \to 0$

Find first those σ_* the absolute values of which increase without bound as $\epsilon_* \to 0$ (first type). It is convenient to employ equation (3.5.9) for Π_*. Clearly, one has in first approximation

$$(L+\epsilon_*\sigma_*^2)\,\Pi_* = 0\,.$$

In essence, this is the problem of the non-rotating spherical layer: $\Pi_* = P_n^k(\mu)$ and

$$\sigma_* = \pm\frac{\sqrt{n(n+1)}}{\epsilon_*^{1/2}}, \qquad n = k, k+1, \ldots \tag{3.5.10}$$

The function V_* is now found from (3.5.6), the function U_* from (3.5.2).

Here and in what follows, for the sake of brevity, not all formulae for the eigenfunctions will be written down (for example, V_* and U_*), and attention will be focussed on the formulae for the eigenvalue curves. An analysis of features of the eigenfunctions is given in [11, 90, 70].

Assume now that σ_* is bounded as $\epsilon_* \to 0$ (second type). In this case, employ equation (3.5.8) for V_*. In first approximation, one has

$$\left(L - \frac{k}{\sigma_*}\right) V_* = 0 ,$$

whence $V_* = P_n^k(\mu)$ and

$$\sigma_* = -\frac{k}{n'(n'+1)}, \qquad n' = k, k+1, \ldots \tag{3.5.11}$$

The index n' has been introduced here in order to be able to distinguish the two types.

Clearly, these formulae must be considered to be first terms of corresponding asymptotic expansions in powers of $\epsilon_* (\epsilon_* \gtrless 0)$.

(2) $\epsilon_* \to \infty$

It is convenient to employ equation (3.5.8) for V_*. Assume that $\sigma_* \to 0$ as $\epsilon_* \to \infty$. However, then at least the term kV_*/σ_* in (3.5.8) will be large. Therefore V_* must have, for large ϵ_*, large derivatives; as $\epsilon_* \to \infty$, the term with $d^2V_*/d\mu^2$ in (3.5.8) turns out to be insignificant and one cannot find a solution V_* which is bounded over the entire interval [−1, 1]. However, if a finite function V_* has a large derivative, then it is natural to assume that V_* differs significantly from zero only over a small segment of μ; it is simplest to assume that V_* is "localized" around the equator. Then $\mu^2 << 1$ and, in first approximation, equation (3.5.8) assumes the form

$$\left[\frac{d^2}{d\mu^2} - \frac{k}{\sigma_*} + \epsilon_*(\sigma_*^2 - \mu^2)\right] V_* = 0 .$$

This equation determines the scale of the variable μ; in fact, require the terms $d^2V_*/d\mu^2$ and $\epsilon_*\mu^2 V_*$ to be of the same order (if this is not so, no "local" functions are obtained); the variable $\theta = \epsilon_*^{1/4}\mu$ will then be a quantity of order unity.

Thus,

$$\left(\frac{d^2}{d\theta^2} + \frac{A}{\epsilon_*^{1/2}} - \theta^2\right) V_* = 0, \qquad A = -\frac{k}{\sigma_*} + \epsilon_*\sigma_*^2 .$$

One must assume that V_* is bounded and determined for all real θ as $\epsilon_* \to \infty$; the problem of determination of such a function is well known (cf.

[62, pp. 323—326]) and one has

$$V_* = \exp(-\tfrac{1}{2}\theta^2)\, H_\nu(\theta), \qquad \theta = \epsilon_*^{1/4}\mu, \qquad \epsilon_*^{-1/2}\left(-\frac{k}{\sigma_*} + \epsilon_*\sigma_*^2\right) = 2\nu + 1,$$
$$\nu = 0, 1, 2, \ldots, \tag{3.5.12}$$

where H_ν is the Hermite polynomial of order ν.

It is not difficult to write down explicitly for large ϵ_* the solution of equation (3.5.12) for σ_*:

$$\sigma_* = \pm\frac{(2\nu+1)^{1/2}}{\epsilon_*^{1/4}} + \frac{k}{4\nu+2}\,\frac{1}{\epsilon_*^{1/2}} + \ldots, \qquad \nu = 0, 1, 2, \ldots, \tag{3.5.13}$$

$$\sigma_* = -\frac{k}{2\nu'+1}\,\frac{1}{\epsilon_*^{1/2}} + \ldots, \qquad \nu' = 1, 2, \ldots \tag{3.5.14}$$

The above reasoning is, in essence, heuristic. However, it may be verified, if one considers the formulae obtained for large ϵ_* to be first terms of corresponding asymptotic expansions. Substitution of such expansions in (3.5.8) really affirms the deductions made, except when, in first approximation, $k^2 = \epsilon_*\sigma_*^2$ (since the denominator of the fraction in (3.5.8) vanishes in this approximation). This is the reason why the value $\nu' = 0$ must be excluded in (3.5.14).

Thus, the case $\sigma_*^2 = k^2/\epsilon_*$ must be studied separately. This is most simply achieved by reverting to equation (3.5.9) for Π_*. Consider first the case when

$$\sigma_* = \frac{k}{\epsilon_*^{1/2}}. \tag{3.5.15}$$

In first approximation, the equation for Π_* will be

$$\left(\frac{d^2}{d\theta^2} - \frac{2}{\theta}\frac{d}{d\theta} - \theta^2 - 1\right)\Pi_* = 0\,.$$

Replacing Π_* by $\Pi_* = \exp(-\theta^2/2)\, Z$, one obtains

$$\frac{d^2Z}{d\theta^2} - 2\left(\theta + \frac{1}{\theta}\right)\frac{dZ}{d\theta} = 0\,.$$

This equation is readily integrated. Finally, in first approximation, the solution of (3.5.9), bounded for all real θ, has the form

$$\Pi_* = \exp(-\theta^2/2)\,. \tag{3.5.16}$$

The function V_* is now found from (3.5.6), and then U_* from (3.5.2):

$$V_* \equiv 0, \qquad U_* = \epsilon_*^{1/2}\Pi_*\,. \tag{3.5.17}$$

It is natural that (3.5.15)—(3.5.17) should yield only first terms of corresponding asymptotic expansions for large ϵ_*.

In the case $\sigma_* = -k/\epsilon_*^{1/2}$, which can be analyzed in an analogous manner, there do not exist solutions of (3.5.9) which are bounded for all real θ.

(3) $\epsilon_* \to -\infty$

Recall that for $\epsilon_* < 0$ the eigenvalue curves of Problem H may only lie in the half-strip $|\sigma_*| < 1$. Study first the question of the behaviour of eigenvalue curves of Type 2 for $\epsilon_* < 0$. Numerical calculations show (cf. [11, 70]) that $\sigma_* \to -1$ for $\epsilon_* \to -\infty$. However, by (3.5.8), the function $1 - \mu^2$ must then be small, and consequently V_* must be "localized" about the poles. Consider the neighbourhood of the North Pole. Let

$$\xi = 2\sqrt{(-\epsilon_*)}(1 - \mu)\,, \tag{3.5.18}$$

$$\sigma_* = -1 + \frac{q}{\sqrt{(-\epsilon_*)}} + O(1/\epsilon_*)\,, \tag{3.5.19}$$

where the coefficient q must be determined.

Substituting into equation (3.5.8) relations (3.5.18) and (3.5.19) and retaining only the principal terms, one obtains

$$\left[\frac{d^2}{d\xi^2} + \left(-\frac{1}{4} + \frac{q}{2\xi} - \frac{k^2 - 2k}{4\xi^2}\right)\right] V_* = 0\,.$$

Letting $U = \xi^{-1/2} V_*$, one has

$$\frac{d}{d\xi}\left(\xi\frac{dU}{d\xi}\right) + \left[\frac{q}{2} - \frac{\xi}{4} - \frac{(\kappa - 1)^2}{4\xi}\right]U = 0\,. \tag{3.5.20}$$

This equation has been studied in great detail (cf. [62, pp. 335—340]): Bounded on the half straight line $0 \leqslant \zeta < \infty$, solutions of this equation exist only when

$$q = 2 - \kappa + 2\nu', \qquad \nu' = \kappa - 1, \kappa, \kappa + 1, \ldots$$

and have the form

$$U = \exp(-\xi/2)\, \xi^{(\kappa-1)/2}\left(\frac{d^{k-1}}{d\xi^{\kappa-1}}\right) L_{\nu'}(\xi),$$

where L_s is the Laguerre polynomial of order s.

Finally, one has

$$q = 2\nu + k, \qquad \nu = 0, 1, 2, \ldots \tag{3.5.21}$$

and

$$V_* = \exp(-\xi/2)\, \xi^{k/2}\left(\frac{d^{k-1}}{d\xi^{k-1}}\right)L_{k-1+\nu}\,. \tag{3.5.22}$$

Next, consider the asymptotic behaviour of the eigenvalue curves of Type 3 of Problem H. Using again results of numerical computations [11,70], seek σ_* in the form

$$\sigma_* = 1 - \frac{q}{(-\epsilon_*)^{1/2}} + O\left(\frac{1}{\epsilon_*}\right) , \qquad (3.5.23)$$

where q must be determined, and the eigenfunction "localized" about the North Pole depends, by assumption, on ξ according to (3.5.18). Following the earlier analysis, one finds

$$q = k + 2\nu + 2, \qquad \nu = 0, 1, 2, \ldots$$

and

$$V_* = \exp(-\xi/2)\, \xi^{(k+2)/2} \left(\frac{\mathrm{d}^{k+1}}{\mathrm{d}\xi^{k+1}}\right) L_{k+\nu+1}(\xi) .$$

Finally, it will be natural to assume that the second branch of eigenvalue curves of the Type 3 approach asymptotically the $\sigma = 0$ axis as $\epsilon_* \to -\infty$. Therefore let

$$\sigma_* = \frac{k}{-\epsilon_*} + \frac{2qk}{(-\epsilon_*)^{3/2}} + O\left(\frac{1}{\epsilon_*^2}\right), \qquad (3.5.24)$$

where q must be determined and V_* is "localized" about the North Pole and depends on ξ according to (3.5.18). One has

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi \frac{\mathrm{d}V_*}{\mathrm{d}\xi}\right) + \left(\frac{q}{2} - \frac{\xi}{4} - \frac{\kappa^2}{4\xi}\right) V_* = 0 .$$

This equation is of the type (3.5.20), whence

$$q = k + 2\nu + 1, \qquad \nu = 0, 1, 2, \ldots$$

and

$$V_* = \exp(-\xi/2)\, \xi^{k/2} \left(\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\right) L_{k+\nu} .$$

Note that one need not study the eigenfunctions "localized" about the South Pole. It is readily shown that solutions of (3.5.8) bounded on $[-1, 1]$ can only be even or odd functions in μ.

These solutions permit, finally, to establish a general presentation of the distribution of the eigenvalue curves of Problem H in the (ϵ, σ)-plane for fixed k. For this purpose, it is useful to keep in mind that, in general, the eigenvalue curves of Problem H for fixed k form an enumerable set of curves (the spectrum is discrete) and that different curves of the set cannot intersect.

The proof of the first statement may be based on considerations of

Morse and Feshbach [83, pp. 668—674]. The second assertion has been proved by Dikii [11, p. 28].

Detailed numerical computations have shown the validity of the asymptotic obtained. Fig. 3.12 shows the complete pattern of the eigenvalue curves of Problem H for $k = 1$. On the basis of this graph one may describe certain general features of the behaviour of the eigenvalue curves of Problem H for fixed k.

First, let $\epsilon_* > 0$ and $\sigma_* > 0$. Then one has the lower curve of first type (closer to the axis); for $\epsilon_* \to 0$, it has the index $n = k$ and is described by (3.5.10) with sign << + >>. Clearly, formula (3.5.15) yields the asymptotic behaviour for $\epsilon_* \to \infty$ of this very curve. Formula (3.5.13) with sign << + >> must then describe the asymptotic behaviour for $\epsilon_* \to \infty$ of the remaining curves of first type; if for $\epsilon_* \to 0$ a curve has index $n = k + l$, then for $\epsilon_* \to \infty$ this curve has the index $\nu = l - 1$.

Now let $\epsilon_* > 0$, but $\sigma_* < 0$. Then one has the lower curve of second type (further away from the ϵ_*-axis); for $\epsilon_* \to 0$, it has index $n' = k$ and is described by (3.5.11). Computations of Dikii [11] and Longuet-Higgins [70] show that its asymptotic behaviour for $\epsilon_* \to \infty$ is given by (3.5.13) with sign << — >> for $\nu = 0$. Now the entire pattern becomes clear. A curve of second type which has for $\epsilon_* \to 0$ index $n' = k + l$, has for $\epsilon_* \to \infty$ index $n' = l$ [formula (3.5.14)]; a first-type curve which has index $n = k + l$ for $\epsilon_* \to 0$, has index $\nu = l + 1$ for $\epsilon_* \to \infty$ [formula (3.5.13) with sign —].

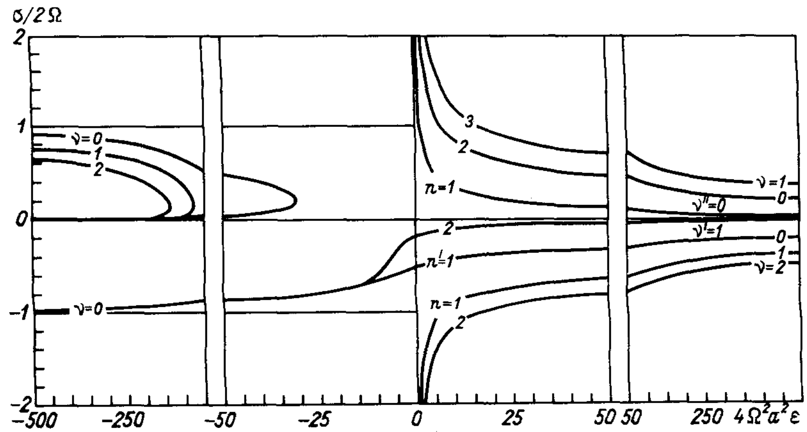


Fig. 3.12. Eigenvalue curves of Problem H for rotating layer for $k = 1$, constructed from results of numerical calculations by Longuet-Higgins [70]. Inscriptions on curves for small and large $|\epsilon|$ indicate applicability of corresponding asymptotic formulae [cf. (3.5.10), (3.5.11), (3.5.13), (3.5.14), (3.5.19) and (3.5.23)]; the asymptotic (3.5.15) is denoted by $\nu'' = 0$.

The asymptotic behaviour of the eigenvalue curves of second type for $\epsilon_* \to -\infty$ is described by (3.5.19). The pairwise "merging" of these curves for $\epsilon_* \to -\infty$, observed in Fig. 3.12, is explained by the fact that for large $|\epsilon_*|$ the eigensolutions of (3.5.8) are "localized" about the poles and therefore, in first approximation, the eigenvalues, corresponding to even and odd eigenfunctions V_*, coincide and differences are only revealed in higher approximations.

Every third-type curve shown in Fig. 3.12 represents two merging "paired" curves (on the graph they do not differ). The reason for the "merging" is the same as in the case of second type curves; besides, third-type curves lie entirely in the region of large $|\epsilon_*|$. The asymptotic behaviour of these curves for $\epsilon_* \to -\infty$ is described by (3.5.23) and (3.5.24).

The case $k = 0$ (purely zonal motion) must be studied specially. Obviously, the location of the eigenvalue curves in the (ϵ_*, σ_*)-plane in this case is symmetric with respect to the straight line $\sigma_* = 0$ and second-type curves do not arise. Analogous analysis permits to find the asymptotic behaviour of the eigenvalue curves of the first and third types.

For $\epsilon_* \to +0$ (first type), one has

$$\sigma_* = \pm \{n(n+1)\}^{1/2}/\epsilon_*^{1/2} + \ldots, \qquad n = 1, 2, \ldots ;$$

for $\epsilon_* \to \infty$ (first type), one has

$$\sigma_* = \pm (2\nu + 1)^{1/2}/\epsilon_*^{1/4} + \ldots, \qquad \nu = 0, 1, 2, \ldots ;$$

for $\epsilon_* \to -\infty$ (third type), one has

$$\sigma_* = \pm 1 \mp q/(-\epsilon_*)^{1/2} + O(1/\epsilon_*),$$

$$q = 2 + 2\nu, \qquad \nu = 0, 1, 2, \ldots .$$

In addition, the axis $\epsilon_* = 0$ and $\sigma_* = 0$ will also be eigenvalue curves ($U_* = 0$, $V_* = 0$, $\Pi_* = 1$).

3.6. CLASSIFICATION OF FREE OSCILLATIONS

Finally, the results of § 3.3. will be generalized to the general case of a rotating spherical layer of a compressible stratified fluid. For this purpose, the eigenvalue curves of Problems V and H will be constructed on one graph in terms of the variables ϵ and σ (Fig. 3.13). Then:

(1) The solutions of System (3.1.3)–(3.1.7) corresponding to points of intersection of the eigenvalue curves $\sigma_n(\epsilon)$, $n = -1, -2, \ldots$ of Problem V with the eigenvalue curves of first type of Problem H for $\epsilon > 0$ are referred to as acoustic waves (Fig. 3.13). The basic physical cause for the generation of these waves is compressibility, the elasticity of the medium (for $c^2 \to \infty$, the eigenvalue curves $\sigma_n(\epsilon)$, $n = -1, -2, \ldots$ of Problem V pull

back in the region $\epsilon < 0$ and points of Type $1^{\pm}$ are missing). Inhomogeneity of the medium, presence of a free surface, effects of gravity forces as well as rotation and the spherical shape of the Earth, of course, affect acoustic waves; however, acoustic waves also exist in a homogeneous weightless ($N \equiv 0$, ρ_0 = constant) compressible medium, located on a non-rotating plane (§ 3.3). Inequality (3.4.23) yields for the frequencies of acoustic waves the simple lower bound

$$\sigma^2 > \sigma_a^2 = N_{\min}^2 + \left(\frac{\pi}{2}\right)^2 \frac{c_{\min}^2}{H^2} \frac{\hat{\rho}_{0\min}}{\hat{\rho}_{0\max}} .$$

Since in the ocean $\sigma_a \sim 1$ rad/sec, acoustic waves are characterized by high frequencies.

(2) Solutions of System (3.1.3)–(3.1.7) corresponding to points of intersection of the eigenvalue curve $\sigma_0(\epsilon)$ of Problem V with eigenvalue curves of Type 1 of Problem H are referred to as surface gravitational waves (Fig. 3.13). The basic physical reason for the occurrence of these waves is the inclination of the free surface of the ocean in the gravity force field.

Kamenkovich and Odulo [47] have shown that replacement of the free surface by a rigid lid destroys the eigenvalue curve $\sigma_0(\epsilon)$ of Problem V, and thereby also surface gravitational waves. In their place, there arise in a compressible fluid another class of waves, so-called Lamb waves.

Compressibility and inhomogeneity of a medium practically do not affect the eigenvalue curve $\sigma_0(\epsilon)$ of Problem V [cf. § 3.3, estimate (3.4.19) and the asymptotic $\sigma_0^2(\epsilon) \sim g^2\epsilon$ for large ϵ]; this means that they also do not affect surface gravitational waves. However, rotation and the Earth's spherical shape prove to be very essential (especially for long waves). Surface gravitational waves on a sphere may have, generally speaking, any frequencies.

Solutions of System (3.1.3)–(3.1.7) corresponding to points of intersection of eigenvalue curves of Type 1 of Problem H are referred to as internal gravitational waves (Fig. 3.13). The basic reason for their generation is the inhomogeneity of sea water in the gravity force field (if $N_{\max} \to 0$, then points of Type $3^{\pm}$ disappear). It will be shown below that internal waves are practically not distorted when the free surface is replaced by a rigid lid; effects of compressibility of sea water likewise are inessential. It is natural that for very long waves one must take account of the Earth's rotation and spherical shape. Frequencies of internal waves may, in general, lie in the range $0 < |\sigma| < N_{\max}$. For all eigenvalues $\epsilon_n(\sigma)$, $n = 1, 2, \ldots$, corresponding to these waves, one has the estimate [cf. (3.4.14)]

$$\epsilon > \epsilon_i = \frac{1}{c_{\max}^2} + \frac{\pi^2 \hat{\rho}_{0\min}}{N_{\max}^2 H^2 \hat{\rho}_{0\max}} . \tag{3.6.1}$$

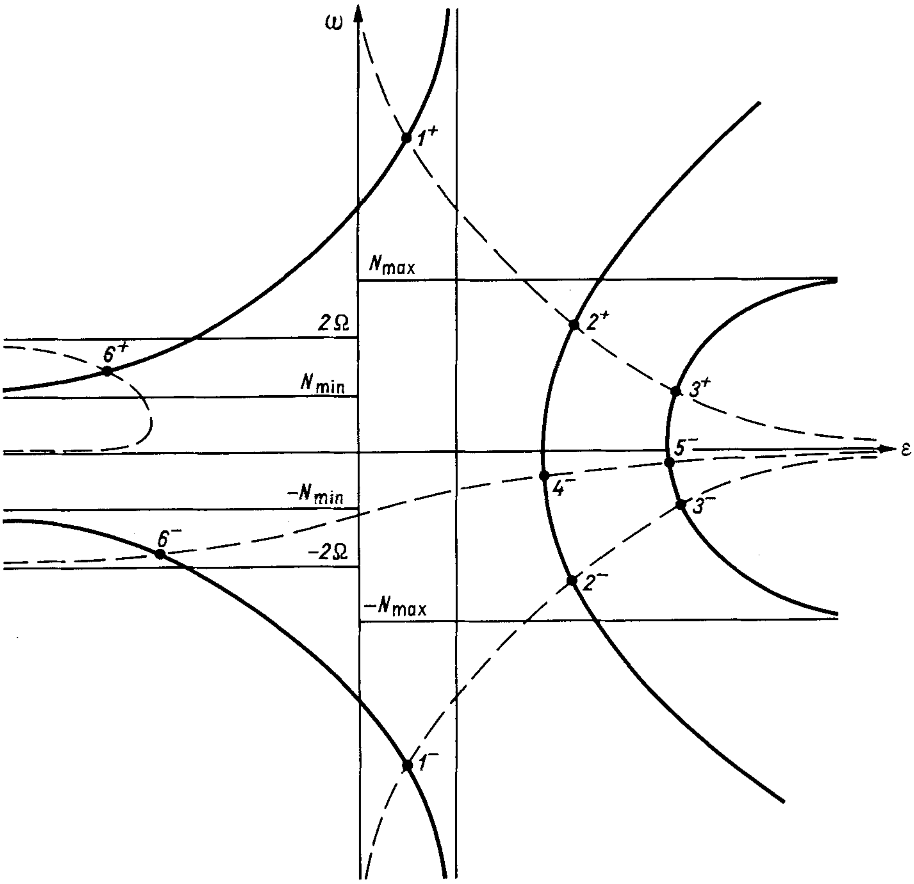


Fig. 3.13. Pattern of intersections of eigenvalue curves of Problem V (solid lines) and Problem H (broken lines) for $2\Omega > N_{min}$. Only one eigenvalue curve of each type is shown. Points of Type $1^{\pm}$ correspond to acoustic waves; points of Type $2^{\pm}$ to surface gravitational waves; points of Type $3^{\pm}$ to internal gravitational waves; points of Type 4^- to barotropic Rossby waves; points of Type 5^- to baroclinic Rossby waves; points of Type $6^{\pm}$ to gyroscopic waves.

(3) Solutions of System (3.1.3)–(3.1.7) corresponding to points of intersection of the eigenvalue curve $\sigma_0(\epsilon)$ of Problem V with the eigenvalue curves of Type 2 of Problem H are referred to as barotropic Rossby waves (Fig. 3.13). The basic physical reason for the occurrence of these waves is the combined effect of the Earth's rotation and spherical shape (condition of existence of eigenvalue curves of Type 2 of Problem H). Compressibility and inhomogeneity of sea water practically do not affect these waves. The range of possible frequencies is $0 < |\sigma| < \Omega$.

Solutions of System (3.1.3)–(3.1.7) corresponding to points of intersection of the eigenvalue curves $\sigma_n(\epsilon)$, $n = 1, 2, \ldots$ of Problem V with eigenvalue curves of Type 2 of Problem H are referred to as baroclinic Rossby waves (Fig. 3.13). The basic physical reason for the occurrence of these waves is the combined effect of the inhomogeneity of sea water in a gravity force field (condition of existence of curves of $\sigma_n(\epsilon)$, $n = 1, 2, \ldots$ of Problem V) and rotation of spherical Earth (condition of existence of curves of Type 2 of Problem H). The range of possible frequencies of these waves is $0 < |\sigma| < \min(\Omega, N_{\max})$.

(4) Solutions of System (3.1.3) through (3.1.7) corresponding to points of intersection of the eigenvalue curves $\sigma_n(\epsilon)$, $n = -1, -2, \ldots$ of Problem V with eigenvalue curves of Types 2 and 3 of Problem H for $\epsilon < 0$ are referred to as gyroscopic waves (Fig. 3.13). The basic physical reason for the occurrence of these waves is the Earth's rotation. However, gyroscopic waves do not arise for any rotation, but only if the condition $2\Omega > N_{\min}$ is fulfilled (otherwise the eigenvalue curves of Problem V and H do not intersect). As also in the case of internal gravitational waves, replacement of the free surface of the ocean by a rigid lid practically does not distort gyroscopic waves; the effect of compressibility of a medium likewise is inessential. The range of possible frequencies is $N_{\min} < |\sigma| < 2\Omega$. It follows from (3.4.22) that for all eigenvalue curves $\epsilon_n(\sigma)$, $n = -1, -2, \ldots$, corresponding to these waves, one has

$$\epsilon < \epsilon_g = \frac{1}{c_{\min}^2} - \frac{(\pi^2/4)\,\hat{\rho}_{0\,\min}}{(4\Omega^2 - N_{\min}^2)\,H^2\hat{\rho}_{0\,\max}}\,. \tag{3.6.2}$$

Gravitational (surface and internal), acoustic and gyroscopic waves may propagate in easterly as well as in westerly directions. Rossby waves (barotropic and baroclinic) propagate only towards the west.

3.7. SOME APPROXIMATIONS AND THEIR ANALYSIS

Under conditions characteristic for the ocean, energies of different types of wave motions may differ strongly from each other. For example, as a rule, the energy of acoustic waves in the ocean is negligibly small compared with that of other types of waves. Therefore there arises often the problem of filtration of those and other waves. In this context, it is expedient to discuss "filtering" properties of approximations which are employed in the theory of wave motions.

Consider first Boussinesq's approximation to the initial equations (3.1.3)–(3.1.7) [replacement of the equation of continuity (3.1.6) by div $v = 0$ and omission of the term $(1/c^2)\partial p'/\partial t$ in (3.1.7)]. For an analysis of wave motions in this approximation, one employs separation of variables (cf. § 3.2); then Problem H does not change, and instead of the system (3.

2.11) and (3.2.12) one obtains

$$\frac{dP}{dz} + (\sigma^2 - N^2)\,\rho_0 W = 0\,, \tag{3.7.1}$$

$$\frac{dW}{dz} + \frac{\epsilon}{\rho_0} P = 0 \tag{3.7.2}$$

with the same boundary conditions (3.2.13). Formally, System (3.7.1) and (3.7.2) is derived from System (3.2.11) and (3.2.12) by omission in the latter of terms containing $1/c^2$. Since in the ocean $c^2_{\min}/gH >> 1$ and $c^2_{\min}/\Omega^2 H^2 >> 1$, then, taking into consideration estimates (3.4.19), (3.6.1) and (3.6.2), it is readily shown that Boussinesq's approximation filters out acoustic waves completely and practically does not distort gravitational, gyroscopic and Rossby waves.

In Boussinesq's approximation, the energy conservation equation (3.1.10) assumes the form

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2} v^2 + \frac{g^2}{2\rho_0 N^2}\rho'^2\right) = -\mathrm{div}(p' v)\,. \tag{3.7.3}$$

Consider now the approximation of the rigid lid (replacement of the free surface of the ocean by a lid). Obviously, in such an approximation, only the second boundary condition (3.2.13) changes, i.e., it is replaced by $W(0) = 0$.

Restrict considerations to analysis of the problem in Boussinesq's approximation. Eliminating P from (3.7.1), (3.7.2) and (3.2.13), one finds

$$\frac{d}{dz}\left(\rho_0 \frac{dW}{dz}\right) + \epsilon(N^2 - \sigma^2)\,\rho_0 W = 0\,, \tag{3.7.4}$$

$$W = 0 \quad \text{for} \quad z = -H, \qquad -\frac{1}{g\epsilon}\frac{dW}{dz} + W = 0 \quad \text{for} \quad z = 0\,. \tag{3.7.5}$$

Derive first the eigenvalue curves $\sigma_n(\epsilon)$, $n = 1, 2, \ldots$ of Problem V and introduce the non-dimensional quantities

$$z = H\bar{z}, \qquad N = N_{\max}\bar{N}, \qquad \rho_0 = \rho_{\max}\bar{\rho}_0, \qquad \sigma = N_{\max}\bar{\sigma}, \qquad \epsilon = \frac{1}{N^2_{\max}H^2}\bar{\epsilon}\,.$$

Then all quantities with bars (except $\bar{\epsilon}$) are of order unity [cf. estimate (3.6. 1)]. Substituting them into (3.7.4) and (3.7.5), one obtains

$$\frac{d}{d\bar{z}}\left(\bar{\rho}_0 \frac{d\bar{W}}{d\bar{z}}\right) + \bar{\epsilon}(\bar{N}^2 - \bar{\sigma}^2)\,\bar{\rho}_0 \bar{W} = 0\,, \tag{3.7.6}$$

$$\bar{W} = 0 \quad \text{for} \quad \bar{z} = -1, \qquad -\frac{\delta}{\bar{\epsilon}}\frac{d\bar{W}}{d\bar{z}} + \bar{W} = 0 \quad \text{for} \quad \bar{z} = 0\,, \tag{3.7.7}$$

where $(1/\bar{\epsilon})\, \mathrm{d}\bar{w}/\mathrm{d}z \sim 1$ and

$$\delta = \frac{N_{\max}^2 H}{g}.$$

Under typical conditions in the ocean $N_{\max} = 10^{-2}$ sec^{-1}, $H = 4$ km and $\delta = 4 \cdot 10^{-2}$, i.e., δ is a small parameter. Therefore, obviously, the relative error arising from replacement of the second condition (3.7.5) by $W(0) = 0$ for the eigenvalue curves $\sigma_n(\epsilon)$, $n = 1, 2, \ldots$ of Problem (3.7.4) and (3.7.5) and for its corresponding eigenfunctions will be of order $O(\delta)$.

In an analogous manner, it may be shown that the approximation of the rigid lid distorts slightly the eigenvalue curves $\sigma_n(\epsilon)$, $n = -1, -2, \ldots$ of Problem (3.7.4) and (3.7.5) and its eigenfunctions for $\epsilon < \epsilon_g$ [cf. (3.6.2)].

Finally, it is obvious that the approximation of the rigid lid annihilates the eigenvalue curve $\sigma_0(\epsilon)$ of Problem V. In its place, System (3.7.1) and (3.7.2) for the conditions $W(-H) = 0$ and $W(0) = 0$ will have the eigenvalue curve $\epsilon = 0$ (P = constant, $W \equiv 0$). Solutions of the initial system of equations corresponding to points of intersection of the eigenvalue curve $\epsilon = 0$ of Problem V and eigenvalue curves of second type of Problem H are naturally called non-divergent Rossby waves. For such waves, one has strictly $w \equiv 0$ and $\mathrm{div}_h(u, v) = 0$; essentially, barotropic Rossby waves were studied in § 3.3 in the approximation of the rigid lid.

Thus, finally, the approximation of the rigid lid completely filters out surface gravitational waves, but practically does not distort internal gravitational, baroclinic Rossby and gyroscopic waves, and likewise converts barotropic Rossby waves into non-divergent Rossby waves (an estimate of the error arising in this step is given in § 3.8). In other words, only surface gravitational waves are linked to deflections of the free surface from its unperturbed position.

In conclusion, consider the quasi-static approximation in the initial equations (3.1.3)–(3.1.7) [omission of the term $\partial w/\partial t$ in (3.1.5)]. This approximation implies the striking out of σ^2 in System (3.2.11) and (3.2.12); System (3.2.8)–(3.2.10) and condition (3.2.13) do not change. In other words, one must simply set $\sigma^2 = 0$ in (3.2.11). But then there remain in the (ϵ, σ^2)-plane only those eigenvalue curves of Problem V which intersect the ϵ-axis; these curves become straight lines parallel to the σ^2-axis. Thus, gravitational and Rossby waves remain, and gyroscopic and acoustic waves vanish.

Clearly, the quasi-static approximation distorts little only low-frequency gravitational and Rossby waves ($\sigma << N_{\max}$).

In the quasi-static approximation, System (3.7.1) and (3.7.2) under condition (3.2.13) conveniently reduces to the single equation for P

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{N^2\rho_0}\frac{\mathrm{d}P}{\mathrm{d}z}\right) + \frac{\epsilon}{\rho_0}P = 0 \qquad (3.7.8)$$

with the conditions

$$\frac{dP}{dz} = 0 \qquad \text{for } z = -H, \qquad P + \frac{g}{N^2}\frac{dP}{dz} = 0 \qquad \text{for } z = 0 \tag{3.7.9}$$

Problem (3.7.8) and (3.7.9) is an ordinary Sturm-Liouville eigenvalue problem. Therefore it is clear that the set of eigenfunctions $P_n(z)$ ($n = 0, -1, -2, \ldots$) of this problem is complete. These functions may be constructed for given values of $N(z)$ and $\rho_0(z)$ once and for all.

At times, instead of the eigenvalues ϵ_n, $n = 0, 1, 2, \ldots$, so called equivalent depths $H_n = 1/g\epsilon_n$, $n = 0, 1, 2, \ldots$ enter into Problem (3.7.8) and (3.7.9). Using estimates (3.4.19) and (3.4.14), one obtains

$$\frac{\rho_0(0)}{\rho_{0\,\max}} < \frac{H_0}{H} < 1\,,$$

$$\frac{1}{[(n+1)\,\pi]^2}\,\frac{\rho_{0\,\min}}{\rho_{0\,\max}}\,\frac{N^2_{\min}H}{g} < \frac{H_n}{H} < \frac{1}{(n\pi)^2}\,\frac{\rho_{0\,\max}}{\rho_{0\,\min}}\,\frac{N^2_{\max}H}{g}\,, \qquad n = 1, 2, \ldots$$

By the first estimate, $H_0 \simeq H$; the estimates of $H_1, H_2, \ldots$ turn out to be rather coarse.

Obviously, in the quasi-static approximation, the energy equation (3.7.3) has the form

$$\frac{\partial}{\partial t}\left\{\rho_0\frac{u^2+v^2}{2} + \frac{g^2}{2\rho_0 N^2}\,\rho'^2\right\} = -\mathrm{div}_h(p'u, p'v)\,. \tag{3.7.10}$$

3.8 APPROXIMATE ANALYSIS OF PROBLEM H. THE CONCEPT OF THE β-PLANE

The asymptotic expansions of § 3.5 for small and large values of $|\epsilon|$ turn out to be very useful for a study of the problem of the disposition of the eigenvalue curves of Problem H in the (ϵ, σ)-plane. However, just now, approximate formulae of another type true for a wider range of values of ϵ will be of interest here. In fact, such formulae permit construction of required dispersion relations for different types of ocean waves.

Taking account of characteristic horizontal scales of the oceans, limit the study to not so very long waves (quarter wavelengths of order 1000 km and less) which correspond to large values of the non-dimensional wavenumbers on the sphere (of order 10 and larger). In this case, the coefficients of system (3.2.8)—(3.2.10) change little over the length of the waves, and for the construction of the corresponding eigensolutions of the problem a short-wave approximation may be applied.

Start with the eigenvalue curves of Type 1 and consider (3.5.9). Let the frequency σ_* be fixed. Then it is clear from the results of § 3.5 that the larger ϵ_* the larger the number of zeros in the interval $(-1, 1)$ of the eigenfunction Π_* and the shorter the corresponding wave length. In other words, short-wave solutions of Problem H correspond to the large values of ϵ_*.

Such solutions will be sought in the form of the asymptotic series

$$\Pi_*(\mu) = [\Pi_0(\mu) + \epsilon_*^{-1/2}\Pi_1(\mu) + \ldots] \exp[i\epsilon_*^{1/2} \int_0^\mu l(\mu)\, \mathrm{d}\mu] \,. \tag{3.8.1}$$

The form of the series (3.8.1) has been chosen on the basis of the following reasoning. If the coefficients in equation (3.5.9) were constant, the $l(\epsilon_*)$ in the series $l = l_0 + l_1\epsilon_*^{-1/2} + \ldots$, one obtains an expression of the form $\exp[i\epsilon_*^{1/2}l(\epsilon_*)\mu]$ with the constant phase coefficient $l(\epsilon_*) = O(1)$. Expanding $l(\epsilon_*)$ in the series $l = l_0 + l_1\epsilon_*^{-1/2} + \ldots$, one obtains an expression of the form (3.8.1) with constant phase coefficient l_0 and amplitude which changes slowly over the wave length, for example, $\Pi_0 = \exp[il_1\mu]$. Clearly, in the case under consideration when the coefficients of equation (3.5.9) are not constant, but change slowly over the wave length, it is reasonable to seek the solution of (3.5.9) in the form of the asymptotic series (3.8.1) assuming the phase coefficient l and the amplitudes $\Pi_0, \Pi_1, \ldots$ to change slowly over the wave length.

Thus, by (3.8.1), the latitudinal wave number for large ϵ_* is of order $\epsilon_*^{1/2}$. The longitudinal wave number k enters into equation (3.5.9) as a parameter. Here special interest attaches to waves for which longitudinal and latitudinal wave numbers are of the same order. Therefore assume that $k \sim \epsilon_*^{1/2}$ and let in equation (3.5.9)

$$k = k_0\epsilon_*^{1/2}, \qquad k_0 = \text{constant} \tag{3.8.2}$$

Substituting (3.8.1) into (3.5.9) and equating to zero coefficients of different powers of ϵ_*, one arrives at a sequence of equations for the determination of $l(\mu)$, $\Pi_0(\mu)$, $\Pi_1(\mu)$, etc. For example, the coefficients of ϵ_* and $\epsilon_*^{1/2}$ yield in this manner the equations

$$\left[-(1-\mu^2)\, l^2 - \frac{k_0^2}{1-\mu^2} + \sigma_*^2 - \mu^2\right]\Pi_0 = 0\,, \tag{3.8.3}$$

$$2i(1-\mu^2)\, l\frac{\mathrm{d}\Pi_0}{\mathrm{d}\mu} + \left[i(1-\mu^2)\frac{\mathrm{d}l}{\mathrm{d}\mu} + 2i\mu\frac{(1-\sigma_*^2)}{\sigma_*^2-\mu^2}l + \frac{k_0}{\sigma_*}\,\frac{\sigma_*^2+\mu^2}{\sigma_*^2-\mu^2}\right]\Pi_0$$
$$+\left[-(1-\mu^2)\, l^2 - \frac{k_0^2}{1-\mu^2} + \sigma_*^2 - \mu^2\right]\Pi_1 = 0 \tag{3.8.4}$$

It follows directly from (3.8.3) that

$$l^2 = E(\mu^2)/(1-\mu^2)^2\,, \tag{3.8.5}$$

where

$$E(\mu^2) = \mu^4 - (1+\sigma_*^2)\,\mu^2 + \sigma_*^2 - k_0^2\,.$$

If $\sigma_*^2 > k_0^2$, one has $E(0) > 0$; since always $E(1) < 0$, one root μ_1^2 of the equation $E(\mu^2) = 0$ will be less than unity, another root larger than unity:

$$\mu_{1,2}^2 = \frac{1+\sigma_*^2}{2} \mp \left\{\frac{(1-\sigma_*^2)^2}{4} + k_0^2\right\}^{1/2} .$$

Thus, one has that $E(\mu^2) > 0$ for $-\mu_1 < \mu < \mu_2$ and $E(\mu^2) < 0$ for $-1 < \mu < -\mu_1$ and $\mu_1 < \mu < 1$, i.e., by (3.8.5), the function Π_* has on the interval $-\mu_1 < \mu < \mu_1$ an oscillatory (wave) character, and on the intervals $-1 < \mu < -\mu_1$ and $\mu_1 < \mu < 1$ an exponential (non-wave) character.

If $\sigma_*^2 < k_0^2$, then the interval $[-1, 1]$ does not contain a section on which the solution Π_* oscillates, and this signifies that in this case there are no short-wave eigensolutions.

By (3.8.3), the coefficient of $\Pi_1(\mu)$ in (3.8.4) vanishes and one obtains a simple equation for the determination of $\Pi_0(\mu)$. One may proceed in an analogous manner to find $\Pi_1, \ldots$.

In order to construct a complete asymptotic expression for the eigenfunction Π_* of Problem H, one must still develop special asymptotic representations for Π_* in the neighbourhood of the transition points $\mu = \pm\mu_1$ and of the ends of the interval $\mu = \pm 1$. Successive matching of all asymptotic expansions leads to asymptotic expansions of the solutions of (3.5.9) which are bounded on $[-1, 1]$ and of the corresponding eigenvalues ϵ_{*_n}.

For the sake of brevity, the complete analysis will be omitted (cf. [49] for such an analysis relating to curves of Types 1—3) and it will be assumed, as an approximation, that *all* values of ϵ_* are admissible (continuous spectrum). Since $(\epsilon_{*_{n+1}} - \epsilon_{*_n})/\epsilon_{*_n} \to 0$ as $n \to \infty$, such an approximation, beginning with large values of n, is completely justified.

Greatest interest relates to asymptotic representation of the eigenfunction Π_* on the interval $[-\mu_1, \mu_1]$, where it has a wave character. Restricting consideration to the first approximation and recalling (3.5.1) and (3.8.2), one finds

$$\Pi_* \simeq \Pi_0(\mu) \exp\{i[k_0\sqrt{\epsilon_*}\lambda + \sqrt{\epsilon_*}\int_0^{\varphi} l(\sin\varphi)\cos\varphi d\varphi]\} .$$

Consider now some point with coordinates λ_0, φ_0 such that $|\sin\varphi_0| < \mu_1$, and introduce local wave numbers

$$k' = k/a\cos\varphi_0 = k_0\sqrt{\epsilon_*}/a\cos\varphi_0, \qquad l' = \sqrt{\epsilon_*}(l/a)(d\mu/d\varphi) = \sqrt{\epsilon_*}(l\cos\varphi_0)/a .$$

Clearly, in the neighbourhood of the point λ_0, φ_0, the above expression may be presented in the form of a plane wave

$$\Pi \simeq \Pi'(\varphi_0)\exp\{i[k'(\varphi_0)x + l'(\varphi_0)y]\} \tag{3.8.6}$$

where Π' is the amplitude and $x = a\cos\varphi_0(\lambda - \lambda_0)$, $y = a(\varphi - \varphi_0)$ are local coordinates in the plane tangent to the sphere at the point λ_0, φ_0.

Using the definitions of the local wave numbers k' and l' and taking into consideration (3.5.5), formula (3.8.5) can be rewritten in the form

$$\sigma^2 = f^2(\varphi_0) + \frac{k'^2 + l'^2}{\epsilon}, \tag{3.8.7}$$

where $f(\varphi_0) = 2\Omega \sin \varphi_0$ is the Coriolis parameter.

By definition, only discrete values k' and l' are admissible; however, in the case of short waves (large values of ϵ_*), one may fix the local wave numbers k' and l' arbitrarily and will thereby not violate the boundary conditions at the poles. Then formulae (3.8.7) and (3.8.6) may be considered to be approximate expressions for eigenvalue curves of Type 1 (k', l' being free parameters) and for eigenfunctions Π_* [in the vicinity of the point λ_0, φ_0] corresponding to the points of these curves, respectively. It must be emphasized that introduction of local wave numbers and the approximate expression (3.8.7) for eigenvalue curves of Type 1 has significance only for short-wave eigensolutions.

The cases of eigencurves of Type 2 for $\epsilon_* < 0$ and of Type 3 can be studied in an analogous manner. For fixed σ_* and large ϵ_*, the solution is sought in the form

$$\Pi_* = [\Pi_0(\mu) + (-\epsilon_*)^{1/2}\, \Pi_1(\mu) + \ldots] \exp[i(-\epsilon_*)^{1/2} \int_c^\mu l(\mu)\, \mathrm{d}\mu],$$

where the lower integration limit c is chosen to suit convenience. Substituting this series into (3.5.9) and setting $k = k_0(-\epsilon_*)^{1/2}$ (latitudinal and longitudinal wave numbers being of the same order), one obtains

$$l^2 = -F(\mu^2)/(1-\mu^2)^2, \tag{3.8.8}$$

where

$$F(\mu^2) = \mu^4 - (1 + \sigma_*^2)\, \mu^2 + \sigma_*^2 + k_0^2.$$

Since $F(0) > 0$ and $F(1) > 0$, there cannot occur for $\sigma^2 > 1 - 2k_0$ short-wave solutions (on the interval $[-1, 1]$ eigensolutions do not oscillate). In the case $\sigma_*^2 < 1 - 2\kappa_0$, the equation $F(\mu^2) = 0$ has on the interval $(-1, 1)$ two roots

$$\mu_{1,2}^2 = \frac{1 + \sigma_*^2}{2} \mp \sqrt{[(1 - \sigma_*^2)^2 - 4k_0^2]/4}$$

and on the intervals $(-\mu_2, -\mu_1)$ and (μ_1, μ_2) the eigensolution will have an oscillatory (wave) character and on the intervals $(-1, -\mu_2)$, $(-\mu_1, \mu_1)$ and $(\mu_2, 1)$ an exponential (non-wave) character.

As in the case of eigencurves of Type 1, the complete asymptotic representation of the solution will not be developed here and it will be assumed that the spectrum is continuous for large $|\epsilon_*|$. Introducing on the wave intervals of the solution local wave numbers k' and l', rewrite (3.8.8) in the form

$$\sigma^2 = f^2(\varphi_0) + \frac{k'^2 + l'^2}{\epsilon},$$

where $\epsilon < 0$. Thus, formula (3.8.7) serves for the description of eigencurves of Type 1 for $\epsilon > 0$ as well as for the description of eigencurves of Types 2 and 3 for $\epsilon < 0$.

Finally, consider eigencurves of Type 2 for $\epsilon > 0$. In this case, it is more convenient to study (3.5.8) for the fixed parameter $\omega = -\sigma_* \epsilon_*^{1/2}$, since eigenfunctions with large latitudinal wave numbers correspond precisely to the points of intersection of the curves $\sigma_* = -\text{constant } \epsilon_*^{-1/2}$ with eigencurves of Type 2 at large ϵ_*.

Seek the solution of the problem in the form

$$V_* = [V_0(\mu) + \epsilon_*^{-1/2} V_1(\mu) + \ldots] \exp[i\epsilon_*^{1/2} \int_0^\mu l(\mu)\, d\mu] .$$

Substituting this expression into (3.5.8) and taking into consideration (3.8.2), one obtains

$$l^2 = \frac{G(\mu^2)}{(1-\mu^2)^2}, \tag{3.8.9}$$

where

$$G(\mu^2) = \mu^4 - \left(1 + \frac{k_0}{\omega}\right) \mu^2 + \frac{k_0}{\omega} - k_0^2 .$$

If $k_0\omega > 1$, one has $G(\mu^2) < 0$ in $(-1, 1)$ and short-wave eigensolutions do not exist. If $k_0\omega < 1$, then $G(\mu^2) > 0$ on the interval $(-\mu_1, \mu_1)$ and $G(\mu^2) < 0$ on the intervals $(-1, \mu_1)$ and $(\mu_1, 1)$.

As before, consider the wave interval of the solution and, introducing local wave numbers k' and l', rewrite (3.8.9) in the form

$$\sigma = \frac{\beta(\varphi_0)\, k'}{k'^2 + l'^2 + f^2(\varphi_0)\, \epsilon} \tag{3.8.10}$$

where the coefficient $\beta(\varphi_0) = (2\Omega/a)\cos\varphi_0$, referred to as latitudinal change of the Coriolis parameter, will play an important role in what follows.

Formula (3.8.10) yields an approximate expression for eigencurves of Type 2 for $\epsilon > 0$ (k' and l' being free parameters).

This analysis may be simplified by introduction of the concept of the so-called β-plane. Assume that the scale factor $\cos\varphi$ in the system of equations (3.2.8)–(3.2.10) may be replaced by the constant $\cos\varphi_0$. Introducing local coordinates $x = a \cos\varphi_0\, (\lambda - \lambda_0)$ and $y = (\varphi - \varphi_0)$, one finds

$$-i\sigma U - fV = -\frac{\partial \Pi}{\partial x}, \qquad -i\sigma V + fU = -\frac{\partial \Pi}{\partial y}, \qquad -i\sigma\epsilon\Pi + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 . \tag{3.8.11}$$

Formally, this system has the same form as in the case of a rotating plane layer. However, it must be recalled that the Coriolis parameter f is here a function of y.

A solution of (3.8.11) will be sought in the form

$$(U,\ V,\ \Pi) = (\hat{U},\ \hat{V},\ \hat{\Pi}) \exp ikx\ , \tag{3.8.12}$$

where $\hat{U}$, $\hat{V}$ and $\hat{\Pi}$ are functions of y. Substitute form (3.8.12) into (3.8. 11) and express $\hat{U}$ and $\hat{\Pi}$ in terms of $\hat{V}$ and $d\hat{V}/dy$, using the first and third equations. After introducing the expressions obtained into the second equation (3.8.11), one finds

$$\frac{d^2\hat{V}}{dy^2} + \left(-k^2 + \sigma^2\epsilon - \frac{k\beta}{\sigma} - f^2\epsilon\right)\hat{V} = 0\ , \tag{3.8.13}$$

where $\beta = df/dy$. It has been assumed in the derivation of (3.8.13) that $\sigma^2 \neq k^2\epsilon$, or, what is the same thing, that $\hat{V} \not\equiv 0$. It is not difficult to show that only for $\sigma = k/\epsilon^{1/2}$, and then only in the neighbourhood of the equator, there exist non-trivial bounded solutions of system (3.8.3).

Finally, a last assumption will be introduced; it relates to moderate latitudes of the oceans where the coefficients f and β in (3.8.13) may be assumed to be constants (variant of the short-wave approximation). The set of these assumptions is usually called the approximation of the β-plane. As a rule, it is assumed that in the equatorial region one has $f = \beta y$, when one speaks of the equatorial β-plane.

If f and β are constants, then a solution of (3.8.13) may be sought in the form $\hat{V} = \exp(ily)$, and one obtains immediately the equation for the eigenvalue curves

$$\epsilon = \frac{k^2 + l^2 + k\beta/\sigma}{\sigma^2 - f^2}. \tag{3.8.14}$$

It determines the set of eigencurves of Problem H in the (ϵ, σ)-plane (Fig. 3.14). It is seen from this figure that qualitatively all three types of eigenvalue curves under consideration are described correctly (for not very large $|\epsilon|$). For these eigenvalue curves, one obtains easily from (3.8.14) simple approximate formulae. For this purpose, let

$$\sigma_g = \sqrt{f^2 + \frac{k^2 + l^2}{\epsilon}}\ , \qquad \sigma_R = \frac{\beta k}{k^2 + l^2 + f^2\epsilon}$$

and write the cubic equation in σ[(3.8.14)] in the form

$$\bar{\sigma}^3 - \bar{\sigma} + \delta = 0\ ,$$

where $\bar{\sigma} = \sigma/\sigma_g$ and $\delta = \sigma_R/\sigma_g$. Since for the ocean δ is a small parameter, one finds $\bar{\sigma}_{1,2} = \pm 1 + O(\delta)$ and $\bar{\sigma}_3 = -\delta + O(\delta^2)$. Reverting to σ and omitting small terms, one has

$$\sigma_{1,2} = \pm\sqrt{f^2 + \frac{k^2 + l^2}{\epsilon}}, \qquad \sigma_3 = -\frac{\beta k}{k^2 + l^2 + f^2\epsilon}\,. \tag{3.8.15}$$

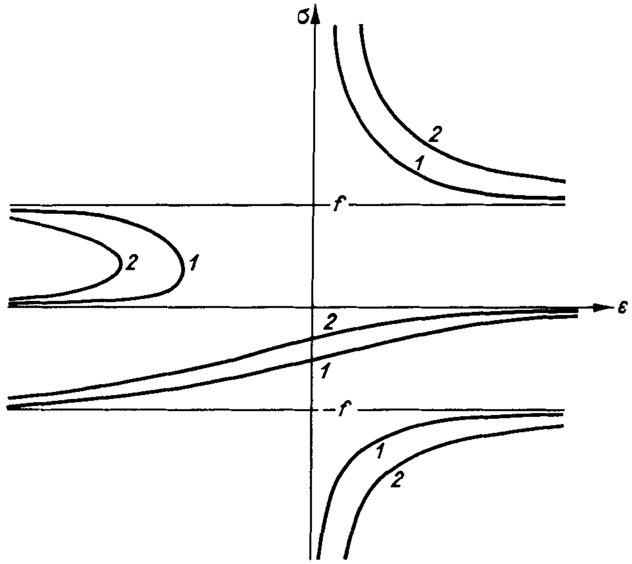


Fig. 3.14. Pattern of eigenvalue curves of Problem H for rotating spherical layer in approximation of β-plane ($k > \beta/f$). The numbers correspond to definite values of $k^2 + l^2$.

Within the notation for the wave numbers, formulae (3.8.15) coincide exactly with (3.8.7) and (3.8.10). Thus, the approximation of the β-plane and the short-wave approximation yield the same formulae for the eigencurves of Problem H.

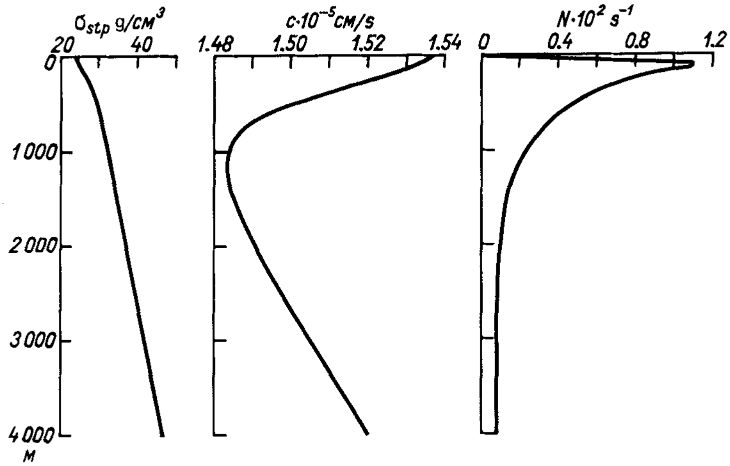


Fig. 3.15. Graphs of $\sigma_{stp}(z)$, $c(z)$ and $N(z)$ typical for the ocean; $N_{max} = 1.1 \cdot 10^{-2}$ sec^{-1}, $c_{min} = 1.483 \cdot 10^5$ cm/sec. $N_{min} = 0$ (chosen so that existence of gyroscopic waves is assured).

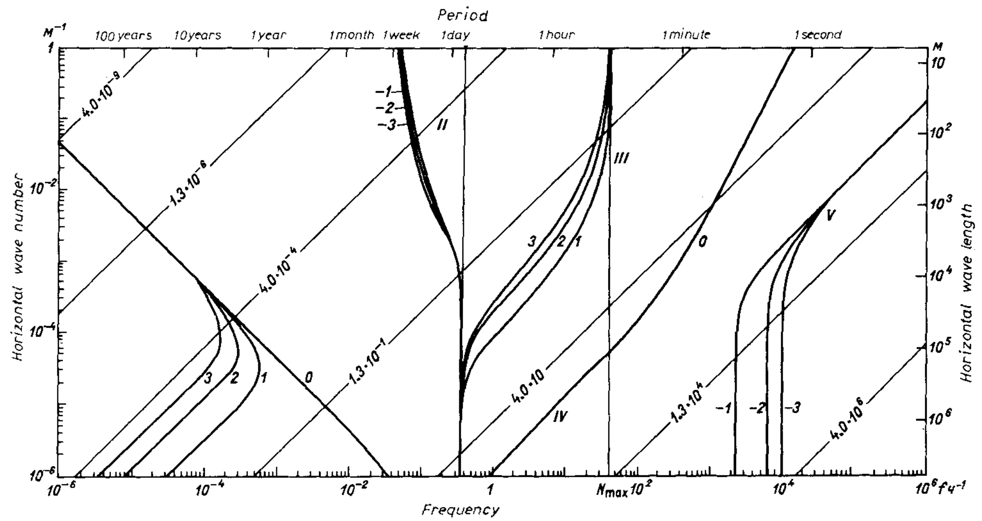


Fig. 3.16. Dispersion curves $\sigma = \sigma(k_h)$ for different types of waves for σ_{stp}, N and c (Fig. 3.15) characteristic for the ocean: $f = 1.0 \cdot 10^{-4}$ sec^{-1}, $\beta = 1.6 \cdot 10^{-11}$ m^{-1} sec^{-1}, $k = k_h$ cos 30°, $l = k_h$ sin 30°. Arabic numerals indicate numbers n of corresponding vertical mode (Fig. 3.11), Roman numerals type of waves: I = Rossby waves, II = gyroscopic waves, III = internal gravitational waves, IV = surface gravitational waves, V = acoustic waves. Equation of inclined straight lines: $c_\phi = \sigma/k_h$, c_ϕ = constant. Numerals on these straight lines are the values of the phase velocity c_ϕ m/sec.

Substitution of the numbers $\epsilon_0, \epsilon_1, \epsilon_2, \ldots$ into (3.8.15) [the eigenvalues of Problem (3.7.8) and (3.7.9)] yields the corresponding dispersion relations for not very long Rossby and gravitational (low-frequency) waves.

Finally, dispersion curves for different classes of not very long ocean waves will be constructed for distributions of $\sigma_{stp}(z)$, $N(z)$ and $c(z)$ typical for the ocean (Fig. 3.15). For this purpose, determine the eigenvalue curves of Problem H from the approximate formula (3.8.14) and the eigenvalue curves of Problem V numerically. Omitting the details of the numerical computations, the final results are shown in Fig. 3.16, constructed in cooperation with A.V. Kulakov. Note that in the stated range of k_h and σ, the frequencies of all types of waves, excluding Rossby waves, depend on the wave numbers k and l only through $k_h = \sqrt{k^2 + l^2}$. In the simplest cases, these relations have been already stated for different types of waves in § 3.3.

In conclusion, the relative error arising from the replacement of barotropic Rossby waves ($\epsilon \simeq 1/gH$) by non-divergent Rossby waves ($\epsilon = 0$) will be estimated from the second formula (3.8.15) [cf. the approximation of the rigid lid in § 3.7]. This error is, obviously, equal to $(f^2/gH)/(k^2 + l^2)$, and for not very long waves it turns out, as a rule, to be insignificant [cf. Fig. 3.16].

3.9 PROBLEM OF FORCED WAVE MOTIONS

It is known that a basic contribution to the variability of the ocean arises from meso- and macro-scale processes with characteristic frequencies of 1 per day and less and wave numbers of 10^{-2}—10^{-3} km^{-1}. In such a range of frequencies and wave numbers, there may only exist barotropic and baroclinic Rossby waves (cf. Fig. 3.16). Therefore, under typical ocean conditions, the portion of energy due to acoustic, gyroscopic and gravitational waves is usually insignificant.

Thus, it is natural to consider the problem of forced waves in Boussinesq's approximation and quasi-statically, filtering out acoustic and gyroscopic waves. Gravitational waves will be filtered out somewhat later.

Forced wave oscillations in the ocean are described by the same system of equations (3.1.3)—(3.1.7) which has been studied in connection with the theory of free oscillations; however, one must now write down on the right-hand sides of (3.1.3) and (3.1.4) the mass forces F_λ and F_φ. Overlook for the present the question of the nature of these forces and assume that F_λ and F_φ have already been expanded in series

$$F_\lambda = \frac{1}{\rho_0} \sum_0^\infty F_\lambda^n(\lambda, \varphi, t)\, P_n(z), \qquad F_\varphi = \frac{1}{\rho_0} \sum_0^\infty F_\varphi^n(\lambda, \varphi, t)\, P_n(z) \tag{3.9.1}$$

with respect to the eigenfunctions $P_n(z)$ of Problem (3.7.8) and (3.7.9).

One of the basic mechanisms of forcing of low-frequency oscillations in the ocean is the action of tangential wind stress τ_λ and τ_φ on the ocean surface (the direct effect of changes of atmospheric pressure p_a, as a rule, is inessential and p_a may be assumed to be constant). This surface force may be replaced by the body force $(\tau_\lambda, \tau_\varphi)/\rho_0 h_0$, acting within the limits of the upper mixed layer of thickness h_0 (cf. [5]).

Then the solution of the problem of forced oscillations may be sought in the form

$$(u, v) = \frac{1}{\rho_0} \sum_0^\infty P_n(z)[U_n(\lambda, \varphi, t),\ V_n(\lambda, \varphi, t)]\,,$$

$$w = -\sum_0^\infty W_n(z)\, \frac{\partial \Pi_n}{\partial t}(\lambda, \varphi, t), \qquad p' = \sum_0^\infty P_n(z)\, \Pi_n(\lambda, \varphi, t),$$

$$\rho' = -\frac{\rho_0 N^2}{g} \sum_0^\infty W_n(z)\, \Pi_n(\lambda, \varphi, t), \quad \zeta = \frac{1}{g\rho_0} \sum_0^\infty P_n(0)\, \Pi_n(\lambda, \varphi, t)\,, \tag{3.9.2}$$

where $W_n(z) = (1/\rho_0 N^2)\, dP_n/dz$; the sea level ζ has been included among the number of unknowns. Obviously, conditions (3.1.8) and (3.1.9) are satisfied. Substitution of (3.9.2) into (3.1.3)—(3.1.7) yields for U_n, V_n and Π_n

the system of Laplace's tidal equations with known body forces. Introducing instead of the eigenvalue ϵ_n of Problem (3.7.8) and (3.7.9) the equivalent depth $H_n = 1/g\epsilon_n$ (cf. end of § 3.7), one has

$$\frac{\partial U_n}{\partial t} - fV_n = -\frac{\partial \Pi_n}{a\cos\varphi\partial\lambda} + F_\lambda^n\,, \qquad \frac{\partial V_n}{\partial t} + fU_n = -\frac{\partial \Pi_n}{a\partial\varphi} + F_\varphi^n\,,$$

$$\frac{\partial \Pi_n}{\partial t} + gH_n\,\mathrm{div}_h(U_n, V_n) = 0\,. \tag{3.9.3}$$

Thus, the general problem of forced oscillations reduces to the solution of a number of problems of the same kind of the determination of normal modes of oscillation (U_n, V_n, Π_n) described by (3.9.3)

The general number of such problems is determined by the given accuracy of representation of the mass forces by the series (3.9.1). The functions $P_n(z)$ and the equivalent depths H_n can be computed for a given stratification once and for all. Analysis of similar problems in the case of n-layer fluids is discussed in [46], as well as in [63, Appendix].

Consider now the question of filtration of gravitational waves. Study System (3.9.3) for moderate latitudes of the ocean. As is already known, the coefficients of System (3.9.3) change very slowly for the short waves of interest here. Therefore, changing to local coordinates $x = a\cos\varphi_0(\lambda - \lambda_0)$ and $y = a(\varphi - \varphi_0)$ [cf. (3.8.11)], one obtains, for example,

$$f = f_0 + \beta_0(y - y_0) + \dots$$

and

$$\frac{\beta_0(y - y_0)}{f_0} \sim \frac{\beta_0}{f_0 k}\,, \tag{3.9.4}$$

where k is a characteristic (dimensional) wave number.

Assume that in the first two equations of System (3.9.3) the Coriolis force (fV, fU) and $\nabla_h\Pi$ have the same orders of magnitude [the index n has been omitted for the sake of simplicity]. Expressing now the characteristic scale Π in terms of the characteristic velocity scales (U, V), one obtains

$$\frac{\partial\Pi/\partial t}{(gH)\partial U/\partial t} \sim \frac{\sigma f_0}{gHk^2}\,, \tag{3.9.5}$$

where σ is a characteristic frequency. Besides, it is obvious that

$$\frac{\partial U/\partial t}{fV} \sim \frac{\sigma}{f_0}\,. \tag{3.9.6}$$

Formulae (3.9.4)–(3.9.6) introduce three non-dimensional parameters; their magnitudes will be estimated for Rossby waves. One has $f_0 = 1.0 \cdot$

TABLE 3.III

Values of the parameters $\sigma_n f_0/gH_n k^2$ and σ_n/f_0

Rossby waves	$k = 10^{-6}$ m^{-1}		$k = 10^{-5}$ m^{-1}	
	$\sigma_n f_0/gH_n k^2$	σ_n/f_0	$\sigma_n f_0/gH_n k^2$	σ_n/f_0
Barotropic: $n = 0$	$2.9 \cdot 10^{-2}$	$1.1 \cdot 10^{-1}$	$3.6 \cdot 10^{-5}$	$1.4 \cdot 10^{-2}$
Baroclinic: $n = 1$	$1.4 \cdot 10^{-1}$	$1.0 \cdot 10^{-4}$	$1.3 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$
$n = 2$	$1.4 \cdot 10^{-1}$	$2.4 \cdot 10^{-5}$	$1.4 \cdot 10^{-2}$	$2.4 \cdot 10^{-4}$

10^{-4} sec^{-1}, $\beta_0 = 1.6 \cdot 10^{-11}$ m^{-1} sec^{-1}; the parameters of stratification σ_{stp}, c and N are given in Fig. 3.15. Numerical calculations yield for the first three eigenvalues $\epsilon_n(0)$: $\epsilon_0 = 0.257 \cdot 10^{-4}$ sec^2/m^2, $\epsilon_1 = 0.140$ sec^2/m^2, $\epsilon_2 = 0.573$ sec^2/m^2, and the equivalent depths $H_0 = 3.97 \cdot 10^3$m, $H_1 = 0.73$ m, $H_2 = 0.18$ m. The parameter $\beta_0/f_0 k = 0.16$ for $k = 10^{-6}$ m^{-1} and $\beta_0/f_0 k = 0.016$ for $k = 10^{-5}$ m^{-1}. The values of the parameters $\sigma_n f_0/gHk^2$ and σ_n/f_0 are given in Table 3.III; the frequencies of the Rossby waves have been computed for the second formula (3.8.15).

Thus, all three parameters $\beta_0/f_0 k$, $\sigma_n f_0/gH_n k^2$ and σ_n/f_0 are small for barotropic as well as for baroclinic Rossby waves. It will be shown below that the order of magnitude of the ratio F/fU is also small. Therefore, presenting the unknown solution in the form

$$U = U^{(0)} + U^{(1)} + \dots, \qquad V = V^{(0)} + V^{(1)} + \dots, \qquad \Pi = \Pi^{(0)} + \Pi^{(1)} + \dots .$$

and neglecting in first approximation all small terms, one finds

$$-f_0 V^{(0)} = -\frac{\partial \Pi^{(0)}}{\partial x}, \qquad f_0 U^{(0)} = -\frac{\partial \Pi^{(0)}}{\partial y}, \tag{3.9.7}$$

$$\frac{\partial U^{(0)}}{\partial x} + \frac{\partial V^{(0)}}{\partial y} = 0 \tag{3.9.8}$$

(it being convenient not to go to the non-dimensional form).

When writing down the second approximation, it is expedient to assume that all small parameters $\beta_0/f_0 k$, $\sigma f_0/gHk^2$ and σ/f_0 and likewise the ratio F/fU are quantities of equal order of magnitude. One has

$$\frac{\partial U^{(0)}}{\partial t} - f_0 V^{(1)} - \beta_0(y - y_0)\, V^{(0)} = -\frac{\partial \Pi^{(1)}}{\partial x} + F_x\,, \tag{3.9.9}$$

$$\frac{\partial V^{(0)}}{\partial t} + f_0 U^{(1)} + \beta_0(y - y_0)\, U^{(0)} = -\frac{\partial \Pi^{(1)}}{\partial y} + F_y\,, \tag{3.9.10}$$

$$\frac{\partial \Pi^{(0)}}{\partial t} + gH\left(\frac{\partial U^{(1)}}{\partial x} + \frac{\partial V^{(1)}}{\partial y}\right) = 0\,. \tag{3.9.11}$$

The relations (3.9.7) are said to be geostrophic. Equation (3.9.8) permits to introduce a stream function

$$U^{(0)} = -\frac{\partial \psi^{(0)}}{\partial y}, \qquad V^{(0)} = \frac{\partial \psi^{(0)}}{\partial x}, \tag{3.9.12}$$

when it follows from (3.9.7) that

$$\Pi^{(0)} = f_0 \psi^{(0)} + \text{constant} . \tag{3.9.13}$$

Eliminating the function $\Pi^{(1)}$ from (3.9.9) and (3.9.10) by cross-differentiation, one obtains the vorticity equation

$$\frac{\partial}{\partial t}\left(\frac{\partial V^{(0)}}{\partial x} - \frac{\partial U^{(0)}}{\partial y}\right) + f_0\left(\frac{\partial U^{(1)}}{\partial x} + \frac{\partial V^{(1)}}{\partial y}\right) + \beta_0(y - y_0)\left(\frac{\partial U^{(0)}}{\partial x} + \frac{\partial V^{(0)}}{\partial y}\right)$$
$$+ \beta_0 V^{(0)} = \mathrm{rot}_z F . \tag{3.9.14}$$

Finally, one finds the required equation for Rossby waves from (3.9.8) and (3.9.11)–(3.9.13)

$$\frac{\partial}{\partial t} \Delta_h \psi - \frac{f_0^2}{gH} \frac{\partial \psi}{\partial t} + \beta_0 \frac{\partial \psi}{\partial x} = \mathrm{rot}_z F , \tag{3.9.15}$$

where Δ_h is the Laplace operator in the (x, y)-plane.

The superscript (0) on the stream function has been omitted here; recall also that (3.9.15) is a first approximation. Likewise, one must keep in mind that in this equation the quantity H has different meanings for barotropic and baroclinic Rossby waves. Once the stream function ψ has been found, then U, V and Π can be determined (in first approximation) from (3.9.12) and (3.9.13).

Equation (3.9.15) determines the at present unknown characteristic scale of the velocity U as $\sim Fk/\beta_0$. Therefore the ratio $F/fU \sim \beta_0/f_0 k$, as it has been assumed, turns out to be small.

COMMENT ON CHAPTER 3

§ § 3.1, 3.2: For the general formulation of the problem, the energy equation and the method of separation of variables, cf. Monin, Obukhov [78], Eckart [13], Dikii [11]; the approximation (3.1.2) is due to Phillips [94].

§ 3.3: Related questions are studied in many places (cf. Eckart [13], Tolstoy [122]). An explanation of the mechanism of the formation of Rossby waves has been given by Longuet-Higgins [69].

§ 3.4: This presentation is due to Kamenkovich, Odulo [47]; cf. also the bibliography in this paper.

§ 3.5: This subject is studied in the papers of Longuet-Higgins [70] and Dikii [11].

§§ 3.6, 3.7: Kamenkovich, Odulo [47].

§ 3.8: The short-wave analysis of Laplace's tidal equations is based on work of Kamenkovich, Tsybanova [49]. Cf. also Phillips [97, Section 3].

§ 3.9: The derivation of the equation for Rossby waves has been taken from the work of Phillips [94, Section 2].

CHAPTER 4

EQUATIONS OF THE THEORY OF OCEAN CURRENTS AND THEIR PROPERTIES

4.1 EQUATION OF EVOLUTION OF POTENTIAL VORTICITY

A start will be made with the derivation of the general vorticity equation. Introducing the vorticity vector $\omega = \mathrm{rot}\boldsymbol{v}$ and employing the well known relation

$$v^{\beta}\nabla_{\beta}v_{\alpha} = \nabla_{\alpha}(\tfrac{1}{2}\boldsymbol{v}^2) + \epsilon_{\alpha\beta\gamma}\omega^{\beta}v^{\gamma} ,$$

Equation (2.3.6) may be rewritten in the form

$$\frac{\partial v_{\alpha}}{\partial t} + \nabla_{\alpha}(\tfrac{1}{2}\boldsymbol{v}^2) + \epsilon_{\alpha\beta\gamma}\omega_A^{\beta}v^{\gamma} = -\frac{1}{\rho}\nabla_{\alpha}p + X_{\alpha} + F_{\alpha} , \tag{4.1.1}$$

where the vector $\omega_A = \omega + 2\Omega$ signifies the vorticity of the absolute motion and is, as a rule, referred to as absolute vorticity (absolute motion comprises motion with velocity $\boldsymbol{v}$ relative to the Earth and rotation with velocity Ω together with the Earth), $\boldsymbol{F}$ is the force of friction per unit mass ($F_{\alpha} = (1/\rho)\nabla_{\beta}\sigma_{\alpha}^{\beta}$) and the remaining notation is known.

Applying to both sides of (4.1.1) the operation $\epsilon^{\chi\nu\alpha}\nabla_{\nu}$ and using a formula of the type (A4.11), one finds

$$\frac{\partial\omega^{\chi}}{\partial t} + v^{\nu}\nabla_{\nu}\omega_A^{\chi} - v^{\chi}\nabla_{\nu}\omega_A^{\nu} + \omega_A^{\chi}\nabla_{\nu}v^{\nu} - \omega_A^{\nu}\nabla_{\nu}v^{\chi} = \frac{1}{\rho^2}\epsilon^{\chi\nu\alpha}\nabla_{\nu}\rho\nabla_{\alpha}p$$
$$-\frac{1}{\rho}\epsilon^{\chi\nu\alpha}\nabla_{\nu}(\nabla_{\alpha}p) + \epsilon^{\chi\nu\alpha}\nabla_{\nu}(X_{\alpha} + F_{\alpha}) .$$

Since $\epsilon^{\chi\nu\alpha}\nabla_{\nu}(\nabla_{\alpha}p) = \mathrm{rot}(\nabla p) = 0$, $\nabla_{\nu}\omega_A^{\nu} = \mathrm{div}\,\omega_A = 0$, $\epsilon^{\chi\nu\alpha}\nabla_{\nu}X_{\alpha} = 0$ (mass forces have a potential) and $\partial\omega/\partial t = \partial\omega_A/\partial t$, this expression may be simplified. Finally, using the equation of conservation of mass (2.2.13), one arrives at the known equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\omega_A^{\chi}}{\rho}\right) - \frac{\omega_A^{\nu}}{\rho}\nabla_{\nu}v^{\chi} = \frac{1}{\rho^3}\epsilon^{\chi\nu\alpha}\nabla_{\nu}\rho\nabla_{\alpha}p + \frac{\epsilon^{\chi\nu\alpha}}{\rho}\nabla_{\nu}F_{\alpha} . \tag{4.1.2}$$

By (4.1.2) [Fridman's equation], the change of absolute vorticity referred to unit mass is caused by the effects of stretching of the absolute vorticity lines, baroclinity of the ocean water and frictional forces.

These effects will now be discussed briefly. Consider in Cartesian coordinates an element of the absolute vortex thread $\delta\boldsymbol{r} = (\delta x, \delta y, \delta z) = (\epsilon/\rho)\omega_A$ (where ϵ is some small quantity). Since this element moves with the liquid, the change in length of the component δx will be

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta x) = \epsilon\frac{\omega_{Ax}}{\rho}\frac{\partial u}{\partial x} + \epsilon\frac{\omega_{Ay}}{\rho}\frac{\partial u}{\partial y} + \epsilon\frac{\omega_{Az}}{\rho}\frac{\partial u}{\partial z}$$

and analogously for δy and δz. Hence one has, in general tensorial form,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\delta r^{\chi}) = \epsilon\frac{\omega_A^{\nu}}{\rho}\nabla_{\nu}v^{\chi} ,$$

but this quantity is exactly proportional to the second term on the left-hand side of (4.1.2). Note that in the case of plane motion the effect of stretching of the absolute vortex lines vanishes (assuming, of course, that the axis of rotation is perpendicular to the plane of motion).

The baroclinic effect is described by the term $(1/\rho^3)\nabla\rho \times \nabla p$. Clearly, vorticity ω arises only from those components of the force ∇p which lie in the plane tangential to the isopycnic surface (in a baroclinic fluid $\rho = \rho\ (p)$ and ∇p does not have such a component), since, in fact, this component of ∇p imparts different accelerations to particles lying on different sides of the isopycnic surface which also arouses their *twisting*. Analogously, the effect of viscosity always causes a couple, twisting the fluid particles (Fig. 4.1).

Thus, even in an ideal baroclinic fluid, absolute vorticity of particles (referred to unit mass) is not conserved during motion (because of stretching of absolute vortex lines).

Consider now a quantity σ the value of which for each fluid particle does not change during adiabatic motion. Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0 . \tag{4.1.3}$$

Quantities $\sigma(x^{\alpha}, t)$ are usually referred to as Lagrangean adiabatic invari-

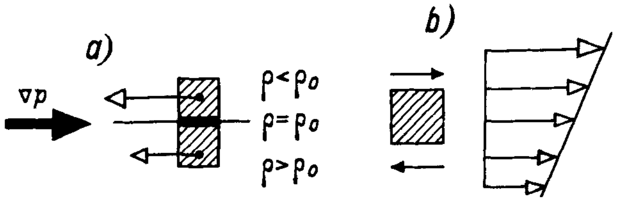


Fig. 4.1. Towards an explanation of the generation of vorticity in a fluid. a) — baroclinic effect; different accelerations of "upper" and "lower" particles leading to deviations from translatory motion are shown; b) — effect of frictional forces, twisting fluid particles; the velocity distribution is shown on the right.

ants or simply as adiabatic invariants. Examples of such invariants are specific entropy η_m and salinity s.

Thus, let σ be some adiabatic invariant. It will be shown that one may obtain a new adiabatic invariant which plays an important role in hydrodynamics (cf., for example, [76, 77, 94, 130]). For this purpose, apply to (4.1.3) the gradient operator ∇_κ. Since in Euclidean space $\nabla_\kappa \nabla_\alpha = \nabla_\alpha \nabla_\kappa$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla_\chi \sigma) + \nabla_\chi v^\alpha \nabla_\alpha \sigma = 0 \,. \tag{4.1.4}$$

For adiabatic motion, one must omit in Fridman's equation (4.1.2) the effect of friction. Multiply then this equation scalarly by $\nabla_\kappa \sigma$, and equation (4.1.4) likewise scalarly by $(1/\rho)\omega_A^\kappa$. Adding the results and using the identity

$$\omega_A^\nu \nabla_\nu v^\chi \nabla_\chi \sigma = \nabla_\chi v^\alpha \nabla_\alpha \sigma \omega_A^\chi \,, \tag{4.1.5}$$

one obtains Ertel's formula

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}_A \, \boldsymbol{\nabla}\sigma}{\rho}\right) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p) \; \boldsymbol{\nabla}\sigma \,. \tag{4.1.6}$$

The quantity $(1/\rho)\omega_A \boldsymbol{\nabla}\sigma$ is called the potential vorticity of the particle. If the thermodynamics of the medium under study are determined by two independent parameters (for example, p and ρ), then

$$(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p) \; \boldsymbol{\nabla}\eta_m = 0 \,,$$

since the vector $\boldsymbol{\nabla}\eta_m$ lies in the plane of the vectors $\boldsymbol{\nabla}\rho$ and $\boldsymbol{\nabla}p$. However, then, by Ertel's formula (4.1.6), it is concluded that the potential vorticity $(1/\rho)\omega_A \boldsymbol{\nabla}\eta_m$ is an adiabatic invariant. However, in the case of sea water, the specific entropy η_m depends on three independent parameters (for example, p, ρ and s) and, generally speaking, $(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p)\boldsymbol{\nabla}\eta_m \neq 0$. Since the velocity of sound in sea water is very large, one has approximately for adiabatic motion $\mathrm{d}\rho/\mathrm{d}t = 0$ [cf. (3.1.1)]. Since always $(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p)\boldsymbol{\nabla}\rho = 0$, the potential vorticity $(1/\rho)\boldsymbol{\omega}_A \boldsymbol{\nabla}\rho$ may be assumed to be an approximate invariant for sea water.

For non-adiabatic motion of a fluid, formula (4.1.6) is readily generalized. Repeating all stages of the derivation of (4.1.6), one obtains

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}_A \, \boldsymbol{\nabla}\sigma}{\rho}\right) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p) \; \boldsymbol{\nabla}\sigma + \boldsymbol{\omega}_A \, \boldsymbol{\nabla}Q + \boldsymbol{\nabla}\sigma \, \mathrm{rot}\, \boldsymbol{F} \,, \tag{4.1.7}$$

where Q is the change of the parameter σ caused by non-adiabatic factors: $\mathrm{d}\sigma/\mathrm{d}t = Q$. Since $\mathrm{div}\,(Q\,\boldsymbol{\omega}_A) = \boldsymbol{\omega}_A \boldsymbol{\nabla}Q$ and $\boldsymbol{\nabla}\sigma \, \mathrm{rot}\, \boldsymbol{F} = \mathrm{div}\,(\sigma \, \mathrm{rot}\, \boldsymbol{F})$, one has finally,

$$\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{\omega}_A \, \boldsymbol{\nabla}\sigma}{\rho}\right) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p) \; \boldsymbol{\nabla}\sigma + \mathrm{div}(Q\boldsymbol{\omega}_A + \sigma \, \mathrm{rot}\, \boldsymbol{F}) \,. \tag{4.1.8}$$

Equation (4.1.8) is referred to as the equation of evolution of potential vorticity.

4.2 BOUSSINESQ'S APPROXIMATION

In Chapter 3, a closed system of equations has been obtained for the determination of the basic parameters T, p, s and $\boldsymbol{v}$. However, in general form, these equations prove to be too complex, and they may be considerably simplified for studies of the motion of sea water.

A start will be made with the equation of evolution of entropy in the form (2.7.1). Transform to the variables T, p and s. By (1.5.4), one has

$$\rho T\left(\frac{c_p}{T}\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\partial(1/\rho)}{\partial T}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial \eta_m}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t}\right) = -\operatorname{div}\boldsymbol{q} + \left(-\frac{\partial \chi_m}{\partial s} + \mu\right)\operatorname{div}\boldsymbol{I}_s + \tfrac{1}{2}\sigma^{\alpha\beta}e_{\alpha\beta}$$

$$+\boldsymbol{I}_s\,\nabla\left(\frac{\partial \chi_m}{\partial s}\right). \tag{4.2.1}$$

Employing the diffusion equation for salt (2.2.14) and the identity $\mu = \partial\chi_m/\partial s - T\partial\eta_m/\partial s$ (cf. end of § 2.7), one derives equation (4.2.1) in the form

$$\rho c_p\left(\frac{\mathrm{d}T}{\mathrm{d}t} - \Gamma\frac{\mathrm{d}p}{\mathrm{d}t}\right) = -\operatorname{div}\boldsymbol{q} + \tfrac{1}{2}\sigma^{\alpha\beta}e_{\alpha\beta} + \boldsymbol{I}_s\,\nabla\left(\frac{\partial\chi_m}{\partial s}\right). \tag{4.2.2}$$

In writing down (4.2.2), the adiabatic temperature gradient Γ has been introduced (cf. § 1.6). It is not difficult to obtain the estimate that on lowering particles from the ocean surface to a depth of 2000 m $\Gamma\mathrm{d}p \sim 1 \div 2°\mathrm{C}$, at the same time as $\mathrm{d}T \sim 10°\mathrm{C}$; for horizontal displacements of particles, the contribution of the term $\Gamma\mathrm{d}p$ is quite insignificant. Therefore in the upper layers of the ocean, the second term on the left-hand side of (4.2.2) may be disregarded deliberately. Further, assuming the gradients of the basic parameters of the medium $\nabla_\alpha T$, $\nabla_\alpha p$, $\nabla_\alpha s$ and $\nabla_\alpha v_\beta$ to be small, neglect the two last terms on the right-hand side of (4.2.2) in comparison with the first term. Finally, one has

$$\rho c_p\frac{\mathrm{d}T}{\mathrm{d}t} = -\operatorname{div}\boldsymbol{q}. \tag{4.2.3}$$

Equation (4.2.3) is usually called the equation of heat conduction for a moving medium.

Next an estimate will be obtained of the change of density $\delta\rho/\rho$ in the ocean. Since $\rho = \rho(T, p, s)$, one has

$$\delta\rho = \frac{\partial\rho}{\partial T}\,\delta T + \frac{\partial\rho}{\partial p}\,\delta p + \frac{\partial\rho}{\partial s}\,\delta s.$$

In the ocean, one has $\partial\rho/\partial T \sim 2 \cdot 10^{-4}$ g/°C, $\delta T \sim 10$°C. Therefore $(\delta\rho)_{p,s} \sim 2 \cdot 10^{-3}$ g. Further, $\partial\rho/\partial s \sim 1$ g and $\delta s \sim 10^{-3}$, whence $(\delta\rho)_{T,p} \sim 10^{-3}$ g. Thus, observable changes in temperature and salinity appear to have roughly identical effects on density changes.

Next, an estimate of the quantity $(\delta\rho)_{T,s}$ will be obtained. The quantity $(\partial\rho/\partial p)_{T,s} = 1/c_T^2$, where c_T^2 is the square of the isothermal velocity of sound (cf. § 1.6). Thus $c_T \sim c$, and

$$(\delta\rho)_{T,s} \simeq \frac{\delta p}{c^2} \, .$$

Estimate the possible pressure drop δp which depends, generally speaking, on the type of motion. For example, for small-scale motions $\delta p/L\rho \sim U^2/L$ (where L is a characteristic horizontal scale and U a characteristic horizontal velocity), and $(1/\rho)(\delta\rho)_{T,s} \simeq v^2/c^2 \simeq 10^{-6} \div 10^{-8}$. For large-scale motions, pressure drops along the horizontal and vertical differ strongly in magnitude. In the first case, using geostrophy (cf. § 3.9), one has $\delta p \sim LfU$ (where f is the Coriolis parameter), i.e. $\delta p \simeq 10^5 \div 10^6$ dyn/cm^2 and $(1/\rho)(\delta\rho)_{T,s} \simeq 10^{-5} \div 10^{-4}$. In the second case, by the hydrostatic condition $\delta p \simeq g\rho H$ (where H is the vertical scale ~1 km). Hence $\delta p \simeq 10^8$ dyn/cm^2 and $(1/\rho)(\delta\rho)_{T,s} \simeq 5 \cdot 10^{-3}$.

These estimates show that for computation of the horizontal density gradient $\nabla_h\rho$ one may assume the density to depend only on temperature and salinity. However, when computing vertical density gradients, the dependence of the density on pressure proves to be very important.

Thus, the density in the ocean changes very little: $\delta\rho/\rho \simeq 10^{-3}$. This fact will now be used to simplify the basic equations. Consider the equation of conservation of mass (2.2.13). Since

$$\frac{\boldsymbol{v}\,\nabla\rho}{\rho\,\partial u/\partial x} \sim \frac{\delta\rho}{\rho} << 1 \, ,$$

equation (2.2.13) may be written in the form

$$\frac{\partial\rho}{\partial t} + \rho \operatorname{div} \boldsymbol{v} = 0 \, . \tag{4.2.4}$$

It has been seen in Chapter 3 that the term $\partial\rho/\partial t$ is essential only for fast acoustic waves. Writing equation (4.2.4) in the form div $\boldsymbol{v} = 0$, one filters out acoustic waves with insignificant energy and practically does not distort other types of waves. Assuming that characteristic velocities of propagation of disturbances in a medium are given by linear theory, it is readily shown that one may neglect the term $\partial\rho/\partial t$ in (4.2.4) also in case of non-linear processes. Thus, with a great degree of accuracy, the equation of conservation of mass may be written in the form

$$\operatorname{div} \boldsymbol{v} = 0 \, . \tag{4.2.5}$$

Finally, it will be assumed that the force of gravity is the only external body force. Consider the expression $-(1/\rho)\nabla p + \boldsymbol{g}$, entering into the equation of motion (2.3.6). Let $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, where $\nabla p_0 = \boldsymbol{g}\rho_0$ and ρ_0 is some mean constant density, $\rho' << \rho_0$. Then, neglecting terms of the second order of smallness, one obtains

$$-\frac{\nabla p}{\rho} + \boldsymbol{g} = -\frac{1}{\rho_0}(\nabla p_0 + \nabla p')\left(1 - \frac{\rho'}{\rho_0} + \ldots\right) + \boldsymbol{g} \simeq -\frac{\nabla p'}{\rho_0} + \boldsymbol{g}\frac{\rho'}{\rho_0},$$

or reverting to total p and ρ,

$$-\frac{\nabla p}{\rho} + \boldsymbol{g} \simeq -\frac{\nabla p}{\rho_0} + \boldsymbol{g}\frac{\rho}{\rho_0}. \tag{4.2.6}$$

One may replace ρ by ρ_0 in (2.2.14) and (4.2.3); likewise, it may be assumed that the thermal conductivity in (4.2.3) is constant.

Taking all these approximations into account, the system of basic equations reduces to the form:

$$\frac{\partial}{\partial t}(\rho_0 v^\alpha) = -\nabla_\beta \Pi^{\alpha\beta} + \rho g^\alpha + 2\rho_0 \epsilon^{\alpha\beta\gamma} v_\beta \Omega_\gamma, \qquad \Pi^{\alpha\beta} = \rho_0 v^\alpha v^\beta + p m^{\alpha\beta} - \sigma^{\alpha\beta}, \tag{4.2.7}$$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{4.2.8}$$

$$\frac{\partial}{\partial t}(\rho_0 s) = -\operatorname{div}(\rho_0 s \boldsymbol{v} + \boldsymbol{I}_s), \tag{4.2.9}$$

$$\frac{\partial}{\partial t}(c_p \rho_0 T) = -\operatorname{div}(c_p \rho_0 T \boldsymbol{v} + \boldsymbol{q}), \tag{4.2.10}$$

where the equation of motion has been rewritten in an equivalent form, taking into consideration (4.2.8).

Since the concrete expressions linking the fluxes $\sigma^{\alpha\beta}$, $\boldsymbol{I}_s$ and $\boldsymbol{q}$ to the gradients of the basic parameters $\boldsymbol{v}$, T, p and s will not be required in what follows (cf. end of § 4.3), no attention will be given here to simplification of the general laws (2.9.1), (2.10.1) and (2.10.2).

The term $\rho\boldsymbol{g}$ in (4.2.7) describes the effect of Archimedes forces. Thus, small changes in density prove to be essential only for the computation of these forces; in all remaining terms entering into the basic equations, the density ρ may be replaced by the constant ρ_0.

Approximations (4.2.5) and (4.2.6) (and likewise replacement of ρ by ρ_0 in all remaining equations) are referred to as Boussinesq approximation. A variant of these approximations has been encountered in Chapter 3 when considering the linear theory of waves. It is not difficult to show that, after introduction of the Väisälä frequency $N(z)$ into the basic equations of the theory of waves, the density $\rho_0(z)$ may be replaced by the constant ρ_0; then all types of waves (except, of course, acoustic waves) will be distorted quite unnoticeably.

According to the approximation adopted, one has for adiabatic motion of a fluid $dT/dt = 0$, $ds/dt =$ [cf. (4.2.8)—(4.2.10)]. Since $d\rho/dt = (1/c_T^2)dp/dt$, one can set approximately $d\rho/dt = 0$, assuming c_T^2 to be very large. Then it is readily shown that, by strength of the equations of motion (4.2.7), the potential vorticity $\omega_A \nabla\rho$ will be an adiabatic invariant. In fact, repeating the derivation of the preceding section and employing the identity

$$\nabla\rho \operatorname{rot} \rho g = \operatorname{div}(\rho \operatorname{rot} \rho g) = \operatorname{div}[\operatorname{rot}(\rho^2/2)\, g] = 0\ ,$$

one finds

$$\frac{d}{dt}(\omega_A \nabla\rho) = 0\ . \tag{4.2.11}$$

Since strictly speaking $d\rho/dt \neq 0$, the quantity $\omega_A\nabla\rho$ for each particle will, generally speaking, change slowly in agreement with a general equation of the type (4.1.8).

It is easily verified, at the same time, that the approximate analogue of Fridman's equation (4.1.2) has the form

$$\frac{d\omega_A^\chi}{dt} - \omega_A^\nu \nabla_\nu v^\chi = \frac{1}{\rho_0}\, \epsilon^{\chi\nu\alpha}\nabla_\nu(\rho g_\alpha) + \epsilon^{\chi\nu\alpha}\nabla_\nu F_\alpha\ , \tag{4.2.12}$$

and, since $\nabla_\alpha p_0 = \rho_0 g_\alpha$, the term $(1/\rho_0^2)\epsilon^{\kappa\nu\alpha}\nabla_\nu(\rho g_\alpha)$ may be rewritten in the form $(1/\rho_0^3)\epsilon^{\kappa\nu\alpha}\nabla_\nu\rho\nabla_\alpha p_0$, which can be usefully compared with the expression $(1/\rho^3)\epsilon^{\kappa\nu\alpha}\nabla_\nu\rho\nabla_\alpha p$ in the exact equation (4.1.2).

In conclusion of this section, the analogue of the equation of mechanical energy transfer (2.6.2) will be written down in Boussinesq approximation. Multiplying the equation of motion (4.2.7) scalarly by $\boldsymbol{v}$, one obtains

$$\frac{\partial}{\partial t}\left(\rho_0\frac{\boldsymbol{v}^2}{2}\right) = -\nabla_\alpha\left(\rho_0\frac{\boldsymbol{v}^2}{2}\, v^\alpha - p^{\alpha\beta}v_\beta\right) + \rho g_\alpha v^\alpha - \tfrac{1}{2}\sigma_{\alpha\beta}e^{\alpha\beta}\ . \tag{4.2.13}$$

Since it has been seen that $\sigma_{\alpha\beta}e^{\alpha\beta} > 0$, then the Boussinesq approximation does not admit conversion of internal energy into mechanical energy (only dissipation of mechanical energy is possible).

4.3 AVERAGING OF BASIC EQUATIONS

Thus, one has found for the determination of the basic parameters of a medium $\boldsymbol{v}$, T, p and s a closed system of equations (4.2.7)—(4.2.10). However, a new circumstance complicates extremely the entire problem: Motion in the ocean is turbulent. Therefore, in essence, the functions $\boldsymbol{v}$, T, p and s turn out to be stochastic fields and for given initial and boundary conditions one has a set of possible realizations of the motion under consideration. Even for small deviations from given external conditions, there develop in the flow finite perturbations (for more detail, cf. [79]). As usually, assume that each concrete realization of the field $\boldsymbol{v}$, T, p and s (real field) satisfies the basic equations (4.2.7)–(4.2.10). Clearly, the problem of description of a concrete realization of the motion under study on the basis of the system of equations (4.2.7)–(4.2.10) is practically insoluble.

It is natural in a study of random fields to take interest, in first place, in their mean values. Each real field a will be represented in the form

$$a = \bar{a} + a' \,, \tag{4.3.1}$$

where $\bar{a}$ is the averaged field and a' the pulsation field. The representation (4.3.1) was first introduced by Reynolds.

What is the significance of the average in the representation (4.3.1)? Strictly speaking, one must understand by average the stochastic average over the set of possible (under given external conditions) realizations of a motion under consideration. In the presence of stationarity or homogeneity of the stochastic field, one may, using the ergodic hypothesis, compute the mean also over individual realizations. However, the method of averaging will not be of concern here. The operation of averaging will be introduced axiomatically; in fact, it will be required that the following conditions are fulfilled (Reynold's conditions):

$$\overline{a+b} = \bar{a} + \bar{b} \,, \qquad \overline{ka} = k\bar{a} \,, \qquad k = \text{constant} \,, \qquad \bar{k} = k \,, \qquad k = \text{constant} \,,$$
$$\overline{\frac{\partial a}{\partial l}} = \frac{\partial \bar{a}}{\partial l} \,, \qquad \frac{\partial}{\partial l} = \frac{\partial}{\partial x^\alpha} \ \text{or} \ \frac{\partial}{\partial l} = \frac{\partial}{\partial t} \,, \qquad \overline{\bar{a}b} = \bar{a}\bar{b} \,. \tag{4.3.2}$$

Note that for a stochastic average all these conditions are satisfied. For so-called time or space averages (practical methods of averaging), the last condition (4.3.2) is only fulfilled approximately.

The following results follow from the last condition (4.3.2), if one sets there consecutively $b = 1$, $b = b'$ and $b = \bar{b}$:

$$\bar{\bar{a}} = \bar{a}, \qquad \bar{a'} = \overline{a - \bar{a}} = 0 \,, \qquad \overline{\bar{a}b'} = \bar{a}\bar{b'} = 0 \,, \qquad \overline{\bar{a}\bar{b}} = \bar{a}\bar{b} \,. \tag{4.3.3}$$

The following important relation follows readily from (4.3.3):

$$\overline{ab} = \bar{a}\bar{b} + \overline{a'b'} \,. \tag{4.3.4}$$

An attempt will now be made to construct a system of equations for the averaged fields $\bar{\boldsymbol{v}}$, $\bar{T}$, $\bar{p}$ and $\bar{s}$. Averaging equations (4.2.7)–(4.2.10) and using rules (4.3.2)–(4.3.4), one finds

$$\frac{\partial}{\partial t}(\rho_0 \bar{v}^\alpha) = -\nabla_\beta \bar{\Pi}^{\alpha\beta} + \bar{\rho} g^\alpha + 2\rho_0 \epsilon^{\alpha\beta\gamma} \bar{v}_\beta \Omega_\gamma \,, \tag{4.3.5}$$

$$\bar{\Pi}^{\alpha\beta} = \rho_0 \bar{v}^\alpha \bar{v}^\beta + \bar{p} m^{\alpha\beta} - (\bar{\sigma}^{\alpha\beta} - \rho_0 \overline{v'^\alpha v'^\beta}) \,,$$

$$\operatorname{div} \bar{\boldsymbol{v}} = 0 \,, \tag{4.3.6}$$

$$\frac{\partial}{\partial t}(c_p\rho_0\overline{T}) = -\mathrm{div}(c_p\rho_0\overline{T}\overline{\boldsymbol{v}} + \overline{\boldsymbol{q}} + c_p\rho_0\overline{T'\boldsymbol{v}'}) . \tag{4.3.7}$$

$$\frac{\partial}{\partial t}(\rho_0\overline{s}) = -\mathrm{div}(\rho_0\overline{s}\overline{v} + \boldsymbol{I}_s + \rho_0\overline{s'\boldsymbol{v}'}) , \tag{4.3.8}$$

Thus, one may write down for the averaged fields $\overline{\boldsymbol{v}}$, $\overline{T}$, $\overline{p}$ and $\overline{s}$ the same equations as for the real fields, if one understands by heat flux $\overline{\boldsymbol{q}} + \boldsymbol{J}_q$, by diffusive flux of salt $\boldsymbol{I}_s + \boldsymbol{J}_s$, and by the tensor of viscous stresses $\sigma^{\alpha\beta} + R^{\alpha\beta}$, where

$$\boldsymbol{J}_q = c_p\rho_0\overline{T'\boldsymbol{v}'} , \qquad \boldsymbol{J}_s = \rho_0\overline{s'\boldsymbol{v}'} , \qquad R^{\alpha\beta} = -\rho_0\overline{v'^\alpha v'^\beta} . \tag{4.3.9}$$

The vector $\boldsymbol{J}_q$ is referred to as vector of density of turbulent flux of heat, the vector $\boldsymbol{J}_s$ as vector of density of turbulent salt flux, and the symmetric tensor $R^{\alpha\beta}$ as tensor of Reynolds stresses. Thus, from the point of view of averaged fields, turbulence leads to a change in the transfer processes, where, as a rule,

$$|\boldsymbol{J}_q| >> |\overline{\boldsymbol{q}}| , \qquad |\boldsymbol{J}_s| >> |\overline{\boldsymbol{I}_s}| , \qquad |R^{\alpha\beta}| >> |\overline{\sigma}^{\alpha\beta}| . \tag{4.3.10}$$

These inequalities may, generally speaking, be violated only for studies of micro-scale processes which will not be considered in this book.

Note now that the system of equations (4.3.5)—(4.3.8) for the averaged fields $\overline{\boldsymbol{v}}$, $\overline{T}$, $\overline{p}$ and $\overline{s}$ appears to be not closed: It contains the new variables $\boldsymbol{J}_q, \boldsymbol{J}_s$ and $R^{\alpha\beta}$ which play, by (4.3.10), a definite role. Formally, this situation is analogous to the position discussed in detail in Chapter 2. However, for averaged motions, there exist no general principles of the type of the second law of thermodynamics on the basis of which one could achieve closure of the systems. The problem of closing Systems (4.3.5)—(4.38) has hitherto not been solved completely (at last, not in visible form) and it represents the basic difficulty encountered in the study of turbulent motions of fluids. A semi-empirical approach to the solution of this problem will be given in § 4.5.

Note that the average $\overline{\rho}$ over known $\overline{T}, \overline{p}$ and $\overline{s}$, by strength of the nonlinearity of the equation of state of sea water, likewise is not clearly defined.

4.4 EQUATION FOR TURBULENT ENERGY

Introduce the important equation for the specific kinetic energy of the mean motion $E_s = (\frac{1}{2})\overline{\boldsymbol{v}}^2$ and the mean specific kinetic energy of the pulsating motion (or simply the turbulent energy) $E_t = (\frac{1}{2})\overline{\boldsymbol{v}'^2}$.

First average the equation of transfer of mechanical energy of the real motion (4.2.13). For this purpose, employ the following useful identities which are readily derived taking into consideration (4.3.2)–(4.3.4):

$$\overline{v_\alpha v^\alpha} = \bar{v}_\alpha \bar{v}^\alpha + \overline{v'_\alpha v'^\alpha} \text{ or } \bar{E} = E_s + E_t\,,$$
$$\overline{v_\alpha v^\alpha v_\beta} = 2E_s \bar{v}_\beta + 2E_t \bar{v}_\beta + 2\bar{v}^\alpha \overline{v'_\alpha v'_\beta} + \overline{v'_\alpha v'^\alpha v'_\beta}\,. \tag{4.4.1}$$

The averaging of (4.2.13) then leads to the relation

$$\frac{\partial}{\partial t}\{\rho_0(E_s + E_t)\} = -\nabla_\beta\{\rho_0(E_s + E_t)\,\bar{v}^\beta - R^{\alpha\beta}\bar{v}_\alpha + \tfrac{1}{2}\rho_0\overline{v'_\alpha v'^\alpha v'^\beta} - \bar{p}^{\alpha\beta}\bar{v}_\alpha$$
$$- \overline{p'^{\alpha\beta}v'_\alpha}\} + \bar{\rho}\bar{v}^\beta g_\beta + \overline{\rho' v'^\beta}g_\beta - \tfrac{1}{2}\bar{\sigma}^{\alpha\beta}\bar{e}_{\alpha\beta} - \tfrac{1}{2}\overline{\sigma'^{\alpha\beta}e'_{\alpha\beta}}\,. \tag{4.4.2}$$

Next, consider the equation of transfer of kinetic energy of the mean motion. Multiplying (4.3.5) scalarly by $\bar{\boldsymbol{v}}$, one obtains after some simple transformations

$$\frac{\partial}{\partial t}(\rho_0 E_s) = -\nabla_\beta\{\rho_0 E_s \bar{v}^\beta - \bar{p}^{\alpha\beta}\bar{v}_\alpha - R^{\alpha\beta}\bar{v}_\alpha\} + \bar{\rho}\bar{v}^\beta g_\beta - \tfrac{1}{2}\bar{\sigma}^{\alpha\beta}\bar{e}_{\alpha\beta} - \tfrac{1}{2}R^{\alpha\beta}\bar{e}_{\alpha\beta}\,. \tag{4.4.3}$$

Subtracting (4.4.3) from (4.4.2), one finds

$$\frac{\partial}{\partial t}(\rho_0 E_t) = -\nabla_\beta\{\rho_0 E_t \bar{v}^\beta + \tfrac{1}{2}\rho_0\overline{v'_\alpha v'^\alpha v'^\beta} - \overline{p'^{\alpha\beta}v'_\alpha}\} + \tfrac{1}{2}R^{\alpha\beta}\bar{e}_{\alpha\beta} + \overline{\rho' v'^\beta}g_\beta\,.$$
$$- \tfrac{1}{2}\overline{\sigma'^{\alpha\beta}e'_{\alpha\beta}}\,. \tag{4.4.4}$$

Equations (4.4.3) and (4.4.4) represent the required equations of the transfer of quantities E_s and E_t.

Basic interest attaches to the expression for the amount of turbulent energy E_t which arises in unit time in unit volume:

$$\tfrac{1}{2}R^{\alpha\beta}\bar{e}_{\alpha\beta} + \overline{\rho' v'^\beta}g_\beta - \tfrac{1}{2}\overline{\sigma'^{\alpha\beta}e'_{\alpha\beta}}\,.$$

Note that the first term on the right-hand side of (4.4.4) describes only redistribution of turbulent energy inside a fluid volume (advection and diffusion of turbulent energy), and under conditions when the velocity pulsations at the boundary of a region vanish this term does not alter the quantity $\int_V \rho_0 E_t \mathrm{d}V$.

The expression $A = (\frac{1}{2})R^{\alpha\beta}\bar{e}_{\alpha\beta}$ enters with opposite signs into equations (4.4.3) and (4.4.4); it describes mutual conversion of the kinetic energies of the mean and pulsating motions. As a rule, this expression is positive, and then the pulsating motion *feeds* on the energy of the mean motion. When $A < 0$, energy transfers from the pulsating motion to the mean motion. Generally speaking, such a case may occur and certain experimental data in support of this fact are available.

The expression $B = \overline{\rho' v'^\beta}g_\beta$ describes the mean work done by the Archimedes forces. Thus, in a stratified fluid, there occurs mutual conversion of potential and turbulent energies. For stable stratification, ρ' and $v'^{\beta}g_{\beta}$ have opposite signs and $B < 0$, which leads to decay of turbulence. For unstable stratification, in contrast, $B > 0$, and the energy of turbulence grows.

The expression $(\frac{1}{2})\overline{\sigma'^{\alpha\beta}e'_{\alpha\beta}}$ describes dissipation of turbulent energy due to viscous forces. By (2.9.1) and (4.2.8), one readily obtains

$$\overline{\sigma'^{\alpha\beta}e'_{\alpha\beta}} = \nu_l\rho_0(\overline{e'^{\alpha\beta}e'_{\alpha\beta}}) > 0 \; .$$

Introduce the dynamic Richardson number

$$R_f = -\frac{B}{A}. \tag{4.4.5}$$

It makes it possible to derive a simple criterion for the development of turbulence (however, rather complex). If at the boundaries of a region there occurs no diffusive flux of turbulent energy and $A > 0$, then for maintenance of already existing turbulence one must have

$$R_f < 1 \; . \tag{4.4.6}$$

A simple expression for R_f may be stated for the case of plane parallel mean motion $[\overline{u} = \overline{u}(z), \overline{v} = 0, \overline{w} = 0, \partial/\partial x = \partial/\partial y = 0]$:

$$R_f = \frac{g\overline{\rho' w'}}{\rho_0\overline{u'w'}\dfrac{\partial\overline{u}}{\partial z}}. \tag{4.4.7}$$

The z-axis is here directed downwards.

4.5 THE BASIC EQUATIONS IN SPHERICAL COORDINATES

As has already been noted in § 3.1, the convenience of the system of spherical coordinates λ, φ and z is due to the fact that at each point the direction of the z-axis coincides with the direction of the gravitational force. Furthermore, it is seen that properties of motions along the vertical differ strongly from those of motions in the horizontal plane.

Employing (2.9.4), (2.9.9) and the approximation (3.1.2), equations (4.3.5)—(4.3.8) may be written in terms of spherical coordinates. It is convenient, first of all, to step over in the equation of motion (4.3.5) with the aid of (4.3.6) to the ordinary form of representation [(2.3.6)]. Omitting for the sake of convenience the averaging signs (bars above symbols), one has

$$\frac{\partial u}{\partial t} + \frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\varphi} + w\frac{\partial u}{\partial z} - uv\frac{\tan\varphi}{a} - 2\Omega w\cos\varphi - 2\Omega v\sin\varphi =$$

$$-\frac{1}{\rho_0}\frac{\partial p}{a\cos\varphi\partial\lambda} + F_\lambda \; , \tag{4.5.1}$$

$$\frac{\partial v}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+w\frac{\partial v}{\partial z}+u^2\frac{\tan\varphi}{a}+2\Omega u\sin\varphi=-\frac{1}{\rho_0}\frac{\partial p}{a\partial\varphi}+F_\varphi\,, \tag{4.5.2}$$

$$\frac{\partial w}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial w}{\partial\lambda}+\frac{v}{a}\frac{\partial w}{\partial\varphi}+w\frac{\partial w}{\partial z}+2\Omega u\cos\varphi=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}+g\frac{\rho}{\rho_0}+F_z\,, \tag{4.5.3}$$

$$\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)+\frac{\partial w}{\partial z}=0\,, \tag{4.5.4}$$

$$\frac{\partial s}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial s}{\partial\lambda}+\frac{v}{a}\frac{\partial s}{\partial\varphi}+w\frac{\partial s}{\partial z}=-\frac{1}{\rho_0}\operatorname{div}J_s\,, \tag{4.5.5}$$

$$\frac{\partial T}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial T}{\partial\lambda}+\frac{v}{a}\frac{\partial T}{\partial\varphi}+w\frac{\partial T}{\partial z}=-\frac{1}{c_p\rho_0}\operatorname{div}J_q\,, \tag{4.5.6}$$

where u, v and w are physical velocity components and z is measured downwards from the undisturbed ocean surface ($0\leqslant z<H$). Therefore the system of coordinates λ, φ, z turns out to be left-handed, recall that in a left-handed coordinate system $\epsilon^{123}=-1/|\sqrt{m}|$, etc.; cf. § A4 and (A9.13), and this fact has been taken into account when writing down the components of the Coriolis force. The quantities F_λ, F_φ, F_z are physical components of the friction force $\rho_0^{-1}\nabla_\alpha R^{\alpha\beta}$ [the terms $\nabla_\beta\bar{\sigma}^{\alpha\beta}$, div $\bar{I}_s$, div $\boldsymbol{q}$ in equations (4.3.5)–(4.3.8) have been omitted in correspondence with (4.3.10)]. By (A.9.5) and (3.1.2), one has

$$\begin{aligned}
\rho_0F_\lambda&=\frac{\partial R_{\lambda\lambda}}{a\cos\varphi\partial\lambda}+\frac{1}{a\cos^2\varphi}\frac{\partial}{\partial\varphi}(R_{\lambda\varphi}\cos^2\varphi)+\frac{\partial R_{\lambda z}}{\partial z},\\
\rho_0F_\varphi&=\frac{\partial R_{\varphi\lambda}}{a\cos\varphi\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(R_{\varphi\varphi}\cos\varphi)+\frac{\partial R_{\varphi z}}{\partial z}+\frac{R_{\lambda\lambda}}{a}\tan\varphi\,,\\
\rho_0F_z&=\frac{\partial R_{z\lambda}}{a\cos\varphi\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(R_{z\varphi}\cos\varphi)+\frac{\partial R_{zz}}{\partial z},
\end{aligned} \tag{4.5.7}$$

where $R_{\lambda\lambda}$, $R_{\lambda\varphi}$, ... are physical components of the tensor $R^{\alpha\beta}$.

In conclusion, expressions for the physical components of the strain rate tensor $e^{\alpha\beta}$ will be written down. By (A9.14) and (3.1.2), one finds

$$\begin{aligned}
&\tfrac{1}{2}e_{\lambda\lambda}=-\frac{v}{a}\tan\varphi+\frac{\partial u}{a\cos\varphi\partial\lambda}\,,\qquad \tfrac{1}{2}e_{\varphi\varphi}=\frac{\partial v}{a\partial\varphi}\,,\qquad \tfrac{1}{2}e_{zz}=\frac{\partial w}{\partial z}\,,\\
&e_{\lambda\varphi}=\frac{\partial v}{a\cos\varphi\partial\lambda}+\cos\varphi\frac{\partial}{a\partial\varphi}\left(\frac{u}{\cos\varphi}\right),\qquad e_{\lambda z}=\frac{\partial u}{\partial z}+\frac{\partial w}{a\cos\varphi\partial\lambda}\,,\\
&e_{\varphi z}=\frac{\partial v}{\partial z}+\frac{\partial w}{a\partial\varphi}\,.
\end{aligned} \tag{4.5.8}$$

4.6 COEFFICIENTS OF TURBULENT EXCHANGE

Consider now the question of how to close the system of equations for the averaged fields. A start will be made with a study of the simplest plane parallel motion $[\bar{u} = \bar{u}(z), \bar{v} = 0, \bar{w} = 0, \partial/\partial x = \partial/\partial y = 0]$. In this case, the system of equations (4.3.5)—(4.3.8) contains only the three characteristics of turbulent transfer $\overline{u'w'}$, $\overline{s'w'}$, $\overline{T'w'}$. By analogy with the laws of molecular transfer (2.9.1), (2.10.1) and (2.10.2), introduce the coefficients of turbulent transfer K, K_s and K_T (employing this analogy, naturally, assuming in (2.10.1) and (2.10.2) $\nabla_\alpha s$ and $\nabla_\alpha T$, respectively, to be the principal terms):

$$K = -\frac{\overline{u'w'}}{\partial\bar{u}/\partial z}, \qquad K_s = -\frac{\overline{s'w'}}{\partial\bar{s}/\partial z}, \qquad K_T = -\frac{\overline{T'w'}}{\partial\bar{T}/\partial z}. \tag{4.6.1}$$

The exchange coefficients K, K_s and K_T may be considered to be new characteristics of turbulence. Linking by means of the averaging hypothesis the coefficients K, K_s and K_T to the parameters of the mean motion, one can arrive at a closed system of equations for the determination of $\bar{\boldsymbol{v}}$, $\bar{T}$, $\bar{p}$ and $\bar{s}$. This is the most commonly used method of closing the system of equations of mean motion. In the sequel, this method will be adopted without giving consideration to any other method of closing the equations. Note that very often it is easier to introduce one or the other assumption relating to K, K_s and K_T rather than to $\overline{u'w'}$, $\overline{s'w'}$ and $\overline{T'w'}$. In essence, this is the reason for the introduction of the exchange coefficients.

Proceed now to the general case of three-dimensional motion and the study of the question of formal determination of the coefficients of turbulent transfer.

For the sake of simplicity, begin with turbulent diffusion. Assuming, in analogy with molecular diffusion, that the components of the vector of turbulent flux density of salt $\rho_0\overline{s'v'_\alpha}$ depend linearly on the components of the vector $\nabla_\beta\bar{s}$, one arrives at the formula

$$\rho_0\overline{s'v'^\alpha} = -\rho_0 D^{\alpha\beta}\nabla_\beta\bar{s}\,. \tag{4.6.2}$$

The components of the second-order tensor $D^{\alpha\beta}$, in the general case, are referred to as coefficients of turbulent diffusion. Apparently, the tensor $D^{\alpha\beta}$ must be assumed to be symmetric. However, then six diffusion coefficients are introduced and (4.6.2) is found to be meaningless without any additional hypothesis regarding the structure of the tensor $D^{\alpha\beta}$. In fact, if at a given point of a flow the quantities $\rho_0\overline{s'v'^\alpha}$ and $\nabla_\beta\bar{s}$ are known, then it is impossible to determine from the three equations represented by (5.6.2) the six components of the tensor $D^{\alpha\beta}$. The whole matter is that the tensor $D^{\alpha\beta}$ defined formally by (4.6.2) is not a physical constant (as, for example, the coefficient of molecular diffusion), but depends, generally speaking, on a motion's character.

For a study of large-scale oceanic motions, one has a natural hypothesis relating to the axial symmetry of the tensor of the exchange coefficients $D^{\alpha\beta}$ about the vertical direction $\boldsymbol{k}$. In fact, large-scale motions in the ocean exhibit sharp differences between the properties of motions along the vertical and in the horizontal plane and the anisotropy of the tensor $D^{\alpha\beta}$ in the horizontal plane can be neglected completely. In general, note that it does not at all follow from the assumptions relating to axi-symmetry of the exchange coefficient tensor that such characteristics of turbulence as $R_{\alpha\beta}$, $\overline{s'v'_\alpha}$ and $\overline{T'v'_\alpha}$ must be axi-symmetric.

Using (A.5.2), rewrite the tensor $D^{\alpha\beta}$ in the form

$$D^{\alpha\beta} = D_L m^{\alpha\beta} + (D_H - D_L)\, k^\alpha k^\beta \,. \tag{4.6.3}$$

In this manner, two scalar exchange coefficients have been derived (which, as a rule, differ strongly from each other).

Employing (4.6.3), rewrite (4.6.2) in the coordinates λ, φ and z:

$$\overline{s'u'} = -D_L \frac{\partial \bar{s}}{a \cos\varphi \partial\lambda}\,, \qquad \overline{s'v'} = -D_L \frac{\partial \bar{s}}{a\partial\varphi}\,, \qquad \overline{s'w'} = -D_H \frac{\partial \bar{s}}{\partial z}\,. \tag{4.6.4}$$

The relations (4.6.4) impose a limitation on the form of the vector $\overline{s'v'}$ (three equations for the two unknowns D_L and D_H; however, it is important that they admit experimental verification). The coefficient D_L is referred to as coefficient of horizontal turbulent diffusion, the coefficient D_H as coefficient of vertical turbulent diffusion.

In a study of turbulent heat transfer, one may almost word for word repeat all the reasoning and write down for the relation between the vectors $\overline{T'v'_\alpha}$ and $\nabla_\beta \bar{T}$ formulae analogous to (4.6.4):

$$\overline{T'u'} = -K_L \frac{\partial \bar{T}}{a \cos\varphi \partial\lambda}\,, \qquad \overline{T'v'} = -K_L \frac{\partial \bar{T}}{a\partial\varphi}\,, \qquad \overline{T'w'} = -K_H \frac{\partial \bar{T}}{\partial z}\,, \tag{4.6.5}$$

where K_L and K_H are the coefficients of horizontal and vertical turbulent thermal conductivity, respectively.

Next, consider the coefficients of turbulent viscosity. By analogy with molecular viscosity, it is natural to assume that the tensor $R_{\alpha\beta}$ is a linear and, generally speaking, inhomogeneous function of the strain rate tensor $\bar{e}^{\gamma\delta}$. Further, assume that for $\bar{e}^{\gamma\delta} = 0$ the tensor $R_{\alpha\beta}$ reduces to an isotropic tensor. Then the general dependence of $R_{\alpha\beta}$ on $\bar{e}^{\gamma\delta}$ has the form

$$R_{\alpha\beta} = -\tfrac{2}{3}\rho_0 E_t m_{\alpha\beta} + \rho_0 K_{\alpha\beta\gamma\delta}\bar{e}^{\gamma\delta}\,. \tag{4.6.6}$$

The components of the fourth-order tensor $K_{\alpha\beta\gamma\delta}$ are called coefficients of turbulent viscosity. Since $R_{\alpha\beta}$ and $\bar{e}^{\gamma\delta}$ are symmetric, one has

$$K_{\alpha\beta\gamma\delta} = K_{\beta\alpha\gamma\delta}\,, \qquad K_{\alpha\beta\gamma\delta} = K_{\alpha\beta\delta\gamma}\,. \tag{4.6.7}$$

Contracting the tensor $R^{\alpha}_{.\beta}$, one obtains still the relation

$$K^{\alpha}_{.\alpha\gamma\delta}\bar{e}^{\gamma\delta} = 0 . \tag{4.6.8}$$

By (4.6.7) and (4.6.8), the tensor $K_{\alpha\beta\gamma\delta}$ consists of 30 different components; the relation (4.6.6) contains only 6 equations. Therefore, as in the case of turbulent diffusion, it is impossible to determine from given tensors $R_{\alpha\beta}$ and $\bar{e}^{\gamma\delta}$ all components of the tensor $K_{\alpha\beta\gamma\delta}$.

By analogy with problems of turbulent diffusion, assume the tensor $K_{\alpha\beta\gamma\delta}$ to be axisymmetric about the vertical direction. Then, using (A.5.2), one derives from (4.6.6) readily the relations

$$R_{\lambda\varphi} = R_{\varphi\lambda} = \rho_0 A_L \bar{e}_{\lambda\varphi} , \qquad R_{\lambda z} = R_{z\lambda} = \rho_0 A_H \bar{e}_{\lambda z} , \qquad R_{\varphi z} = R_{z\varphi} = \rho_0 A_H \bar{e}_{\varphi z} ,$$

$$R_{\lambda\lambda} = -\tfrac{2}{3}\rho_0 E_t + \rho_0 A_L \bar{e}_{\lambda\lambda} + \frac{\rho_0}{2}(A_L - A)\,\bar{e}_{zz} ,$$

$$R_{\varphi\varphi} = -\tfrac{2}{3}\rho_0 E_t + \rho_0 A_L \bar{e}_{\varphi\varphi} + \frac{\rho_0}{2}(A_L - A)\,\bar{e}_{zz} , \qquad R_{zz} = -\tfrac{2}{3}\rho_0 E_t + \rho_0 A \bar{e}_{zz} . \tag{4.6.9}$$

The formulae (4.6.9) contain the three coefficients of turbulent viscosity A_L, A_H and A and, in principle, admit experimental verification. As a rule, the coefficients A_L and A_H are called horizontal and vertical turbulent viscosities, respectively (and they differ strongly in orders of magnitude from each other).

Compute now the amount of energy $(1/\rho_0)\,\mathscr{A} = (\tfrac{1}{2}\rho_0)R^{\alpha\beta}\bar{e}_{\alpha\beta}$, transferred from the mean motion to the pulsating motion. By (4.6.9), after some simple transformations, one finds

$$\left(\frac{1}{\rho_0}\right)\mathscr{A} = A_L \bar{e}^2_{\lambda\varphi} + A_H(\bar{e}^2_{\lambda z} + \bar{e}^2_{\varphi z}) + \frac{3AA_L}{3A + A_L}\,\bar{e}^2_{\varphi\varphi} + \tfrac{1}{4}(A_L + 3A) \times \left[\bar{e}_{\lambda\lambda} + \frac{3A - A_L}{3A + A_L}\,\bar{e}_{\varphi\varphi}\right]^2 . \tag{4.6.10}$$

Hence it is seen that for $A_L > 0$, $A_H > 0$ and $A > 0$ energy will be transferred from the mean motion to the pulsating motion. Note that probably only for $\mathscr{A} > 0$ the concept of turbulent viscosity and the hypothesis of dependence of $R_{\alpha\beta}$ on $\bar{e}^{\gamma\delta}$ makes sense.

Thus, in the general case of large-scale motions, it may be assumed that turbulence is characterized by two coefficients of turbulent diffusion (and two thermal conductivities) and three coefficients of turbulent viscosity (E_t may be included in $\bar{p}$). Concrete hypotheses on the structure of these coefficients will be discussed later.

Revert now to the case of plane parallel motion. Using (4.6.4) and (4.6.9) and neglecting the dependence of ρ on p, rewrite (4.4.7) in the form ($D_H \simeq K_H$)

$$R_f = \frac{1}{\mathrm{Pr}_t}\,\mathrm{Ri}$$

where

$$\mathrm{Ri} = \frac{g}{\rho_0}\,\frac{\partial\bar{\rho}/\partial z}{(\partial\bar{u}/\partial z)^2}$$

is the ordinary Richardson number and $\mathrm{Pr}_t = A_H/D_H$ is the turbulent Prandtl number.

4.7 BOUNDARY CONDITIONS

Consider first the free surface of the ocean $z = \zeta(\lambda, \varphi, t)$. The question of formulation of boundary conditions at a random surface in itself is quite difficult; besides, one must take into account the reverse influence of irregular perturbations of this surface on the boundary layer of the atmosphere. Proceeding approximately, replace formally the fluxes $\sigma^{\alpha\beta}$, $\boldsymbol{I}_s$ and $\boldsymbol{q}$ by the turbulent fluxes $R^{\alpha\beta}$, $\boldsymbol{J}_s$ and $\boldsymbol{J}_q$ and assume the ordinary *laminar* boundary conditions to hold true for the averaged fields $\bar{v}$, $\bar{T}$, $\bar{p}$ and $\bar{s}$ at the mean free surface $z = \bar{\zeta}(\lambda, \varphi, t)$.

Consider now the dynamic boundary conditions. At the surface $z = \bar{\zeta}$, the forces $(-\bar{p}m_{\alpha\beta} + R_{\alpha\beta})n^\beta$ acting per unit area with normal n^β must be continuous. Since the inclinations of the level $\bar{\zeta}$ are very small, the external normal n^β in the coordinate system λ, φ, z is approximately $(0, 0, -1)$. Hence

$$-R_{\lambda z} = \rho_0\bar{\tau}_\lambda\,, \qquad -R_{\varphi z} = \rho_0\bar{\tau}_\varphi\,, \qquad \bar{p} = \bar{p}_a \text{ for } z = \bar{\zeta}\,, \tag{4.7.1}$$

where $\rho_0(\bar{\tau}_\lambda, \bar{\tau}_\varphi)$ are the components of the tangential wind and $\bar{p}_a$ is the atmospheric pressure. In writing down the last condition, the component R_{zz} has been neglected in comparison with $\bar{p}$.

In what follows, the averaging signs (bars above symbols) will be omitted. Using the smallness of ζ compared with H, condition(4.7.1) may be written down for $z = 0$. Clearly, $R_{\lambda z}(\zeta) \simeq R_{\lambda z}(0)$ and $R_{\varphi z}(\zeta) \simeq R_{\varphi z}(0)$. By (4.5.8), one has $e_{\lambda z} \simeq \partial u/\partial z$, $e_{\varphi z} \simeq \partial v/\partial z$ (it will be seen in the next section that $|w| << |u|$ for the motions under consideration). Finally, using (4.6.9) and the first formula (3.1.8), one obtains for $z = 0$

$$-A_H\frac{\partial u}{\partial z} = \tau_\lambda\,, \qquad -A_H\frac{\partial v}{\partial z} = \tau_\varphi\,, \qquad p = p_a - g\rho_0\zeta\,. \tag{4.7.2}$$

This is the final form of the dynamic boundary condition.

One may likewise write down the kinematic boundary condition (2.2.17) for $z = 0$. Since the difference of precipitation—evaporation represents roughly 50 cm per year, or 10^{-6} cm/sec, which is by two orders of magnitude smaller than characteristic values of vertical velocities in the ocean, the flux of fresh water in the first of the conditions (2.2.17) will be neglected. Thus

$$w = \frac{\partial \zeta}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial \zeta}{\partial \lambda} + \frac{v}{a} \frac{\partial \zeta}{\partial \varphi} \qquad \text{for } z = 0 . \tag{4.7.3}$$

Replacing the molecular diffusive flux I_s in the second conditions (2.2.17) by the turbulent flux J_s, taking into account the smallness of the inclinations of the sea level ζ, one has

$$J_{sz} = -bs \text{ for } z = 0 . \tag{4.7.4}$$

Using (4.6.4), this condition may be rewritten in the form

$$-D_H \frac{\partial s}{\partial z} = -(b/\rho_0)\, s \qquad \text{for } z = 0 . \tag{4.7.5}$$

As it had to be expected, condition (4.7.5) turns out to be homogeneous. This reflects the fact that the total amount of dissolved salt in the ocean remains unchanged; changes in salinity in the ocean are only caused by influx of fresh water.

In writing down the boundary condition for the temperature, one must set up the equation of heat balance at the ocean surface. Assume that all radiation (short wave from the Sun and long wave from the atmosphere) is absorbed by a very thin surface layer of the ocean and must be taken into consideration only in the boundary conditions (in fact, this was the reason why in the treatment of thermodynamics certain parameters which characterize the electromagnetic field in the fluid had not been introduced). Further, long-wave radiation of the ocean surface must be taken into account as well as heat of evaporation and turbulent heat flux from the atmosphere. Denote the total heat flux in unit time per unit area of the ocean surface by Q^*; it depends on the temperature of the ocean surface and atmospheric parameters. Taking into account (4.6.5), the boundary condition for the temperature may be written in the form

$$-K_H \frac{\partial T}{\partial z} = \frac{1}{c_p \rho_0} Q^* (T, \ldots) \qquad \text{for } z = 0 . \tag{4.7.6}$$

Naturally, this condition turns out to be inhomogeneous, since there exist external energy fluxes (for example, short-wave radiation from the Sun) which form the temperature field in the ocean.

The boundary condition at the ocean floor $z = H$ is obvious; this is conditions of no-slip and absence of fluxes of heat and salt:

$$u = v = w = 0 , \qquad \frac{\partial s}{\partial z} = \frac{\partial T}{\partial z} = 0 \qquad \text{for } z = H. \tag{4.7.7}$$

A generalization of the last two conditions (4.7.7) to the case of variable ocean depths presents no difficulties.

Conditions at the shores may be written down in an analogous manner.

Note that for the purpose of this study of the boundary conditions, the new unknown function $\zeta(\lambda, \varphi, t)$ has been introduced. This function does not depend on z and is determined from the boundary conditions, as will be seen below.

4.8 QUASI-STATIC APPROXIMATION

For studies of meso- and macro-scale processes in the ocean (vertical scales of order $H \sim 100$ m $\div$ 1 km, horizontal scales of order $L \sim 100 \div 1000$ km, characteristic frequencies of order 1 day^{-1} and less), vertical velocities of motions, as a rule, turn out to be small compared with horizontal velocities. In fact, assuming that all terms in (4.4.5) have the same order of magnitude, one obtains for w the upper bound: $W = (H/L)U$ or $W = 10^{-3}U$. The smallness of the vertical velocities in the ocean permits to write the equation of motion along the vertical (4.5.3) in the form

$$\frac{\partial p}{\partial z} = g\rho \ . \tag{4.8.1}$$

Equation (4.8.1) has been written down as if the fluid were at rest; therefore such an approximation is said to be quasi-static. The filtering properties of this approximation have been explained in detail in § 3.7. Naturally, it does not at all follow from this fact that (4.5.3) may be written approximately in the form (4.8.1) that the vertical velocity component w is altogether inessential. For example, the term $\partial w/\partial z$ proves to be very important in the equation of continuity (4.5.4).

Integrating (4.8.1) with respect to z from 0 to z and taking the last condition (4.7.2) into account, one has

$$p(\lambda, \varphi, z, t) = p_a - g\rho_0\zeta + g\int_0^z \rho \mathrm{d}z \ .$$

As a rule, the atmospheric pressure gradient is inessential:

$$\nabla_h p = -g\rho_0 \nabla_h\zeta + g\int_0^z \nabla_h\rho \mathrm{d}z \ . \tag{4.8.2}$$

This formula clearly displays the factors which form horizontal pressure gradients. It is easily seen that changes $\nabla_h p$ along z are only caused by density inhomogeneity of the ocean.

Since $|w| << |u|$, the term $2\Omega w \cos\varphi$ in (4.5.1) will be neglected (a traditional approximation when writing down Coriolis forces; cf. § 3.1). Then one obtains in quasi-static approximation instead of the equations of motion (4.5.1) and (4.5.2) the equations

$$\frac{\partial u}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a}\frac{\partial u}{\partial\varphi}+w\frac{\partial u}{\partial z}-uv\frac{\tan\varphi}{a}-2\Omega v\sin\varphi=g\frac{\partial\zeta}{a\cos\varphi\partial\lambda}$$

$$-\frac{g}{\rho_0}\int_0^z\frac{\partial\rho}{a\cos\varphi\partial\lambda}\,\mathrm{d}z+F_\lambda\,, \tag{4.8.3}$$

$$\frac{\partial v}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{v}{a}\frac{\partial v}{\partial\varphi}+w\frac{\partial v}{\partial z}+u^2\frac{\tan\varphi}{a}+2\Omega u\sin\varphi=g\frac{\partial\zeta}{a\partial\varphi}$$

$$-\frac{g}{\rho_0}\int_0^z\frac{\partial\rho}{a\partial\varphi}\,\mathrm{d}z+F_\varphi\,. \tag{4.8.4}$$

Assume that estimates of $\delta\rho/\rho$ (cf. § 4.2) for the averaged field are correct. Then it may be assumed for computation of $\nabla_h p$ that the density ρ does not depend on the pressure p. Therefore it is concluded from (4.8.3) and (4.8.4) that for meso-macro-scale motions under consideration the dependence of ρ on p is inessential.

This circumstance makes it possible to employ as potential vorticity the quantity $\omega_A\nabla_\rho$ (cf. §§ 4.1 and 4.2). It will only be shown here how one must *simplify* this quantity in order that it will not change during abiabatic motions of the liquid studied in quasi-static approximation (more exactly, for zero friction forces in (4.8.3) and (4.8.4) and under the condition $\mathrm{d}\rho/\mathrm{d}t = 0$).

First of all, expressions for the components of absolute vorticity in the left-handed coordinate system λ, φ, z will be written down. By (A.9.13) and (3.1.2), one has $\omega_A=(\omega_\lambda, \omega_\varphi+2\Omega\cos\varphi, \omega_z-2\Omega\sin\varphi)$, where

$$\omega_\lambda=\frac{\partial v}{\partial z}-\frac{\partial w}{a\partial\varphi},\qquad \omega_\varphi=\frac{\partial w}{a\cos\varphi\partial\lambda}-\frac{\partial u}{\partial z},$$

$$\omega_z=\frac{1}{\cos\varphi}\frac{\partial}{a\partial\varphi}(u\cos\varphi)-\frac{\partial v}{a\cos\varphi\partial\lambda}\,. \tag{4.8.5}$$

Since the vertical velocities are small in comparison with horizontal velocities, it is natural to introduce for the components of absolute vorticity approximate expressions of the form $\tilde\omega_A=(\tilde\omega_\lambda, \tilde\omega_\varphi, \tilde\omega_z-f)$, where

$$\tilde\omega_\lambda=\frac{\partial v}{\partial z},\qquad \tilde\omega_\varphi=-\frac{\partial u}{\partial z},\qquad \tilde\omega_z=\frac{1}{\cos\varphi}\frac{\partial}{a\partial\varphi}(u\cos\varphi)-\frac{\partial v}{a\cos\varphi\partial\lambda}\,. \tag{4.8.6}$$

Differentiating equations (4.8.4) and (4.8.3) with respect to z and using (4.5.4) and (4.8.1), one finds after some simple transformations

$$\frac{\partial\tilde\omega_\lambda}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial\tilde\omega_\lambda}{\partial\lambda}+\frac{v}{a}\frac{\partial\tilde\omega_\lambda}{\partial\varphi}+w\frac{\partial\tilde\omega_\lambda}{\partial z}-\frac{u\tilde\omega_\varphi}{a}\tan\varphi$$

$$-\left[\tilde{\omega}_\lambda \frac{\partial u}{a \cos \varphi \partial \lambda} + \tilde{\omega}_\varphi \frac{\partial u}{a \partial \varphi} + (\tilde{\omega}_z - f)\frac{\partial u}{\partial z} - \frac{\tilde{\omega}_\lambda v}{a} \tan \varphi\right] = -\frac{g}{\rho_0}\frac{\partial \rho}{a \partial \varphi} + \frac{\partial F_\varphi}{\partial z}, \tag{4.8.7}$$

$$\frac{\partial \tilde{\omega}_\varphi}{\partial t} + \frac{u}{a \cos \varphi}\frac{\partial \tilde{\omega}_\varphi}{\partial \lambda} + \frac{v}{a}\frac{\partial \tilde{\omega}_\varphi}{\partial \varphi} + w\frac{\partial \tilde{\omega}_\varphi}{\partial z} + \frac{u \tilde{\omega}_\lambda}{a} \tan \varphi$$

$$-\left[\tilde{\omega}_\lambda \frac{\partial v}{a \cos \varphi \partial \lambda} + \tilde{\omega}_\varphi \frac{\partial v}{a \partial \varphi} + (\tilde{\omega}_z - f)\frac{\partial v}{\partial z} + \frac{\tilde{\omega}_\lambda u}{a} \tan \varphi\right] = +\frac{g}{\rho_0}\frac{\partial \rho}{a \cos \varphi \partial \lambda} - \frac{\partial F_\lambda}{\partial z} \tag{4.8.8}$$

Further, differentiate (4.8.4) with respect to λ and (4.8.3) with respect to φ after multiplying it first by $\cos\varphi$. Subtracting the relations obtained term by term, one obtains after division of $\cos\varphi$ the equation for the vertical component of absolute vorticity $\omega_{Az} = \omega_z - f$. Obviously, the result may be written down directly, projecting equation (4.2.12) on to the z-axis. By (4.8.5), (4.8.6) and the fact that

$$(\omega_\lambda / a \cos \varphi)\, \partial w/\partial\lambda + (\omega_\varphi / a)\, \partial w/\partial\varphi = (\tilde{\omega}_\lambda / a \cos \varphi)\, \partial w/\partial\lambda + (\tilde{\omega}_\varphi / a)\, \partial w/\partial\varphi\,,$$

one finds

$$\frac{\partial \tilde{\omega}_{Az}}{\partial t} + \frac{u}{a \cos \varphi}\frac{\partial \tilde{\omega}_{Az}}{\partial \lambda} + \frac{v}{a}\frac{\partial \tilde{\omega}_{Az}}{\partial \varphi} + w\frac{\partial \tilde{\omega}_{Az}}{\partial z} - \left[\tilde{\omega}_{A\lambda}\frac{\partial w}{a \cos \varphi \partial \lambda} + \tilde{\omega}_{A\varphi}\frac{\partial w}{a \partial \varphi} + \tilde{\omega}_{Az}\frac{\partial w}{\partial z}\right]$$

$$= \frac{1}{\cos \varphi}\frac{\partial}{a \partial \varphi}(F_\lambda \cos \varphi) - \frac{\partial F_\varphi}{a \cos \varphi \partial \lambda}\,. \tag{4.8.9}$$

Equations (4.8.7) through (4.8.9) represent nothing else but projections of Fridman's approximate equations on to the axes λ, φ and z. It is interesting to note that equations (4.8.7)—(4.8.9) may be obtained from Fridman's general equation (4.2.12) by replacing in it the exact expression for the absolute vorticity $\boldsymbol{\omega}_A$ by the approximate expression $\tilde{\boldsymbol{\omega}}_A$ [cf. (A.9.7) and the approximation (3.1.2)]. Furthermore, apply the operation $\nabla\cdot$ to the equation $\mathrm{d}\rho/\mathrm{d}t = 0$; it may be shown by straight differentiation that in spherical coordinates with the approximation (3.1.2) the physical components of the vectors $\nabla_\kappa(\mathrm{d}\rho/\mathrm{d}t)$ and $\mathrm{d}(\nabla_\kappa\rho)/\mathrm{d}t + \nabla_\kappa v^\alpha \nabla_\alpha \rho$ are equal to each other [generally speaking, this is not obvious; cf. derivation of (4.1.4) and the note preceding (3.1.10)]. Then, setting F_λ and F_φ equal to zero and using an identity of the type (4.1.5), one obtains directly

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\boldsymbol{\omega}}_A \nabla\rho) = 0\,. \tag{4.8.10}$$

This is a sought form of the equation of the potential vorticity in quasi-static approximation.

In conclusion, the equation for the kinetic energy will be written down in quasi-static approximation. Reverting first to pressure p, multiply (4.8.3) and (4.8.4) by u and v, respectively. Adding the results and employing (4.5.4) and (4.8.1), one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_0\frac{u^2+v^2}{2}\right) = -\mathrm{div}(p\boldsymbol{v}) + g\rho w + \rho_0(uF_\lambda + vF_\varphi) \ . \tag{4.8.11}$$

4.9 GEOSTROPHIC MOTION

As a rule, in the open ocean, frictional forces and accelerations of fluid particles are small and the horizontal pressure gradient $\nabla_h p$ is balanced by the Coriolis force. By (4.8.2)–(4.8.4), one has

$$-2\Omega v \sin\varphi = -\frac{1}{\rho_0}\frac{\partial p}{a\cos\varphi\partial\lambda} \ , \qquad +2\Omega u \sin\varphi = -\frac{1}{\rho_0}\frac{\partial p}{a\partial\varphi} \ . \tag{4.9.1}$$

Motions described by equations (4.9.1) are called geostrophic. Such motions are very uncommon. For example, it is seen from (4.9.1) that the horizontal velocity will be directed along the isobar or perpendicular to $\nabla_h p$. It has been shown in § 3.9 that large- and meso-scale Rossby waves may be assumed to be geostrophic in first approximation.

Compute the vertical velocity for geostrophic motion. Equations (4.9.1) and (4.5.4) readily yield

$$f\frac{\partial w}{\partial z} = \beta v \ . \tag{4.9.2}$$

This equation demonstrates the effect of the Earth's spherical shape. If the Earth were plane, then $\beta = 0$ and for geostrophic motion $\partial w/\partial z = 0$, which would be in sharp contradiction to observations in the ocean.

Equation (4.9.2) yields a more exact estimate of W than that derived in § 4.8. One has

$$W = \frac{\beta H}{f}U = 2\cdot 10^{-4}\ U \ , \tag{4.9.3}$$

since $\beta = 2\cdot 10^{-13}\ \mathrm{cm}^{-1}\ \mathrm{sec}^{-1}$, $f = 10^{-4}\ \mathrm{sec}^{-1}$, $H \simeq 10^5$ cm.

Differentiate (4.9.2) with respect to z. Using (4.8.1) and (4.9.1), one obtains

$$\frac{\partial^2 w}{\partial z^2} = \frac{g\beta}{\rho_0 f^2}\frac{\partial\rho}{a\cos\varphi\partial\lambda} \ . \tag{4.9.4}$$

It is interesting to note that in the ocean $\partial^2 w/\partial z^2$ is linked to the zonal density gradient. It is a fact that on the surface of the ocean, on the whole, density varies in meridional directions. By (4.9.2), the zonal density gradient is very essential in the deep parts of the ocean.

Differentiating (4.9.1) with respect to z and using (4.8.1), one finds

$$f\frac{\partial u}{\partial z} = -\frac{g}{\rho_0}\frac{\partial \rho}{a\partial \varphi}, \qquad f\frac{\partial v}{\partial z} = \frac{g}{\rho_0}\frac{\partial \rho}{a\cos\varphi\partial\lambda}. \tag{4.9.5}$$

These formulae form the basis of the dynamic method for the computation of current velocities u and v from the observed temperature and salinity fields, which is used widely in oceanographical practice. In fact, it has already been stated that for evaluation of $\nabla_h \rho$ the density may be assumed to be a function of T and s only. Therefore, one may compute from observed values of $\nabla_h T$ and $\nabla_h s$ the quantity $\nabla_h \rho$ and from (4.9.5) the quantity $\partial u/\partial z$ and $\partial v/\partial z$. Thus, in order to find u and v on a given vertical, one has to know their values only at one level. For a construction of current charts, this reference level is chosen, as a rule, on the basis of certain supplementary considerations (usually, of an empirical character). It must be emphasized that the dynamic method is an indirect method for the computation of currents, and that the necessity of knowing certain reference values of u and v is its essential deficiency.

The geostrophic character of the motion of the water in the open ocean is a fundamental fact. In what follows, different simple models will be employed to study in what sense and under what conditions the motion in the ocean is approximately geostrophic. At this stage, a brief reference is only made to Fig. 4.2 which demonstrates well the agreement between instrumental observations and geostrophic current velocities.

Fig. 4.2. Graph of geostrophic velocity and direct measurement of velocity (from observations by means of floats, given by points, according to Swallow [121]. Instrumental observations of velocities represent mean values over two days. Reference level for geostrophic velocities was chosen in such a manner as to ensure best possible agreement with instrumental observations.

COMMENT ON CHAPTER 4

§ 4.1. Equation (4.1.2) has been proposed in the work of Fridman, cf. [25, § 18 and Appendix § 2]. The explanation of the effect of stretching of vortex lines has been given by Lamb [58, p. 257]; on the discussion of the generation of vorticity in a fluid, cf., for example, Yih [134, p. 78]. Formula (4.1.6) was obtained by Ertel [17]; equations (4.1.7) and (4.1.8) have been derived by Obukhov [89].

§ 4.2. Related questions are discussed in Landau and Lifshits [59, §§ 10, 56] and Phillips [98, Chap. II].

§§ 4.3, 4.4. On the whole these results follow from Monin and Yaglom [42, §§ 3.1–3.3, 5.1, 5.8, 6.2, 6.3].

§ 4.6. Cf. Kamenkovich [45].

§ 4.8. On the derivation of equation (4.8.10), cf., for example, Monin [76], Phillips [94] and Welander [130].

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CHAPTER 5

EKMAN THEORY, WIND-DRIVEN CURRENTS IN A HOMOGENEOUS OCEAN

5.1 PURE DRIFT CURRENT

Consider the single exact solution of the basic equations. Let an unbounded plane layer of fluid of depth H rotate about the z-axis with constant angular velocity Ω. On the surface of the fluid acts a tangential wind stress which is independent of time and horizontal coordinates. It is natural to assume that all basic parameters of the motion likewise do not depend on time and horizontal coordinates. Then the equations of motion along the x- and y-axes and corresponding boundary conditions follow from relations of the type (4.5.1), (4.5.2), (4.5.7), (4.6.9), (4.7.2), and (4.7.7):

$$-2\Omega v = A_H \frac{\partial^2 u}{\partial z^2}\,, \tag{5.1.1}$$

$$2\Omega u = A_H \frac{\partial^2 v}{\partial z^2}\,, \tag{5.1.2}$$

$$A_H \frac{\partial u}{\partial z} = -\tau_x, \qquad A_H \frac{\partial v}{\partial z} = -\tau_y \qquad \text{for } z = 0\,, \tag{5.1.3}$$

$$u = v = 0 \qquad \text{for } z = H \tag{5.1.4}$$

As the simplest hypothesis, let the coefficient of vertical turbulent exchange A_H in equations (5.1.1) and (5.1.2) be constant. It is readily seen that $w \equiv 0$ and the dynamic equations are separated completely from the equations for temperature and salinity.

Problem (5.1.1)—(5.1.4) describes so-called pure drift current in the ocean. It is convenient for the solution of this problem to introduce the complex velocity $\boldsymbol{U} = u + iv$ and the complex wind stress $\boldsymbol{\tau} = \tau_x + i\tau_y$ and rewrite (5.1.1)—(5.1.4) in the form

$$2i\Omega \boldsymbol{U} = A_H \frac{\partial^2 \boldsymbol{U}}{\partial z^2}\,; \tag{5.1.5}$$

$$A_H \frac{\partial \boldsymbol{U}}{\partial z} = -\tau \qquad \text{for } z = 0\,, \qquad \boldsymbol{U} = 0 \qquad \text{for } z = H\,. \tag{5.1.6}$$

The solution of this problem is readily found:

$$U = \frac{\tau}{\mu A_H} \frac{\sinh \mu(H-z)}{\cosh \mu H}, \qquad \mu = \sqrt{\frac{\Omega}{A_H}}(1+i)\,. \tag{5.1.7}$$

It is seen that during an analysis of pure drift current there arises an internal scale of length $h_E = \sqrt{A_H/\Omega}$. Introduce the Ekman number E

$$E = \frac{h_E}{H}\,. \tag{5.1.8}$$

Let the depth H of the ocean be small and $E >> 1 (h_E >> H)$. Then (5.1.7) assumes the form

$$U \simeq \frac{\tau}{A_H}(H-z)\,. \tag{5.1.9}$$

This is the ordinary formula for plane parallel Couette flow. The rotation of the fluid in this case does not exert an appreciable effect on the current.

Consider now another extreme case of a deep ocean: $E << 1 (h_E << H)$. Then

$$U \simeq \frac{\tau}{\mu A_H} e^{-\mu z}\,, \quad \text{or:}$$

$$U = \frac{\tau}{\sqrt{2 A_H \Omega}} \exp\left\{-\frac{z}{h_E} - \mathrm{i}\left(\frac{\pi}{4} + \frac{z}{h_E}\right)\right\}. \tag{5.1.10}$$

It follows from (5.1.10) that for $z = 0$ the vector of the surface velocity U_s is inclined at an angle $\pi/4$ in a clockwise direction to the vector τ. With increasing z, the modulus of the vector U decreases exponentially, while the vector itself rotates in a clockwise direction. At depth $z = \pi h_E = D$, the vector U will already point in the opposite direction to U_s. The solution (5.1.10) is often referred to as Ekman spiral.

Formula (5.1.10) could have been obtained directly, by solving Problem (5.1.5) and (5.1.6) for the infinitely deep ocean.

The physical significance of the scale h_E (or D) is now completely clear. It characterizes the depth of the layer of water involved in motion in the case of a deep ocean. The quantity h_E (or D) is referred to as Ekman depth friction.

The solution obtained depends parametrically on the coefficient of vertical turbulent exchange A_H. Since $A_H = \Omega h_E^2$ and, according to observations in the deep ocean, $h_E \simeq 100$ m, then $A_H \simeq 10^2$ cm^2/sec. It is customary to estimate by means of such a "fit" to observations of certain characteristics the coefficient of turbulent exchange.

Compute the total flux S of pure drift motion. Since in the ocean h_E

$<< H$, one has

$$S = \int_0^H U\mathrm{d}z = \frac{\tau}{2i\Omega}. \tag{5.1.11}$$

It is interesting to note that, if one assumes from the outset the ocean to be infinitely deep, then (5.1.11) may be obtained also for arbitrary A_H [for arbitrary A_H the right-hand side of (5.1.5) may be written in the form $\partial(A_H \partial U/\partial z)/\partial z$]. Then (5.1.5) must be integrated from 0 to ∞ and the first boundary condition (5.1.6) used.

5.2 THE BASIC EQUATIONS OF EKMAN THEORY

It has been seen that pure drift flow penetrates to a not large depth ($h_E \simeq 100$ m). This is caused by the fact that for such motion the horizontal pressure gradient $\nabla_h p$ vanishes (by strength of the assumptions of a shoreless ocean and uniformity of wind). Consider now the simplest model of wind flow in which the horizontal pressure gradient plays an essential role.

Thus, consider the Ekman model which is based on the following premises:

(1) The ocean is homogeneous in density ($\rho = \rho_0 =$ constant).

(2) Non-linear inertial terms in the equations of motion (4.5.1) and (4.5.2) are negligibly small.

(3) There exists essentially only vertical turbulent friction.

(4) The motion is steady.

Naturally, such a model cannot explain all phenomena observed in the ocean. However, its simplicity permits at least to explain qualitatively a number of important features of the dynamics of the real ocean (geostrophic flow in the open ocean, Ekman boundary layer, necessity of formation of intense boundary flow along western shores, etc.).

The basic equations of the model are obtained from the general relations of Chapter 4. First of all, the equations of motion (4.5.1) and (4.5.2) will be written down with due consideration to the initial premises of the model and relations (4.5.7), (4.6.9) and (4.8.2). The equations of motion will be augmented by the equation of conservation of mass in the form (4.5.4). Finally, the first two of the dynamic boundary conditions (4.7.2) will be written down as well as the kinematic condition (4.7.3) (omitting in it non-linear terms) and the boundary condition (4.7.7) at the bottom for the velocity $\boldsymbol{v}$. Thus one obtains

$$-fv = g\frac{\partial\zeta}{a\cos\varphi\partial\lambda} + A_H\frac{\partial^2 u}{\partial z^2}, \tag{5.2.1}$$

$$+fu = g\frac{\partial\zeta}{a\partial\varphi} + A_H\frac{\partial^2 v}{\partial z^2}, \tag{5.2.2}$$

$$\frac{\partial u}{a\cos\varphi\partial\lambda}+\frac{1}{\cos\varphi}\frac{\partial}{a\partial\varphi}(v\cos\varphi)+\frac{\partial w}{\partial z}=0\,, \tag{5.2.3}$$

$$A_H\frac{\partial u}{\partial z}=-\tau_\lambda\,,\qquad A_H\frac{\partial v}{\partial z}=-\tau_\varphi\,,\qquad w=0\qquad \text{for } z=0\,, \tag{5.2.4}$$

$$u=v=w=0\qquad \text{for } z=H. \tag{5.2.5}$$

The boundary conditions over the horizontal will be posed later.

The unknown functions of Problems (5.2.1)–(5.2.5) are $u(\lambda, \varphi, z)$, $v(\lambda, \varphi, z)$, $w(\lambda, \varphi, z)$ and $\zeta(\lambda, \varphi)$. This is the mathematical formulation of the Ekman model of wind-induced flow in a homogeneous ocean.

The problem formulated will now be solved. Equations (5.2.1) and (5.2.2) and the first two equations (5.2.4) and (5.2.5) may be rewritten in the form

$$if\boldsymbol{U}=g\boldsymbol{P}+A_H\frac{\partial^2\boldsymbol{U}}{\partial z^2}\,,\qquad A_H\frac{\partial\boldsymbol{U}}{\partial z}=-\boldsymbol{\tau}\qquad \text{for } z=0\,,\qquad \boldsymbol{U}=0\qquad \text{for } z=H\,, \tag{5.2.6}$$

where

$$\boldsymbol{U}=u+iv\,,\qquad \boldsymbol{P}=\frac{\partial\zeta}{a\cos\varphi\partial\lambda}+i\frac{\partial\zeta}{a\partial\varphi}\,,\qquad \boldsymbol{\tau}=\tau_\lambda+i\tau_\varphi\,. \tag{5.2.7}$$

The solution $\boldsymbol{U}$ of Problem (5.2.6) is readily found ($\boldsymbol{P}$ does not depend on z). Restricting consideration to the northern hemisphere ($f > 0$), one has

$$\boldsymbol{U}=\frac{\boldsymbol{\tau}}{\mu A_H}\frac{\sinh\mu(H-z)}{\cosh\mu H}+\frac{g\boldsymbol{P}}{if}\left(1-\frac{\cosh\mu z}{\cosh\mu H}\right),\qquad \mu=\sqrt{\frac{f}{2A_H}}\,(1+i)\,. \tag{5.2.8}$$

Naturally, this general expression contains a pure drift component of the motion [compare the first term in (5.2.8) with (5.1.7)], but now $\boldsymbol{\tau}$ is already variable and instead of Ω one has $\Omega\sin\varphi$; the second term in (5.2.8) will be referred to as gradient component of the motion. If the pure drift component is a result of the direct action of the wind on the ocean surface, then the gradient component arises from the presence of the horizontal pressure gradient $\nabla_h p$. Obviously, the reason for the development of the pressure gradient is the following: Space inhomogeneity of the wind and presence of shores lead to piling up and removal of water which is accompanied by deflection of the water level from its unperturbed position.

From (5.2.3) and the last condition (5.2.4), one finds

$$w=-\int_0^z \operatorname{div}_h\boldsymbol{U}dz=-\operatorname{div}_h\Big(\int_0^z\boldsymbol{U}dz\Big)=-\operatorname{div}_h\left\{\frac{\boldsymbol{\tau}}{if}\frac{\cosh\mu H-\cosh\mu(H-z)}{\cosh\mu H}\right.$$

$$\left.+\frac{g\boldsymbol{P}}{if}\left(z-\frac{1}{\mu}\frac{\sinh\mu z}{\cosh\mu H}\right)\right\}, \tag{5.2.9}$$

with the new notation

$$\operatorname{div}_h \boldsymbol{Z} = \operatorname{div}_h(\operatorname{Re} Z, \operatorname{Im} Z) . \tag{5.2.10}$$

Only one of the basic relations of the theory (5.2.1)—(5.2.5) has not been used, namely $w(\lambda, \varphi, H) = 0$. Since the expression (5.2.9) for w contains second derivatives of the level ζ, the condition $w(\lambda, \varphi, H) = 0$ yields the second-order equation required for the determination of $\zeta(\lambda, \varphi)$. For the solution of this equation one must have a "two-dimensional" (i.e., independent of z) boundary condition along the horizontal. Clearly, at shores of a basin, one may not demand fulfillment either of a condition of non-slip or of a no-flow condition (since non-linear inertial terms and horizontal turbulent exchange were neglected in the equations of motion). However, assuming the shores to be sheer cliffs, one may require fulfillment of an integral no-flow condition

$$(\boldsymbol{S}, \boldsymbol{n})|_{\Gamma} = 0 , \tag{5.2.11}$$

where $\boldsymbol{n}$ is the normal to the shoreline Γ and $\boldsymbol{S}$ is the total flow vector

$$S_\lambda = \int_0^H u \mathrm{d}z , \qquad S_\varphi = \int_0^H v \mathrm{d}z . \tag{5.2.12}$$

The vector $\boldsymbol{S}$ is easily found from (5.2.8). Integrating, one obtains

$$\boldsymbol{S} = \frac{\tau}{if} \frac{\cosh \mu H - 1}{\cosh \mu H} + \frac{g\boldsymbol{P}}{if}\left(H - \frac{1}{\mu} \tanh \mu H\right) . \tag{5.2.13}$$

Separating here real and imaginary parts and constructing the expression (5.2.11), one arrives at the required boundary condition for the determination of $\zeta(\lambda, \varphi)$. After determination of ζ, formulae (5.2.8) and (5.2.9) permit to find the velocity vector $\boldsymbol{v}$ at all horizons, i.e., the complete solution of problem (5.2.1)—(5.2.5) has been constructed.

This method of solution of problem (5.2.1)—(5.2.5) and (5.2.11) is not always the most convenient. Sometimes it is expedient not to write down directly the equation for ζ, but to investigate the system of equations for S_λ, S_φ, ζ

$$S_\lambda = A \frac{\partial \zeta}{a \cos \varphi \partial \lambda} - B \frac{\partial \zeta}{a \partial \varphi} + F\tau_\lambda - G\tau_\varphi , \tag{5.2.14}$$

$$S_\varphi = B \frac{\partial \zeta}{a \cos \varphi \partial \lambda} + A \frac{\partial \zeta}{a \partial \varphi} + G\tau_\lambda + F\tau_\varphi . \tag{5.2.15}$$

$$\frac{\partial S_\lambda}{\partial \lambda} + \frac{\partial}{\partial \varphi} (S_\varphi \cos \varphi) = 0 , \tag{5.2.16}$$

for the boundary condition (5.2.11).

Equations (5.2.14) and (5.2.15) are obtained from (5.2.13) after separation of real and imaginary parts. Therefore the functions A, B, F and G are known. Their expressions in the general case will not be written down; for deep and shallow seas, expressions for A, B, F and G can be obtained from (5.3.6), (5.3.7), (5.3.10) and (5.3.11). Equation (5.2.16) is derived by integration of (5.2.3) taking into account the last conditions (5.2.4) and (5.2.5).

Thus, the study of the initial system of equations (5.2.1)—(5.2.3) for the conditions (5.2.4), (5.2.5) and (5.2.11) has been reduced to a study of the "two-dimensional" system of equations (5.2.14)—(5.2.16) for the condition (5.2.11).

5.3 VERTICAL STRUCTURE OF THE FLOW; EKMAN BOUNDARY LAYERS

As in § 5.1, introduce the friction depth $h_E = \sqrt{A_H/\Omega}$ and the Ekman number E [cf. (5.1.8)]. Consider two limiting cases:

(1) $E << 1$ ($h_E << H$): deep ocean

In this case $|\mu H| \sim 1/E >> 1$ and one has

$$\frac{\sinh \mu(H-z)}{\cosh \mu H} \simeq e^{-\mu z}, \qquad \frac{\cosh \mu z}{\cosh \mu H} \simeq e^{-\mu(H-z)}, \qquad \frac{\cosh \mu(H-z)}{\cosh \mu H} \simeq e^{-\mu z},$$

$$\frac{\sinh \mu z}{\cosh \mu H} \simeq e^{-\mu(H-z)}, \qquad 1-\frac{1}{\cosh \mu H} \simeq 1,$$

$$\frac{1}{\mu H} \tanh \mu H \simeq 1 - i\frac{h_E}{2H} \sin^{1/2}\varphi .$$

Using these relations, rewrite (5.2.8) and (5.2.9) in the form

$$U = \frac{\tau}{\mu A_H} e^{-\mu z} + \frac{gP}{if} - \frac{gP}{if} e^{-\mu(H-z)}, \tag{5.3.1}$$

$$w = \mathrm{div}_h\left(\frac{\tau}{if} e^{-\mu z}\right) + \mathrm{rot}_z\left(\frac{\tau}{f}\right) - \frac{g\beta z}{f^2}\frac{\partial\zeta}{a\cos\varphi\partial\lambda} + \mathrm{div}_h\left(\frac{gP}{if\mu} e^{-\mu(H-z)}\right), \tag{5.3.2}$$

where $\beta = (2\Omega/a)\cos\varphi$.

It is easily seen that over the main deeps of the ocean (outside boundary layers of width of order h_E located at the ocean surface and floor) formula (5.3.1) may be rewritten with great accuracy in the form

$$U = \frac{gP}{if}. \tag{5.3.3}$$

This formula describes geostrophic flow. It is not difficult to understand why this formula does not hold near the surface and bottom of the ocean. In fact, the formula for geostrophic flow is obtained from (5.2.6) by discarding in these equations frictional forces; however, these forces play an essential role at the surface of the ocean (otherwise it is not possible to transfer momentum from the atmosphere to the ocean) and at the ocean floor (condition of no slip). Mathematically this means that the solution (5.3.3) cannot satisfy the boundary conditions (5.2.6) of the problem.

Thus, it is clear why the solution (5.3.1) of Problem (5.2.6) contains, besides the geostrophic term, yet two terms: $\boldsymbol{U}_d = (\tau/\mu A_H) \exp(-\mu z)$ and $\boldsymbol{U}_b = -(g\boldsymbol{P}/if) \exp[-\mu(H-z)]$. The first of these terms describes pure drift flow, the second bottom flow. The boundary layer within the limits of which the term $\boldsymbol{U}_d$ (or $\boldsymbol{U}_b$, respectively) is essential is referred to as Ekman surface (or bottom, respectively) friction layer.

The analogue of (5.3.1) is (5.3.2). Here one has in the interior of the ocean outside the boundary layers (recalling the z-axis points downwards)

$$w_g = \mathrm{rot}_z\left(\frac{\tau}{f}\right) - \frac{g\beta z}{f^2}\frac{\partial\zeta}{a\cos\varphi\partial\lambda}, \tag{5.3.4}$$

$$\mathrm{rot}_z\left(\frac{\tau}{f}\right) = \frac{1}{\cos\varphi}\frac{\partial}{a\partial\varphi}\left(\frac{\tau_\lambda}{f}\cos\varphi\right) - \frac{\partial\tau_\varphi}{af\cos\varphi\partial\lambda}.$$

Analogous terms in (5.3.2) give "corrections" which are required for the description of vertical velocities in the surface and bottom boundary layers.

Note the following important circumstances. Rewrite (5.3.4) first in the form

$$w_g = -\mathrm{div}_h\left(\frac{\tau}{if}\right) - \int_0^z \mathrm{div}_h\left(\frac{g\boldsymbol{P}}{if}\right) gz \tag{5.3.5}$$

and pose the following question: Is it possible, after neglecting the friction terms in (5.2.1) and (5.2.2) [and some of the boundary conditions (5.2.4) and (5.2.5)], to determine all characteristics of geostrophic flow? It has been shown that the horizontal velocity components u_g and v_g may be determined in this manner. Furthermore, one may find $\partial w_g/\partial z$ from (5.2.3). However, as it follows from (5.3.5), neither of the boundary conditions (5.2.4) and (5.2.5) can yield w_g when $\partial w_g/\partial z$ is known; $w_g(0)$ is completely determined by the structure of the surface friction layer.

Write down equations (5.2.14) and (5.2.15) for the case of the deep ocean. One has

$$S_\lambda = \frac{1}{\sqrt{\sin\varphi}}\frac{gh_E}{2f}\frac{\partial\zeta}{a\cos\varphi\partial\lambda} + \frac{gH}{f}\frac{\partial\zeta}{a\partial\varphi} + \frac{\tau_\varphi}{f}, \tag{5.3.6}$$

$$S_\varphi = -\frac{gH}{f}\frac{\partial\zeta}{a\cos\varphi\partial\lambda} + \frac{1}{\sqrt{\sin\varphi}}\frac{gh_E}{2f}\frac{\partial\zeta}{a\partial\varphi} - \frac{\tau_\lambda}{f}. \tag{5.3.7}$$

(2) $E >> 1 (h_E >> H)$: shallow sea

In this case $|\mu H| \sim 1/E << 1$ and (5.2.8) and (5.2.9) may be rewritten in the form

$$U = \frac{\tau}{A_H}(H - z) + \frac{gP}{2A_H}(H^2 - z^2), \tag{5.3.8}$$

$$w = \mathrm{div}_h\left\{\frac{\tau}{A_H}\left(\frac{z^2}{2} - Hz\right) + \frac{gP}{2A_H}\left(\frac{z^3}{3} - H^2 z\right)\right\}. \tag{5.3.9}$$

The effect of rotation is completely missing from these formulae. They are readily obtained by omitting the Coriolis force during solution of Problem (5.2.6).

Equations (5.2.14) and (5.2.15) for the shallow sea have the form

$$S_\lambda = \frac{H^2}{2A_H}\tau_\lambda + \frac{1}{3}\frac{gH^3}{A_H}\frac{\partial\zeta}{a\cos\varphi\partial\lambda}, \tag{5.3.10}$$

$$S_\varphi = \frac{H^2}{2A_H}\tau_\varphi + \frac{1}{3}\frac{gH^3}{A_H}\frac{\partial\zeta}{a\partial\varphi}. \tag{5.3.11}$$

5.4 GENERAL METHOD OF SOLUTION OF PROBLEM

Consider now a general analytical method for the solution of the System (5.2.14)–(5.2.16) for the boundary conditions (5.2.11). On the basis of (5.2.16), it is convenient to introduce the total flow function ψ:

$$S_\varphi = \frac{\partial\psi}{a\cos\varphi\partial\lambda}, \qquad S_\lambda = -\frac{\partial\psi}{a\partial\varphi}. \tag{5.4.1}$$

Seek the solution of the System (5.2.14)–(5.2.16) in the multiply connected region D (a basin with islands). Let $\Gamma_0, \Gamma_1, ..., \Gamma_m$ be closed boundary contours of the region D (Γ_0 is the external contour, $\Gamma_1, ..., \Gamma_m$ are internal boundaries), τ the tangent vector to a contour with positive direction leaving D on the left and n the internal normal (Fig. 5.1).

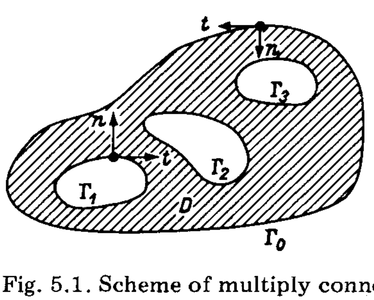


Fig. 5.1. Scheme of multiply connected region D.

First solve equations (5.2.14) and (5.2.15) with respect to the inclinations of the sea level $\partial\zeta/\partial\lambda$ and $\partial\zeta/\partial\varphi$:

$$g\frac{\partial\zeta}{a\cos\varphi\partial\lambda} = KS_\lambda - LS_\varphi + M\tau_\lambda - N\tau_\varphi\,, \tag{5.4.2}$$

$$g\frac{\partial\zeta}{a\partial\varphi} = LS_\lambda + KS_\varphi + N\tau_\lambda + M\tau_\varphi\,, \tag{5.4.3}$$

and then eliminate ζ by cross-differentiation. Substituting into the final result from (5.4.1), one finds

$$K\Delta_h\psi + \frac{\partial\psi}{a\cos\varphi\partial\lambda}\left(\frac{\partial K}{a\cos\varphi\partial\lambda} + \frac{\partial L}{a\partial\varphi}\right) + \frac{\partial\psi}{a\partial\varphi}\left(\frac{\partial K}{a\partial\varphi} - \frac{\partial L}{a\cos\varphi\partial\lambda}\right) = \mathrm{rot}_z(M\boldsymbol{\tau}) - \mathrm{div}_h(N\boldsymbol{\tau}), \tag{5.4.4}$$

where Δ_h is the Laplace operator on a sphere of radius a.

In (5.4.2)–(5.4.4), the functions K, L, M and N are known and formulae for their determination are readily written down. It is important that $K > 0$.

According to the boundary conditions (5.2.11), the function ψ must be constant on each boundary contour. Since the function ψ is determined exactly apart from an arbitrary constant, the boundary condition(5.2.11) may be rewritten in the form

$$\psi|_{\Gamma_0} = 0\,, \qquad \psi|_{\Gamma_k} = Q_k\,, \qquad k = 1, \ldots, m\,, \tag{5.4.5}$$

where the differences between the unknown contour constants Q_k have the significance of total transports of water through the straits. Obviously, all Q_k must be found during the process of solution for a given wind field $\boldsymbol{\tau}$.

Note that certain of the unknown contour constants do not arise, if the region of integration D is simply connected; equation (5.4.4) may be solved for the first of the conditions (5.4.5).

It is impossible to find the constants Q_k by considering only one of the equations (5.4.4). However, one must keep in mind that one requires a solution of the system of equations for S_λ, S_φ, ζ. Assume that one has succeeded in finding the function ψ for the problem under consideration. Then one finds S_φ and S_λ from (5.4.1) and, substituting them into (5.4.2) and (5.4.3), obtains a problem for the determination of ζ from known derivatives $\partial\zeta/\partial\lambda$ and $\partial\zeta/\partial\varphi$. Equation (5.4.4) which must be satisfied by ψ expresses that the two mixed second derivatives of ζ must be equal. However, one of these conditions is insufficient for the construction of a single-valued function ζ; in addition, one must fulfill the conditions

$$\oint_{\Gamma_k} d\zeta = 0\,, \qquad k = 1, \ldots, m\,. \tag{5.4.6}$$

Note that, if these conditions are satisfied, than an analogous relation holds true for any closed contour which lies entirely in D, including likewise the external contour Γ_0. In other words, one has only m (i.e., as many as there are internal contours) independent conditions (5.4.6).

Substituting in (5.4.6) the expressions (5.4.2) and (5.4.3) and using (5.4.1) and (5.2.11) one obtains

$$\oint_{\Gamma_k} K\frac{\partial\psi}{\partial n}\,\mathrm{d}s = \oint_{\Gamma_k} (M\tau_t - N\tau_n)\,\mathrm{d}s\ , \qquad k = 1\ , \ldots, m\ , \tag{5.4.7}$$

where $\mathrm{d}s$ is the element of arc of the contour Γ_k, τ_t and τ_n are projections of $\boldsymbol{\tau}$ on to the directions of the vectors $\boldsymbol{t}$ and $\boldsymbol{n}$, respectively.

Thus, the unknown function ψ must satisfy (5.4.4) and the boundary conditions (5.4.5) and (5.4.7). It will now be shown that the solution of such a problem exists and is unique.

Uniqueness will be proved first. It is sufficient for this purpose to show that the homogeneous problem (5.4.4), (5.4.5) and (5.4.7) has only a solution which is identically equal to zero. If this is not so, then, by strength of properties of elliptic equations, there exists a contour Γ_l on which ψ has a maximum value. However, then (cf. [91])

$$\oint_{\Gamma_l} K\frac{\partial\psi}{\partial n}\,\mathrm{d}s < 0$$

which contradicts the condition (5.4.7).

Now seek a solution of the problem in the form

$$\psi = \psi_0 + \sum_{k=1}^{m} Q_k\psi_k\ , \tag{5.4.8}$$

where $\psi_0, \psi_1, \ldots, \psi_m$ are auxiliary functions which are determined in the following manner. The function ψ_0 is a solution of the non-homogeneous equation (5.4.4); $\psi_0|_{\Gamma_j} = 0 (j = 0, 1, \ldots, m)$. All functions $\psi_k (k = 1, \ldots, m)$ satisfy the homogeneous equation (5.4.4); $\psi_k|_{\Gamma_j} = 0$ for $j \neq k$ and $\psi_k|_{\Gamma_k} = 1$. Since the Dirichlet problem for the elliptic equation (5.4.4) has a unique solution, the functions $\psi_1, \psi_2, \ldots, \psi_m$ are completely defined.

Thus, the expression (5.4.8) satisfies (5.4.4) and the conditions (5.4.5). Substituting it into (5.4.7), one obtains m linear algebraic equations for the unknown contour constants $Q_1, \ldots, Q_m$. The determinant of this system depends only on the functions $\psi_1, \ldots, \psi_m$, which satisfy the homogeneous equation (5.4.4) for the boundary conditions stated above. Therefore it is clear that, by strength of the uniqueness of Problem (5.4.4), (5.4.5) and (5.4.7) proved above, this determinant is non-zero and all contour constants $Q_1, \ldots, Q_m$ can be determined.

This method of solution has been employed to compute the total transport of the Antarctic Circumpolar Current [36,43].

5.5 CERTAIN VERY SIMPLE SOLUTIONS

Consider a shallow sea. The coefficients in equations (5.4.2) and (5.4.3) are given by

$$K = \frac{3A_H}{H^3}\,, \qquad L = 0\,, \qquad M = -\frac{3}{2H}\,, \qquad N = 0\,. \tag{5.5.1}$$

Equation (5.4.4) for the total flow function ψ then becomes

$$\Delta_h \psi = -\frac{H^3}{2A_H}\,\mathrm{rot}_z(\tau/H)\,. \tag{5.5.2}$$

For the sake of simplicity, assume that the Earth's curvature may be neglected, and consider a basin of constant depth H of rectangular shape: $0 \leqslant x \leqslant L_x$; $0 \leqslant y \leqslant L_y$. Let the tangential wind stress be given by

$$\tau_x = \tau_0\left(c + \cos\frac{\pi y}{L_y}\right), \qquad \tau_y = 0\,, \qquad c = \text{constant}\,. \tag{5.5.3}$$

Seek the solution of equation (5.5.2), vanishing on the edges of the basin, in the form

$$\psi = X(x)\sin\frac{\pi y}{L_y}\,. \tag{5.5.4}$$

Substituting this expression into (5.5.2), one arrives at the following problem for the determination of the function X:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} - \kappa^2 X = \frac{H^2}{2A_H}\kappa\tau_0\,, \qquad X(0) = X(L_x) = 0\,, \qquad \kappa = \frac{\pi}{L_y}\,, \tag{5.5.5}$$

with the solution

$$X(x) = \frac{H^2\tau_0}{2\kappa A_H}\left[-1 + \frac{\mathrm{e}^{-\kappa L_x} - 1}{\mathrm{e}^{-\kappa L_x} - \mathrm{e}^{\kappa L_x}}\,\mathrm{e}^{\kappa x} - \frac{\mathrm{e}^{\kappa L_x} - 1}{\mathrm{e}^{-\kappa L_x} - \mathrm{e}^{\kappa L_x}}\,\mathrm{e}^{-\kappa x}\right]. \tag{5.5.6}$$

It is interesting to note the case when the basin is strongly elongated: $L_x >> L_y$. Clearly, formula (5.5.6) can then be written with sufficient accuracy in the form

$$X(x) = \frac{H^2\tau_0}{2\kappa A_H}\,[-1 + \mathrm{e}^{-x/\epsilon L_x} + \mathrm{e}^{-(L_x - x)/\epsilon L_x}]\,, \qquad \epsilon = \frac{L_y}{\pi L_x}\,, \tag{5.5.7}$$

and far away from the edges $x = 0$ and $x = L_x$ one finds

$$\psi = -\frac{H^2}{2A_H}\left(\int_0^y \tau_x \mathrm{d}y - y\bar{\tau}_x\right), \qquad S_x = \frac{H^2}{2A_H}(\tau_x - \bar{\tau}_x)\,,$$

$$S_y = 0\,, \qquad \frac{\partial\zeta}{\partial x} = -\frac{3}{2}\frac{\bar{\tau}_x}{gH}\,, \qquad \frac{\partial\zeta}{\partial y} = 0\,, \tag{5.5.8}$$

where

$$\bar{\tau}_x = \frac{1}{L_y} \int_0^{L_y} \tau_x \mathrm{d}y \ . \tag{5.5.9}$$

In essence, formulae (5.5.8) give the first terms of corresponding asymptotic series for expansions of the solution of the problem under consideration with respect to the small parameter ϵ. These formulae are valid not only for the wind field (5.5.3).

In fact, consider a rectangular basin elongated along the x-axis and a wind field of the form $\tau_x = \tau_x(y)$, $\tau_y = 0$, where $\tau_x(y)$ is an arbitrary function. Then, far away from the boundaries $x = 0$ and $x = L_x$, it may be assumed that in first approximation S_x, S_y, $\partial\zeta/\partial x$ and $\partial\zeta/\partial y$ do not depend on x. However, by (5.2.16) written in Cartesian coordinates, one has $\partial S_y/\partial y = 0$, and since $S_y = 0$ for $y = 0$ and $y = L_y$, it follows that $S_y \equiv 0$. It follows from (5.3.11) that $\partial\zeta/\partial y \equiv 0$, and hence that $\partial\zeta/\partial x$ = constant. This constant is readily determined from the condition

$$\int_0^{L_y} S_x \mathrm{d}y = 0 \ ,$$

which follows from (5.2.16) and the boundary conditions of the problem. As a result one obtains (5.5.8). Clearly, the method stated can also be applied to solve the problem in the case when τ_x and H are arbitrary functions of y.

The solution of (5.5.8) permits to find the vertical velocity distribution

$$u = \frac{3H\bar{\tau}_x}{4A_H}\left(1 - \frac{z}{H}\right)\frac{z_0 - z}{H} \ , \qquad v = 0 \ , \tag{5.5.10}$$

where

$$\frac{z_0(y)}{H} = \frac{4\tau_x(y) - 3\bar{\tau}_x}{3\bar{\tau}_x} \ . \tag{5.5.11}$$

This formula yields the value of the depth at which $u = 0$ (inside the

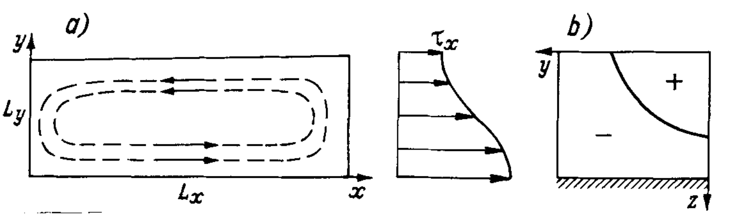


Fig. 5.2. Pattern of flow in shallow sea for variable wind stress. a — planform; tangential wind stress to the right; b — vertical section at the middle of the basin, plus (+) sign indicates that $u > 0$, minus (—) sign that $u < 0$.

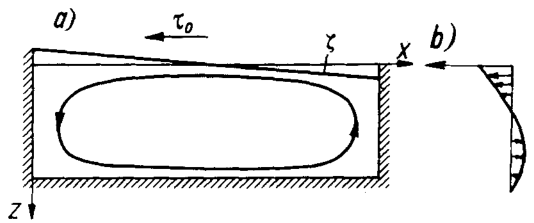


Fig. 5.3. Pattern of flow in shallow sea for constant wind stress. a — vertical section; b — distribution of velocity $u(z)$ at middle of basin.

basin). More important is the fact that for wind directed everywhere to one side a counter current may occur on the surface of the ocean (cf. Fig. 5.2). In general, presence of a counter current follows from the condition

$$\int_0^H \int_0^L u \mathrm{d}y \mathrm{d}z = 0 \ ;$$

however, appearance of the counter current on the ocean surface is caused by non-uniform wind. This fundamental fact lies at the base of the theory of equatorial counter currents (cf. [114]).

Now let $\tau_x = \tau_0$ = constant. Formula (5.5.10) then assumes the form

$$u = \frac{3H\tau_0}{4A_H}\left(1 - \frac{z}{H}\right)\left(\frac{1}{3} - \frac{z}{H}\right). \tag{5.5.12}$$

The simplest explanation of equatorial undercurrents in the oceans (Fig. 5.3) is based on this formula (cf. [19]).

5.6 WESTERN BOUNDARY CURRENT

Consider an ocean in moderate latitude and assume that its depth is large compared with the depth of the Ekman layer ($E << 1$). Neglecting in (5.4.4) terms of order E which do not contain second-order derivatives of ψ, one finds

$$\frac{fh_E}{2H^2\sqrt{\sin\varphi}}\,\Delta_h\psi + \frac{\partial\psi}{a\cos\varphi\partial\lambda}\,\frac{\partial}{a\partial\varphi}\left(\frac{f}{H}\right) - \frac{\partial\psi}{a\partial\varphi}\,\frac{\partial}{a\cos\varphi\partial\lambda}\left(\frac{f}{H}\right) = -\mathrm{rot}_z\frac{\tau}{H}. \tag{5.6.1}$$

Analysis of this equation will be started with the case of an ocean of constant depth. Obviously, the first term on the left-hand side of (5.6.1) describes the effect of bottom friction. Since it may be expected that in the open ocean friction is inessential, one has

$$\beta \frac{\partial \psi}{a \cos \varphi \partial \lambda} = -\mathrm{rot}_z \boldsymbol{\tau} , \qquad \beta = \frac{2\Omega}{a} \cos \varphi . \tag{5.6.2}$$

Equation (5.6.2) is referred to as Sverdrup's relation. It permits introduction of a characteristic scale ψ_0 for the total flow function ψ depending on the determining parameters of the problem: τ_0, a, $\beta_0 = 2\Omega/a$, A_H and the characteristic magnitude of $\mathrm{rot}_z \boldsymbol{\tau}$, representable as τ_0/a:

$$\Psi_0 = \frac{\tau_0}{\beta_0} . \tag{5.6.3}$$

Let $\tau_0 \simeq 1$ cm^2/sec^2, $\beta_0 \simeq 2 \cdot 10^{-13}$ cm^{-1} sec^{-1}. Then $\psi_0 \simeq 0.6 \cdot 10^{13}$ cm^3/sec. Generally speaking, for such a choice of scales, the quantity ψ_0 is somewhat too small (the non-dimensional value of $|\mathrm{rot}_z \boldsymbol{\tau}|$ is of order $5 \div 7$; however, for the asymptotic theory presented below, this is not important).

Proceeding in (5.6.1) to non-dimensional variables and retaining, for the sake of simplicity, the same notation for non-dimensional ψ and $\mathrm{rot}_z \boldsymbol{\tau}$, one obtains

$$Ek(\varphi)\Delta_h \psi + \frac{\partial \psi}{\partial \lambda} = -\mathrm{rot}_z \boldsymbol{\tau} , \tag{5.6.4}$$

where

$$k(\varphi) = \tfrac{1}{2}\sqrt{\sin \varphi} \tag{5.6.5}$$

and Δ_h is the Laplace operator on the unit sphere.

Equation (5.6.4) belongs to the class of equations with a small parameter as factor of the highest derivatives. It is known that presence of boundary layers is, generally speaking, characteristic for solutions of such equations.

The solution of (5.6.4) will be sought in the form

$$\psi = \psi_0 + E\psi_1 + \ldots + \tilde{\psi}_0 + E\tilde{\psi}_1 + \ldots \tag{5.6.6}$$

where the functions $\psi_0, \psi_1, \ldots$ and their first order-derivatives are of order O (1). It follows from (5.6.4) that these functions will satisfy equations of first order in λ. However, then the functions $\psi_0, \psi_1, \ldots$ cannot satisfy the boundary conditions on both sides of the ocean (assuming there to be no islands). Therefore introduce side by side with $\psi_0, \psi_1, \ldots$ "rapidly" varying "correction" functions $\tilde{\psi}_0, \tilde{\psi}_1, \ldots$ which are non-zero only in a narrow band of width E near the shore line (boundary layer). These functions are also of order O (1), and thus both terms on the left-hand side of (5.6.4) will be of the same order within the limits of the boundary layer. It is readily shown that in the problem under consideration a boundary layer can arise only on the western shore of the ocean [cf. note following (5.6.12)].

Substitution of (5.6.6) into (5.6.4) leads to two groups of terms: Those which change "normally" and those which change "quickly". Obviously, the group of "normally" changing terms must be equal to $\mathrm{rot}_z\,\boldsymbol{\tau}$, and the group of "quickly" changing terms equal to zero.

First write down the equations for the functions $\psi_0, \psi_1, \ldots$. Setting the coefficients of different powers of E equal to zero, one finds

$$\frac{\partial \psi_0}{\partial \lambda} = -\mathrm{rot}_z\,\boldsymbol{\tau}\,, \qquad \ldots\ldots\ldots\ldots \tag{5.6.7$_0$}$$

Let $\lambda = \lambda_E(\varphi)$ be the equation of the eastern boundary. Since it has been assumed that a boundary layer is only formed along the western shore, one has $\psi_0 = 0$, $\psi_1 = 0, \ldots$ for $\lambda = \lambda_E(\varphi)$. These boundary conditions are sufficient for solution of equations (5.6.7), i.e., all functions $\psi_0, \psi_1, \ldots$ may be found. For example,

$$\psi_0 = \int\limits_{\lambda}^{\lambda_E} (\mathrm{rot}_z\boldsymbol{\tau})\, d\lambda\,. \tag{5.6.8}$$

In order to construct equations for the functions $\tilde{\psi}_0, \tilde{\psi}_1, \ldots$, it is convenient to step over to curvilinear orthogonal coordinates $x = x(\lambda, \varphi)$, $y = y(\lambda, \varphi)$ such that the line $x = 0$ coincides with the western shore $\lambda = \lambda_w(\varphi)$ and the coordinates x and y increase for motion into the region and towards the north, respectively (Fig. 5.4) (where, naturally, it has been assumed that the shore line is sufficiently smooth and does not contain segments with large curvature). Then the functions $\tilde{\psi}_0, \tilde{\psi}_1, \ldots$ will change "normally" along the coordinate y and "quickly" along x. Introducing the "stretched" variable $\zeta = x/E$, derivatives of $\tilde{\psi}_0, \tilde{\psi}_1, \ldots$ with respect to ζ are converted to quantities of order $O(1)$, just as the same functions and their derivatives with respect to y. (Later, it will be shown that the width of the boundary layer will be of order E; therefore the functions describing the structure of the boundary current

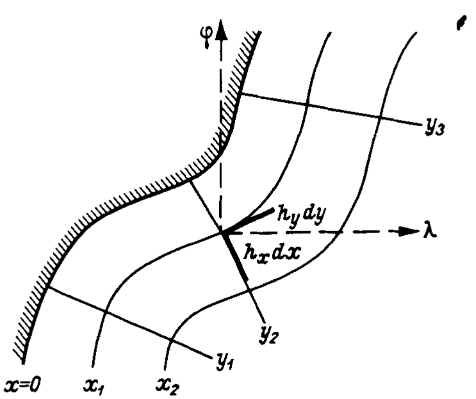


Fig. 5.4. Orthogonal curvilinear coordinates x and y for analysis of the boundary current along the western shore of an ocean.

must depend on x/E. Note that the notation ζ for x/E must not be confused with the sea level).

It will be natural to write now also (5.6.4) in the new coordinates x and y. Denote the Lamé coefficients of the system x, y by h_x, h_y:

$$a^2 h_x^2 = \left(\frac{\partial \lambda}{\partial x}\right)^2 h_\lambda^2 + \left(\frac{\partial \varphi}{\partial x}\right)^2 h_\varphi^2 , \qquad a^2 h_y^2 = \left(\frac{\partial \lambda}{\partial y}\right)^2 h_\lambda^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 h_\varphi^2 .$$

Proceeding in (5.6.4) to x and y, one finds

$$E k \Delta_h \psi + \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \lambda} = -\mathrm{rot}_z\, \boldsymbol{\tau} , \tag{5.6.9}$$

where

$$\Delta_h = \frac{1}{h_x h_y}\left\{\frac{\partial}{\partial x}\left(\frac{h_y}{h_x}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{h_x}{h_y}\frac{\partial}{\partial y}\right)\right\} .$$

Expand the coefficients of this equation in series of powers of E. One obtains within the limits of the boundary layer, for example,

$$h_x(x, y) = h_x(E\zeta, y) = h_x(0, y) + E\frac{\partial h_x}{\partial x}(0, y)\zeta + \dots \tag{5.6.10}$$

Substituting into (5.6.9) the expansion (5.6.6) and setting equal to zero the coefficients of different powers of E in the group of "quickly" changing terms, one finds the required equations for $\tilde{\psi}_0$, $\tilde{\psi}_1$, ...:

$$\frac{k_0}{h_{x0}^2}\frac{\partial^2 \tilde{\psi}_0}{\partial \zeta^2} + \left(\frac{\partial x}{\partial \lambda}\right)_0 \frac{\partial \tilde{\psi}_0}{\partial \zeta} = 0 , \tag{5.6.11$_0$}$$

. . . .

with the notation

$$h_{x0} = h_x(0, y) , \qquad \left(\frac{\partial x}{\partial \lambda}\right)_0 = \frac{\partial x}{\partial \lambda}(0, y) , \qquad k_0 = k(0, y) .$$

Since the coordinate x grows inwards, one has $(\partial x/\partial \lambda)_0 > 0$.

Consider equation (5.6.11), an ordinary differential equation with respect to ζ; the dependence of $\tilde{\psi}_0$ on y turns out to be parametric. Since the functions $\tilde{\psi}_0$, $\tilde{\psi}_1$, ... describe the structure of the boundary layer, they must, in essence, vanish anywhere else but near the shore. Mathematically, this may be achieved by a condition of exponential decay of the functions $\tilde{\psi}_0$, $\tilde{\psi}_1$, ... for large ζ.

The functions $\tilde{\psi}_0$, $\tilde{\psi}_1$, ... have been introduced in order to remove the inaccuracy in the fulfillment of the boundary conditions on the western shore in the functions ψ_0, ψ_1, ..., respectively. Therefore the second boundary condition for (5.6.11$_0$) will be

$$\psi_0(0, y) + \tilde{\psi}_0(0, y) = 0 ,$$

and the required solution ($5.6.11_0$) have the form

$$\tilde{\psi}_0 = -\int_{\lambda_w}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, \mathrm{d}\lambda \cdot \exp\left\{-\frac{h_{x0}^2}{k_0}\left(\frac{\partial x}{\partial \lambda}\right)_0 \frac{x}{E}\right\}. \tag{5.6.12}$$

The functions $\tilde{\psi}_1$, $\tilde{\psi}_2$, ... are constructed in the same manner. Thus all terms in the expansion (5.6.6) can be determined.

A similar procedure may also be adopted in the vicinity of an eastern shore. Then one arrives at an equation analogous to ($5.6.11_0$), but with an opposed sign in front of the term $\partial \tilde{\psi}_0/\partial \zeta$. Such an equation does not have a non-zero solution which decays exponentially for large ζ. Hence follows the conclusion that there is no boundary layer at the eastern shore.

It is an important fact that currents in moderate latitude may be considered independently from currents in tropical and polar regions. In fact, the construction of solutions of the problem, by strength of properties of (5.6.7), employed only boundary conditions on western and eastern shores. "Liquid" boundaries remind one only of "fitting" together of solutions for moderate latitudes with solutions for equatorial and polar regions. However, the effect of such a boundary will be essential only in a "narrow" zonal strip the width of which with respect to φ is of order $\sqrt{E}$.

Consider now the solution obtained. The function $\tilde{\psi}_0$ differs essentially from zero only in the vicinity of the western shore for $x \sim E$, or in dimensional variable for $x \sim aE$, i.e., the scale aE actually characterizes the thickness of the boundary layer.

Using (5.6.8) and (5.6.12), present the solution of the problem in the following form:

In the open ocean, outside the boundary layer at the western shore, but up to the eastern shore,

$$\psi = \int_{\lambda}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, \mathrm{d}\lambda + O(E)\, ; \tag{5.6.13}$$

in the vicinity of the western shore,

$$\psi = \int_{\lambda_w}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, \mathrm{d}\lambda \left\{1 - \exp\left[-\frac{h_{x0}^2}{k_0}\left(\frac{\partial x}{\partial \lambda}\right)_0 \frac{x}{E}\right]\right\} + O(E)\, ,$$

$$\frac{\partial \psi}{\partial x} = \frac{1}{E}\frac{h_{x0}^2}{k_0}\left(\frac{\partial x}{\partial \lambda}\right)_0 \int_{\lambda_w}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, \mathrm{d}\lambda \exp\left[-\frac{h_{x0}^2}{k_0}\left(\frac{\partial x}{\partial \lambda}\right)_0 \frac{x}{E}\right] + O(1)\, . \tag{5.6.14}$$

Fig. 5.5 shows lines of constant ψ, computed by means of (5.6.13). Obviously, these lines cannot "run into" shore. Therefore (5.6.13) turns out to be unreal at a western shore of an ocean and must be replaced by (5.6.14),

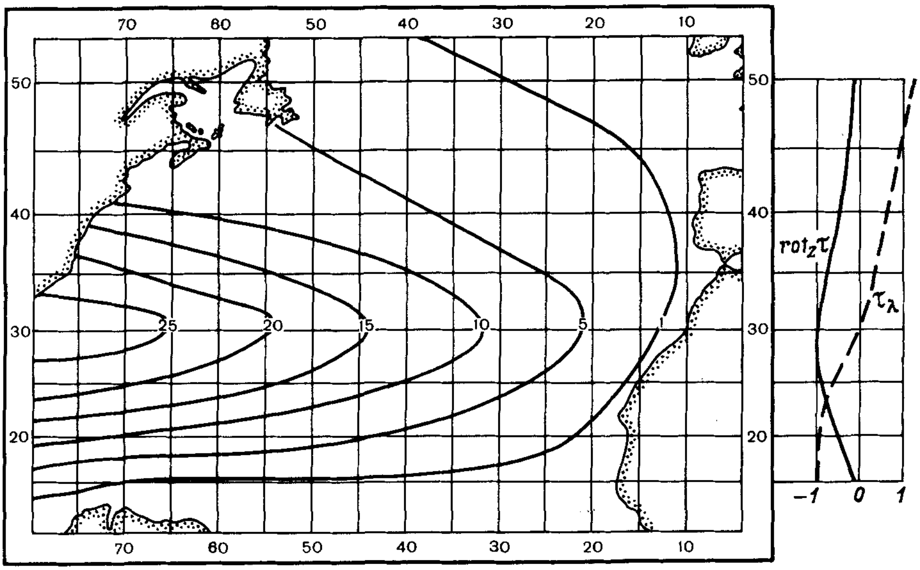


Fig. 5.5. Lines of equal total flow $\psi \cdot 10^{-12}$ cm^3/sec for the North Atlantic, by Sverdrup's formula

$$\psi = (a^2/2\Omega) \int_{\lambda}^{\lambda_E} (\mathrm{rot}_z \tau)\, d\lambda \,.$$

On the right-hand side are given the mean over many years of the tangential wind stress τ_λ cm^2/sec^2 and the corresponding distribution of 10^{-8} rot$_z$ τ cm/sec^2 (according to Munk [84]).

"closing" the stream lines at a western shore of an ocean.

For non-dimensional components of total flow within the bounds of the boundary layer, one finds that $S_x = O\,(1)$, $S_y = O\,(1/E)$. Thus, one actually has obtained a strong coastal flow at the western shore of the ocean. This is the simplest analogue of the Gulfstream in the North Atlantic and the Kuroshio in the Pacific Oceans. It is readily seen that at an eastern shore of an ocean there does not develop any strong coastal flow. It is clear from the analysis above that such asymmetric horizontal structure of the current is caused entirely by the latitudinal variation of the Coriolis parameter.

The necessity for formulation of boundary flow follows from the condition of conservation of mass of sea water. In fact, by Sverdrup's dimensional relation (5.5.2), one has in the open ocean

$$S_\varphi = -\frac{a}{2\Omega}\frac{\mathrm{rot}_z \tau}{\cos\varphi}, \qquad Q = \int_{\lambda_w}^{\lambda_E} S_\varphi a \cos\varphi d\lambda = -\frac{a^2}{2\Omega}\int_{\lambda_w}^{\lambda_E} (\mathrm{rot}_z \tau)\, d\lambda \,. \quad (5.6.15)$$

Since the total transport Q depends on φ, the solution of (5.6.13) violates the condition of conservation of mass for an ocean as a whole (cf. Fig. 5.5). Therefore one must form a boundary layer, in order to compensate for this defect at western shores. However, the total transport of the boundary layer Q_b must then be equal to $-Q$, and one arrives at Munk's formula

$$Q_b = \frac{a^2}{2\Omega} \int\limits_{\lambda_w}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, \mathrm{d}\lambda\ . \tag{5.6.16}$$

It is remarkable that, by (5.6.16), the total transport of the boundary current Q_b does not depend on the coefficient of turbulent exchange and may be computed from the field $\mathrm{rot}_z\, \boldsymbol{\tau}$.

It is natural that (5.6.16) should be obtained formally also on the basis of (5.6.14). However, the derivation presented here has the advantage of generality. In the sequel (Chapter 6), more complex models will be designed for the coastal boundary layer which take into consideration non-linearity, horizontal turbulent exchange, etc. However, all the same, if the current in the open ocean is described by (5.6.13), then (5.6.16) remains true independently of the form of the equation which describes the structure of the boundary layer. Note only that it does not follow from the reasoning based on the condition of conservation of mass that a boundary layer must form at the western shore of an ocean (cf. § 6.2).

Formula (5.6.12) permits to obtain an estimate of the coefficient of turbulent exchange A_H. Since the order of the width of the Gulfstream is known ($aE \simeq 50$ km), one finds $A_H \simeq 10^2 \div 10^3$ cm^2/sec.

The influence of the shape of the coast line on the boundary layer is described by the coefficient $h_{x0}^2\ (\partial x/\partial\lambda)_0$ in (5.6.12) and (5.6.14). Therefore the conclusion may be drawn that deformation of a coast line leads only to a certain change of the width of the boundary layer and of its velocity, without, however, being reflected in the magnitude of the total transport Q_b.

5.7 EFFECT OF BOTTOM RELIEF ON BOUNDARY CURRENT

Consider now the analysis of (5.6.1) in the general case of an ocean of variable depth. Neglecting, as in the case of an ocean of constant depth, the effect of bottom current, one obtains Sverdrup's generalized relation

$$\frac{\partial\psi}{a\cos\varphi\partial\lambda}\,\frac{\partial}{a\partial\varphi}\left(\frac{f}{H}\right) - \frac{\partial\psi}{a\partial\varphi}\,\frac{\partial}{a\cos\varphi\partial\lambda}\left(\frac{f}{H}\right) = -\mathrm{rot}_z\frac{\boldsymbol{\tau}}{H}\,.$$

Lines of constant f/H serve as characteristics of this first-order equation. Therefore, in essence, not the very depth H, but the function f/H displays the influence of the bottom on the current in a homogeneous ocean.

In the case under consideration, the equation for the boundary current is readily derived. It is not difficult to show that the boundary current at a shore of an ocean [an exponentially decaying solution of an equation of the type [5.6.11)] exists only along those segments of a western shore line where

$$\frac{\partial}{\partial y}\left(\frac{f}{H}\right) > 0$$

and of an eastern shore where

$$\frac{\partial}{\partial y}\left(\frac{f}{H}\right) < 0\,,$$

where the coordinate axis y coincides with the shore line and points to the north.

The mean inclination of the ocean floor is of order 10^{-3} which, generally speaking, is sufficient for changes in sign of the quantity $\partial(f/H)/\partial y$. Therefore consider a problem for which this quantity changes sign as one moves along a western shore; its solution yields the simplest model of the separation of a boundary current from a shore, caused by the effect of the bottom relief.

Consider the problem

$$E\Delta_h\psi - y\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y} = -1\,. \tag{5.7.1}$$

$$\psi = 0 \qquad \text{for } x = 0 \text{ and } x = 1\,, \tag{5.7.2}$$

and construct a bounded solution of this problem in the strip $0 \leqslant x \leqslant 1$, $|y| < \infty$. The choice of the strip for the analysis of the model is justified by the fact that one wishes to concentrate attention on a study of separation of a boundary current from a western coast. The characteristics of the limiting equation ($E = 0$) will be parabolae $x - y^2/2$ = constant (lines of constant f/H in the present problem). North of the point (0, 0), there is no coastal boundary flow; it will be shown below that at this point it separates from the west coast.

In the open ocean (outside the boundary layer), the term $E\Delta_h\psi$ in (5.7.1) is small and the geostrophic solution ψ_g is given by the series

$$\psi_g = \psi_{g0}(x, y) + E\psi_{g1}(x, y) + \ldots + 0(E)\,. \tag{5.7.3}$$

Since for $y < 0$ a boundary layer may exist only at a western shore ($x = 0$), and for $y > 0$ only at an eastern shore ($x = 1$), it is clear that $\psi_g = 0$ for $x = 1$, $y < 0$ and $x = 0, y > 0$. The functions ψ_{g0}, ψ_{g1}, ... are readily found. For example:

$$\psi_{g0} = \begin{cases} y - \sqrt{y^2 - 2x} & \text{for } y > \sqrt{2x}\,, \\ \\ y + \sqrt{2 - 2x + y^2} & \text{for } y < \sqrt{2x}\,. \end{cases} \tag{5.7.4}$$

Since $\psi_g \neq 0$ for $x = 0, y < 0$ and $x = 1, y > 0$, one finds coastal boundary layers in Regions 1 and 4 of Fig. 5.6. Further, obviously, the function ψ_g undergoes a discontinuity along the line $y = \sqrt{2x}$; this discontinuity is obviated by the boundary layer (with large velocities along $y = \sqrt{2x}$ in Region 3). Besides, near (0, 0) and (1, 0), asymptotic expansions in Regions 1 and 4 turn out to be invalid, and therefore transition region 2 is formed (a region of separation) and Region 5 with special asymptotics.

The solution of problem (5.7.1) and (5.7.2) in Regions 1, 2 and 3 may be represented in the form of asymptotic series:

In Region 1

$$\psi = \psi_1 = \psi_{10}(\zeta, y) + E\psi_{11} + o(E) \ldots , \qquad \zeta = \frac{x}{E}, \qquad y < 0 , \qquad \zeta \geqslant 0 ; \tag{5.7.5}$$

in Region 2

$$\psi = \psi_2 = \psi_{20}(\xi, \eta) + E^{1/3}\psi_{21} + o(E^{1/3}) \ldots , \qquad \xi = \frac{x}{E^{2/3}}, \qquad \eta = \frac{y}{E^{1/3}},$$
$$\xi \geqslant 0 , \qquad |\eta| < \infty ; \tag{5.7.6}$$

in Region 3

$$\psi = \psi_3 = \psi_{30}(\sigma, y) + o(1), \qquad \sigma = \frac{x - y^2/2}{E^{1/2}}, \qquad y \geqslant 0 , \qquad |\sigma| < \infty . \tag{5.7.7}$$

As always, the scale E for the variable x in Region 1 is determined from the condition that the terms $E\partial^2\psi/\partial x^2$ and $y\partial\psi/\partial x$ must be of equal orders of magnitude. In Region 2, these terms are also of equal order; however, the term $\partial\psi/\partial y$ must be here of the same order. Otherwise the expansion (5.7.6) would not differ from the expansion (5.7.5) and the effect of separation of the boundary layer could not be described. These two conditions yield for the scales along x and y the quantities $E^{2/3}$ and $E^{1/3}$, respectively. Finally, the scale $E^{1/2}$ of the coordinate $x_1 = x - \frac{1}{2}y^2$, changing across boundary layer 3, will be determined by the condition that the terms $E\partial^2\psi/\partial x_1^2$, and $\partial\psi/\partial y$ must be of equal order.

Note that it is not obligatory to introduce for an analysis of a boundary layer orthogonal coordinates (the coordinates in Region 3: $x_1 = x - y^2/2, y$ are not orthogonal). It is only important that one of the families of coordinate lines should be directed "along" the boundary layer. In Region 3, this line is x_1 = constant, and therefore $\partial/\partial x_1 \sim E^{-1/2}$, $\partial/\partial y \sim E^0$.

Construction of the equations for the individual terms of asymptotic series presents no great difficulties. One obtains in the ordinary manner the following equations and boundary conditions:

$$L_1\psi_{10} = 0 , \qquad L_1\psi_{11} = -1 + \frac{\partial\psi_{10}}{\partial y}, \qquad L_1 = \frac{\partial^2}{\partial\zeta^2} - y\frac{\partial}{\partial\zeta}, \tag{5.7.8}$$

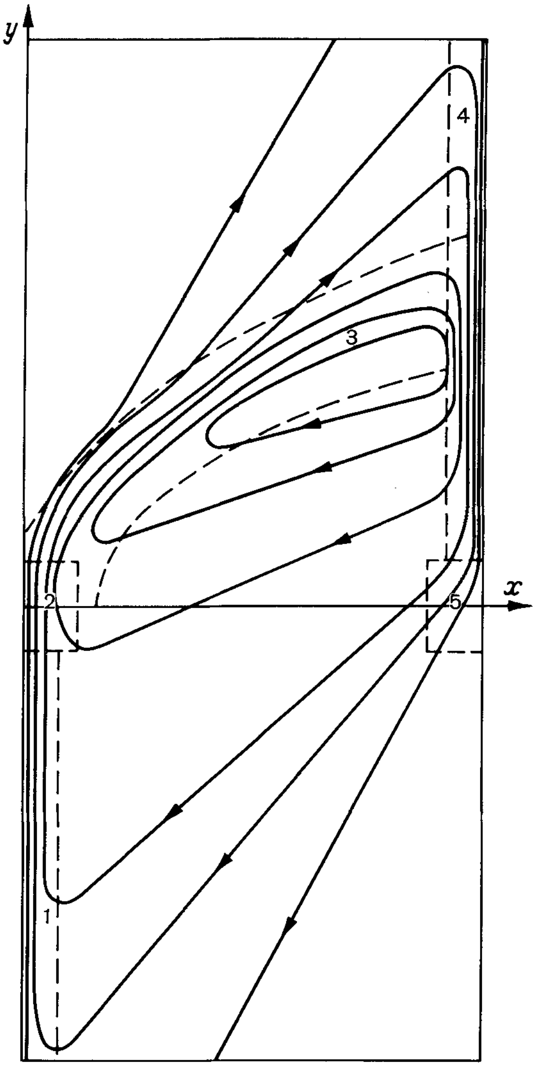


Fig. 5.6. Boundary layers 1, 3, and 4, transition regions 2 and 5 and the pattern of stream lines (of constant ψ) for problems (5.7.1) and (5.7.2). For the sake of clearity, the boundary layers have been strongly enlarged.

$$L_2\psi_{20} = 0\,, \qquad L_2\psi_{21} = -1\,, \qquad L_2 = \frac{\partial^2}{\partial\xi^2} - \eta\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\,, \tag{5.7.9}$$

$$L_3\psi_{30} = -1\,, \qquad L_3 = (1+y^2)\frac{\partial^2}{\partial\sigma^2} - \frac{\partial}{\partial y}\,, \tag{5.7.10}$$

$$\psi_{10} = \psi_{11} = 0\,, \qquad \psi_{20} = \psi_{21} = 0 \qquad \text{for } x = 0\,. \tag{5.7.11}$$

Note that the study of Problems (5.7.1) and (5.7.2) employed a method which differed somewhat from that employed in § 5.6 (although for an analogous type of problem). In § 5.6, "correction" functions have been introduced within the confines of the boundary layers, which compensate for the inaccuracy in the fulfillment of the boundary conditions for the functions describing the solution in the open ocean. For such "correction" functions, special equations were constructed and a search was made only for exponential decaying with the distance from the boundary solutions of these equations.

However, in the case under consideration, we will not apply this method, since separate terms of expansion (7.7.3) have singularities on the boundary, for example, in the neighbourhood of the point (0, 0). Therefore, the complete region of integration will be subdivided into individual parts in each of which special asymptotic expansions of the solutions of the problem will be written down; these expansions must now be matched in a definite manner. The basic idea of this general method of matched asymptotic expansions is explained in Appendix B. Note that in those cases when the method of "correction" functions is applicable, it leads automatically to matched asymptotic expansions.

Consider the problem of matching of asymptotic expansions in the geostrophic region and Region 1. Applying Procedure 1 (cf. Appendix B), one arrives without difficulties at the condition

$$\psi_{10} \to \psi_{g0}(0, y)\,, \qquad \psi_{11} \sim -\frac{\zeta}{\sqrt{2+y^2}} + \psi_{g1}(0, y) \qquad \text{for } \zeta \to \infty\,. \quad (5.7.12)$$

Equation (5.7.8) and conditions (5.7.11) and (5.7.12) permit easy construction of the functions ψ_{10} and ψ_{11}. These functions are analogous to (5.6.14) and will therefore not be written down here.

Analogously, application of Procedure 1 (cf. Appendix B) to matching of the asymptotic expansions in the geostrophic region and Region 3 yields

$$\psi_{30} \to \sqrt{2} + y \qquad \text{for } \sigma \to \infty\,, \qquad \psi_{30} \to y \qquad \text{for } \sigma \to -\infty\,. \quad (5.7.13)$$

Consider the matching of asymptotic expansions in the geostrophic region and Region 2. According to Procedure 1, rewrite the partial sum $\psi_{g0}(x, y) + E\psi_{g1}(x, y)$ in the form

$$\psi_{g0}(E^{2/3}\xi, E^{1/3}\eta) + E\,\psi_{g1}(E^{2/3}\xi, E^{1/3}\eta)$$

and expand in a series of powers $1, E^{1/3}, \ldots$ for fixed ξ and η. Using (5.7.4), one finds

$$\psi_{g0}(E^{2/3}\xi, E^{1/3}\eta) + E\,\psi_{g1}(E^{2/3}\xi, E^{1/3}\eta) = \sqrt{2} + E^{1/3}\eta + o(E^{1/3})\,, \quad (5.7.14)$$

whence

$$\psi_{20} \to \sqrt{2}\,, \qquad \psi_{21} \sim \eta \qquad \text{for } \xi \to \infty\,, \qquad \eta \to -\infty\,. \quad (5.7.15)$$

Equations (5.7.9) and conditions (5.7.11) and (5.7.15) determine the functions ψ_{20} and ψ_{21} in a unique manner.

Next, the matching of the asymptotic expansions in Regions 1 and 2 will be checked. Rewrite the partial sum $\psi_{10}(\zeta, y) + E\psi_{11}(\zeta, y)$ in the form $\psi_{10}(\xi E^{-1/3}, \eta E^{1/3}) + E\psi_{11}(\xi E^{-1/3}, \eta E^{1/3})$ and expand in a series of powers $1, E^{1/3}, \ldots$ for fixed ξ and η. Explicit formulae for ψ_{10} and ψ_{11} yield

$$\psi_{10}(\xi E^{-1/3}, \eta E^{1/3}) + E\,\psi_{11}(\xi E^{-1/3}, \eta E^{1/3}) = \sqrt{2}(1 - \mathrm{e}^{\xi\eta})$$
$$- \sqrt{2}\frac{1}{\eta}\left(\frac{1}{2}\xi^2 - \frac{\xi}{\eta}\right)\mathrm{e}^{\xi\eta} + \left[\eta(1 - \mathrm{e}^{\xi\eta}) - \frac{\xi^2}{2}\mathrm{e}^{\xi\eta}\right]E^{1/3} + o(E^{1/3}). \tag{5.7.16}$$

Using the asymptotic expansion of the solution of problem (5.7.9), (5.7.11), (5.7.15) for $\xi \to 0$ and $\eta \to -\infty$, it may be shown that the matching conditions for the expansions (5.7.5) and (5.7.6) are fulfilled (cf. [38]).

In order to determine the function ψ_{30}, one must still find initial condition for $y = 0$ for equation (5.7.10). For $\sigma > 0$, these conditions are "dictated" by the asymptotic expansion in the geostrophic region, for $\sigma < 0$ by the boundary conditions at the shore for $x = 0$ (cf. the scales of the corresponding regions in Fig. 5.7). The condition for ψ_{30} will be written in the form

$$\psi_{30}(\sigma, 0) = 0 \qquad \text{for } \sigma < 0\,; \qquad \psi_{30}(\sigma, 0) = \sqrt{2} \qquad \text{for } \sigma \geqslant 0\,. \tag{5.7.17}$$

Equation (5.7.10) is readily solved for conditions (5.7.13) and (5.7.17) :

$$\psi_{30} = y + \sqrt{2} - \sqrt{\frac{2}{\pi}} \int\limits_{\sigma/2\sqrt{t}}^{\infty} e^{-s^2} \mathrm{d}s\,, \qquad t = y + \tfrac{1}{3}y^3\,. \tag{5.7.18}$$

Only the first term in the expansion for ψ_3 will be determined. However, it must be kept in mind that for construction of a unique solution of Problem (5.7.10), (5.7.13) and (5.7.17), because of the discontinuity of the initial conditions (5.7.17) for $y = 0$, one must, in general, know the character of the singularity of the required function at the point $\sigma = 0$, $y = 0$. This singularity is determined from the matching condition of the asymptotic expansions (5.7.6) and (5.7.7).

It will now be shown that ψ_{30} is actually found in this manner. For this purpose, change in Region 2 to the variables $\theta = (x - \frac{1}{2}y^2)/E^{2/3}$, $\eta = y/E^{1/3}$; then the operator L_2 assumes the form $L_2 = \partial^2/\partial\theta^2 - \partial/\partial\eta$. Using the asymptotic expansion of ψ_{20} for $|\theta|, \eta \to \infty$ one finds

$$\psi_{20}(\sigma E^{-1/6}, yE^{-1/3}) = y + \sqrt{2} - \sqrt{2/\pi} \int\limits_{\sigma/2\sqrt{y}}^{\infty} \mathrm{e}^{-s^2} \mathrm{d}s + o(1).$$

Thus, formula (5.7.18) yields the true character of the singularity of ψ_{30} for $(\sigma, y) \to (0, 0)$. Construction of the subsequent terms of expansion (5.7.7) is given in [37,38].

Thus, asymptotic expansions have been constructed for Regions 1, 2 and 3. In essence, determination of the asymptotic in Regions 4 and 5 does not differ (in Region 5, the expansion begins with terms of order $E^{1/3}$).

The velocities of the current (derivatives of ψ with respect to x and y) computed from the asymptotic expansion for ψ will be continuous. Although, as it follows from (5.7.4) and (5.7.18), the derivatives of ψ_g and ψ_{30} have singularities for $(x, y) \to (0, 0)$ and $\partial\psi_g/\partial x$ also for $(x, y) \to (1, 0)$, these singularities are removed by the transition regions 2 and 5.

These asymptotic expansions of the solution of Problem (5.7.1) and (5.7.2) permit readily construction of the pattern of stream lines (lines of constant ψ) and the distributions of velocities in the boundary flows (Figs. 5.6 and 5.8).

Note that, although $\partial(f/H)/\partial y$ changes sign at both points (0, 0) and (1, 0), separation of boundary flow 1 from the shore and its continuation in the form of an internal jet 3 has been observed only at the point (0, 0); on approach to the point (1, 0), boundary current 4 gradually becomes exhausted.

The method of solution of Problems (5.7.1) and (5.7.2) presented above is readily generalized to the case of arbitrary right-hand sides in (5.7.1) and likewise to arbitrary shore lines. Note that the axis of the internal boundary current in Problems (5.7.1) and (5.7.2) is the curve $y = \sqrt{2x}$. In the general case, this axis will be a line of constant f/H which touches the western shore at some point (point of separation) and turns back to the east coast.

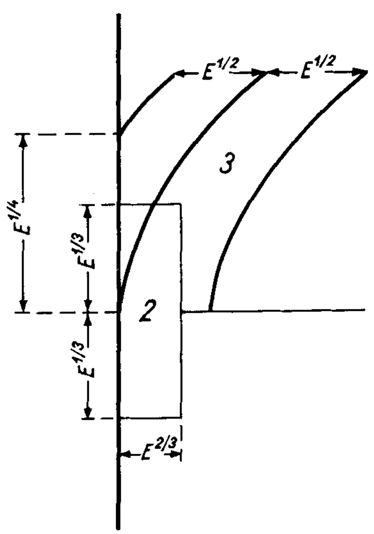


Fig. 5.7. Scales of Regions 2 and 3.

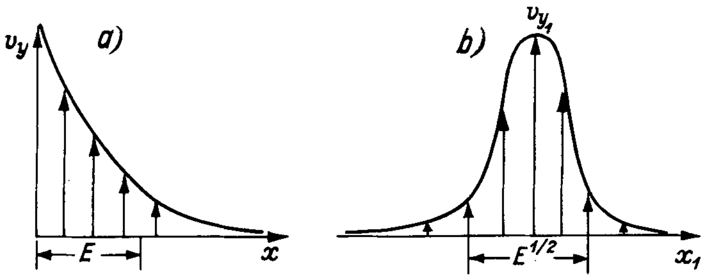


Fig. 5.8. Schematic distributions of velocities at section perpendicular to the axes of boundary currents 1 and 3: a — Layer 1, $v_y = 0\ (1/E)$; b — Layer 3, $v_{y_1} = 0\ (1/\sqrt{E})$, $x_1 = x - y^2/2$ (x_1, y_1 — orthogonal coordinates).

COMMENT ON CHAPTER 5

The material of §§ 5.1—5.3 has been studied repeatedly in the literature, cf. Ekman [15,16] and also Felsenbaum's book [18], where it has been proposed to construct an equation for the total flow function ψ and not for the sea level, as this was done by Ekman.

§ 5.4. This section follows the work of Kamenkovich [42,43].

§ 5.5. The important role of the non-uniformity of the wind field in ocean dynamics, in particular for the formation of equatorial counter currents, was first pointed out by Stokman [114].

§ 5.6. The basic theory of western boundary currents was developed by Stommel [116]. Sverdrup's relation was proposed in [119] and Munk's formula in [84]. The methods of analysis of the boundary layer have been exposed in detail, for example, in Cole [7] and Carrier [3]; cf. also Appendix B of this book.

§ 5.7. Criteria for the existence of boundary currents for oceans of variable depth have been discussed by Welander [128]. Problem (5.7.1) and (5.7.2) has been considered by Kamenkovich and Reznik [48]; cf. also Ilyin et al. [37,38] which presents a complete mathematical analsysis of the problem.

CHAPTER 6

TWO-DIMENSIONAL MODELS OF OCEAN CURRENTS

6.1 METHOD OF TOTAL FLOWS

In order to arrive in a natural manner at the idea of averaging over depth of the equations of motion and at an exposition of the method of total flows, consider first wind-induced currents in a homogeneous deep ocean. Integrating the initial equations of Ekman's model (5.2.1) and (5.2.2) with respect to z from 0 to H and taking (5.2.4) into consideration, one has

$$fS_\lambda = gH \frac{\partial \zeta}{a\partial \varphi} + \tau_\varphi - \tau_\varphi^b \,, \qquad -fS_\varphi = gH \frac{\partial \zeta}{a \cos \varphi \partial \lambda} + \tau_\lambda - \tau_\lambda^b \,, \tag{6.1.1}$$

where $(\tau_\lambda^b, \tau_\varphi^b) = -A_H(\partial u/\partial z, \partial v/\partial z)\,|_H$ are the components of the bottom friction. Eliminating from (6.1.1) the level ζ and introducing the total flow function ψ, one finds

$$-\mathrm{rot}_z \frac{\boldsymbol{\tau}_b}{H} + \frac{\partial \psi}{a \cos \varphi \partial \lambda} \frac{\partial}{a \partial \varphi}\left(\frac{f}{H}\right) - \frac{\partial \psi}{a \partial \varphi} \frac{\partial}{a \cos \varphi \partial \lambda}\left(\frac{f}{H}\right) = -\mathrm{rot}_z \frac{\boldsymbol{\tau}}{H} \,. \tag{6.1.2}$$

Comparing equation (6.1.2) with (5.6.1), one obtains the approximate [recall that small terms of order E were neglected in (5.6.1)] formula

$$-\mathrm{rot}_z \frac{\boldsymbol{\tau}_b}{H} \simeq \frac{fh_E}{2H^2} \frac{1}{\sqrt{\sin \varphi}} \Delta_h \psi \,. \tag{6.1.3}$$

Consider now as an example the Ekman model with variable transfer coefficient $A_H = A_H(z)$. It is readily seen that equations (6.1.1) and (6.1.2) will also be valid in this case; however, it is difficult to find a general expression for $\mathrm{curl}_z\,(\tau_b/H)$ in terms of ψ. If it is assumed that (6.1.3) holds true approximately also in this case, then one derives immediately an equation for the total flow function ψ in the study of which the parameter h_E may be considered to be the mean characteristic of the bottom friction in the ocean. Naturally, this equation lends itself to analysis which is considerably simpler than that of the initial system of equations, and thus there arises the possibility of investigating velocities of currents averaged over depth for more complex models.

Apply such an approach in the general case. Of course, one has to introduce additional hypotheses modelling one or the other new effect in order to obtain a closed system of equations for parameters of a motion averaged over depth. However, it turns out that the initial equations are simplified significantly, and, most important of all, that such a theory explains in a sufficiently reasonable manner important features of the horizontal structure of ocean currents.

Attention will now be given to stationary currents in moderate latitudes. Using (4.5.4), (4.5.7) and (4.6.9), find first of all a general expression for the horizontal components of the friction force. Integrating then the equations of motion (4.8.3) and (4.8.4) with respect to z from 0 to H, one finds

$$\int_0^H \left\{ \frac{u}{a\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\varphi} + w\frac{\partial u}{\partial\varphi} - \frac{uv}{a}\tan\varphi \right\} dz - fS_\varphi = -\frac{1}{\rho_0}\int_0^H \frac{\partial p}{a\cos\varphi\partial\lambda}\,dz$$

$$+ A_L \int_0^H \left\{ \Delta_h u + \frac{\cos 2\varphi}{a^2\cos^2\varphi}u - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda} \right\} dz + \tau_\lambda - \tau_\lambda^b +$$

$$+ A_H \frac{\partial w}{a\cos\varphi\partial\lambda}\bigg|_0^H - \frac{1}{a\cos\varphi}\int_0^H \frac{\partial}{\partial\lambda}\left(A\frac{\partial w}{\partial z}\right) dz, \tag{6.1.4}$$

$$\int_0^H \left\{ \frac{u}{a\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{v}{a}\frac{\partial v}{\partial\varphi} + w\frac{\partial v}{\partial z} + \frac{u^2}{a}\tan\varphi \right\} dz + fS_\lambda = -\frac{1}{\rho_0}\int_0^H \frac{\partial p}{a\partial\varphi}\,dz$$

$$+ A_L \int_0^H \left\{ \Delta_h v + \frac{\cos 2\varphi}{a^2\cos^2\varphi}v + \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda} \right\} dz + \tau_\varphi - \tau_\varphi^b$$

$$+ A_H \frac{\partial w}{a\partial\varphi}\bigg|_0^H - \frac{1}{a}\int_0^H \frac{\partial}{\partial\varphi}\left(A\frac{\partial w}{\partial z}\right) dz\,. \tag{6.1.5}$$

In these formulae, the quantity E_t has been included with the pressure p; since, as a rule, $E_t << p$, no further consideration will be given to terms with E_t. The symbol Δ_h denotes the Laplace operator on a sphere with radius a.

Normally, in the ocean, one has $|\tau_b| \sim 0.01$ cm^2/sec^2 (cf. [133]). Since $|\tau| \sim 1$ cm^2/sec^2, one may neglect the effect of bottom friction in the averaged equations (6.1.4) and (6.1.5) in comparison with the effect of the tangential wind stress. However, in that case, the averaging of the initial equations permits to eliminate from the consideration the coefficient of turbulent vertical transfer A_H which has only been studied little to date.

In the Ekman model, the role of bottom friction in the region of a western boundary current is overstated artificially, since the model does not contain other factors which are suitable for *closing* the isolines of the total flow function (cf. § 5.6).

Introducing $P = \int_0^H p\,dz$, the pressure averaged over the depth, and the pressure p_b at the ocean floor, one has

$$\int_0^H (\nabla_h p)\,dz = \nabla_h P - p_b\,\nabla_h H\,.$$

In essence, the term $p_b \nabla_h H$ describes the effect of the inhomogeneity of the density of sea water on the distribution of total flows in the ocean; it is very difficult to obtain for this term an objective estimate. All the same, this term will be neglected below. Otherwise one does not succeed in the construction of a closed system of equations for the study of features of the distributions S_λ, S_φ and P in the ocean.

Furthermore, assume that the integrals of the non-linear inertial terms and of the horizontal turbulent transfer in (6.1.4) may be represented approximately in the form

$$\frac{1}{H}\left\{\frac{S_\lambda}{a\cos\varphi}\frac{\partial S_\lambda}{\partial\lambda}+\frac{S_\varphi}{a}\frac{\partial S_\lambda}{\partial\varphi}-\frac{S_\lambda S_\varphi}{a}\tan\varphi\right\}, \qquad A_L\left\{\Delta_h S_\lambda+\frac{\cos 2\varphi}{a^2\cos^2\varphi}S_\lambda-\right.$$
$$\left.-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial S_\varphi}{\partial\lambda}\right\},$$

and in (6.1.5) analogously.

Finally, if the transfer coefficient A is assumed to be constant, then one may, to the same approximation, not take into consideration, by the boundary conditions $w(\lambda, \varphi, H) = 0$ and $w(\lambda, \varphi, 0) = 0$, the two last terms on the right-hand sides of (6.1.4) and (6.1.5). In this manner, the as yet unknown transfer coefficient has been excluded completely.

Hence one arrives at the equations

$$\frac{1}{H}\left\{\frac{S_\lambda}{a\cos\varphi}\frac{\partial S_\lambda}{\partial\lambda}+\frac{S_\varphi}{a}\frac{\partial S_\lambda}{\partial\varphi}-\frac{S_\lambda S_\varphi}{a}\tan\varphi\right\}-fS_\varphi=-\frac{1}{\rho_0}\frac{\partial P}{a\cos\varphi\partial\lambda}+A_L\times$$
$$\left\{\Delta_h S_\lambda+\frac{\cos 2\varphi}{a^2\cos^2\varphi}S_\lambda-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial S_\varphi}{\partial\lambda}\right\}+\tau_\lambda\ , \tag{6.1.6}$$

$$\frac{1}{H}\left\{\frac{S_\lambda}{a\cos\varphi}\frac{\partial S_\varphi}{\partial\lambda}+\frac{S_\varphi}{a}\frac{\partial S_\varphi}{\partial\varphi}+\frac{S_\lambda^2}{a}\tan\varphi\right\}+fS_\lambda=-\frac{1}{\rho_0}\frac{\partial P}{a\partial\varphi}+A_L\times$$
$$\left\{\Delta_h S_\varphi+\frac{\cos 2\varphi}{a^2\cos^2\varphi}S_\varphi+\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial S_\lambda}{\partial\lambda}\right\}+\tau_\varphi\ . \tag{6.1.7}$$

These equations will yet be augmented by the equation of conservation of mass (5.2.16), viz.

$$\frac{\partial S_\lambda}{\partial\lambda}+\frac{\partial}{\partial\varphi}(S_\varphi\cos\varphi)=0\ . \tag{6.1.8}$$

Thus, the closed system of equations (6.1.6)—(6.1.8) for the determination of the unknown functions S_λ, S_φ and P has been constructed. The hypotheses introduced have permitted to derive instead of the complex, initial, three-dimensional model of currents caused by surface effects (wind, heat flux, etc.) a two-dimensional model with an external mass force. This approach to the study of the horizontal structure of ocean currents is also referred to as the method of total flows.

In equations (6.1.6)—(6.1.8), the single external force is the tangential wind stress. In other words, the method of total flows permits to study only horizontal features of wind-induced currents in the ocean. The thermo-haline components of a current, caused by action of factors directly changing the temperature and salinity of sea water, does not contribute in the analysis of currents averaged over the depth of the ocean (within the framework of the approximations introduced).

The following sections show that the analysis of the system (6.1.6)—(6.1.8) makes it possible to explain important peculiarities of the horizontal structure of ocean currents.

6.2 GENERAL ANALYSIS OF A TWO-DIMENSIONAL MODEL

Generally speaking, the equations derived hold true for an ocean of variable depth. However, in order to simplify the ensuing analysis, assume that the depth is constant, i.e., H = constant. Then, dividing (6.1.6) through (6.1.8) by H and introducing velocities u and v, averaged over the depth of the ocean, and pressure p (employing the former symbols), one obtains after a number of transformations the equation of vorticity for two-dimensional currents on a sphere (with the z-axis directed downwards)

$$\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}(\omega-f)+\frac{v}{a}\frac{\partial}{\partial\varphi}(\omega-f)=A_L\left(\Delta_h\omega+\frac{2}{a^2}\omega\right)+\frac{1}{H}\operatorname{rot}_z\boldsymbol{\tau}\,, \tag{6.2.1}$$

where

$$\omega=\frac{1}{\cos\varphi}\frac{\partial}{a\partial\varphi}(u\cos\varphi)-\frac{\partial v}{a\cos\varphi\partial\lambda}\,. \tag{6.2.2}$$

The left-hand side of equation (6.2.1) describes advection of absolute vorticity $\omega-f$ [since the model is two-dimensional, there does not occur stretching of vortex lines in (6.2.1)], the right-hand side diffusion of relative vorticity $A_L\Delta_h\omega$, an internal source of vorticity $(2A_L/a^2)\omega$ and an external source of vorticity $(1/H)\operatorname{rot}_z \bar{\tau}$ (in a two-dimensional model, there is no baroclinic effect).

In the equation of vorticity transfer for two-dimensional motions on a plane, friction leads only to diffusion of vorticity, described by the term $A_L\Delta_h\omega$. Mathematically speaking, this is a consequence of the fact that in Euclidean space the result of repeated covariant differentiations does not depend on the sequence of differentiations. The development of the internal source of vorticity $(2A_L/a^2)\omega$ in (6.2.1) is caused by the curvature of the spherical surface (non-Euclidean space) (cf. § A.8).

At the same time, inclusion of bottom friction in (6.1.3) likewise leads to an internal vorticity source.

Important features of the motions under consideration are the closed nature of the stream lines and the constant sign of the intensity of the external vorticity source (for example, in the region of the Azores Anti-cyclone, $\mathrm{rot}_z\, \boldsymbol{\tau} > 0$). Therefore one must take into consideration in (6.2.1) diffusion of relative vorticity; otherwise there would arise for motion of particles along closed trajectories a surplus of relative vorticity and stationary motion would prove to be impossible. The formal proof follows from the identity

$$\oint_\Gamma \boldsymbol{v}_h\, \nabla_h(\omega - f)\, ds = \oint_\Gamma \mathrm{div}_h[\boldsymbol{v}_h(\omega - f)]\, ds = 0\ ,$$

where the integral is taken along the closed trajectory Γ and $\boldsymbol{v}_h = (u, v)$. If friction is not included in the equations of motion, then one obtains on integration of (6.2.1) along a closed trajectory a contradiction, since $\oint_\Gamma (\mathrm{rot}_z\, \boldsymbol{\tau}) ds > 0$.

Consider an open ocean and let $\mathrm{rot}_z\, \boldsymbol{\tau} > 0$. Simple estimates show that in the open ocean changes in f essentially exceed changes in ω; however, then diffusion and advection of relative vorticity are here insignificant, and one obtains Sverdrup's relation

$$\beta v = -\frac{1}{H}\mathrm{rot}_z \boldsymbol{\tau}\ . \tag{6.2.3}$$

According to Sverdrup's relation, all particles must move southwards to larger values of the planetary vortex $-f$, since flux of vorticity from the external source is positive (in the left-handed coordinate system λ, φ and z). Obviously, the total meridional transport of such motion Q will depend on the latitude (intensity of the external vorticity source is not constant) and there exist lines which cross the ocean from eastern to western shores on which $\mathrm{rot}_z\, \boldsymbol{\tau} = 0$ (for example, the southern and northern boundaries of the Azores Anti-cyclone). But such motion violates the law of conservation of mass (the liquid being incompressible). Therefore there must arise at shores of an ocean northerly currents which remove violations of the law of conservation.

At what shores do there arise boundary currents? Clearly, within the limits of boundary flow advection and diffusion of relative vorticity ω must be essential (otherwise, the fluid particles would be moving south). Consequently, the width of such a current must be small compared with the characteristic dimensions of the basin (the expressions for advection and diffusion of relative vorticity ω contain derivatives of higher order than the expression βv for advection of the planetary vortex $-f$). However, then one has within the limits of a narrow boundary current $\omega \simeq -(1/h_x)\partial v_y/\partial x$, where v_y is the velocity of the current along the shore [cf. Fig. 5.4], and the total advection of vorticity ω by the coastal current is (Fig. 6.1)

$$\int_A^B v_y \omega h_x \mathrm{d}x = \tfrac{1}{2} v_y^2(A) - \tfrac{1}{2} v_y^2(B)\,.$$

The point A lies on the shore and by strength of the condition of no slip $v_y(A) = 0$; the point B has been chosen at the outer edge of the boundary layer and therefore $v_y(B) \simeq 0$, since the velocity v_y within the limits of the boundary layer exceeds significantly the speed v_y in the open ocean. This follows from the fact that the total transport of the narrow coastal current must be equal in order of magnitude to the total transport of fluid in the open ocean. Furthermore, the total contribution of the internal vorticity source $(2A_L/a^2)\omega$ within the limits of the boundary current is likewise readily seen to be equal to zero. Thus, it is clear that the diffusive flux of vorticity from the wall must play a fundamental role.

Consider first a northerly boundary current along a western shore (Fig. 6.1a). Since the wall is at rest, it acts on the fluid with a force which counteracts the motion. Since the absolute magnitude of the vorticity generated at the wall decreases with distance from the wall, it may be asserted that due to diffusion negative vorticity is added to the fluid. This flux also gives the required excess negative vorticity which is carried northward by the boundary current because of advection of planetary vorticity $-f$. Thus, a strong current is possible at a western shore which does not violate the equation of vorticity transfer (since in the open ocean $\beta v = -(1/H)\,\mathrm{rot}_z\,\tau$, it is obvious that $(1/H)\,\mathrm{rot}_z\,\tau$ makes an insignificant contribution to the vorticity balance within the limits of a strong boundary current).

It is important to note that the reasoning presented does not depend on

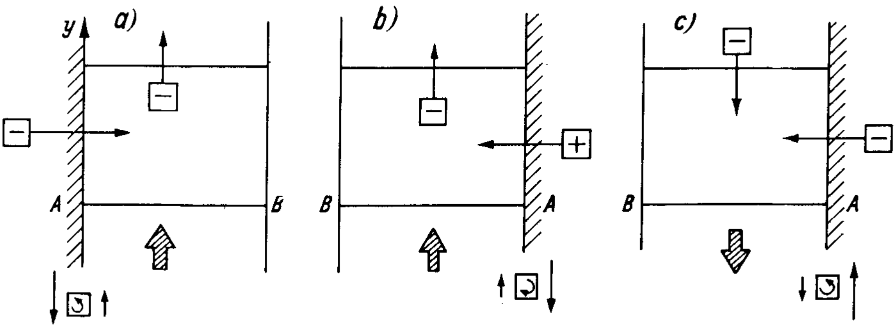


Fig. 6.1. Pattern of vorticity transfer in boundary currents. Shown is advective transfer of planetary vorticity and diffusive transfer of relative vorticity as well as the couple of forces which twist water particles at a shore. View from above; the axis z is directed downwards. a). Western shore, current to the north. b). Eastern shore, current to the north. c). Eastern shore, current to the south.

the concrete form in which the terms of turbulent viscosity have been written down and is based only on sufficiently general and reasonable concepts of turbulent diffusion of vorticity.

Next, consider the analysis of a boundary current which runs northward along an eastern ocean shore (Fig. 6.1b). Clearly, this time positive vorticity is supplied to the current because of diffusion from the wall. However, such a current, in general, cannot exist, even when the action of an external source is taken into account. If the boundary current is directed towards the south, then negative vorticity is added because of diffusion, and then balance of vorticity is possible, but only on participation of the external vorticity source $(1/H) \operatorname{rot}_z \boldsymbol{\tau}$. As it has already been seen, such a current cannot be strong. In this manner, it has been proved that formation of a strong boundary current is only possible at a western shore; the no slip condition at the shore and features of vorticity diffusion do not admit formation of a strong boundary current at an eastern ocean shore. Analogous reasoning also applies to the case $\operatorname{rot}_z \boldsymbol{\tau} < 0$.

A very essential result has been derived which explains the pronounced asymmetry in the total flows observed in moderate latitudes of the Atlantic and Pacific Oceans (Fig. 6.2). The result is even more surprising as the tangential wind field $(\tau_\lambda, \tau_\varphi)$ does not exhibit in these regions any special asymmetry and nevertheless strong coastal currents (Gulfstream and Kuroshio) are only observed along western shores. It must be emphasized that in the above reasoning the effect of latitudinal changes of the Coriolis parameter f played the determining role.

Thus, a boundary current is only formed at a western shore of an ocean. However, then one obtains for its total transport immediately Munk's formula (5.6.16) (recall that this formula was assumed to be true also for $H \neq$ constant). Recall again that this formula does not contain matching turbulent transfer coefficients. It is useful to compare computations based on Munk's formula with observations [southwards from the point of separation of the Gulfstream from shore ($\varphi \simeq 33°$N)]. As shown by Fig. 6.3, the order of magnitude of the total transport of the boundary current is given correctly by the two-dimensional theory (the theoretical values are about half those observed). Taking into consideration the approximate character of the two-dimensional theory (for example, stationary theory) as well as of the observations (because of the difficulty of objectively estimating the transport of the Gulfstream averaged over many years), one cannot but admit that the results of such a comparison are the *quantitative* success of the theory under study.

As it follows from the analysis above, direct action of the tangential wind stress in the region of the boundary current does not affect its dynamics; the boundary current is determined by the action of the wind on the entire ocean [cf. Munk's formula (5.6.16)].

Features of boundary currents are studied in detail in the next sections.

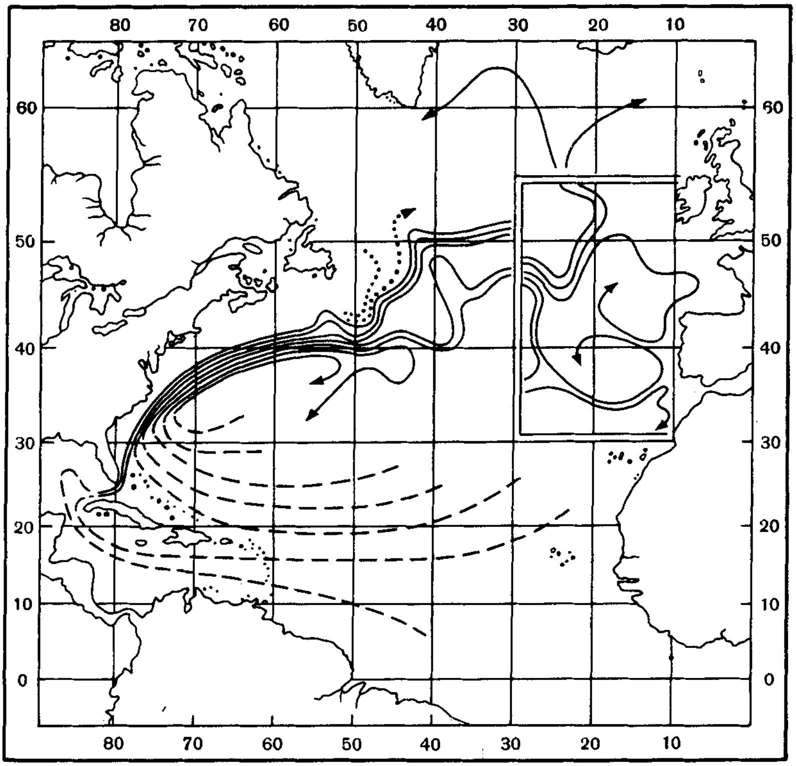


Fig. 6.2. Schematic chart of the total flows in the Atlantic Ocean. (According to Iselin [86]). Water transport between adjoining lines on the western side of the ocean is $12 \cdot 10^{12}$ cm^3/sec.

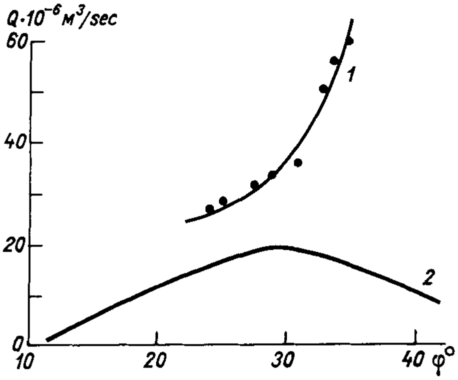


Fig. 6.3. Dependence of transport of Gulfstream Q [m^3/sec] on latitude φ (according to Gill [27]). *1* = Knauss' observations [51]; *2* = Munk's formula (5.6.16); $\mathrm{rot}_z\,\tau$ has been computed from Hellerman [31].

The following work demands writing down of the basic equations of the two-dimensional model in non-dimensional form. The characteristic scale for the velocity U_0 (having in mind the velocity averaged over the depth of the ocean) is determined, according to Sverdrup's relation (6.2.3), by

$$U_0 = \frac{\tau_0/a}{\beta_0 H}, \tag{6.3.1}$$

where τ_0/a is taken as a characteristic value of $\mathrm{rot}_z\, \boldsymbol{\tau}$ (cf. § 5.6) and $\beta_0 = 2\Omega/a$ is the characteristic value of the change of the Coriolis parameter with latitude.

Taking as characteristic horizontal scale of the ocean basin the radius a of the Earth, one finds

$$\epsilon^2\left(\frac{u}{\cos\varphi}\frac{\partial u}{\partial\lambda} + v\frac{\partial u}{\partial\varphi} - uv\tan\varphi\right) - fv =$$

$$-\frac{\partial p}{\cos\varphi\partial\lambda} + \delta^3\left(\Delta_h u + \frac{\cos 2\varphi}{\cos^2\varphi}u - \frac{2\sin\varphi}{\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right) + \tau_\lambda\,, \tag{6.3.2}$$

$$\epsilon^2\left(\frac{u}{\cos\varphi}\frac{\partial v}{\partial\lambda} + v\frac{\partial v}{\partial\varphi} + u^2\tan\varphi\right) + fu =$$

$$-\frac{\partial p}{\partial\varphi} + \delta^3\left(\Delta_h v + \frac{\cos 2\varphi}{\cos^2\varphi}v + \frac{2\sin\varphi}{\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) + \tau_\varphi\,, \tag{6.3.3}$$

$$\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\varphi}(v\cos\varphi) = 0\,, \tag{6.3.4}$$

where, in order to avoid complicated notation, u and v are non-dimensional components of the horizontal velocity, averaged over the depth of the ocean [scale (6.3.1)], f is the non-dimensional Coriolis parameter (scale 2Ω), p is the non-dimensional pressure averaged over the depth of the ocean (scale $2\Omega a\rho_0 U_0$), τ_λ and τ_φ are non-dimensional components of tangential wind stress (scale τ_0) and Δ_h is the Laplace operator for the sphere of unit radius.

In writing down (6.3.2)—(6.3.4), two parameters have been introduced: ϵ and δ; it will be convenient to define them in terms of two internal length scales L_i and L_v:

$$\epsilon = \frac{L_i}{a}, \qquad L_i = \sqrt{\frac{U_0}{\beta_0}}\,; \qquad \delta = \frac{L_v}{a}, \qquad L_v = \sqrt[3]{\frac{A_L}{\beta_0}}\,. \tag{6.3.5}$$

For the external parameters of the problem one has the values: $\tau_0 \simeq 1\ \mathrm{cm^2/sec^2}$, $a = 6.4\cdot 10^8$ cm, $H \simeq 1$ km, $\beta_0 \simeq 2\cdot 10^{-13}\ \mathrm{cm^{-1}\,sec^{-1}}$, $A_L = 0.3\,.\,10^8$

cm^2/sec. Then $U_0 \simeq 0.1$ cm/sec, $L_i \simeq 8$ km, $L_0 \simeq 50$ km; thus, the parameters ϵ and δ prove to be very small. (Note that characteristic values of U_0 and L_i are somewhat too low because of the choice of τ_0/a as scale for $\mathrm{rot}_z\, \boldsymbol{\tau}$.)

It is seen that equations (6.3.2)—(6.3.4) contain two small parameters which are coefficients of the highest derivatives in the system. The asymptotic methods of § § 5.6 and 5.7 will be applied to the study of such problems.

Proceeding step by step, this section will be concerned with the viscous boundary layer and it will be assumed that the non-linear inertial terms in (6.3.2) and (6.3.3) may be neglected (clearly, this depends on the magnitude of the exchange coefficient A_L, cf. § 6.5). Eliminating the function p from (6.3.2) and (6.3.3) and introducing the non-dimensional stream function

$$\frac{\partial \psi}{\cos \varphi \partial \lambda} = v \, , \qquad \frac{\partial \psi}{\partial \varphi} = -u \, , \tag{6.3.6}$$

one finds [cf. (6.2.1)]

$$-\delta^3(\Delta_h \Delta_h \psi + 2\Delta_h \psi) + \beta \frac{\partial \psi}{\cos \varphi \partial \lambda} = -\mathrm{rot}_z\, \boldsymbol{\tau} \, , \tag{6.3.7}$$

where $\beta = \cos \varphi$ is the non-dimensional latitudinal change of the Coriolis parameter (scale $2\Omega/a$).

On western and eastern shores of a basin, there must be satisfied the no-slip conditions

$$\psi = 0, \qquad \frac{\partial \psi}{\partial n} = 0 \qquad \text{for} \qquad \lambda = \lambda_W(\varphi) \, , \tag{6.3.8}$$

$$\psi = 0, \qquad \frac{\partial \psi}{\partial n} = 0 \qquad \text{for} \qquad \lambda = \lambda_E(\varphi) \, , \tag{6.3.9}$$

where $\lambda = \lambda_E\ (\varphi)$, $\lambda = \lambda_W\ (\varphi)$ are the equations of the eastern and western shore lines, respectively, and $\boldsymbol{n}$ is the normal to the shore.

Equation (6.3.7) will now be studied for the case of moderate latitudes. Since, as will be shown below, the solution in the open ocean may be constructed without consideration of boundary conditions at "liquid" boundaries, attention will be limited to conditions (6.3.8) and (6.3.9). In other words, during an analysis of currents in moderate latitudes, the conditions at "liquid" boundaries do not affect the solution of a problem. This important fact was already noted earlier during the analysis of the Ekman model (cf. § 5.6).

Seek the solution of (6.3.7) in the form

$$\psi = \psi_0 + \delta \cdot \psi_1 + \ldots + \tilde{\psi}_0 + \delta \cdot \tilde{\psi}_1 + \ldots + \tilde{\tilde{\psi}}_0 + \delta \cdot \tilde{\tilde{\psi}}_1 + \ldots \tag{6.3.10}$$

The functions ψ_0, ψ_1, ... describe the solution in the open ocean, but in view of the structure of equation (6.3.7) they may not satisfy the boundary conditions (6.3.8) and (6.3.9) on both shores of the basin. In order to remove these discrepancies at the western and eastern shores, respectively, the "correction" functions $\tilde{\psi}_0$, $\tilde{\psi}_1$, ... and $\tilde{\tilde{\psi}}_0$, $\tilde{\tilde{\psi}}_1$, ... have been introduced. These functions differ essentially from zero only near the corresponding coast. For the functions ψ_0, ψ_1, ..., one obtains readily the equations

$$\frac{\partial \psi_0}{\partial \lambda} = -\mathrm{rot}_z \boldsymbol{\tau} \,, \tag{6.3.11$_0$}$$

$$\frac{\partial \psi_1}{\partial \lambda} = 0 \,, \tag{6.3.11$_1$}$$

. . . .

In order to construct equations for the "rapidly" changing functions $\tilde{\psi}_0$, ... and $\tilde{\tilde{\psi}}_0$, ..., change over to a new coordinate system which permits to separate out the "rapid" variable for these functions. For this purpose, introduce in the neighbourhood of the western shore the curvilinear orthogonal coordinate system x_1 and y_1 for which the curve $x_1 = 0$ coincides with the shore line and the coordinates x_1 and y_1 increase for motion into the basin and northward, respectively (cf. Fig. 5.4). Denote the Lamé coefficients of the coordinate system by $h_x^{(1)}$ and $h_y^{(1)}$. Proceed analogously in the neighbourhood of the eastern shore: the coordinate system x_2 and y_2 has Lamé coefficients $h_x^{(2)}$, $h_y^{(2)}$ with corresponding properties.

In these new coordinate systems, the functions $\tilde{\psi}_0$, ... and $\tilde{\tilde{\psi}}_0$, ... have the forms: $\tilde{\psi}_0 = \tilde{\psi}_0\ (\zeta_1, y_1)$, $\zeta_1 = x_1/\delta$, ... and $\tilde{\tilde{\psi}}_0 = \tilde{\tilde{\psi}}_0\ (\zeta_2, y_2)$, $\zeta_2 = x_2/\delta$, ..., and they must decay exponentially for large ζ_1 and ζ_2. Next, rewrite equation (6.3.7) in the new coordinate system. It is not difficult to represent the operator $\Delta_h \Delta_h + 2\Delta_h$ in the form

$$\Delta_h \Delta_h + 2\Delta_h = \frac{1}{h_x^4} \frac{\partial^4}{\partial x^4} + M \tag{6.3.12}$$

where the operator M contains derivatives with respect to x of not higher than third order. Obviously, the representation (6.3.12) is true at western as well as at eastern shores.

The required equations for $\tilde{\psi}_0$, ... and $\tilde{\tilde{\psi}}_0$, ... are readily found (cf. § 5.7):

$$-\frac{1}{(h_{x0}^{(1)})^4} \frac{\partial^4 \tilde{\psi}_0}{\partial \zeta_1^4} + \left(\frac{\partial x_1}{\partial \lambda}\right)_0 \frac{\partial \tilde{\psi}_0}{\partial \zeta_1} = 0 \,, \tag{6.3.13$_0$}$$

. . . .

$$-\frac{1}{(h_{x0}^{(2)})^4} \frac{\partial^4 \tilde{\tilde{\psi}}_0}{\partial \zeta_2^4} + \left(\frac{\partial x_2}{\partial \lambda}\right)_0 \frac{\partial \tilde{\tilde{\psi}}_0}{\partial \zeta_2} = 0 \,, \tag{6.3.14$_0$}$$

where

$$h_{x0}^{(1)} = h_x^{(1)}(0, y_1), \qquad h_{x0}^{(2)} = h_x^{(2)}(0, y_2)\,,$$

$$\left(\frac{\partial x_1}{\partial \lambda}\right)_0 = \frac{\partial x_1}{\partial \lambda}(0, y_1), \qquad \left(\frac{\partial x_2}{\partial \lambda}\right)_0 = \frac{\partial x_2}{\partial \lambda}(0, y_2) \tag{6.3.15}$$

Note that the functions $\tilde{\psi}_0, \ldots$ and $\tilde{\tilde{\psi}}_0, \ldots$ satisfy ordinary differential equations in ζ_1 and ζ_2; the dependence of these functions on y_1 and y_2 is parametric. This is the result of the transition to the "fast" variable ζ in equation (6.3.7).

Write down the boundary conditions. Substituting the expansion (6.3.10) into (6.3.8) and (6.3.9), one finds:

for $\lambda = \lambda_W(\varphi)$

$$\psi_0|_{\lambda_W} + \tilde{\psi}_0(0, y_1) = 0\,. \tag{6.3.16$_0$}$$

$$\psi_1|_{\lambda_W} + \tilde{\psi}_1(0, y_1) = 0\,, \tag{6.3.16$_1$}$$

. . . .

$$\frac{\partial \tilde{\psi}_0}{\partial \zeta_1}(0, y_1) = 0\,, \tag{6.3.17$_0$}$$

$$\left.\frac{\partial \psi_0}{\partial x_1}\right|_{\lambda_W} + \frac{\partial \tilde{\psi}_1}{\partial \zeta_1}(0, y_1) = 0\,, \tag{6.3.17$_1$}$$

. . . .

for $\lambda = \lambda_E(\varphi)$

$$\psi_0|_{\lambda_E} + \tilde{\tilde{\psi}}_0(0, y_2) = 0\,, \tag{6.3.18$_0$}$$

$$\psi_1|_{\lambda_E} + \tilde{\tilde{\psi}}_1(0, y_2) = 0\,, \tag{6.3.18$_1$}$$

. . . .

$$\frac{\partial \tilde{\tilde{\psi}}_0}{\partial \zeta_2}(0, y_2) = 0\,, \tag{6.3.19$_0$}$$

$$\left.\frac{\partial \psi_0}{\partial x_2}\right|_{\lambda_E} + \frac{\partial \tilde{\tilde{\psi}}_1}{\partial \zeta_2}(0, y_2) = 0\,, \tag{6.3.19$_1$}$$

. . . .

Thus, separate equations have been obtained for the functions ψ_0, $\tilde{\psi}_0$, $\tilde{\tilde{\psi}}_0$, ...; however, the boundary conditions for these functions have turned out to be "interlinked". In other words, the solution ψ_0 for the open ocean depends through the boundary conditions on the structure of the boundary layers.

Introduce $l_1(y)$ and $l_2(y)$:

$$l_{1,2}^3(y) = [h_{x0}^{(1,2)}]^4 \left(\frac{\partial x_{1,2}}{\partial \lambda}\right)_0 , \tag{6.3.20}$$

and write down the characteristic equations for ($6.3.13_0$) and ($6.3.14_0$), respectively, in the forms

$$\nu^3 - l_1^3 = 0 , \tag{6.3.21}$$

$$\nu^3 - l_2^3 = 0 . \tag{6.3.22}$$

Equation (6.3.22) will be considered first. Since x_2 increases when one moves into the basin, then $(\partial x_2/\partial \lambda)_0 < 0$ and $l_2 < 0$. Hence, equation (6.3.22) has only the single root $\nu = l_2$, which lies in the left half-plane of the complex variable ν. Consequently, the exponentially decaying solution of equation ($6.3.14_0$) has the form

$$\tilde{\tilde{\psi}}_0 = C(y) \exp(l_2 \zeta_2) \tag{6.3.23}$$

and, by strength of the boundary condition ($6.3.19_0$), one has $C \equiv 0$. But then $\tilde{\tilde{\psi}}_0 \equiv 0$. Hence, by ($6.3.18_0$), one has $\psi_0 = 0$ for $\lambda = \lambda_E(\varphi)$ and equation ($6.3.11_0$) may be solved:

$$\psi_0 = \int_{\lambda}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, d\lambda . \tag{6.3.24}$$

Thus, the current in the open ocean is described by the same Sverdrup formula which was obtained during the analysis of the Ekman model.

Consider now equation (6.3.21). Since $l_1 > 0$, this equation has two roots ν_1 and ν_2 which lie in the left half-plane of the complex variable ν. The exponentially decaying solution of equation ($6.3.13_0$) which satisfies conditions ($6.3.16_0$) and ($6.3.17_0$) has the form

$$\tilde{\psi}_0 = -\int_{\lambda_W}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, d\lambda \left\{\frac{2}{\sqrt{3}} \exp\left(-\frac{l_1 \zeta_1}{2}\right) \sin\left(\frac{\sqrt{3}}{2} l_1 \zeta_1 + \frac{\pi}{3}\right)\right\} . \tag{6.3.25}$$

It is easily seen that the function $\tilde{\tilde{\psi}}_1$ must satisfy equation ($6.3.14_0$) and condition ($6.3.19_1$). Hence

$$\tilde{\tilde{\psi}}_1 = -\frac{1}{l_2} \frac{\partial \psi_0}{\partial x_2}\bigg|_{\lambda_E} \exp(l_2 \zeta_2). \tag{6.3.26}$$

Once $\tilde{\tilde{\psi}}_1$ is known, the values of ψ_1 at the eastern shore can be found from ($6.3.18_1$) and the function ψ_1 inside the basin determined by solution of equation ($6.3.11_1$). As soon as ψ_1 is known, formulae ($6.3.16_1$) and ($6.3.17_1$) yield the necessary boundary conditions for the construction of $\tilde{\psi}_1$, etc. Thus, all terms of the series (6.3.10) may be constructed.

Note that the expansion (6.3.10) bears an asymptotic character and, as a rule, consideration will be restricted to its first terms. By (6.3.24)–(6.3.26), the solution of the problem under consideration has been obtained in the following form:

In the open ocean, outside the boundary layers along the western and eastern shores,

$$\psi = \int_{\lambda}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, d\lambda + O(\delta)\ ; \tag{6.3.27}$$

In the neighbourhood of the western shore of the ocean ($x_1 \sim \delta$)

$$\psi = \int_{\lambda_W}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, d\lambda \left\{1 - \frac{2}{\sqrt{3}} \exp\left(-\frac{l_1 \zeta_1}{2}\right) \sin\left(\frac{\sqrt{3}}{2}\, l_1 \zeta_1 + \frac{\pi}{3}\right)\right\} + O(\delta)\ , \tag{6.3.28}$$

$$\frac{\partial \psi}{\partial x_1} = \frac{1}{\delta} \int_{\lambda_W}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, d\lambda \cdot \frac{2}{\sqrt{3}}\, l_1 \exp\left(-\frac{l_1 \zeta_1}{2}\right) \sin \frac{\sqrt{3}}{2}\, l_1 \zeta_1 + O(1)\ ;$$

In the neighbourhood of the eastern shore ($x_2 \sim \delta$)

$$\psi = O(\delta), \qquad \frac{\partial \psi}{\partial x_2} = \frac{\partial}{\partial x_2}\left\{\int_{\lambda}^{\lambda_E} (\mathrm{rot}_z \boldsymbol{\tau})\, d\lambda\right\}\Bigg|_{\lambda_E} [1 - \exp(l_2 \zeta_2)] + O(\delta)\ . \tag{6.3.29}$$

As had been assumed in determining the characteristic scale for the velocity from (6.3.1), the non-dimensional velocity in the open ocean turns out to be of order unity. However, it should not be concluded that the choice of scale (6.3.1) "enforces" the Sverdrup solution (6.3.27) for the open ocean. Choice of one or the other scales for unknown functions for solution of a problem is in a definite sense an assumption the truth of which can only be established after solution of the problem.

Attention will now be turned on the structure of the boundary layer at the western shore of an ocean. Using (6.3.28) and formulae of the type (6.3.6), one finds readily that within the limits of the boundary layer $v_{x_1} \sim 1$, $v_{y_1} \sim 1/\delta$ and both velocity components vanish at the shore.

It is not difficult to show that Munk's formula (5.6.16) is obtained for the magnitude of the total transport of the boundary current, as in the case of the Ekman model (cf. the discussion of this question in § 6.2). However, the structure of the boundary current obtained above differs from that of the Ekman theory. The meridional velocity component (or the total flow) change sign within the boundary layer, an indication that weak counter currents arise near the outer edge of the boundary layer.

It is clear that the thickness of the viscous boundary layer is of order L_v and that, once L_v has been determined from observations, the coefficient A_L may be estimated; if $L_v \simeq 100$ km, then $A_L \simeq 10^8$ cm^2/sec. Note that L_v

changes little when A_L is varied by one order of magnitude [cf. (6.3.5)].

Next, consider the boundary layer at an eastern shore of an ocean. Using (6.3.29), one finds that the boundary layer which forms at an eastern coast does not lead to a strong coastal current; it only removes the inability to fulfill the boundary condition $v_{y_2} = 0$ for $x_2 = 0$ to the Sverdrup solution.

Furthermore, as in the case of the Ekman model, it is seen that a change in the configuration of the shore line leads only to a quantitative change in the velocity within the limits of the boundary layer, and likewise of the width of this layer, but that it does not bring about any qualitative change in the structure of the boundary current.

It is interesting to note that current velocities averaged over the depth of the ocean (or total flows) are shown by the solution obtained to depend only on $\mathrm{rot}_z\, \boldsymbol{\tau}$, and not on the field $\boldsymbol{\tau}$ itself.

Thus, a general linear theory of the viscous boundary layer at ocean coasts has been developed which explains a number of interesting features of such coastal currents as the Gulfstream, et al. However, this theory has an essential defect. The boundary current described by the linear theory exists all along the shore; it first "saturates" to maximum total transport and then gradually "exhausts" itself to zero transport (cf. Fig. 5.5). Hence also Sverdrup's relation appears to be valid throughout the oceans outside coastal boundary layers.

Such patterns are not observed in Nature. The Gulfstream, on attaining maximum transport, leaves the shore in the form of a narrow concentrated jet which gradually dissipates in the open ocean. A more interesting feature of the current in the open ocean to the north of the Gulfstream's point of separation from shore is meandering. Therefore also Sverdrup's relation which does not describe such meanders cannot be true in this region. Note that a similar pattern is also observed in the northern part of the Pacific Ocean.

Furthermore, the large magnitude of A_L must be noted which is required by the linear theory. According to estimates, obtained by Stommel [117, p. 98, 97, 99], $A_L \simeq 10^6$ cm^2/sec. However, for such a value of A_L, inertial terms which have not been taken into account in the model above appear to be essential in the equations of motion.

All these circumstances lead to the necessity of giving consideration to the more complicated models of §§ 6.4—6.6.

6.4 INERTIAL BOUNDARY LAYER

The effect of the non-linear inertial terms on the structure of the boundary layer will now be studied, neglecting the action of forces of turbulent friction. Combined effects of viscous and inertial terms will be considered in §§ 6.5 and 6.6.

A beginning will be made immediately with the equation for a western boundary current. Introducing, as in § 6.3, curvilinear coordinates x and y, rewrite equations (6.3.2) and (6.3.3) without frictional forces:

$$\epsilon^2\left(\frac{u}{h_x}\frac{\partial u}{\partial x}+\frac{v}{h_y}\frac{\partial u}{\partial y}+\frac{uv}{h_x h_y}\frac{\partial h_x}{\partial y}-\frac{v^2}{h_x h_y}\frac{\partial h_y}{\partial x}\right)-fv=-\frac{\partial p}{h_x\partial x}+\tau_x\ , \tag{6.4.1}$$

$$\epsilon^2\left(\frac{u}{h_x}\frac{\partial v}{\partial x}+\frac{v}{h_y}\frac{\partial v}{\partial y}+\frac{uv}{h_x h_y}\frac{\partial h_y}{\partial x}-\frac{u^2}{h_x h_y}\frac{\partial h_x}{\partial y}\right)+fu=-\frac{\partial p}{h_y\partial y}+\tau_y\ , \tag{6.4.2}$$

where u and v are the components of the velocity in the directions x and y.

On the basis of (6.3.4), define a stream function ψ by

$$\frac{\partial\psi}{h_x\partial x}=v,\qquad \frac{\partial\psi}{h_y\partial_y}=-u\ . \tag{6.4.3}$$

At the shore, the no-flow condition

$$u=0\qquad x=0 \tag{6.4.4}$$

must be fulfilled.

In the analysis of the problem under consideration, it will be convenient not to introduce "correction" functions for separate terms of the asymptotic expansion of the solution of the problem in the open ocean, but to represent the required solution directly in the region of the boundary current in the form of asymptotic series (cf. § 5.7):

$$u=u_0(\zeta,y)+\epsilon u_1(\zeta,y)+\ldots,\qquad v=\frac{1}{\epsilon}v_0(\zeta,y)+v_1(\zeta,y)+\ldots,$$

$$p=p_0(\zeta,y)+\epsilon p_1(\zeta,y)+\ldots,\qquad \psi=\psi_0(\zeta,y)+\epsilon\psi_1(\zeta,y)+\ldots, \tag{6.4.5}$$

where $\zeta=x/\epsilon$.

Substitute the series (6.4.5) into (6.4.1)—(6.4.3) and collect terms of equal powers of ϵ. One finds for the first approximation

$$-f_0v_0=-\frac{\partial p_0}{h_{x0}\partial\zeta}\ , \tag{6.4.6}$$

$$\frac{u_0}{h_{x0}}\frac{\partial v_0}{\partial\zeta}+\frac{v_0}{h_{y0}}\frac{\partial v_0}{\partial y}+f_0u_0=-\frac{\partial p_0}{h_{y0}\partial y}+\tau_{y0}\ , \tag{6.4.7}$$

$$\frac{\partial\psi_0}{h_{x0}\partial\zeta}=v_0,\qquad \frac{\partial\psi_0}{h_{y0}\partial y}=-u_0\ , \tag{6.4.8}$$

where the notation

$$h_{x0}=h_x(0,y),\qquad h_{y0}=h_y(0,y),\qquad f_0=f(0,y)$$

has been introduced.

In the open ocean, the solution of the problem becomes an asymptotic series of the normal type. It will now be assumed that all terms of these series have been found.

Substituting the expansion (6.4.5) for u into (6.4.4), one obtains in first approximation

$$u_0 = 0 \qquad \text{for} \qquad \zeta = 0\,. \tag{6.4.9}$$

The asymptotic expansions in the regions of the boundary layer and the open ocean must be matched in a definite manner. Methods for such matching have been explained in Appendix B (cf. also § 5.7). Without difficulties, one finds for the first approximation

$$u_0 = u_{g0}, \qquad v_0 = 0, \qquad p_0 = p_{g0}, \qquad \psi_0 = \psi_{g0} \qquad \text{for} \qquad \zeta = \infty\,, \tag{6.4.10}$$

where u_{g0}, p_{g0} and ψ_{g0} are the values of the first terms of the asymptotic expansion in the open ocean for $x = 0$.

Thus the unknown functions u_0, v_0, p_0 and ψ_0 satisfy the system of equations (6.4.6)–(6.4.8) for the boundary conditions (6.4.9) and (6.4.10).

Equations (6.4.6) and (6.4.7) are conveniently rewritten in the form

$$-v_0\left(f_0 + \frac{1}{h_{x0}}\frac{\partial v_0}{\partial \zeta}\right) = -\frac{1}{h_{x0}}\frac{\partial}{\partial \zeta}\left(\frac{v_0^2}{2} + p_0 + Y\right), \tag{6.4.11}$$

$$u_0\left(f_0 + \frac{1}{h_{x0}}\frac{\partial v_0}{\partial \zeta}\right) = -\frac{1}{h_{y0}}\frac{\partial}{\partial y}\left(\frac{v_0^2}{2} + p_0 + Y\right), \tag{6.4.12}$$

where

$$Y = -\int_0^y (h_{y0}\tau_{y0})\, dy\,.$$

Multiply equation (6.4.11) by u_0, equation (6.4.12) by v_0 and add the results to obtain

$$u_0\frac{\partial}{h_{x0}\partial \zeta}\left(\frac{v_0^2}{2} + p_0 + Y\right) + v_0\frac{\partial}{h_{y0}\partial y}\left(\frac{v_0^2}{2} + p_0 + Y\right) = 0\,.$$

This equation yields directly the Bernoulli type integral

$$\frac{v_0^2}{2} + p_0 + Y = Q(\psi_0)\,, \tag{6.4.13}$$

where $Q(\psi_0)$ is an arbitrary function.

Expressing the contents of the brackets on the right-hand sides of (6.4.11) and (6.4.12) with the aid of (6.4.13) in terms of Q, one obtains the absolute vorticity integral

$$f_0 + \frac{1}{h_{x0}} \frac{\partial v_0}{\partial \zeta} = Q'(\psi_0) . \tag{6.4.14}$$

Equations (6.4.13) and (6.4.14) are equivalent to the equations of motion (6.4.6) and (6.4.7).

In order to determine $Q(\psi_0)$, attention will be turned on the boundary conditions. Since the current regime is known for $\zeta = \infty$, the value of Q on any stream line is readily found. The formal procedure is as follows. For $\zeta = \infty$, the Bernoulli type integral (6.4.13) assumes, by strength of the boundary conditions (6.4.10), the form

$$p_{g0} + Y = Q(\psi_{g0}) . \tag{6.4.15}$$

Furthermore, inverting the function $\psi_{g0} = \psi_{g0}(y)$, one finds $y = y(\psi_{g0})$. Substituting this expression for y on the left-hand side of (6.4.15), one obtains $Q(\psi_{g0})$. Since ψ_{g0} covers the same interval of values as ψ_0, the unknown function Q has been constructed.

Note that the function p_0 is defined exactly apart from an arbitrary constant $-c$; the constant c drops out of equations (6.4.13) and (6.4.14) and need not be written down.

Note that for $\zeta = \infty$ the absolute vorticity integral (6.4.14) may be written in the form

$$f_0 = Q'(\psi_{g0}) . \tag{6.4.16}$$

It follows from this relation that inertial boundary layer cannot exist along an entire coast line. In fact, the values of the Coriolis parameter $f_0(y)$ at points of entry and exit of a stream line from a boundary layer cannot coincide, since $\mathrm{d}f_0/\mathrm{d}y > 0$. This question will now be studied in greater detail.

First, an equation for the function ψ_0 will be constructed. For this purpose, one may combine relation (6.4.14) and the first equation (6.4.8) to arrive at a second-order ordinary differential equation the order of which may be reduced. However, it is simpler to proceed in the following manner.

Substituting for v_0 from the first relation (6.4.8) into (6.4.6), one finds

$$f_0 \frac{\partial \psi_0}{\partial \zeta} = \frac{\partial p_0}{\partial \zeta} .$$

Integrating this equation with respect to ζ and taking into account the boundary condition for $\zeta = \infty$, one obtains

$$p_0 = p_{g0} + f_0(\psi_0 - \psi_{g0}) . \tag{6.4.17}$$

Substituting this expression into the Bernoulli integral (6.4.13), one arrives at the equation

$$\frac{v_0^2}{2} = Q(\psi_0) - f_0(\psi_0 - \psi_{g0}) - p_{g0} - Y . \tag{6.4.18}$$

Finally, substituting into the first equation (6.4.8) for v_0 from (6.4.18), one finds the required first-order ordinary differential equation for $\psi_0(\zeta, y)$ which has for the boundary condition $\psi_0(0, y) = 0$ the solution

$$\frac{1}{h_{x0}} \int\limits_0^{\psi_0} \frac{d\psi_0}{v_0} = \zeta \,. \tag{6.4.19}$$

After ψ_0 has been found, the function v_0 may be found from (6.4.18), the function p_0 from (6.4.17) and the function u_0 from the second equation (6.4.8).

There arises now the following question: When is it possible to determine v_0 from (6.4.18) (or, in other words, when will the right-hand side of (6.4.18) be positive) and obtain from (6.4.19) a function ψ_0 which tends exponentially to the function ψ_{g0} for $\zeta \to \infty$? In order to answer this question, rewrite (6.4.18), taking account of (6.4.15) and (6.4.16), in the form

$$\frac{v_0^2}{2} = Q(\psi_0) - Q(\psi_{g0}) - Q'(\psi_{g0})(\psi_0 - \psi_{g0}) \,, \tag{6.4.20}$$

whence it follows that the function Q must be concave, i.e.,

$$Q''(\psi_{g0}) > 0 \,. \tag{6.4.21}$$

If a boundary layer exists, then $v_0 > 0$ and for large ζ

$$v_0 \simeq \sqrt{Q''(\psi_{g0})}(\psi_{g0} - \psi_0) + \ldots \tag{6.4.22}$$

When condition (6.4.21) is fulfilled, then (6.4.18) and (6.4.19) make sense. Furthermore, if $|\psi_0 - \psi_{g0}| << 1$, then the expansion (6.4.22) holds true and by (6.4.19) the variable $\zeta \sim -[1/(h_{x0})\sqrt{Q''(\psi_{g0})}] \log (\psi_{g0} - \psi_0)$; but this means that for large ζ the function ψ_0 tends exponentially to ψ_{g0}. Thus, it has turned out that condition (6.4.21) is also sufficient for the existence of an inertial boundary layer.

An expression will now be found for $Q''(\psi_{g0})$. Since

$$\frac{1}{h_{y0}} \frac{\partial \psi_{g0}}{\partial y} = -u_{g0} \,,$$

one finds

$$Q''(\psi_{g0}) = \frac{d}{d\psi_{g0}} Q'(\psi_{g0}) = -\frac{1}{h_{y0}u_{g0}} \frac{df_0}{dy} \,. \tag{6.4.23}$$

Since $df_0/dy > 0$, the condition for existence of an inertial boundary layer (6.4.21) may be written in the form

$$u_{g0} < 0 \,. \tag{6.4.24}$$

Thus, it is seen that the normal to shore component of the geostrophic velocity must be negative at the shore. In other words, at the outer edge of a boundary layer, the fluid must flow into the boundary layer. Therefore condition (6.4.24) may be formulated as follows: At a western shore of an ocean, there exists only an inertial boundary current the transport of which increases along its flow. Such a boundary current will be said to be accelerating.

Consider a region of a boundary current. It follows from (6.4.7) that for $\zeta = \infty$

$$f_0 u_{g0} = -\frac{\partial p_{g0}}{h_{y0}\partial y} + \tau_{y0} \,. \tag{6.4.25}$$

Setting in (6.4.20) the quantity $\psi_0 = 0$ and differentiating the relation obtained with respect to y, one obtains, by (6.4.15), (6.4.16), (6.4.23) and (6.4.25), that

$$\frac{\partial}{\partial y}\left(\frac{v_0^2}{2}\right) = \frac{\mathrm{d}f_0}{\mathrm{d}y}\,\psi_{g0} \qquad \text{for} \qquad \psi_0 = 0\,. \tag{6.4.26}$$

Since $v_0 > 0$ in the region of the boundary current and, by (6.4.23) and (6.4.24), $\psi_{g0} > 0$, then the velocity v_0 at the shore increases along the flow.

Note that the criterion of existence of an inertial boundary layer at a western coast of an ocean is determined by an integral characteristic of the wind field over the entire basin and is not linked directly to local features of the wind field near the coast itself.

Consider now Fig. 5.5. It is seen that in the southern part of the region under consideration $u_{g0} < 0$ (the stream lines of the current in the open ocean "run" into the shore) and criterion (6.4.24) is satisfied. This is the region of formation of the Gulfstream. As it moves along shore northwards, the velocity u_{g0} becomes zero and then positive; in this region, by (6.4.24), there does not exist an inertial boundary layer.

It is interesting to note that the configuration of the west coast of an ocean actually does not exert any influence on the existence criterion of a boundary layer.

Thus, an important step has been taken along the path towards a study of the structure of the Gulfstream: It has been shown that it is impossible for an inertial boundary layer to exist along an entire shore line. This is the basic non-linear effect of the model under consideration.

It is readily shown that at an eastern shore of the ocean there may only exist an inertial boundary current for which the transport decreases along the flow. Such a boundary current will be said to be decelerating.

6.5 INERTIAL—VISCOUS BOUNDARY LAYER

After this study of the limiting cases of viscous and inertial boundary layers, consider now the general problem of a coastal boundary current in the presence of viscous and inertial effects. In order to study this difficult problem, recourse will be taken to maximum simplification of the model. Firstly, introduce the β-plane approximation: Consider motion on a plane rotating Earth with a Coriolis parameter which depends linearly on latitude (in non-dimensional variables $f = f_0 + {}_a y$) (cf. § 3.8). Secondly, assume that shore boundaries pass along meridians and consider a very simple zonal wind field which depends only on latitude. It follows from §§ 6.3 and 6.4 that these simplifications do not lead to qualitative changes in the structure of viscous and inertial boundary currents. For example, if a shore nowhere touches a line φ = constant, then its configuration prescribes only a definite direction of the boundary current without exerting influence on the criterion of its existence.

Write down the boundary conditions. By strength of the assumptions relating to the shape of the coastal boundaries, the equation of the western shore will be $x = 0$, and that of the eastern shore $x = 1$ (in non-dimensional variables). Therefore conditions of no slip at rigid boundaries assume the form

$$\psi = 0, \qquad \frac{\partial \psi}{\partial x} = 0 \qquad \text{for} \qquad x = 0\,, \tag{6.5.1}$$

$$\psi = 0, \qquad \frac{\partial \psi}{\partial x} = 0 \qquad \text{for} \qquad x = 1\,. \tag{6.5.2}$$

Thus, the solution of the problem under consideration depends on two small parameters ϵ and δ which are coefficients of the highest derivatives of system (6.3.2)–(6.3.4). In other words, there arise two characteristic length scales L_i and L_v which describe the structure of the boundary layer [cf. (6.3.5)], and, at a first glance, it is not clear how to introduce a "stretched" variable and in what form to write down the asymptotic expansion of the solution of the problem.

The following approach will be adopted. Consider a boundary layer within the limit of which inertial and viscous terms have the same orders of magnitude (inertial–viscous boundary layer). As characteristic thickness of the boundary layer, select the scale L_i (by assumption, the scale L_v has the same order, but the choice of L_i will be more convenient below). Then the inertial terms will have within the limits of the boundary layer order $0(1)$ [in equation (6.3.3)], but the viscous terms order $0(\delta^3/\epsilon^3)$. It is convenient to introduce a Reynolds number for the boundary layer as a ratio of a characteristic magnitude of inertial terms to a characteristic magnitude of viscous terms in equation (6.3.3). One has

$$R = \left(\frac{\epsilon}{\delta}\right)^3 = \left(\frac{L_i}{L_v}\right)^3 = \frac{UL_i}{A_L}\,, \tag{6.5.3}$$

where U is the magnitude of the velocity at the outer edge of the boundary layer.

TABLE 6.I

Parameters A_L, R, L_v, δ

A_L cm^2/sec	R	L_v km	δ
10^8	$2.2 \cdot 10^{-2}$	79	$1.2 \cdot 10^{-2}$
10^7	$2.2 \cdot 10^{-1}$	37	$0.6 \cdot 10^{-2}$
10^6	2.2	17	$0.3 \cdot 10^{-2}$
10^5	$2.2 \cdot 10$	7.9	$1.2 \cdot 10^{-3}$

All determining parameters of the problem (a, β_0, τ_0) with the exception of A_L may be assumed to be sufficiently well known. The range of possible values of A_L is very wide. Therefore the structure of boundary layers will be studied for different values of R, and in this manner values of A_L will be found for which the theory is closest to observations.

For example, let $U = 1$ cm/sec; then, since $\beta_0 = 2 \cdot 10^{-13}$ cm^{-1} sec^{-1}, one finds $L_i = 22$ km and $\epsilon = 3.4 \cdot 10^{-3}$; the following table is readily constructed (Table 6.I).

Replacing the parameter δ^3 by ϵ^3/R, rewrite equations (6.3.2)—(6.3.4), taking account of the assumptions formulated at the beginning of this section, in the form

$$\epsilon^2\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - fv = -\frac{\partial p}{\partial x} + \frac{\epsilon^3}{R}\,\Delta_h u + \tau\,, \tag{6.5.4}$$

$$\epsilon^2\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + fu = -\frac{\partial p}{\partial y} + \frac{\epsilon^3}{R}\,\Delta_h v\,, \tag{6.5.5}$$

$$\frac{\partial \psi}{\partial x} = v, \qquad \frac{\partial \psi}{\partial y} = -u\,. \tag{6.5.6}$$

The structure of the inertial—viscous boundary layer will now be studied. Fix the Reynolds number R and assume it to be of order unity. Then the system (6.5.4)—(6.5.6) will contain only one small parameter ϵ multiplying the highest derivative and known asymptotic methods may be applied. Thus, seek the solution of the problem for the entire basin in the form

$$u = u_0(x, y) + \ldots + \tilde{u}_0\left(\frac{x}{\epsilon}, y\right) + \ldots + \tilde{\tilde{u}}_0\left(\frac{1-x}{\epsilon}, y\right) + \ldots,$$

$$v = v_0(x, y) + \ldots + \frac{1}{\epsilon}\,\tilde{v}_0\left(\frac{x}{\epsilon}, y\right) + \ldots + \frac{1}{\epsilon}\tilde{\tilde{v}}_0\left(\frac{1-x}{\epsilon}, y\right) + \ldots,$$

$$p = p_0(x, y) + \ldots + \tilde{p}_0\left(\frac{x}{\epsilon}, y\right) + \ldots + \tilde{\tilde{p}}_0\left(\frac{1-x}{\epsilon}, y\right) + \ldots,$$

$$\psi = \psi_0(x, y) + \ldots + \tilde{\psi}_0\left(\frac{x}{\epsilon}, y\right) + \ldots + \tilde{\tilde{\psi}}_0\left(\frac{1-x}{\epsilon}, 1\right) + \ldots, \tag{6.5.7}$$

Obviously, exponential decay of the functions $\tilde{u}_0, \tilde{v}_0, \tilde{p}_0, \tilde{\psi}_0, \ldots$ is required for $\zeta_1 \to \infty$ ($\zeta_1 = x/\epsilon$) and of the functions $\tilde{\tilde{u}}_0, \tilde{\tilde{v}}_0, \tilde{\tilde{p}}_0, \tilde{\tilde{\psi}}_0, \ldots$ for $\zeta_2 \to \infty$ $[\zeta_2 = (1-x)/\epsilon]$. Note that, on the basis of the assumptions made, there is no need to have a special coordinate system for the description of the boundary layer; the role of the "rapid" variable is played by x.

Substitution of the expansions (6.5.7) into (6.5.4)–(6.5.6) yields after simple transformations for the first approximation the equations

$$-fv_0 = -\frac{\partial p_0}{\partial x} + \tau , \tag{6.5.8}$$

$$fu_0 = -\frac{\partial p_0}{\partial y} , \tag{6.5.9}$$

$$\frac{\partial \psi_0}{\partial x} = v_0, \qquad \frac{\partial \psi_0}{\partial y} = -u_0 , \tag{6.5.10}$$

$$-f\tilde{v}_0 = -\frac{\partial \tilde{p}_0}{\partial \zeta_1} , \tag{6.5.11}$$

$$(u_{00} + \tilde{u}_0)\frac{\partial \tilde{v}_0}{\partial \zeta_1} + \tilde{v}_0\frac{\partial \tilde{v}_0}{\partial y} + f\tilde{u}_0 = -\frac{\partial \tilde{p}_0}{\partial y} + \frac{1}{R}\frac{\partial^2 \tilde{v}_0}{\partial \zeta_1^2} , \tag{6.5.12}$$

$$\frac{\partial \tilde{\psi}_0}{\partial \zeta_1} = \tilde{v}_0, \qquad \frac{\partial \tilde{\psi}_0}{\partial y} = -\tilde{u}_0 , \tag{6.5.13}$$

$$-f\tilde{\tilde{v}}_0 = \frac{\partial \tilde{\tilde{p}}_0}{\partial \zeta_2} , \tag{6.5.14}$$

$$-(u_{01} + \tilde{\tilde{u}}_0)\frac{\partial \tilde{\tilde{v}}_0}{\partial \zeta_2} + \tilde{\tilde{v}}_0\frac{\partial \tilde{\tilde{v}}_0}{\partial y} + f\tilde{\tilde{u}}_0 = -\frac{\partial \tilde{\tilde{p}}_0}{\partial y} + \frac{1}{R}\frac{\partial^2 \tilde{\tilde{v}}_0}{\partial \zeta_2^2} , \tag{6.5.15}$$

$$-\frac{\partial \tilde{\tilde{\psi}}_0}{\partial \zeta_2} = \tilde{\tilde{v}}_0, \qquad \frac{\partial \tilde{\tilde{\psi}}_0}{\partial y} = -\tilde{\tilde{u}}_0 . \tag{6.5.16}$$

Here and below, $u_{00} = u_0(0, y)$, $p_{00} = p_0(0, y)$, $\psi_{00} = \psi_0(0, y)$, $u_{01} = u_0(1, y)$, $p_{01} = p_0(1, y)$, $\psi_{01} = \psi_0(1, y)$.

Note that in a description of inertial–viscous boundary layers inertial and viscous effects are essential only in the equation of motion along the y-axis (along the current); the equation of motion along the x-axis (across the current) is geostrophic. This comment applies also with respect to viscous and inertial boundary layers. It follows from this remark that the analogue for equation (6.4.17) for the inertial boundary is likewise true in the case under consideration. Combining (6.5.11) and (6.5.13), and likewise (6.5.14) and (6.5.16), and taking the conditions at infinity into consideration, one has

$$\tilde{p}_0 = f\tilde{\psi}_0 , \tag{6.5.17}$$

$$\tilde{\tilde{p}}_0 = f\tilde{\tilde{\psi}}_0 \,. \tag{6.5.18}$$

Substituting this expression for $\tilde{p}_0$ into (6.5.12), the system of equations (6.5.11)—(6.5.13) is readily reduced to the equation for $\tilde{\psi}_0$

$$\left(u_{00} - \frac{\partial \tilde{\psi}_0}{\partial y}\right) \frac{\partial^2 \tilde{\psi}_0}{\partial \zeta_1^2} + \frac{\partial \tilde{\psi}_0}{\partial \zeta_1} \frac{\partial^2 \tilde{\psi}_0}{\partial \zeta_1 \partial y} + \tilde{\psi}_0 = \frac{1}{R} \frac{\partial^3 \tilde{\psi}_0}{\partial \zeta_1^3} \,. \tag{6.5.19}$$

Equations (6.5.14)—(6.5.16) are reduced in an analogous manner to the equation for $\tilde{\tilde{\psi}}_0$

$$\left(u_{01} - \frac{\partial \tilde{\tilde{\psi}}_0}{\partial y}\right) \frac{\partial^2 \tilde{\tilde{\psi}}_0}{\partial \zeta_2^2} + \frac{\partial \tilde{\tilde{\psi}}_0}{\partial \zeta_2} \frac{\partial^2 \tilde{\tilde{\psi}}_0}{\partial \zeta_2 \partial y} + \tilde{\tilde{\psi}}_0 = -\frac{1}{R} \frac{\partial^3 \tilde{\tilde{\psi}}_0}{\partial \zeta_2^3} \,. \tag{6.5.20}$$

The system (6.5.8)—(6.5.10) is readily reduced to the equation for ψ_0

$$\frac{\partial \psi_0}{\partial x} + \tau' = 0 \,. \tag{6.5.21}$$

Note that the terms with Coriolis parameter drop out from equations (6.5.19)—(6.5.21) and only the latitudinal change of the Coriolis parameter is essential.

Thus, one has arrived at the required equations for the functions ψ_0, $\tilde{\psi}_0$ and $\tilde{\tilde{\psi}}_0$ which describe the motion in the open ocean and within the limits of the boundary layers. Note that the equations for $\tilde{\psi}_0$ and $\tilde{\tilde{\psi}}_0$ depend for the present still on the unknown function u_0. Likewise, it is seen that, in contrast to what happens in the case of the viscous boundary layer, equations (6.5.19) and (6.5.20) are non-linear also in the partial derivatives. It is also obvious that the equations of the inertial—viscous boundary layer do not have integrals of the Bernoulli and absolute vorticity type. Therefore numerical analysis must be employed for their solution.

The boundary conditions for equations (6.5.19)—(6.5.21) will be derived next. Substitution of the expansion (6.5.7) into (6.5.1) and (6.5.2) yields

$$\psi_0(0, y) + \tilde{\psi}_0(0, y) = 0, \qquad \psi_1(0, y) + \tilde{\psi}_1(0, y) = 0 \,, \tag{6.5.22}$$

$$\frac{\partial \tilde{\psi}_0}{\partial \zeta_1}(0, y) = 0, \qquad \frac{\partial \psi_0}{\partial x}(0, y) + \frac{\partial \tilde{\psi}_1}{\partial \zeta_1}(0, y) = 0 \,, \tag{6.5.23}$$

$$\tilde{\tilde{\psi}}_0(0, y) + \psi_0(1, y) = 0, \qquad \tilde{\tilde{\psi}}_1(0, y) + \psi_1(1, y) = 0 \,, \tag{6.5.24}$$

$$\frac{\partial \tilde{\tilde{\psi}}_0}{\partial \zeta_2}(0, y) = 0, \qquad \frac{\partial \psi_0}{\partial x}(1, y) - \frac{\partial \tilde{\tilde{\psi}}_1}{\partial \zeta_2}(0, y) = 0 \,. \tag{6.5.25}$$

It is seen that the equations and boundary conditions for the functions ψ_0, $\tilde{\psi}_0$ and $\tilde{\tilde{\psi}}_0$ are "interlinked". They are "disentangled" in the following manner.

Revert to equations (6.5.19) and (6.5.20). Their solutions in the half-strip $0 \leqslant y \leqslant 1$, $0 \leqslant \zeta_1, \zeta_2 < \infty$ $[\tau'(0) = \tau'(1) = 0]$ will be considered. By sense of the expansions, the functions $\tilde{\psi}_0$, $\tilde{\tilde{\psi}}_0$, ... must decay exponentially for large ζ_1 and ζ_2. It is natural to assume that the behaviour of such functions for large ζ_1 and ζ_2 is determined by the linear terms of equations (6.5.19) and (6.5.20). Under those circumstances, one will have for large ζ_1 and ζ_2, respectively,

$$u_{01}\frac{\partial^2\tilde{\tilde{\psi}}_0}{\partial\zeta_2^2} + \tilde{\tilde{\psi}}_0 = -\frac{1}{R}\frac{\partial^3\tilde{\tilde{\psi}}_0}{\partial\zeta_2^3}. \tag{6.5.26}$$

$$u_{00}\frac{\partial^2\tilde{\psi}_0}{\partial\zeta_1^2} + \tilde{\psi}_0 = \frac{1}{R}\frac{\partial^3\tilde{\psi}_0}{\partial\zeta_1^3}. \tag{6.5.27}$$

Equations (6.5.26) and (6.5.27) are ordinary differential equations with the characteristic equations

$$u_{00}\nu^2 + 1 = \frac{1}{R}\nu^3, \tag{6.5.28}$$

$$u_{01}\nu^2 + 1 = -\frac{1}{R}\nu^3. \tag{6.5.29}$$

Note equations (6.5.28) and (6.5.29) do not have purely imaginary roots for any positive R. However, then equation (6.5.28) has always two roots in the left half-plane of the complex variable ν, and equation (6.5.29) only one such root. Therefore equation (6.5.26) has two linearly independent solutions which are damped at infinity, while equation (6.5.27) has only one such solution. Hence it may be concluded that one must have for the determination of solutions of the non-linear equations (6.5.19) and (6.5.20), which decrease for large ζ_1 and ζ_2, two conditions for $\zeta_1 = 0$ in the case of equation (6.5.19) and a single condition for $\zeta_2 = 0$ in the case of equation (6.5.20).

However, under these conditions, by the first condition (6.5.25) and the homogeneity of equation (6.5.20), one has $\tilde{\tilde{\psi}}_0 \equiv 0$ and, according to the first condition (6.5.24), $\psi_0(1, y) = 0$; hence equation (6.5.21) readily yields the solution

$$\psi_0(x, y) = \tau'(1 - x). \tag{6.5.30}$$

Note that for all functions $\tilde{\tilde{\psi}}_1$, $\tilde{\tilde{\psi}}_2$, $\tilde{\tilde{\psi}}_3, \ldots$ simple linear equations are obtained. Only the formula for $\tilde{\tilde{\psi}}_1$ will be written down here:

$$\tilde{\tilde{\psi}}_1 = -\frac{\tau'}{\nu}e^{\nu\zeta_2}, \qquad \nu < 0. \tag{6.5.31}$$

The problem has now been reduced to analysis of the following non-linear equation for the inertial—viscous boundary layer at a western shore of an ocean:

$$-\frac{1}{R}\frac{\partial^3\tilde{\psi}_0}{\partial\psi_1^3}+\frac{\partial\tilde{\psi}_0}{\partial\zeta_1}\frac{\partial^2\tilde{\psi}_0}{\partial\zeta_1\partial y}-\left(\tau''+\frac{\partial\tilde{\psi}_0}{\partial y}\right)\frac{\partial^2\tilde{\psi}_0}{\partial\zeta_1^2}+\tilde{\psi}_0=0 \tag{6.5.32}$$

$$\tilde{\psi}_0(0,y)+\tau'=0,\qquad \frac{\partial\tilde{\psi}_0}{\partial\zeta_1}(0,y)=0,\qquad \tilde{\psi}_0(\infty,y)=0\,. \tag{6.5.33}$$

The function $\tilde{p}_0$ is readily found from (6.5.17), once $\tilde{\psi}_0$ has been determined. The arbitrary constant in the formula for $\tilde{p}_0$ is not essential.

Thus, after construction of $\tilde{\psi}_0$, the solution of the problem under consideration may be presented in the form:

In the open ocean

$$\psi=\tau'(1-x)+O(\epsilon)\,; \tag{6.5.34}$$

At a western shore of an ocean ($x\sim\epsilon$)

$$\psi=\tau'+\tilde{\psi}_0(\zeta_1,y)+O(\epsilon),$$

$$\frac{\partial\psi}{\partial x}=\frac{1}{\epsilon}\frac{\partial\tilde{\psi}_0}{\partial\zeta_1}+O(1),\qquad \zeta_1=\frac{x}{\epsilon}\,; \tag{6.5.35}$$

At an eastern shore of an ocean ($1-x\sim\epsilon$)

$$\psi=O(\epsilon),\qquad \frac{\partial\psi}{\partial x}=-\tau'(1-e^{\nu\zeta_2})+O(\epsilon),\qquad \zeta_2=\frac{1-x}{\epsilon},\qquad \nu<0\,. \tag{6.5.36}$$

In this way, strong currents arise only at western shores of an ocean. The transport of such a boundary current is readily computed and turns out to be equal to τ'. It should be noted specially that the validity of Sverdrup's solution (6.5.30) in the open ocean has again been demonstrated, but this time in a more general case; speaking more exactly, it is true in those regions of the ocean where one may "concoct" a boundary layer described by problem (6.5.32) and (6.5.33). Then the proof given is based essentially on consideration of the terms of horizontal turbulent transfer (cf. § 6.2). If these terms were not present, then the characteristic equation (6.5.29) in the general case might not have a root with a negative real part and, above all, the boundary condition (6.5.25) would be absent. In other words, it could not be proved that $\tilde{\tilde{\psi}}_0\equiv 0$, and one would not obtain (6.5.30).

However, it must be emphasized again that the current in the open ocean is linked to the structure of the coastal boundary conditions at the shores. This peculiar fact does not take place, for example, in the study of Prandtl's boundary layer in classical hydrodynamics.

The results of the "linear" analysis of Problem (6.5.32) and (6.5.33) will be presented first. Linearizing equation (6.5.32), one obtains

$$-\frac{1}{R}\frac{\partial^3\tilde{\psi}_0}{\partial\zeta_1^3}-\tau''\frac{\partial^2\tilde{\psi}_0}{\partial\zeta_1^2}+\tilde{\psi}_0=0 \tag{6.5.37}$$

with the boundary conditions (6.5.33). The solution of Problems (6.5.37) and (6.5.33) has the form

$$\tilde{\psi}_0 = C_1(y)\, e^{\nu_1 \zeta_1} + C_2(y)\, e^{\nu_2 \zeta_1} \,,$$

where ν_1 and ν_2 are roots of the polynomial

$$-\frac{1}{R}\nu^3 - \tau''\nu^2 + 1 = 0 \,,$$

for which $Re\ \nu_{1,2} < 0$ (such ν_1 and ν_2 have been seen to always exist) and C_1 and C_2 are easily found from (6.5.33).

It is of interest to investigate problem (6.5.37) and (6.5.33) for limiting values of the Reynolds number R. For $R << 1$, one obtains readily

$$\tilde{\psi}_0 = -\frac{2}{\sqrt{3}}\tau' \exp\left(-\frac{\zeta_1\sqrt[3]{R}}{2}\right)\sin\left(\frac{\sqrt{3}}{2}\zeta_1\sqrt[3]{R} + \frac{\pi}{3}\right) + O(R^{2/3}) \,. \tag{6.5.38}$$

Recalling the definitions of ζ_1 and R, one has

$$\zeta_1\sqrt[3]{R} = \frac{x}{\delta} \,.$$

As had to be expected, this is the solution for the viscous boundary layer. Note that (6.5.38) holds true for any R in a narrow strip with respect to y near a shore, where $\tau'' = 0$.

Next, let $R >> 1$. In this case, three regions in terms of y may be distinguished:

(1) The region where $\tau'' > 0$ or $u_{00} < 0$; one easily finds for large R

$$\tilde{\psi}_0 = -\tau' \exp\left(-\frac{\zeta_1}{\sqrt{\tau''}}\right) + \frac{\tau'}{R(\tau'')^{3/2}} \exp\left(-\frac{\tau''\zeta_1}{1/R}\right) + \ldots$$

Thus, the inertial—viscous boundary layer would split into an inertial layer with thickness of order ϵ and a viscous sub-layer with thickness of order ϵ/R (recall that $\zeta_1 = x/\epsilon$).

(2) The region where $\tau'' = 0$; on approaching this region, the thickness of the inertial layer decreases as the viscous sub-layer "becomes fat"; in the very region where $\tau'' = 0$ the role of the "inertial" terms is insignificant, and the solution is given by (6.5.38).

(3) The region where $\tau'' < 0$ or $u_{00} > 0$; for large R, one has

$$\tilde{\psi}_0 = -\tau' \exp\left(-\frac{\zeta_1}{2R\tau''^2}\right)\cos\frac{\zeta_1}{\sqrt{|\tau''|}} + \ldots$$

It is seen that splitting into inertial and viscous regions does not exist; the effect of diffusion of relative vorticity is essential within the limits of the boundary layer the thickness of which is of order $R\epsilon$ (it increases with growing R).

These are the results of "linear" analysis. Naturally, the linearized equation (6.5.37) is true only for large values of ζ_1. However, it will be seen below that from a qualitative point of view many effects are correctly assessed by the "linear" theory. Therefore the deductions from "linear" theory will serve as a lead.

6.6 THE BOUNDARY LAYER FOR LARGE AND SMALL REYNOLDS NUMBERS

The coefficient of turbulent transfer A_L (and thereby also the Reynolds number R) will now be varied. Assume the equation (6.5.32) for the inertial—viscous boundary layer is true not only for finite values of R, but also for large and small R (it is not difficult to verify this assumption, once the corresponding solutions have been constructed).

Consider first the region of an accelerating boundary current ($\tau'' > 0$). Relying on the results of the "linear" analysis, it is natural to expect for $R >> 1$ splitting of the inertial—viscous boundary layer into an inertial region and a viscous sub-layer. Therefore the solution of problem (6.5.32) will be sought in the form

$$\tilde{\psi}_0(\zeta_1, y) = \varphi_0(\zeta_1, y) + \frac{1}{\sqrt{R}}\varphi_1(\zeta_1, y) + \ldots + \frac{1}{\sqrt{R}}\tilde{\varphi}_1(\zeta_1\sqrt{R}, y) + \ldots \qquad (6.6.1)$$

The choice of an expansion in powers of $1/\sqrt{\mathrm{R}}$ may be explained as follows. If one disregards completely the viscous term in equation (6.5.32), then one obtains an inertial boundary layer which satisfies the condition $u_0 + \tilde{u}_0 = 0$ for $x = 0$. The velocity component $\tilde{v}_0$ does not vanish for this case at the shore, and, in order to remove such discrepancy, one requires a viscous sublayer at the wall. Within the limits of such a viscous sub-layer, the term $-(1/R)\partial^3\tilde{\psi}_0/\partial\zeta_1^3$ must have the same order as the inertial term $(\partial\tilde{\psi}_0/\partial\zeta_1)(\partial^2\tilde{\psi}_0/\partial\zeta_1\partial y)$. However, since the viscous sublayer is described by a term of the form $(1/R^\alpha)\psi_0(\zeta_1 R^\alpha, y)$, it is clear that one has $\alpha = \frac{1}{2}$.

Substituting the expansion (6.6.1) into (6.5.32) and the boundary condition (6.5.33), one arrives at the equations

$$\frac{\partial\varphi_0}{\partial\zeta_1}\frac{\partial^2\varphi_0}{\partial\zeta_1\partial y} - \left(\tau'' + \frac{\partial\varphi_0}{\partial y}\right)\frac{\partial^2\varphi_0}{\partial\zeta_1^2} + \varphi_0 = 0\,, \qquad (6.6.2_0)$$

. . . .

$$-\frac{\partial^3\tilde{\varphi}_1}{\partial\sigma^3} + \left.\frac{\partial\varphi_0}{\partial\zeta_1}\right|_0\frac{\partial^2\tilde{\varphi}_1}{\partial\sigma\partial y} + \left.\frac{\partial^2\varphi_0}{\partial\zeta_1\partial y}\right|_0\frac{\partial\tilde{\varphi}_1}{\partial\sigma} + \frac{\partial\tilde{\varphi}_1}{\partial\sigma}\frac{\partial^2\tilde{\varphi}_1}{\partial\sigma\partial y}$$

$$-\left[\left.\frac{\partial^2\varphi_0}{\partial\zeta_1\partial y}\right|_0\sigma + \left.\frac{\partial\varphi_1}{\partial y}\right|_0 + \frac{\partial\tilde{\varphi}_1}{\partial y}\right]\frac{\partial^2\tilde{\varphi}_1}{\partial\sigma^2} = 0, \qquad \sigma = \zeta\sqrt{R} \qquad (6.6.3_1)$$

. . . .

and the boundary conditions

$$\varphi_0(0, y) + \tau' = 0, \tag{6.6.4$_0$}$$

. . . .

$$\varphi_1(0, y) + \tilde{\varphi}_1(0, y) = 0, \qquad \frac{\partial \varphi_0}{\partial \zeta_1}(0, y) + \frac{\partial \tilde{\varphi}_1}{\partial \sigma}(0, y) = 0, \qquad \tilde{\varphi}_1(\infty, y) = 0\,. \tag{6.6.5$_1$}$$

. . . .

Obviously, equation (6.6.2$_0$) for the boundary condition (6.6.4$_0$) describes the structure of the inertial boundary layer. This problem has been studied in detail in § 6.4, where existence of an inertial layer and its degeneration have been proved for $\tau'' > 0$ and for $\tau'' = 0$, respectively. It has also been shown that, by (6.4.26), the velocity $\tilde{v}_0$ at the wall grows with increasing y.

Next, consider equation (6.6.3$_1$). Introduce the new function

$$\chi = \tilde{\varphi}_1(\sigma, y) + \varphi_1(0, y) + \sigma \frac{\partial \varphi_0}{\partial \zeta_1}(0, y)\,, \tag{6.6.6}$$

which yields, by (6.6.1), the values of the stream function within the limits of the viscous boundary layer with an accuracy of up to $1/\sqrt{R}$. Proceeding in (6.6.3$_1$) and (6.6.5$_1$) to the function χ, one obtains

$$-\frac{\partial^3 \chi}{\partial \sigma^3} + \frac{\partial \chi}{\partial \sigma}\frac{\partial^2 \chi}{\partial \sigma \partial y} - \frac{\partial \chi}{\partial y}\frac{\partial^2 \chi}{\partial \sigma^2} = \left.\left(\frac{\partial \varphi_0}{\partial \zeta_1}\frac{\partial^2 \varphi_0}{\partial \zeta_1 \partial y}\right)\right|_0, \tag{6.6.7}$$

$$\frac{\partial \chi}{\partial \sigma}(0, y) = 0, \qquad \chi(0, y) = 0, \qquad \frac{\partial \chi}{\partial \sigma}(\infty, y) = \left.\frac{\partial \varphi_0}{\partial \zeta_1}\right|_0. \tag{6.6.8}$$

where $(\partial \varphi_0/\partial \zeta_1)|_0 = \tilde{v}_{00}$ is the velocity of the inertial boundary layer at the shore.

Problem (6.6.7) and (6.6.8) is nothing else but Prandtl's problem of a viscous boundary layer along a wall, where the role of the velocity of the outer flow is played by $\tilde{v}_{00}$. Since the velocity $\tilde{v}_{00}$ increases when one moves along the flow, the solution of problem (6.6.7) and (6.6.8) must exist. Thus, "linear" analysis predicted accurately the effect of splitting of the inertial–viscous boundary layer for $\tau'' > 0$ and $R >> 1$; it must only be noted that the viscous sub-layer has thickness of order $1/\sqrt{R}$, and not $1/R$, as in the "linear" theory.

Problems (6.5.32) and (6.5.33) have been analyzed numerically for $\tau' = 4y(1 - y)$ (cf. [44]). Figs. 6.4 and 6.5 demonstrate the phenomenon of splitting of the inertial–viscous boundary layer for $0 \leqslant y \leqslant 0.5$. Already for $R \sim 10$, the region is sufficiently clearly subdivided into inertial flow and viscous sub-layer (this is seen more distinctly in Fig. 6.5).

It is useful to study the effect of splitting of the inertial–viscous boundary layer for large R from the point of view of vorticity transfer (§ 6.2).

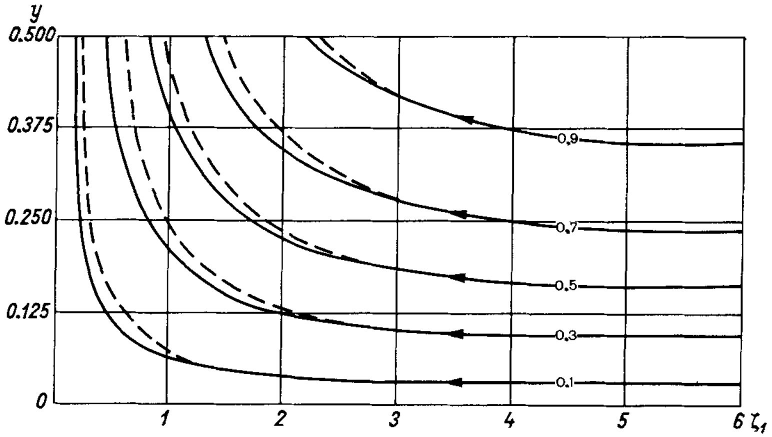


Fig. 6.4. Isolines of the stream function ψ for boundary flow [44]. Solid line = inertial flow $\psi = \tau'(y) + \varphi_0$; broken line = inertial viscous flow $\psi = \tau'(y) + \tilde{\psi}_0$; $R = 16$, $\tau' = 4y(1-y)$.

Since diffusion of relative vorticity has been seen to be essential only in the general vorticity balance, viscosity may only play a role in a narrow sub-layer near shore (for $R >> 1$). However, in such a case, the basic region of the boundary layer will be free from viscous effects, and the absolute vorticity of particles $\omega - f$ will not change as it moves in this region [cf. 6.4.14)]. However, then also the stream lines must be crowded towards shore (the more so, the larger is ω in the inertial region) and such flow must be accelerated (its total transport must increase when one moves along the flow).

As the boundary flow slows down, the stream lines must abandon the boundary layer. However, then vorticity balance is only possible under conditions when viscosity acts throughout the entire layer. Since for large R the action of viscosity in the region of acceleration of boundary flow is concentrated within the limits of a very narrow coastal layer, right up to $y = 0.5$, and on slowing down of the boundary current viscosity must be essential throughout the entire layer even for very large R, it is clear that such "jumps" in the thickness of the viscous layer are impossible and therefore there cannot exist a slowed-down boundary layer for large R.

Thus, one has arrived at a basis on which to assume that in the northern half of a basin ($\tau'' < 0$) there does not exist an inertial—viscous boundary layer for $R >> 1$. However, the results of numerical analysis (omitted here because of its complexity, cf. [44]) show that this layer is already not present for $R \sim 10$. It will only be noted that inclusion of non-linear terms in (6.5.32) completely alters the deductions of the "linear" analysis relating to the behaviour of the inertial—viscous boundary layer for $y \to \frac{1}{2}$. Recall that the linearized equation (6.5.37) has a solution of the boundary layer type

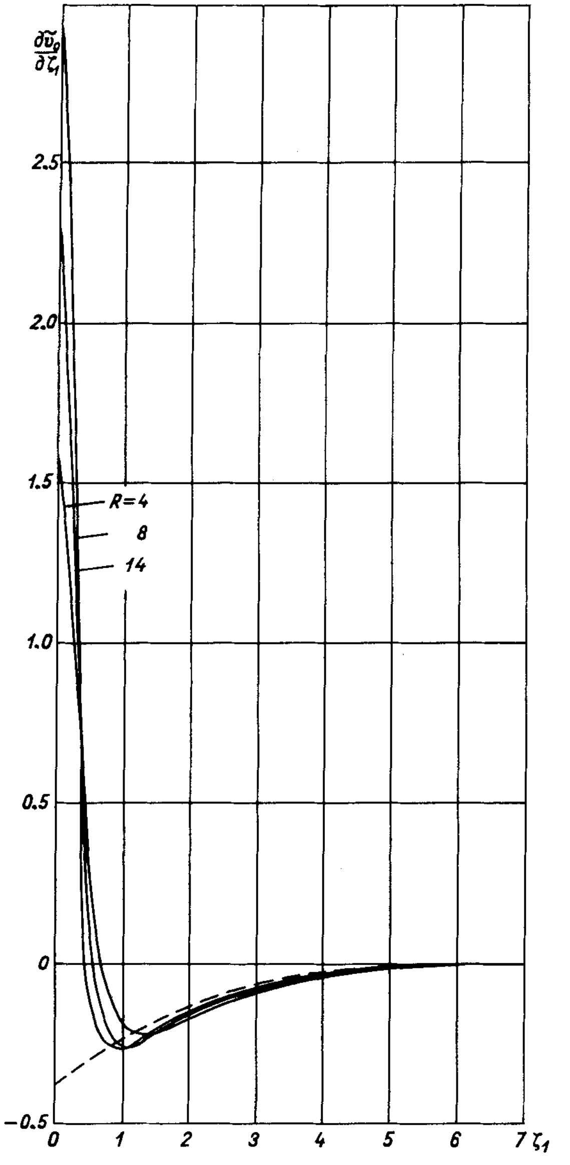


Fig. 6.5. Distribution of vorticity $\partial\tilde{v}_0/\partial\zeta_1$ for inertial—viscous flow for $y = \frac{3}{8}$ and different values of R [44]. Broken line = corresponding vorticity distribution $\partial^2\varphi_0/\partial\zeta_1^2$ for inertial current; $\tau' = 4y(1 - y)$.

for all y and R; the complete equation (6.5.32) does not have this property.

It is important to note that, since there does not exist a boundary layer for $\tau'' > 0$ and $R >> 1$, Sverdrup's relation for the open ocean for $\tau'' < 0$ and $R >> 1$ may not be fulfilled.

Consider now the solution of Problem (6.5.32) and (6.5.33) for small R. Introduce the new variable $\eta = \zeta_1\sqrt[3]{R} = x/\delta$, then the function $\tilde{\psi}_0(\eta, y)$ may be sought in the form

$$\tilde{\psi}_0(\eta, y) = \tilde{\psi}_{00}(\eta, y) + R^{2/3}\tilde{\psi}_{01}(\eta, y) + \dots \tag{6.6.9}$$

Construction of the functions $\tilde{\psi}_{00}$, $\tilde{\psi}_{01}$, ... which are solutions of ordinary linear differential equations does not encounter principal difficulties. The function $\tilde{\psi}_{00}$ describes the viscous boundary layer:

$$\tilde{\psi}_{00} = -\frac{2}{\sqrt{3}}\tau' \exp\left(-\frac{\eta}{2}\right) \sin\left(\frac{\sqrt{3}}{2}\eta + \frac{\pi}{3}\right). \tag{6.6.10}$$

Consider the character of the boundary layer for various Reynolds numbers. For not large R, it is possible to find numerically the solution of problem (6.5.32) and (6.5.33) for all y $(0 \leqslant y \leqslant 1)$[for $\tau' = 4y(1 - y)$ this was possible for $R < 0.5$]. For R of order 1/512 and 1/256, the function $\tilde{\psi}_0$ was sufficiently close to the solution (6.6.10). Since in this case the thickness of the boundary layer is of order δ (and not ϵ), then one obtains in terms of ζ_1 a sufficiently "thick" layer. As R increases, the boundary layer loses symmetry with respect to $y = 0.5$ and the centre of rotation shifts upwards along the flow. The thickness of the layer for $\tau'' < 0$ is somewhat reduced and the stream lines acquire a distinct oscillatory character with a wave length which is less than for very small R. For $\tau'' > 0$, the thickness of the boundary layer decreases as R increases more sharply, and the oscillations of the stream lines become insignificant (Figs. 6.6 and 6.7). When R increases further, as is already known, in the region $\tau'' > 0$ the boundary layer becomes basically inertial; in the region $\tau'' < 0$ there does not exist a boundary layer for $R >> 1$.

Table 6.II gives an idea of the dependence of the thickness of the boundary layer ζ_b on R for $\tau' = 4y(1 - y)$ for all values of y. [Equation (6.5.32)

TABLE 6.II

Dependence of ζ_b on R

R	ζ_b			γ_b	R	ζ_b			γ_b
	$y = 1/16$	$y = 1/2$	$y = 15/16$			$y = 1/16$	$y = 1/2$	$y = 15/16$	
1/512	50	52	54	$2 \cdot 10^{-6}$	1/4	10	20	43	2^{-14}
1/128	32	50	50	$2 \cdot 10^{-6}$	1/2	8	12	—	2^{-14}
1/64	32	40	40	$2 \cdot 10^{-6}$	1	8	10	—	2^{-14}
1/32	10	21	29	2^{-14}	4	7	7.8	—	2^{-14}
1/16	9	23	24	2^{-14}	8	6.8	7.8	—	2^{-14}
1/8	8	19	27	2^{-14}	16	6.8	7.8	—	2^{-14}

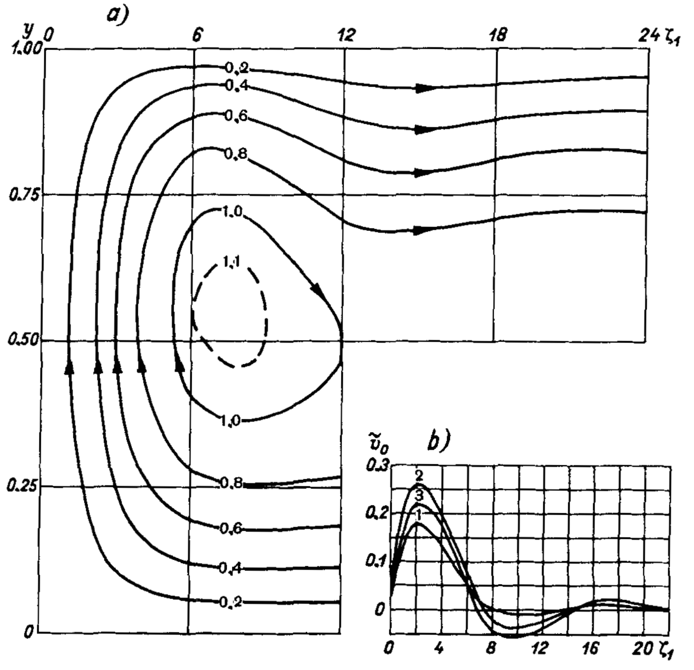


Fig. 6.6. Stream lines of $\psi = \tau'(y) + \widetilde{\psi}_0$ for the boundary flow $R = \frac{1}{8}$, $\tau' = 4y(1-y)$. a). Corresponding graph of $\widetilde{v}_0(\zeta_1)$ for different values of y. b). [44]: $1 - y = \frac{1}{4}$, $2 - y = \frac{1}{2}$, $3 - y = \frac{3}{4}$.

with the conditions (6.5.33) has been solved numerically for $\zeta = 0$ and $\widetilde{\psi}_0(\zeta_b, y) = 0$ with ζ_b chosen in such a manner that $(\partial \widetilde{\psi}_0/\partial \zeta_1)^2 + (\partial \widetilde{\psi}_0/\partial y)^2 < \gamma_b$ for $\zeta_1 = \zeta_b$ and given γ_b. Therefore ζ_b must be assumed to be the thickness of the boundary layer].

Thus, for large Reynolds number R, there does not exist a stationary inertial—viscous boundary layer in the northern half of a basin. Therefore it may be asserted that also Sverdrup's relation is not fulfilled there in the open ocean. This result appears to be quite natural, since it was hard to expect that the ordinary Sverdrup relation for H = constant could describe such features of currents in the open ocean as the meanders of the continuation of the Gulfstream, etc. Apparently, it is necessary to take into account for this purpose new factors: Bottom relief, density inhomogeneity, etc. Note that the described pattern is already existing for values of $R \sim 10$ which corresponds to $A_L = 10^5 \div 10^6$ cm^2/sec (cf. Table 6.I).

This is an example of estimation of the coefficient of turbulent transfer A_L based on qualitative agreement of theoretical patterns and observations. Previously, exchange coefficients had been assessed by means of "fitting" of theoretical values of one or the other quantity (for example, the width of

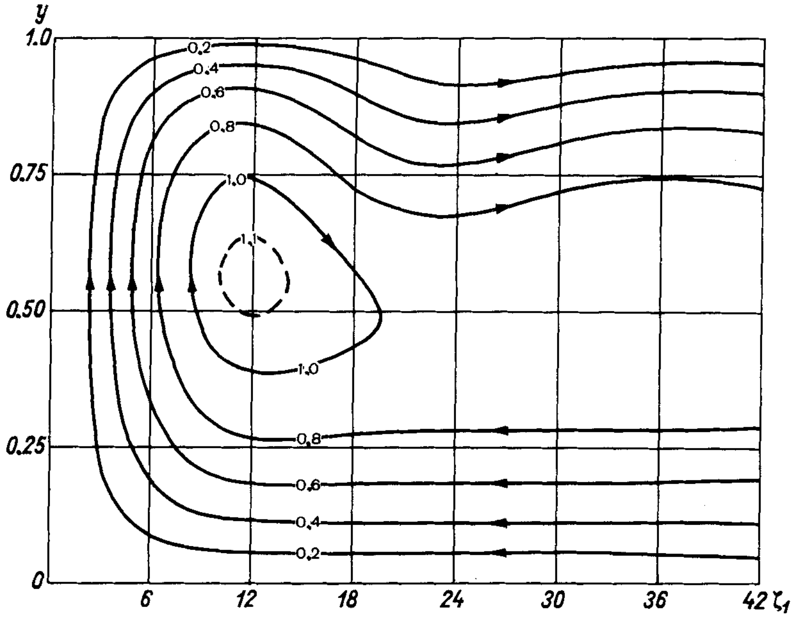


Fig. 6.7. Stream lines $\psi = \tau'(y) + \tilde{\psi}_0$ for the boundary flow $R = \frac{1}{4}$, $\tau' = 4y(1 - y)$.

the boundary current) to observations. With regard to estimates of A_L, cf. also [118, pp. 98, 97, 99].

Speaking generally, it has been assumed that the two-dimensional model is also valid for an ocean with variable depth [incidentally, the term $p_b \nabla_h H$ which describes the effect of the density inhomogeneities of sea water on the distribution of total flows vanishes for $H =$ constant (cf. § 6.1)], and the results of § 6.7 indicate that the relief of the bottom may be one of the factors causing separation of the boundary current from shore. However, a solution of such a problem which takes into account horizontal turbulent friction and inertial terms has not yet been obtained.

6.7 NON-STATIONARY BOUNDARY LAYER

It will be useful to consider now the method of total flows (two-dimensional models) from another point of view. It has been shown in Chapter 3 that low-frequency Rossby waves in an ocean of constant depth split up into barotropic and baroclinic modes which do not interact between each other. Therefore averaging of the basic equations over the depth is in this case, in essence, equivalent to separation of the barotropic mode. As soon as the effect of bottom relief is taken into consideration, there arises interaction between baroclinic and barotropic modes of oscillation, and it is now difficult to assess to what extent this interaction is essential. It may be said that §§ 6.2—6.6, by separating the barotropic component of the motion from the baroclinic components, were concerned with viscous and non-linear effects caused by the barotropic components. In the general case, there is apparently a need to take also into consideration, in addition to the non-linear interaction of the barotropic and baroclinic components of a motion, the linear interaction described by the term $p_b \nabla_h H$ in equations (6.1.4) and (6.1.5).

Consider the problem of propagation of barotropic Rossby waves in an ocean of constant depth. Assuming the excited waves not to be very long, the problem will be studied in the β-plane approximation. Recalling results of § 3.9, one has

$$\frac{\partial}{\partial t} \Delta_h \psi + \beta \frac{\partial \psi}{\partial x} = -\mathrm{rot}_z(\tau/h_0) \, . \tag{6.7.1}$$

As follows from results of §§ 3.7 and 3.8, barotropic Rossby waves may be assumed with high accuracy to be non-divergent ($\mathrm{div}_h\, v = 0$). This circumstance has been taken into account in the writing down of (6.7.1).

First of all, assume that the ocean has no shores. Let at time $t = 0$ arise over the ocean a tangential wind stress field τ such that the right-hand side of (6.7.1) may be written in the form

$$-\mathrm{rot}_z(\tau/h_0) = \mathrm{Re}\{F(x, t) \exp ily\} \, , \tag{6.7.2}$$

where the wave number l is assumed to be known. Under these conditions, it will be natural to represent the reaction of the ocean in the form

$$\psi = \mathrm{Re}(\Psi \exp ily) \, . \tag{6.7.3}$$

In the general case of arbitrary changes of τ with respect to y, the function F may be assumed to be the Fourier transform (with respect to y) of the right-hand side of (6.7.1) and the function Ψ the Fourier transform (with respect to y) of the solution ψ of the problem in hand. Thus, one obtains for the function $\Psi(x, t)$ the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \Psi}{\partial x^2} - l^2 \Psi\right) + \beta \frac{\partial \Psi}{\partial x} = F(x, t) \, . \tag{6.7.4}$$

Consideration will now be limited to a definite class of functions $F(x, t)$. Firstly, interest attaches to the case when the wind field arises suddenly and then does not change in the course of time; it is known that the solution of the general problem when the wind field varies arbitrarily in time may be expressed in terms of the solution of this problem in a sufficiently simple manner. Secondly, as a rule, the wind over the oceans is very close to a zonal distribution; at least for all significant Fourier components of the function

$F(x)$ $k^2 << l^2$ (where k is the wave number in the x-direction).

Consider a simple auxiliary problem. Let

$$F(x, t) = \begin{cases} \exp(ikx) \text{ for } t > 0, \\ 0 \text{ for } t \leqslant 0 . \end{cases}$$

Then one finds readily

$$\Psi = \frac{i}{k\beta}\left\{\exp\left[ik\left(x + \frac{\beta t}{k^2 + l^2}\right)\right] - \exp(ikx)\right\},$$

or

$$\Psi = -\frac{1}{\beta}\int\limits_x^{x+[\beta t/(k^2+l^2)]} \exp(ik\xi)\,\mathrm{d}\xi \,. \tag{6.7.5}$$

Thus, each Fourier component of $F(x)$ with wave number k excites in the ocean a disturbance described by (6.7.5). Since $k^2 << l^2$, the upper integration limit in (6.7.5) does not depend on k for all significant Fourier components of $F(x)$. Thus the solution of the problem under consideration will then have the form

$$\Psi = -\frac{1}{\beta}\int\limits_x^{x+(\beta t/l^2)} F(\xi)\,\mathrm{d}\xi \,. \tag{6.7.6}$$

Naturally, formula (6.7.6) is approximate (for a more detailed derivation and discussion, cf. [63]). It follows from (6.7.6) that

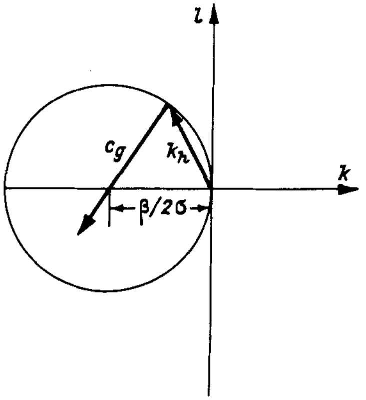
$$\frac{\partial \Psi}{\partial x} = \frac{1}{\beta}F(x) - \frac{1}{\beta}F\left(x + \frac{\beta}{l^2}\,t\right). \tag{6.7.7}$$

This expression is of special interest. The general reaction of the ocean is seen to split up into a *stationary* reaction $(1/\beta)F(x)$ in accordance with Sverdrup's relation and a *wave* reaction $(1/\beta)F(x + (\beta/l^2)t)$ which moves to the west with velocity β/l^2. For $l = 1/200$ km, $\beta = 1.6 \cdot 10^{-13}$ cm^{-1} sec^{-1}, one has $\beta/l^2 = 0.64$ cm/sec.

This result will now be discussed. Recall that the dispersion relation for the waves under consideration has the form

$$\sigma = -\frac{k\beta}{k^2 + l^2} \,.$$

Since, as a rule, the vector (k, l) points in the direction of the motion of the wave crests, it is convenient to assume that $\sigma > 0$ and $k < 0$. For fixed frequency σ, all possible values of k and l lie on a circle with centre $(-\beta/2\sigma, 0)$ and radius $\beta/2\sigma$ (Fig. 6.8). The group velocity of the wave packet with wave numbers (k, l) is known to be equal to $(\partial\sigma/\partial k,\, \partial\sigma/\partial l)$; it is directed along the



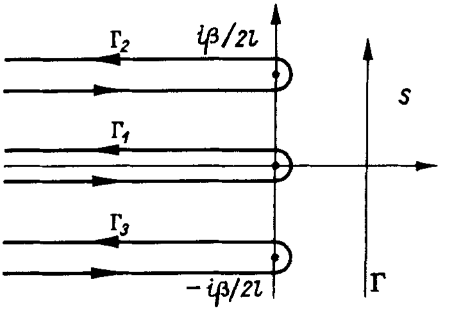


Fig. 6.8. Circle which is locus of wave numbers (k, l) of barotropic Rossby waves of given frequency $\sigma > 0$; $k_h = (k, l)$; c_g = group velocity; $|c_g| = \beta/|k_h|^2$ (according to Longuet-Higgins [68]).

Fig. 6.9. Integration path for inversion of Laplace transform.

radius of the circle to its centre and has the magnitude $\beta/(k^2 + l^2)$. Since $k^2 << l^2$ for all wave packets aroused, all of them will have the same group velocity β/l^2, and therefore the profile F, by (6.7.7), will move with velocity β/l^2 without changing its shape.

All points (k, l) for which $k^2 << l^2$ lie around the point $(0, 0)$ and the group velocity for the corresponding wave packets is practically orientated towards the west. Now it is clear why intensive boundary currents are formed at western shores of oceans.

However, how is such a current formed? Assume that at $x = 0$ lies a straight coast line and that at a certain instant of time there begins to arrive from the open ocean a disturbance. The solution of such a problem may be presented in the form $\psi = \psi_U + \psi_B$, where ψ_U is the solution for the shoreless ocean and ψ_B that for the "reflected" disturbance (reaction of shore). The function ψ_U has already been determined, while for ψ_B one has the homogeneous equation (6.7.1) with the boundary condition

$$\psi_B = -\psi_U \qquad \text{for} \qquad x = 0 . \tag{6.7.8}$$

Let

$$\psi_U(0, y, t) = \mathrm{Re}(\Psi_U \exp(ily)) ,$$

where

$$\Psi_U = \text{const.} \qquad \text{for} \qquad t > 0 ,$$

$$\Psi_U = 0 \qquad \text{for} \qquad t \leqslant 0 .$$

As soon as this problem for Ψ_U has been solved, the solution of the problem for the general case may be written down in the form of a convolution integral with respect to t over functions which are already known.

Represent ψ_B in the form (6.7.3) and apply the Laplace transformation with respect to t. One finds

$$\tilde{\Psi}_B = \int_0^\infty e^{-st}\, \Psi_B(x, t)\, dt\,.$$

Equation (6.7.1) readily yields

$$\tilde{\Psi}_B = -\Psi_U \frac{\exp \nu x}{s}\,,$$

$$\nu = -\frac{\beta}{2s} - \sqrt{\frac{\beta^2}{4s^2} + l^2}\,. \tag{6.7.9}$$

The asymptotic form of the function $\tilde{\Psi}_B$ will be sought for large t. For this purpose, replace the ordinary integration path Γ in the inversion formula for $\tilde{\Psi}_B$ in accordance with Jordan's lemma, and properties of the integrand, by three "loops" Γ_1, Γ_2 and Γ_3 (Fig. 6.9). Consider the integral

$$\frac{1}{2\pi i} \int_{\Gamma_1} \exp\left\{st - \left(\frac{\beta}{2s} + \sqrt{\frac{\beta^2}{4s^2} + l^2}\right) x\right\} \frac{ds}{s}\,. \tag{6.7.10}$$

For large t, the major contribution to the value of the integral (6.7.10) arises from integration in the neighbourhood of the point $s = 0$; therefore the exponent in the integrand may be replaced by $st - (\beta/s + l^2 s/\beta)x$. However, this integral is tabulated and one finds readily

$$J_0\left\{2\sqrt{\beta x\left(t - \frac{l^2 x}{\beta}\right)}\right\}.$$

An asymptotic estimate of integrals of the type (6.7.10) along "loops" Γ_2 and Γ_3 is readily obtained by known methods (cf. [12, § 35]). Collecting all estimates, one may write down the asymptotic formula

$$\psi_B = -\psi_U\left\{J_0\left[2\sqrt{\beta x\left(t - \frac{l^2 x}{\beta}\right)}\right] \cos ly - \sqrt{\frac{2}{\pi}}\,(lx)\left(\frac{2l}{\beta t}\right)^{3/2} \cos\left(lx + ly + \frac{\beta}{2l}t + \frac{\pi}{4}\right)\right\} + \ldots \tag{6.7.11}$$

The first term in this formula dominates. The second term in curly brackets may be omitted for large t; it is interesting to note that it represents a reflected "short" Rossby wave (for incident waves $k^2 << l^2$). The amplitude of such a wave decreases with time; its group velocity is directed along shore (the crest moves at an angle of 45° to the negative x-axis).

Formula (6.7.11) describes the formation of a non-stationary boundary layer at a western shore of an ocean. Introduce the thickness L_t of this layer as distance to the first zero of $J_0(\xi)$. One obtains then for large t

$$L_t = \frac{1.4}{\beta t} . \tag{6.7.12}$$

Hence it is seen that in the course of approximately one week the thickness of the boundary layer stays at an order of 100 km. For large times, one must take into account friction and non-linear terms; otherwise, by (6.7.12), the quantity L_t will decrease continuously with growing t.

Revert once more to Fig. 6.8. Consider a certain wave packet with wave number (k, l) and frequency σ. Since a chord is always smaller than a diameter of a circle, one has

$$\sqrt{k^2 + l^2} < \frac{\beta}{\sigma} . \tag{6.7.13}$$

However, the wave number (k, l) of this wave packet has been seen to be determined entirely by the characteristic horizontal scales of the atmospheric system. Therefore it may be said that (6.7.13) selects the class of external effects which are able to excite in the ocean significant barotropic motions of given frequencies. For example, if $\sigma = 1/\text{week}$, then one must have $k_h < \frac{1}{100}$ km $(k_h = \sqrt{k^2 + l^2})$.

COMMENT ON CHAPTER 6

§ 6.1. The method of total flows was first proposed by Stockman [112] and has been discussed repeatedly in the literature (cf., for example, [20]).

§ 6.2. Basically, this section follows the work of Stewart [111].

§ 6.3. The basic results were obtained by Munk [84], Munk and Carrier [87].

§ 6.4. The basic results were derived by Charney [6] and Morgan [81], cf. also [22].

§§ 6.5 and 6.6. The basic results are given in Il'in and Kamenkovich [34, 35] and Kamenkovich [44], cf. also [86].

§ 6.7. The basic results are due to Pedlosky [92], Il'in [33] and Lighthill [63,64]. The presentation follows basically [63].

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CHAPTER 7

THREE-DIMENSIONAL MODELS OF OCEAN CURRENTS

7.1 BOUNDARY CURRENTS IN A HOMOGENEOUS FLUID

In Chapter 5, in a study of the Ekman model of wind-driven currents, only vertical turbulent viscosity has been taken into account. Consideration will now be given to horizontal turbulent viscosity; this effect causes a number of new phenomena (especially in the inshore regions).

Assume that the density of the water is constant and study the problem in the β-plane approximation. As it has been seen in § § 4.5, 4.6 and 6.1, the general expression for the forces of turbulent friction is very complicated and depends on the three exchange coefficients: A_L, A_H and A. As a rule, in the β-plane approximation, the friction force is represented in the form

$$(F_x, F_y) = A_L \Delta_h(u, v) + A_H \frac{\partial^2}{\partial z^2}(u, v)\,, \qquad (7.1.1)$$

where the coefficient A_L and A_H are assumed to be constant and Δ_h is the Laplace operator in the x, y-plane.

Formula (7.1.1) may be derived from relations of the type (4.5.4), (4.5.7) and (4.6.9), rewritten in Cartesian coordinates if it is assumed that the terms $A_H(\partial^2 w/\partial x \partial z)$ and $A(\partial^2 w/\partial x \partial z)$ in the expression for F_x (and analogously for F_y) may be neglected (E_t may always be included in the pressure term). The term $A_H(\partial^2 w/\partial x \partial z)$ is actually small compared with the term $A_H(\partial^2 u/\partial z^2)$; an estimate of the term $A(\partial^2 w/\partial x \partial z)$ is not readily obtained, since there exist no data on the exchange coefficient A.

Incidentally, it will be noted that the approximate manner of presenting turbulent friction in the form $A_L \Delta_h(u, v)$ for a study of motion on a sphere is not correct, since, for example, it leads to non-zero friction force for rigid rotation of a fluid $u = a \cos\varphi$, $v = 0$, $w = 0$ as $\Delta_h u = \cos 2\varphi / a \cos\varphi$ [cf. (6.1.4) and (6.1.5) for exact expressions for horizontal friction].

Thus, the equations of motion for the problem in question have the form [to be compared with (5.2.1)–(5.2.3)]:

$$-fv = g\frac{\partial \zeta}{\partial x} + A_L \Delta_h u + A_H \frac{\partial^2 u}{\partial z^2}\,, \qquad +fu = g\frac{\partial \zeta}{\partial y} + A_L \Delta_h v + A_H \frac{\partial^2 v}{\partial z^2}\,,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.$$

Denoting by L a characteristic horizontal length scale and by H the depth of the ocean assumed to be constant, one arrives at non-dimensional variables (dashed) through the formulae

$$(u, v) = U_0(u', v') , \qquad (x, y) = L(x', y') , \qquad w = \frac{H}{L} U_0 w' , \qquad z = Hz' ,$$

$$\zeta = \left(\frac{f_0 U_0 L}{g}\right) \zeta' , \qquad \tau = \tau_0 \tau' , \qquad f = f_0(1 + \beta y') ,$$

where τ_0 is a characteristic value of the tangential wind stress, $U_0 = \tau_0(A_H f_0/2)^{-1/2}$ [according to (5.3.10)], f_0 is a characteristic value of the Coriolis parameter and β is the non-dimensional latitudinal change of the Coriolis parameter.

Rewriting the equations of the problem in non-dimensional form and omitting strokes on non-dimensional variables, one has

$$-fv = \frac{\partial \zeta}{\partial x} + \frac{E_L^2}{2} \Delta_h u + \frac{E_H^2}{2} \frac{\partial^2 u}{\partial z^2} , \tag{7.1.2}$$

$$+fu = \frac{\partial \zeta}{\partial y} + \frac{E_L^2}{2} \Delta_h v + \frac{E_H^2}{2} \frac{\partial^2 v}{\partial z^2} , \tag{7.1.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \tag{7.1.4}$$

where the horizontal and vertical Ekman numbers

$$E_L^2 = \frac{2A_L}{f_0 L^2} , \qquad E_H^2 = \frac{2A_H}{f_0 H^2} \tag{7.1.5}$$

have been introduced.

For the real ocean, both these numbers are small, but they may be of different orders. The problem is greatly simplified if it is assumed that the numbers E_L and E_H are equal [in other words, $A_H/A_L = (H/L)^2$]. It may be shown that the basic features of the structure of boundary layers for $E_L = E_H$ are characteristic also for the general problem when the two small parameters E_L and E_H are independent (cf. [101]).

Consider a current in a basin in the form of the cube $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, $0 \leqslant z \leqslant 1$, in non-dimensional variables. The boundary conditions of the problem have the form

$$u = v = 0 \qquad \text{for } x = 0 \text{ and } x = 1 , \tag{7.1.6}$$

$$u = v = 0 \qquad \text{for } y = 0 \text{ and } y = 1 , \tag{7.1.7}$$

$$\frac{\partial u}{\partial z} = -\frac{1}{E} \tau_x , \qquad \frac{\partial v}{\partial z} = -\frac{1}{E} \tau_y , \qquad w = 0 \qquad \text{for } z = 0 , \tag{7.1.8}$$

$$u = v = w = 0 \qquad \text{for } z = 1\,. \tag{7.1.9}$$

Since $E_L = E_H = E$, the System (7.1.2)–(7.1.4) contains one small parameter at the highest derivative and ordinary asymptotic methods may be used to solve it. Assume that far away from shores horizontal turbulent friction is small and the motion is described by the Ekman formulae (of § 5.3); then the solution of the problem there may be written in the form

$$\begin{aligned}
u &= Eu_g(x, y) + \ldots + u_E(x, y, \xi) + \ldots + Eu_b(x, y, \eta) + \ldots\,,\\
v &= Ev_g(x, y) + \ldots + v_E(x, y, \xi) + \ldots + Ev_b(x, y, \eta) + \ldots\,,\\
w &= Ew_g(x, y, z) + \ldots + Ew_E(x, y, \xi) + \ldots + E^2 w_b(x, y, \eta) + \ldots\,,\\
\zeta &= E\zeta_g(x, y) + \ldots\,,
\end{aligned} \tag{7.1.10}$$

where $\xi = z/E$, $\eta = (1 - z)/E$, and u_g, v_g, w_g and ζ_g describe geostrophic motion and the "correction" functions u_E, v_E, w_E, u_b, v_b, and w_b Ekman boundary layers at the surface and the bottom of the ocean, respectively.

It is readily shown that the determination of the functions u_E and v_E reduces to the well-known problem of pure drift motion in an infinitely deep ocean (§ 5.1). One has

$$u_E + iv_E = \frac{\tau}{\sqrt{2f}} \exp\left\{-\sqrt{f}\xi - i\left(\sqrt{f}\xi + \frac{\pi}{4}\right)\right\}. \tag{7.1.11}$$

The vertical velocity w_E is easily found from the equation $\partial w_E/\partial \xi + \partial u_E/\partial x + \partial v_E/\partial y = 0$, since $w_E \to 0$ for $\xi \to \infty$. However, the expression for $w_E(x, y, 0)$ required below may be obtained directly by integrating this equation with respect to ξ from 0 to ∞:

$$w_E(x, y, 0) = -\tfrac{1}{2}\, \mathrm{rot}_z\left(\frac{\tau}{f}\right). \tag{7.1.12}$$

It now follows from the third boundary condition (7.1.8) that

$$w_g(x, y, 0) = \tfrac{1}{2}\, \mathrm{rot}_z\left(\frac{\tau}{f}\right). \tag{7.1.13}$$

Thus, the analysis of the surface boundary layer yields the required boundary conditions for the determination of the vertical velocity of the geostrophic current.

Note that (7.1.12), and also (7.1.13) is true for an arbitrary change of A_H with depth. This follows from the fact that $w_E(x, y, 0)$ is determined by the divergence of the total flow of a pure drift current (cf. end of § 5.1).

For the velocities u_g, v_g and w_g of the geostrophic motion, one finds the usual relations

$$-fv_g = \frac{\partial \zeta_g}{\partial x}\,, \qquad fu_g = \frac{\partial \zeta_g}{\partial y}\,, \tag{7.1.14}$$

$$\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial y}+\frac{\partial w_g}{\partial z}=0\,. \tag{7.1.15}$$

Eliminating the level ζ_g from (7.1.4) and utilizing (7.1.15), one arrives at Sverdrup's relation

$$\beta v_g = f\frac{\partial w_g}{\partial z}\,. \tag{7.1.16}$$

The bottom Ekman layer (u_b, v_b, w_b) may be considered in an analogous manner to the surface layer. It must only be noted that, as a consequence of the first two boundary conditions (7.1.9), the expansions (7.1.10) for u_b and v_b begin with terms of order $O(E)$. Therefore, by (7.1.4), the expansion for w_b begins with a term of order $O(E^2)$; however, then, by the third condition (7.1.9), one finds that $w_g(x, y, 1) = 0$. Since u_g and v_g do not depend on z, one has by (7.1.13) and (7.1.16) that

$$v_g = -\frac{f}{2\beta}\,\mathrm{rot}_z\left(\frac{\tau}{f}\right), \qquad w_g = \tfrac{1}{2}\,\mathrm{rot}_z\left(\frac{\tau}{f}\right)(1-z)\,. \tag{7.1.17}$$

From (7.1.15), one obtains

$$u_g = -\int_x^1 \left\{\frac{\partial}{\partial y}\left[\frac{f}{2\beta}\,\mathrm{rot}_z\left(\frac{\tau}{f}\right)\right] + \tfrac{1}{2}\,\mathrm{rot}_z\left(\frac{\tau}{f}\right)\right\} \mathrm{d}x + k(y)\,, \tag{7.1.18}$$

where the function $k(y)$ must still be determined. Once this function has been found, formula (7.1.14) permits to determine the level ζ_g (exactly apart from a constant).

In order to find $k(y)$, one must study the inshore boundary layer. Consider the vicinity of the western shore ($x = 0$) and present certain qualitative arguments. By (7.1.10), in the open ocean, the total flows $(S_{gx}, S_{gy}) = O(E)$, $(S_{Ex}, S_{Ey}) = O(E)$, $(S_{bx}, S_{by}) = O(E^2)$. Using (7.1.13) and (7.1.16), one derives Sverdrup's integral relation

$$S_{gy} + S_{Ey} = -\frac{E}{2\beta}\,\mathrm{rot}_z\,\tau\,.$$

Hence the total meridional transport of water in the open ocean

$$\int_0^1 (S_{gy} + S_{Ey})\,\mathrm{d}x$$

is not equal zo zero (for definiteness, it may be assumed that $\mathrm{rot}_z\tau > 0$). Relying on the results of analysis of the viscous boundary layer in the two-dimensional theory (§ 6.3), it may be assumed that at the shore ($x = 0$) a boundary layer of thickness $O(E^{2/3})$ is formed the total transport of which is of $O(E)$ and compensates the transport in the open ocean. Then the total

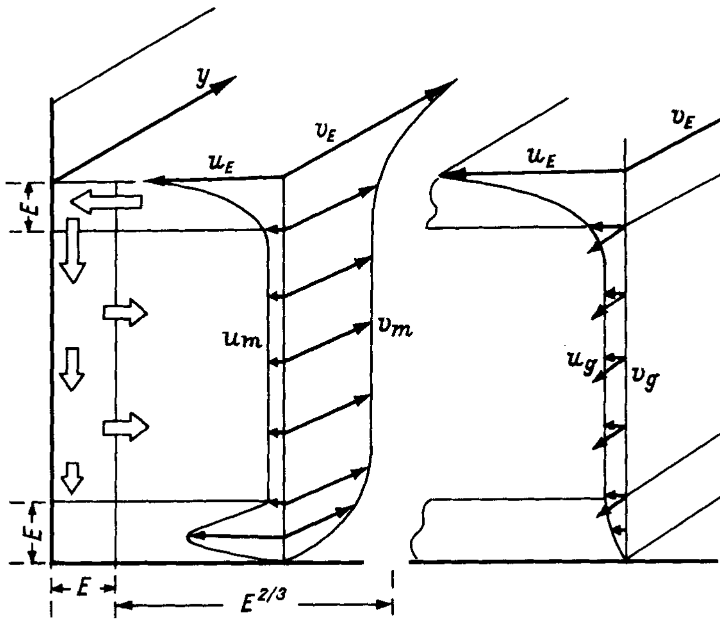


Fig. 7.1. Pattern of disposition of boundary layers and flows at a western shore which are taken into consideration in the first approximation to the solution of the problem. Schematic distributions of velocities in the boundary layer of thickness $O(E^{2/3})$ and in the open ocean are shown.

flows and velocities in this layer are of order $S_x = O(E)$, $S_y = O(E^{1/3})$, $u = O(E)$ and $v = O(E^{1/3})$. Obviously, by (7.1.2) and (7.1.3), the order of the level is $O(E)$. It is readily seen that the vertical friction in the basic thickness of this boundary layer is negligibly small and that it may be shown that the "correction" functions for the horizontal velocities u and v do not depend on z. Note that the boundary layer under consideration, as will be shown below, exists only at a western shore of an ocean.

There still remains to determine the scale of the vertical velocity w. The "correction" functions for u and v in a boundary layer of thickness $O(E^{2/3})$ will not vanish at the bottom. In order to remove this discrepancy, one must take account of vertical friction. In the vicinity of the line $x = 0, z = 1$, one may single out a "transition" region with scales $E^{2/3}$, 1, E along the x, y, z-axes, respectively, within the limits of which the vertical friction and the Coriolis force are of the same order. Since, on the basis of the estimates already derived, $v = O(E^{1/3})$, the same order is also obtained for u: $u = O(E^{1/3})$. However, then $\partial u/\partial x = O(E^{-1/3})$, and since $\partial w/\partial z = O(E^{-1/3})$, by (7.1.4), one has $w = O(E^{2/3})$. It is natural to assume that the order of w will be this within the limits of the entire boundary layer.

Consider now the Ekman friction layer at the ocean surface. The velocities of pure drift current u_E and v_E are readily verified not to vanish for $x = 0$. In order to remove this discrepancy, one must take into consideration horizontal friction. Then one may single out in the vicinity of the line $x = 0, z = 0$ a transition region within which the effects of horizontal and vertical friction are of the same order: hence the scales of this region along the x, y, z-axes are E, 1, E, respectively (Fig. 7.1). One has for this region: $u = O(1)$, $v = O(1)$, $\partial u/\partial x = O(E^{-1})$, $\partial v/\partial y = O(1)$; therefore the equation of continuity yields $\partial w/\partial z = O(E^{-1})$, and hence $w = O(1)$.

Since in the transition region the vertical velocity is of order $O(1)$, one requires at the shore ($x = 0$) an additional sublayer of thickness $O(E)$ within which the vertical velocity likewise is of order $O(1)$. Assume now that here, as in the transition region, the terms $\partial w/\partial z$ and $\partial u/\partial x$ in (7.1.4) are of the same order; then $u = O(E)$. It is natural to assume that in such a boundary layer horizontal turbulent exchange plays an essential role in the general balance of forces; hence $v = O(E)$. Besides, comparing the orders of the level gradient $\partial\zeta/\partial x$ and the forces of horizontal friction $E^2(\partial^2 u/\partial x^2)$, one finds that $\zeta = O(E^2)$.

Thus, at the shore ($x = 0$), there arises a new boundary layer of thickness $O(E)$ with intense vertical motion the existence of which is not linked to an effect of latitudinal variation of the Coriolis parameter [as occurred in the case of the boundary layer of thickness $O(E^{2/3})$]. Naturally, such a boundary layer exists also near an eastern ocean coast. Thus, asymmetry of the horizontal structure of ocean currents is "created" basically by the boundary layer of thickness $O(E^{2/3})$.

The procedure will now be formalized. Consider a region outside the bottom boundary layer, the near shore boundary layers at the eastern shore ($x = 1$) and the zonal boundaries ($y = 0$ and $y = 1$). By strength of the above, the solution of the problem in this region must be sought in the form

$$
\begin{aligned}
u &= Eu_g(x, y) + \ldots + u_E(x, y, \xi) + \ldots + Eu_m(\sigma, y) + \ldots + Eu_s(\kappa, y, z) + \ldots \\
&\quad + u_c(\kappa, y, \xi) + \ldots , \\
v &= Ev_g(x, y) + \ldots + v_E(x, y, \xi) + \ldots + E^{1/3}v_m(\sigma, y) + \ldots + Ev_s(\kappa, y, z) + \ldots \\
&\quad + v_c(\kappa, y, \xi) + \ldots , \\
w &= Ew_g(x, y, z) + \ldots + Ew_E(x, y, \xi) + \ldots + E^{2/3}w_m(\sigma, y, z) + \ldots + w_s(\kappa, y, z) + \ldots \\
&\quad + w_c(\kappa, y, \xi) + \ldots , \\
\zeta &= E\zeta_g(x, y) + \ldots + E\zeta_m(\sigma, y) + \ldots + E^2\zeta_s(\kappa, y) + \ldots ,
\end{aligned}
\tag{7.1.19}
$$

where $\sigma = x/E^{2/3}$, $\kappa = x/E$, $\xi = z/E$, and the functions u_g, v_g, w_g, ζ_g and u_E, v_E, w_E are already assumed to be known. Note that *correction* functions ζ_E and ζ_c have not been introduced into the expansion for the level.

Substitution of the expansions (7.1.19) into the original system of equations (7.1.4) yields the system of equations:

For the functions u_m, v_m, w_m, ζ_m:

$$-fv_m = \frac{\partial \zeta_m}{\partial \sigma}, \qquad fu_m = \frac{\partial \zeta_m}{\partial y} + \frac{1}{2}\frac{\partial^2 v_m}{\partial \sigma^2}, \qquad \frac{\partial u_m}{\partial \sigma} + \frac{\partial v_m}{\partial y} = 0\,; \tag{7.1.20}$$

For the functions u_c, v_c, w_c:

$$-fv_c = \frac{1}{2}\left(\frac{\partial^2 u_c}{\partial \kappa^2} + \frac{\partial^2 u_c}{\partial \xi^2}\right), \qquad +fu_c = \frac{1}{2}\left(\frac{\partial^2 v_c}{\partial \kappa^2} + \frac{\partial^2 v_c}{\partial \xi^2}\right), \qquad \frac{\partial u_c}{\partial \kappa} + \frac{\partial w_c}{\partial \xi} = 0\,; \tag{7.1.21}$$

For the functions u_s, v_s, w_s, ζ_s:

$$-fv_s = \frac{\partial \zeta_s}{\partial \kappa} + \frac{1}{2}\frac{\partial^2 u_s}{\partial \kappa^2}, \qquad fu_s = \frac{1}{2}\frac{\partial^2 v_s}{\partial \kappa^2}, \qquad \frac{\partial u_s}{\partial \kappa} + \frac{\partial w_s}{\partial z} = 0\,. \tag{7.1.22}$$

The necessity of matching corresponding asymptotic expansions leads to the following boundary conditions which are readily verified:

$$u_c, v_c, w_c \to 0 \quad \text{for } \kappa, \xi \to \infty\,, \qquad u_m, v_m, w_m, \zeta_m \to 0 \quad \text{for } \sigma \to \infty\,,$$
$$u_s, v_s, w_s, \zeta_s \to 0 \quad \text{for } \kappa \to \infty\,. \tag{7.1.23}$$

Throughout conditions (7.1.23), it is assumed that the functions approach zero exponentially. Furthermore, substitution of the expansions (7.1.19) into the boundary conditions (7.1.6) and (7.1.8) yields

$$u_E(0, y, \xi) + u_c(0, y, \xi) = 0\,, \qquad v_E(0, y, \xi) + v_c(0, y, \xi) = 0\,, \tag{7.1.24}$$

$$v_m(0, y) = 0\,, \tag{7.1.25}$$

$$u_g(0, y) + u_m(0, y) + u_s(0, y, z) = 0\,, \tag{7.1.26}$$

$$w_s(\kappa, y, 0) + w_c(\kappa, y, 0) = 0\,, \tag{7.1.27}$$

$$\frac{\partial u_c}{\partial \xi}(\kappa, y, 0) = 0\,, \qquad \frac{\partial v_c}{\partial \xi}(\kappa, y, 0) = 0\,. \tag{7.1.28}$$

Solution of systems (7.1.20)–(7.1.22) may be achieved in the following manner. First consider system (7.1.20). Eliminating ζ_m from the first two equations, one has

$$\frac{\partial^3 v_m}{\partial \sigma^3} - 2\beta v_m = 0\,.$$

The solution of this equation, satisfying (7.1.25) and decaying exponentially for large σ, is

$$v_m = C(y)\exp[-(2\beta)^{1/3}\sigma/2]\sin\left[(2\beta)^{1/3}\frac{\sqrt{3}}{2}\,\sigma\right], \tag{7.1.29}$$

where the function $C(y)$ must still be determined. By the third equation (7.1.20) and the condition for $\sigma \to \infty$, one finds

$$u_m = (2\beta)^{-1/3}\frac{dC}{dy}\exp[-(2\beta)^{1/3}\sigma/2]\sin\left[(2\beta)^{1/3}\frac{\sqrt{3}}{2}\sigma + \frac{\pi}{3}\right]. \tag{7.1.30}$$

The formula for ζ_m will not be derived here.

An analogous construction at the eastern shore shows that $u_m = 0$ and $v_m = 0$, i.e., that S_y in the eastern boundary layer of thickness $O(E^{2/3})$ is of order $O(E^{1/3})$. Therefore only the boundary layer of thickness $O(E^{2/3})$ at the western shore takes part in the total meridional transport of mass, and one finds

$$\int_0^1 (S_{gy} + S_{Ey})\,dx + \int_0^1 \left(\int_0^\infty v_m(\sigma, y)\,d\sigma\right) dz = 0\,. \tag{7.1.31}$$

Hence follows the expression for the function $C(y)$

$$C(y) = -\frac{2}{\sqrt{3}}(2\beta)^{-2/3}\int_0^1 (\mathrm{rot}_z\tau)\,dx\,. \tag{7.1.32}$$

Consider next system (7.1.21). Integrating these equations with respect to ξ from 0 to ∞, one finds, by (7.1.28) and (7.1.23), for $\xi \to \infty$

$$-fS_{cy} = \frac{1}{2}\frac{\partial^2 S_{cx}}{\partial\kappa^2}\,, \qquad fS_{cx} = \frac{1}{2}\frac{\partial^2 S_{cy}}{\partial\kappa^2}\,, \qquad \frac{\partial S_{cx}}{\partial\kappa} - w_c(\kappa, y, 0) = 0\,, \tag{7.1.33}$$

where

$$S_{cx} = \int_0^\infty u_c d\xi\,, \qquad S_{cy} = \int_0^\infty v_c d\xi\,.$$

Integrating conditions (7.1.24) with respect to ξ from 0 to ∞, one obtains readily from the first two equations (7.1.33)

$$S_{cx} = -[S_{Ey}\sin\sqrt{f}\kappa + S_{Ex}\cos\sqrt{f}\kappa]\exp(-\sqrt{f}\kappa)\,,$$
$$S_{cy} = -[S_{Ey}\cos\sqrt{f}\kappa - S_{Ex}\sin\sqrt{f}\kappa]\exp(-\sqrt{f}\kappa)\,. \tag{7.1.34}$$

Since, by (7.1.11), one has $S_{Ex} + iS_{Ey} = \tau/(2if)$, the third equation (7.1.33) yields an expression for $w_c(\kappa, y, 0)$. Substituting it into (7.1.27), one obtains

$$w_s(\kappa, y, 0) = -\frac{\partial}{\partial\kappa}\left\{\left[\frac{\tau_x}{2f}\sin\sqrt{f}\kappa - \frac{\tau_y}{2f}\cos\sqrt{f}\kappa\right]\exp(-\sqrt{f}\kappa)\right\}. \tag{7.1.35}$$

A complete determination of the functions u_c, v_c, w_c will not be given here (they are only essential in the transition region E, 1, E). Note that an analogous treatment of a transition region with scale E, 1, E in the vicinity of the line $x = 0$, $z = 1$ leads to the condition $w_s(\kappa, y, 1) = 0$.

Proceed now to the study of system (7.1.22). By (7.1.26), it may be assumed that u_s does not depend on z. It follows then from (7.1.22) that v_s likewise does not depend on z, and w_s is linear in z. One has, by (7.1.35) and the condition $w_s(\kappa, y, 1) = 0$, that

$$w_s = w_s(\kappa, y, 0)\,(1 - z)\,. \tag{7.1.36}$$

It is now easy to obtain from (7.1.22) and (7.1.23), by integration with respect to κ, for $\kappa \to \infty$ formulae for u_s, v_s, ζ_s. They will not be derived here as only the expression for $u_s(0, y)$ is required. Integrating the third equation (7.1.22) with respect to κ from 0 to ∞, taking into account the condition $u_s \to 0$ for $\kappa \to \infty$, one finds

$$u_s(0, y) = \frac{\tau_y}{2f}\,. \tag{7.1.37}$$

As is already known, there does not arise a boundary layer of thickness $O(E^{2/3})$ at the eastern shore in the first approximation. Therefore the condition for the zonal velocity at $x = 1$ assumes the form

$$u_g(1, y) + u_s(0, y) = 0\,. \tag{7.1.38}$$

Since the constructions of the function u_s for the eastern and western shores do not at all differ, formula (7.1.37) is also true for the eastern shore, and combining (7.1.18), (7.1.37) and (7.1.38), one finds

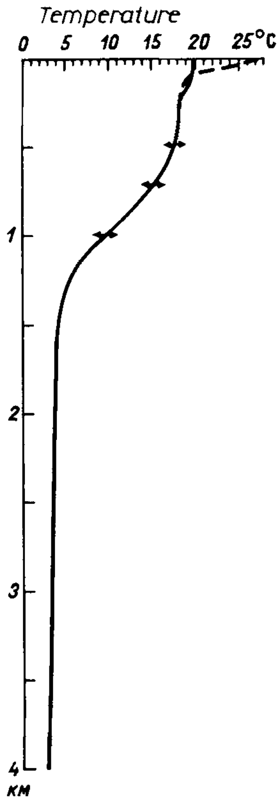
$$k(y) = -\frac{\tau_y(1, y)}{2f}\,.$$

Thus, the solution in the geostrophic region has been constructed. It is interesting to note that the zonal geostrophic velocity u_g does not vanish at the eastern shore (however, the zonal component of the total flux vanishes). Likewise, it must be emphasized that the velocity u_g is determined not only by $\mathrm{rot}_z\boldsymbol{\tau}$, as in the two-dimensional theory, but it depends also on the field $\boldsymbol{\tau}$ itself.

Since the total meridional transport of water vanishes [cf. (7.1.31)] and, by (7.1.38), the water does not escape through the eastern shore, it also must not escape through the western shore. Therefore condition (7.1.26) must now be fulfilled automatically, as is readily verified by substitution.

Analysis of the bottom boundary layer and the corresponding transition regions, and likewise of the boundary layers at the boundaries $y = 0$ and $y = 1$, will not be presented here (cf. [101,93]).

It is readily shown that bottom friction plays a secondary role in the problem under consideration. Therefore the distribution derived above for the total flows (outside the boundary layers at the boundaries $y = 0$ and $y = 1$) coincide completely with the results of the two-dimensional theory.



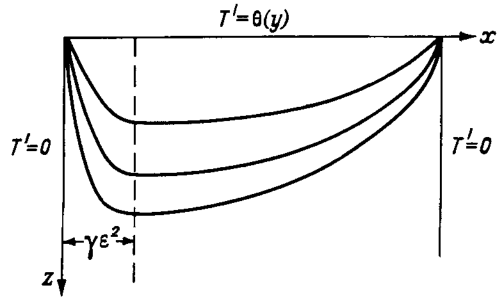


Fig. 7.2. Typical graph of temperature changes with depth in moderate latitudes of the ocean (during two different seasons) (according to Stommel [117]).

Fig. 7.3. Distribution of isolines of temperature perturbations T' in the x, z-plane (according to Barcilon [1]). The thickness of the western boundary layer is shown.

However, it must be emphasized that the boundary layer of thickness $O(E)$ is lost in a study of the problem by the method of total flows. The currents within the limits of these layers may be of great interest, especially in connection with the problem of upwelling and downwelling motions.

Recall that, for the analysis of wind driven currents in a homogeneous ocean within the framework of the Ekman model, use has been made of approximate horizontal boundary conditions. [Cf. conditions (5.2.11).] The analysis above justifies the formulation adopted.

7.2. SIMPLEST LINEAR MODEL OF THERMOCLINE

The temperature (and likewise salinity and density) field in the ocean has one clearly pronounced feature: practically all temperature changes (in vertical as well as horizontal directions) are concentrated in the upper kilometre layer which is usually referred to as main thermocline or simply as thermocline (Fig. 7.2). The bottom thickness of the water of the ocean (below the thermocline) has almost constant temperature which, in essence, does not depend on the thermal conditions at the surface of the ocean. Such a pattern is true for the entire world ocean, except, may be, in high latitudes. Starting from general reasoning, it may be assumed that the thermocline is nothing else but a specific thermal boundary layer of the ocean. It is the task of theory to explain, first of all, the parameters on which the characteristic thickness of this boundary layer depends. Besides, the existence of a thermal boundary layer in the open ocean raises naturally a number of new problems also for the theory of coastal boundary currents. It has already been shown in the last section how complicated the structure of coastal boundary layers becomes (even in the case of a homogeneous ocean) when one steps over from two-dimensional to three-dimensional models.

A start will be made with the simplest possible model. Assume that there is no wind, i.e., the motion is due to purely thermal causes. However, then one need not take into consideration vertical turbulent exchange (at least, outside the bottom Ekman boundary layer). Disregard also non-linear inertial terms and horizontal turbulent transfer; for the sake of simplicity, restrict consideration to the β-plane approximation, so that

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}, \tag{7.2.1}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}, \tag{7.2.2}$$

$$\frac{\partial p}{\partial z} = g\rho, \tag{7.2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7.2.4}$$

where, as usually, the x-axis is directed to the east, the y-axis to the north, the z-axis downwards, ρ_0 is the mean density in the ocean, $f = f_0 + \beta(y - y_0)$ and the remaining notation is as before.

It will be assumed that the density depends only on the temperature, and besides linearly, so that

$$\rho = \rho_0[1 - \alpha(T - T_0)], \tag{7.2.5}$$

where T_0 is the temperature averaged over the entire ocean and α the constant coefficient of thermal expansion.

The equation of heat transfer is non-linear, and this fact is the cause of basic difficulties encountered in the construction of a theory. As a first step, linearization will be introduced (the non-linear theory will be considered in the next section). Thus, let the temperature field in the ocean assume the form

$$T = T_s - Gz + T'(x, y, z)\,, \qquad G > 0\,, \tag{7.2.6}$$

where $T = T_s - Gz (T_s,\ G = \text{constant})$ is a certain mean distribution of temperature in the ocean and T' is a perturbation. Assuming the perturbation of the temperature field to be small, linearize the equation of heat transfer (4.5.6). In addition, assume the coefficients of vertical and horizontal heat conductivity (K_H, K_L) to be constant, to obtain, finally,

$$-Gw = K_H \frac{\partial^2 T'}{\partial z^2} + K_L \Delta_h T'\,. \tag{7.2.7}$$

The boundary conditions will be formulated next. Thermal processes at the ocean surface are extremely complicated, and their discussion is not required with the problem under consideration. As regards a study of the effect of the formation of a thermal boundary layer, it is sufficiently simple to specify the temperature at the ocean surface. Recalling likewise the kinematic boundary condition, one has

$$T' = \theta(y)\,, \qquad w = 0 \qquad \text{for } z = 0\,, \tag{7.2.8}$$

where $\theta(y)$ is a known function (the temperature at the surface of the ocean changes chiefly in a meridional direction).

Let it be assumed that there is no perturbation of the temperature at the ocean floor for $z = H$. Furthermore, since vertical turbulent exchange has been neglected in the equations of motion, one may impose at the bottom only a condition of no flow. Thus,

$$T' = 0\,, \qquad w = 0 \qquad \text{for } z = H\,. \tag{7.2.9}$$

A formulation of boundary conditions at shores which are assumed to be sheer cliffs is very specific. In general, there is no heat flux at a shore (for example, for $x = 0$ and $x = L$) and the horizontal velocity must also vanish. However, since horizontal turbulent exchange is absent from the equations of motion as well as are non-linear terms, one is forced to forget about fulfillment of all conditions at the shore. Clearly, it is impossible to violate the no-flow condition: The total mass of fluid in the basin must remain constant. However, if, for example, one has $u = 0$ for $x = 0$, then, by (7.2.2), $\partial p/\partial y = 0$, which means, by (7.2.3) and (7.2.5), that also $\partial T'/\partial y = 0$ for $x = 0$. Therefore it will be simplest to write

$$T' = 0 \qquad \text{for } x = 0\,, \qquad T' = 0 \qquad \text{for } x = L\,. \tag{7.2.10}$$

Thus, the thermal and dynamic boundary conditions could be consolidated in the single condition (7.2.10). The conditions at the zonal boundaries of the region will not be considered here.

It will be convenient to reduce the problem to a single equation for T'.

Eliminating from (7.2.1) and (7.2.2) the pressure p and employing (7.2.4), one obtains

$$\beta v = f\frac{\partial w}{\partial z}\,. \tag{7.2.11}$$

Differentiating this equation with respect to z and taking (7.2.3) and (7.2.5) into account, one finds

$$\frac{\partial^2 w}{\partial z^2} = -\frac{g\alpha\beta}{f^2}\frac{\partial T'}{\partial x}\,.$$

Finally, substituting into this equation, in accordance with (7.2.7), the expression for w in terms of T', one obtains

$$K_H\frac{\partial^4 T'}{\partial z^4} + K_L\Delta_h\left(\frac{\partial^2 T'}{\partial z^2}\right) - \frac{g\alpha\beta G}{f^2}\frac{\partial T'}{\partial x} = 0\,. \tag{7.2.12}$$

Conditions (7.2.8), (7.2.9) and (7.2.10) will now be rewritten so that they only involve the temperature perturbation T':

$$T' = \theta(y)\,, \qquad K_H\frac{\partial^2 T'}{\partial z^2} + K_L\Delta_h T' = 0 \qquad \text{for } z = 0\,,$$

$$T' = 0\,,\ K_H\frac{\partial^2 T'}{\partial z^2} + K_L\Delta_h T' = 0 \qquad \text{for } z = H\,,$$

$$T' = 0 \qquad \text{for } x = 0\,, \qquad T' = 0 \qquad \text{for } x = L\,. \tag{7.2.13}$$

Thus, the study of the thermal boundary layer has been reduced to analysis of the problem (7.2.12) and (7.2.13). These equations will now be written in non-dimensional form.

Choose L and H as characteristic scales in horizontal and vertical directions, respectively. Let θ_0 be the characteristic value of the function $\theta(y)$; it will be quite natural to adopt θ_0 as characteristic scale for T'. Writing (7.2. 12) and (7.2.13) in terms of non-dimensional quantities (denoted below by the former symbols), one obtains

$$\epsilon^4\frac{\partial^4 T'}{\partial z^4} + \gamma\epsilon^4\Delta_h\left(\frac{\partial^2 T'}{\partial z^2}\right) - \frac{1}{f^2(y)}\frac{\partial T'}{\partial x} = 0\,,$$

$$T' = \theta(y)\,,\ \frac{\partial^2 T'}{\partial z^2} + \gamma\Delta_h T' = 0 \qquad \text{for } z = 0\,, \tag{7.2.14}$$

$$T' = 0\,, \qquad \frac{\partial^2 T'}{\partial z^2} + \gamma\Delta_h T' = 0 \qquad \text{for } z = 1\,,$$

$$T' = 0 \qquad \text{for } x = 0\,, \qquad T' = 0 \qquad \text{for } x = 1\,, \tag{7.2.15}$$

where

$$\epsilon = \frac{H_0}{H}, \qquad H_0 = \left(\frac{K_H f_0^2 L}{g\alpha\beta G}\right)^{1/4}, \qquad \gamma = \frac{K_L}{K_H}\left(\frac{H}{L}\right)^2. \tag{7.2.16}$$

If $K_H = 1$ cm^2/sec, $K_L = 10^7$ cm^2/sec, $H = 4$ km, $L = 5000$ km, $\alpha = 2.5 \cdot 10^{-4}$ $(^0\mathrm{C})^{-1}$, $G = 10^{-4}$ $(^0\mathrm{C})$/cm, then $\gamma \simeq 6$, $H_0 \simeq 0.3$ km, $\epsilon \simeq 0.08$. These estimates are very approximate; however, it will be assumed in the sequel that the parameter γ is finite and the parameter ϵ small. The asymptotic of the solution of problem (7.2.14) and (7.2.15) for small ϵ yields completely satisfactory understanding of the peculiarities of the solution of the problem also for not very small ϵ.

It is already clear, starting from these considerations, that the internal characteristic scale H_0 gives the order of magnitude of the thickness of the thermal boundary layer, or of the thermocline, in the ocean. Therefore it may be assumed that qualitatively the model under consideration actually describes the effect of formation of the thermal boundary layer in the ocean. The structure of the boundary layers will now be studied in greater detail.

The temperature perturbation T' outside the boundary layers vanishes according to (7.2.14) and (7.2.15) (internal solution). Obviously, the non-dimensional thickness of the thermal boundary layer is of order ϵ. It is not difficult to determine also the order of the non-dimensional thickness of the coastal boundary layer $O(\epsilon^2)$ from the condition that the terms $\gamma\epsilon^4\Delta_h(\partial^2 T'/\partial z^2)$ and $\partial T'/\partial x$ must be of equal order of magnitude. Thus, the solution of Problems (7.2.14) and (7.2.15) outside the boundary layers at zonal boundaries will be sought in the form

$$T' = T_s(x, y, \xi) + \ldots + T_W(\zeta, y, \xi) + \ldots + T_E(\eta, y, \xi) + \ldots, \tag{7.2.17}$$

where $\xi = z/\epsilon$, $\zeta = x/\epsilon^2$, $\eta = (1 - x)/\epsilon^2$ and all functions T_s, T_W, T_E must decay exponentially for large ξ, ζ and η.

Substitution of (7.2.17) into (7.2.14) and (7.2.15) leads to the relations

$$\frac{\partial^4 T_s}{\partial \xi^4} - \frac{1}{f^2}\frac{\partial T_s}{\partial x} = 0, \qquad 0 < x < 1, \qquad \xi > 0, \tag{7.2.18}$$

$$T_s = \theta(y), \qquad \frac{\partial^2 T_s}{\partial \xi^2} = 0 \qquad \text{for } \xi = 0, \tag{7.2.19}$$

$$\gamma\frac{\partial^4 T_W}{\partial \xi^2 \partial \zeta^2} - \frac{1}{f^2}\frac{\partial T_W}{\partial \zeta} = 0, \qquad \xi, \zeta > 0, \tag{7.2.20}$$

$$T_s + T_W = 0 \qquad \text{for } \zeta = 0, \tag{7.2.21}$$

$$T_W = 0 \qquad \text{for } \xi = 0, \tag{7.2.22}$$

$$\gamma\frac{\partial^4 T_E}{\partial \xi^2 \partial \eta^2} + \frac{1}{f^2}\frac{\partial T_E}{\partial \eta} = 0, \qquad \xi, \eta > 0, \tag{7.2.20$'$}$$

$$T_s + T_E = 0 \qquad \text{for } \eta = 0\ , \tag{7.2.21'}$$

$$T_E = 0 \qquad \text{for } \xi = 0\ . \tag{7.2.22'}$$

A start will be made with problems (7.2.18) and (7.2.19). Introduce the Fourier sine transform with respect to ξ

$$\widetilde{T}_s = \int\limits_0^\infty T_s \sin(\xi\sigma)\,\mathrm{d}\xi\ .$$

Then, by (7.2.18) and (7.2.19), one obtains for $\widetilde{T}_s$ the equation

$$\sigma^4 \widetilde{T}_s - \sigma^3 \theta(y) - \frac{1}{f^2}\frac{\partial \widetilde{T}_s}{\partial x} = 0\ ,$$

with the solution

$$\widetilde{T}_s = A(y)\,\mathrm{e}^{f^2\sigma^4 x} + \frac{1}{\sigma}\theta(y)\ , \tag{7.2.23}$$

where the function $A(y)$ must still be determined.

For solution of problems (7.2.20)—(7.2.22) and (7.2.20′)—(7.2.22′), the Fourier sine transform will again be employed. After single integrations with respect to ζ and η, respectively, one obtains from (7.2.20) and (7.2.20′) the equations

$$-f^2\sigma^2\gamma\frac{\partial \widetilde{T}_W}{\partial \zeta} - \widetilde{T}_W = 0\ , \tag{7.2.24}$$

$$-f^2\sigma^2\gamma\frac{\partial \widetilde{T}_E}{\partial \eta} + \widetilde{T}_E = 0\ . \tag{7.2.24'}$$

Equation (7.2.24′) has no non-zero solution which decays exponentially for large η. However, then the function T_E likewise vanishes identically, and, by (7.2.21′), one finds that $\widetilde{T}_s(1, y, \xi) = 0$; hence the function $A(y)$ entering into (7.2.23) has been determined, and

$$\widetilde{T}_s = \frac{\theta(y)}{\sigma}\{1 - \exp[-(1-x)f^2\sigma^4]\} \tag{7.2.25}$$

or, reverting to the original function,

$$T_s = \frac{2\theta(y)}{\pi}\int\limits_0^\infty \frac{1 - \exp[-(1-x)f^2\sigma^4]}{\sigma}\sin(\sigma\xi)\,\mathrm{d}\sigma\ . \tag{7.2.26}$$

The substitution $\sigma = \tau\xi^{1/3}(1-x)^{-1/3}$ reduces (7.2.26) to the form

$$T_s = \frac{2\theta(y)}{\pi}\int\limits_0^\infty \frac{1 - \exp(-\chi f^2\tau^4)}{\tau}\sin(\chi\tau)\,\mathrm{d}\tau\ , \tag{7.2.27}$$

where

$$\chi = \xi^{4/3}(1-x)^{-1/3} . \tag{7.2.28}$$

Thus, the function T_s depends on the variables x and ξ only through the combination χ.

The function $\widetilde{T}_W$ is found from (7.2.24) for the condition $T_c(0, y, \sigma) + T_W(0, y, \sigma) = 0$ [cf. (7.2.21)]. Using (7.2.25) and inverting the transform, one obtains

$$T_W = -\frac{2\theta(y)}{\pi} \int_0^\infty \frac{1-\exp(-f^2\sigma^4)}{\sigma} \exp\left(-\frac{\zeta}{f^2\sigma^2\gamma}\right) \sin(\sigma\xi)\, d\sigma . \tag{7.2.29}$$

The integrals in (7.2.27) and (7.2.29) apparently must be evaluated numerically. However, it is not difficult to establish qualitatively the behaviour of the solution of the problem. Outside the western boundary layer, one has $T' = T_s(\chi, y)$. Therefore the lines T' = constant coincide in the x, z-plane with the lines χ = constant. By (7.2.28), these curves are given by the simple equation z^4 = constant $(1-x)$, according to which all lines χ = constant "come out of" the point $x = 1$, $z = 0$. The "correction" function $\tau_W(\zeta, y, \xi)$ "turns upwards" these curves within the limits of a western boundary layer and forces them "into" the point $x = 0, z = 0$ (Fig. 7.3 on p. 202).

Thus, due to the β-effect, the pattern of the curves T' = constant is sharply asymmetric with respect to the plane $x = 1/2$, although the motion generating factor $\theta(y)$ does not at all depend on x. This phenomenon has already been encountered repeatedly.

The width of the western boundary layer is here overestimated (for the adopted values of the determining parameters, of order 200 km). However, recall again that the problem under consideration only bears a qualitative character. Besides, in essence, the parameter G has been introduced solely for linearization of the problem; for a real ocean, its estimate is very indefinite. In general, linearization of the equation of heat transfer introduces a series of artificial aspects. For example, since practically there does not occur below the thermal boundary layer a change in temperature, the vertical velocity w, by (7.2.7), will likewise be equal to zero there, and consequently also the horizontal velocity will vanish [cf., for example, (7.2.11)]. It is important to emphasize that this result follows from (7.2.7) and it does not depend on the form of the other equations. If one admits that the quantity G is not constant, then the basic state $T = T(z)$ will not satisfy the equation of heat transfer with constant exchange coefficients and the method of perturbations will then not be very sensible. All this suggests the necessity of studying non-linear models.

Non-linear heat advection will now be completely taken into account in the equation of heat transfer (4.5.6). Then, limiting consideration to motion in the open ocean, disregard in this equation terms describing horizontal turbulent exchange of heat. Let the remaining equations be the same as in § 7.2 (cf. also § 7.2 regarding the formulation of the problem). In spherical coordinates, these equations assume the form

$$-fv = -\frac{1}{\rho_0}\frac{\partial p'}{a\cos\varphi\partial\lambda}\,, \tag{7.3.1}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p'}{a\partial\varphi}\,, \tag{7.3.2}$$

$$\frac{\partial p}{\partial z} = -g\alpha\rho_0(T - T_0)\,, \tag{7.3.3}$$

$$\frac{\partial u}{a\cos\varphi\partial\lambda} + \frac{1}{\cos\varphi}\frac{\partial}{a\partial\varphi}(v\cos\varphi) + \frac{\partial w}{\partial z} = 0\,, \tag{7.3.4}$$

$$\frac{u}{a\cos\varphi}\frac{\partial T}{\partial\lambda} + \frac{v}{a}\frac{\partial T}{\partial\varphi} + w\frac{\partial T}{\partial z} = K_H\frac{\partial^2 T}{\partial z^2}\,, \tag{7.3.5}$$

where p' is the deviation of the pressure from its equilibrium value $p_a + g\rho_0 z$.

It will be convenient to reduce this system of equations to a single equation for some auxiliary function $M(\lambda, \varphi, z)$. For this purpose, eliminate first by cross-differentiation the pressure p from (7.3.1) and (7.3.2). Using (7.3.4), one has

$$v = a\tan\varphi\frac{\partial w}{\partial z}\,. \tag{7.3.6}$$

By (7.3.1), equation (7.3.6) may be rewritten in the form

$$\frac{1}{2\Omega a^2\sin^2\varphi}\frac{\partial(p'/\rho_0)}{\partial\lambda} = \frac{\partial w}{\partial z}\,.$$

Based on analogy with the stream function for two-dimensional motion of an incompressible fluid, introduce now a function $M(\lambda, \varphi, z)$ such that

$$p' = g\frac{\partial M}{\partial z}\,, \qquad w = \frac{g}{2\Omega\rho_0 a^2\sin^2\varphi}\frac{\partial M}{\partial\lambda}\,. \tag{7.3.7}$$

However, then, by (7.3.1)—(7.3.3), one has

$$u = -\frac{g}{2\Omega\rho_0 a\sin\varphi}\frac{\partial^2 M}{\partial\varphi\,\partial z}\,, \qquad v = \frac{g}{2\Omega a\rho_0\sin\varphi\cos\varphi}\frac{\partial^2 M}{\partial\lambda\,\partial z}\,,$$

$$T - T_0 = -\frac{1}{\alpha\rho_0}\frac{\partial^2 M}{\partial z^2}\,. \tag{7.3.8}$$

Substituting (7.3.8) into (7.3.5), one arrives at the required equation for M

$$\frac{\partial\left(\frac{\partial M}{\partial z}, \frac{\partial^2 M}{\partial z^2}\right)}{\partial(\lambda, \varphi)} + \cot\varphi \frac{\partial M}{\partial \lambda}\frac{\partial^3 M}{\partial z^3} = \frac{\Omega a^2 \rho_0 K_H}{g} \sin 2\varphi \frac{\partial^4 M}{\partial z^4} . \tag{7.3.9}$$

The boundary conditions along the vertical at $z = 0$ and $z = H$ remain the same as in the case of the linear model [cf. (7.2.8), (7.2.9)]; generally speaking, the effect of wind may be taken into account by replacing the second condition (7.2.8) by a condition of the type (7.1.13). However, formulation of the boundary conditions along the horizontal demands special analysis, since system (7.3.1)—(7.3.5) [or, what is the same thing, equation (7.3.9)] describes motion in the open ocean and the boundary conditions required for its solution must be derived by study of the structure of nearshore boundary layers. Therefore only a certain particular solution of the nonlinear equation (7.3.9) will be presented here, in order to demonstrate the possibility of formation of a thermal boundary layer and to discuss its properties.

Consider equation (7.3.9) in non-dimensional form. Let θ_0 be a characteristic scale of temperature changes at the ocean surface. Then, by the third equation (7.3.8), the scale M_0 for the function M will be

$$M_0 = \theta_0 \alpha \rho_0 H_0^2 , \tag{7.3.10}$$

where H_0 is a characteristic thickness of the thermal boundary layer.

In equation (7.3.9), all terms describing heat advection have the same order of magnitude M_0^2/H_0^3. Assuming that the term describing vertical turbulent heat exchange has the same order, the quantity H_0 may be found:

$$H_0 = \left(\frac{a^2 K_H \Omega}{g \alpha \theta_0}\right)^{1/3} . \tag{7.3.11}$$

Let $a = 6.4 \cdot 10^3$ km, $K_h = 1$ cm^2/sec, $\Omega = 10^{-4}$ sec^{-1}, $g = 10^3$cm^2/sec, $\alpha = 2.5 \cdot 10^{-4}$ ^{0}C^{-1}, $\theta_0 = 10°$C. Then $H_0 = 250$ m. Note specially that by comparison with (7.2.16) the term derived for H_0 contains only a single vague parameter — the coefficient of vertical turbulent temperature transfer K_H; however, since $H_0 \sim \sqrt[3]{K_H}$, changes in K_H do not effect strongly the magnitude of H_0. Finally, using (7.3.7), (7.3.8), (7.3.10) and (7.3.11), characteristic magnitudes of velocities may be estimated: $u = 1$ cm/sec and $w = 0.4 \cdot 10^{-4}$ cm/sec.

Selecting H_0 as characteristic vertical scale and M_0 as scale for the function M, equation (7.3.9) may be written in the form

$$\frac{\partial\left(\frac{\partial M}{\partial z}, \frac{\partial^2 M}{\partial z^2}\right)}{\partial(\lambda, \varphi)} + \cot\varphi \frac{\partial M}{\partial \lambda}\frac{\partial^3 M}{\partial z^3} = \sin 2\varphi \frac{\partial^4 M}{\partial z^4} , \tag{7.3.12}$$

where the previous notation has been retained for non-dimensional M and z.

A particular solution of this equation will be sought in the form

$$M = A(\lambda, \varphi)\, z + B(\lambda, \varphi) + \frac{m(\lambda, \varphi)}{k(\lambda, \varphi)}\, e^{zk(\lambda,\varphi)} . \tag{7.3.13}$$

Substituting (7.3.13) into (7.3.12), one finds conditions under which (7.3.13) actually yields a solution of (7.3.12):

$$\frac{\partial k}{\partial \lambda} = 0 , \tag{7.3.14}$$

$$\frac{\partial m}{\partial \lambda}\left(\frac{\partial k}{\partial \varphi} + k \cot \varphi\right) = 0 , \tag{7.3.15}$$

$$\frac{\partial A}{\partial \lambda}\left(\frac{\partial k}{\partial \varphi} + k \cot \varphi\right) = 0 , \tag{7.3.16}$$

$$\frac{\partial B}{\partial \lambda} = \frac{mk^3 \sin 2\varphi - m\dfrac{\partial A}{\partial \lambda}\dfrac{\partial k}{\partial \varphi} - k\dfrac{\partial(A, m)}{\partial(\lambda, \varphi)}}{k^2 m \cot \varphi} . \tag{7.3.17}$$

The case when $\partial k/\partial\varphi + k \cot\varphi \neq 0$ is not of interest: it gives the solution

$$u = u(\varphi, z) , \qquad v = 0 , \qquad w = w(\lambda, \varphi) , \qquad T = T(\varphi, z) ,$$

for which horizontal advection of heat vanishes identically. Consequently

$$k = \frac{c}{\sin \varphi} , \tag{7.3.18}$$

where c is some constant.

If the functions A and m are known, then the function B can be found from (7.3.17). Hence the solution (7.3.13) contains two "free" functions: A and m.

Rewriting (7.3.7) and (7.3.8) in non-dimensional form, one finds u, v, w, and T. Employing (7.3.17), one has

$$u = -\frac{1}{\sin \varphi}\left\{\frac{\partial A}{\partial \varphi} + \left(\frac{\partial m}{\partial \varphi} - czm\frac{\cos \varphi}{\sin^2 \varphi}\right) e^{cz/\sin\varphi}\right\} ,$$

$$v = \frac{1}{\sin \varphi \cos \varphi}\left\{\frac{\partial A}{\partial \lambda} + \frac{\partial m}{\partial \lambda}\, e^{cz/\sin\varphi}\right\} ,$$

$$w = \frac{z}{\sin^2\varphi}\frac{\partial A}{\partial \lambda} + \frac{2c}{\sin\varphi} + \frac{1}{c \sin\varphi}\frac{\partial A}{\partial \lambda} - \frac{1}{cm \cos\varphi}\frac{\partial(A, m)}{\partial(\lambda, \varphi)} + \frac{1}{c \sin\varphi}\frac{\partial m}{\partial \lambda}\, e^{cz/\sin\varphi} ,$$

$$T - T_0 = -\frac{cm}{\sin\varphi}\, e^{cz/\sin\varphi} . \tag{7.3.19}$$

It is seen that one must select $c < 0$ for description of the thermocline (on the northern hemisphere). Furthermore, the quantity $m(\lambda, \varphi)$ is immediately determined, as soon as the temperature at the ocean surface $z = 0$ is given. Setting $w = 0$ for $z = H$ (non-dimensional depth), one finds

$$\frac{H}{\sin^2\varphi}\frac{\partial A}{\partial\lambda} + \frac{2c}{\sin\varphi} + \frac{1}{c\sin\varphi}\frac{\partial A}{\partial\lambda} - \frac{1}{cm\cos\varphi}\frac{\partial(A, m)}{\partial(\lambda, \varphi)} = 0 . \tag{7.3.20}$$

where it has been assumed that the constant c has been given a value such that the term with exponent may be neglected ($|c|H >> 1$). Since the function m is already known, relation (7.3.20) permits determination of $A(\lambda, \varphi)$. Then formula (7.3.19) for w may be rewritten in the form

$$w = \frac{z - H}{\sin^2\varphi}\frac{\partial A}{\partial\lambda} + \frac{1}{c\sin\varphi}\frac{\partial m}{\partial\lambda}\,e^{cz/\sin\varphi} . \tag{7.3.21}$$

Assume that the wind has been selected such that the function $w(\lambda, \varphi, 0)$ satisfies a condition of the type (7.1.13). Then conditions of the type (7.2.8) and (7.2.9) will be fulfilled and if one forgets for the time being about the boundary conditions along the horizontal which are unknown, it may be said that the solution derived describes the velocity and temperature fields in the open ocean. Note that side by side with the thermal boundary layer the model under consideration describes also barotropic motion below the thermocline (the horizontal velocity does not depend on z). This is an important non-linear effect of the model. Recall that by equation (7.2.7) of the linear model in those regions where $T \equiv 0$, necessarily also $w \equiv 0$; the complete non-linear equation (7.3.5) is free from this deficiency.

Thus, the models of §§ 7.2. and 7.3 have demonstrated the possibility of explaining the thermocline in the ocean as a specific thermal boundary layer. However, answers to many questions remain still unclear, for example, relating to the relative roles of turbulent exchange and advection of heat in the formation of the thermocline, etc.

COMMENT ON CHAPTER 7

§ 7.1 Basically, this follows Pedlosky [93], cf. also [101].

§ 7.2 In essence, the first linear model of the thermocline was proposed by Lineikin [65]. Formula (7.2.16) was obtained by Stommel and Veronis [118]; cf. also [41, 66] and Lineikin's survey [67]. The problem studied here was solved by Barcilon [1].

§ 7.3 Non-linear models have been studied in many papers (cf. the survey by Veronis [127]). The problem investigated here was solved by Needler [88]. Cf. Welander [129] for a survey of further developments of the problem.

APPENDIX A

ELEMENTS OF TENSOR ANALYSIS

A.1 CURVILINEAR COORDINATES

Consider three-dimensional Euclidean space. Let y^1, y^2, y^3 be rectangular Cartesian coordinates in a region V of this space. The formulae

$$x^\alpha = x^\alpha(y^1, y^2, y^3) \tag{A.1.1}$$

determine curvilinear coordinates x^α in the region V, if the functions $x^\alpha(y^1, y^2, y^3)$ have a single-valued inverse in this region so that

$$y^\alpha = y^\alpha(x^1, x^2, x^3)\ . \tag{A.1.2}$$

Here and in what follows, subscripts or superscripts which are letters may assume any of the values 1, 2, 3, i.e., relation (A.1.1) represents not one but three equations. Employment of upper and lower indices proves to be a convenient formalism.

In the particular case when the functions $x^\alpha(y^1, y^2, y^3)$ are linear, the coordinates x^α are said to be *affine*.

Surfaces x^α = constant are called coordinate surfaces, and lines along which only one of the coordinates x^α changes, coordinate lines. For example, the coordinate surfaces x^1 = constant and x^2 = constant intersect along a coordinate line x^3.

Let $\boldsymbol{O}$ be a fixed point. Consider some point $\boldsymbol{M}(x^\alpha)$ of V and its radius vector $\boldsymbol{OM}$. The vector

$$\boldsymbol{e}_\alpha = \frac{\partial \boldsymbol{OM}}{\partial x^\alpha} \tag{A.1.3}$$

is known to be directed along the tangent to the coordinate line x^α. According to (A.1.3), each coordinate system x^α gives rise at the point $\boldsymbol{M}$ to three vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ which form a local basis there. This basis plays a fundamental role in what follows.

Components a^α of a vector $\boldsymbol{a}$ with respect to the basis $\boldsymbol{e}_\alpha$ are said to be contravariant

$$\boldsymbol{a} = a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3 = a^\alpha\boldsymbol{e}_\alpha\ . \tag{A.1.4}$$

Here and below (unless stated otherwise) the customary convention is adopted that, if in a monomial one and the same Greek index is encountered twice, once as a superscript and once as a subscript, then the expression denotes the sum of monomials for the values 1, 2, 3 of the index.

The basis $\boldsymbol{e}_\alpha$ at a point M is closely linked to the basis $\boldsymbol{e}^\alpha$ the vectors of which are defined as

$$\boldsymbol{e}^1 = \frac{\boldsymbol{e}_2 \times \boldsymbol{e}_3}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)}, \qquad \boldsymbol{e}^2 = \frac{\boldsymbol{e}_3 \times \boldsymbol{e}_1}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)}, \qquad \boldsymbol{e}^3 = \frac{\boldsymbol{e}_1 \times \boldsymbol{e}_2}{(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)}, \tag{A.1.5}$$

where $\boldsymbol{e}_\alpha \times \boldsymbol{e}_\beta$ denote vector products and $(\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)$ a triple product. Note that $\boldsymbol{e}_\alpha$ can be expressed in terms of $\boldsymbol{e}^\alpha$ by formulae analogous to (A.1.5).

Consider the scalar product of vectors $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\beta$

$$\boldsymbol{e}_\alpha \boldsymbol{e}^\beta = \delta^\beta_{.\alpha}, \tag{A.1.6}$$

where $\delta^\beta_{.\alpha} = 0$ for $\alpha \neq \beta$ and $\delta^\beta_{.\alpha} = 1$ for $\alpha = \beta$.

The components a_α of a vector $\boldsymbol{a}$ with respect to the basis $\boldsymbol{e}_\alpha$ are said to be covariant

$$\boldsymbol{a} = a_\alpha \boldsymbol{e}^\alpha . \tag{A.1.7}$$

It is seen from (A.1.5) that in rectangular Cartesian coordinates (and only in those) contravariant and covariant components of a vector $\boldsymbol{a}$ coincide.

The bases $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}^\alpha$ generate at a point M two important symmetric matrices $m_{\alpha\beta}$ and $m^{\alpha\beta}$ (α referring to rows, β to columns):

$$m_{\alpha\beta} = \boldsymbol{e}_\alpha \boldsymbol{e}_\beta, \qquad m^{\alpha\beta} = \boldsymbol{e}^\alpha \boldsymbol{e}^\beta . \tag{A.1.8}$$

It is not difficult to obtain from (A.1.4)—(A.1.8) the relations

$$a_\beta = m_{\alpha\beta} a^\alpha, \qquad a^\alpha = m^{\alpha\beta} a_\beta, \tag{A.1.9}$$

and

$$\boldsymbol{e}_\beta = m_{\beta\alpha} \boldsymbol{e}^\alpha, \qquad \boldsymbol{e}^\alpha = m^{\alpha\beta} \boldsymbol{e}_\beta, \qquad m_{\alpha\beta} m^{\omega\beta} = \delta^\omega_{.\alpha} . \tag{A.1.10}$$

Obviously, the matrices $m_{\alpha\beta}$ and $m^{\alpha\beta}$ are inverses of each other.

Next, find the expression for the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. By (A.1.4), (A.1.7) and (A.1.8), one has

$$\boldsymbol{ab} = m_{\alpha\beta} a^\alpha b^\beta = m^{\alpha\beta} a_\alpha b_\beta = a_\alpha b^\alpha . \tag{A.1.11}$$

Consider a point $M'(x^\alpha + \mathrm{d}x^\alpha)$, close to the point $M(x^\alpha)$, and find the distance between them:

$$\mathrm{d}\boldsymbol{s} = \boldsymbol{OM}' - \boldsymbol{OM} = \frac{\partial \boldsymbol{OM}}{\partial x^\alpha} \mathrm{d}x^\alpha = \mathrm{d}x^\alpha \boldsymbol{e}_\alpha$$

Hence

$$\mathrm{d}s^2 = m_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta, \tag{A.1.12}$$

from which follows that the matrix $m_{\alpha\beta}$ is positive definite.

A.2 TRANSFORMATION OF COORDINATES

Let new curvilinear coordinates $x^{\alpha'}$ be introduced in the region V which are related to x^{α} by the formulae

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, x^3)\,, \qquad x^{\alpha} = x^{\alpha}(x^{1'}, x^{2'}, x^{3'})\,. \tag{A.2.1}$$

It will be convenient to denote this coordinate system by dashed indices.

At a point M under consideration, the coordinate system $x^{\alpha'}$ generates new bases $\boldsymbol{e}_{\alpha'}$ and $\boldsymbol{e}^{\alpha'}$ and matrices $m_{\alpha'\beta'}$ and $m^{\alpha'\beta'}$. In this context, there arise also new contravariant and covariant components $a^{\alpha'}$ and $a_{\alpha'}$ of a vector $\boldsymbol{a}$. How are the new quantities $\boldsymbol{e}_{\alpha'}$, $\boldsymbol{e}^{\alpha'}$, $m_{\alpha'\beta'}$, $m^{\alpha'\beta'}$, $a^{\alpha'}$ and $a_{\alpha'}$ related to the old quantities $\boldsymbol{e}^{\alpha}$, $\boldsymbol{e}_{\alpha}$, $m_{\alpha\beta}$, $m^{\alpha\beta}$, a_{α} and a^{α}?

First establish the link between the vectors of the local bases $\boldsymbol{e}_{\alpha'}$ and $\boldsymbol{e}_{\alpha}$. One has

$$\boldsymbol{e}_{\alpha'} = \frac{\partial \boldsymbol{OM}}{\partial x^{\alpha'}} = \frac{\partial \boldsymbol{OM}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\boldsymbol{e}_{\alpha}\,, \tag{A.2.2}$$

$$\boldsymbol{e}_{\alpha} = \frac{\partial \boldsymbol{OM}}{\partial x^{\alpha}} = \frac{\partial \boldsymbol{OM}}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\boldsymbol{e}_{\alpha'}\,. \tag{A.2.3}$$

where the index α' in the derivative $\partial x^{\alpha}/\partial x^{\alpha'}$ is assumed to be subscript.

Two basic matrices of the transformation have been introduced:

$$A^{\alpha}_{\cdot\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\,, \qquad A^{\alpha'}_{\cdot\alpha} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\,. \tag{A.2.4}$$

For fixed α and α', the number $A^{\alpha}_{\cdot\alpha'}$ may be interpreted as that element of the matrix which stands at the intersection of row α and column α'. Since the correspondence (A.2.1) is one-to-one, the matrices $A^{\alpha}_{\cdot\alpha'}$ and $A^{\alpha'}_{\cdot\alpha}$ are not singular. It follows from the obvious identity

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\alpha'}}{\partial x^{\beta}} = \delta^{\alpha}_{\cdot\beta} \tag{A.2.5}$$

that they are inverses of each other.

Consider a vector $\boldsymbol{a}$ at the point M. Its components a^{α} or a_{α} depend on the choice of the coordinate system. However, the vector $\boldsymbol{a}$ is invariant with respect to this choice; consequently,

$$a^{\alpha'}\boldsymbol{e}_{\alpha'} = a^{\alpha}\boldsymbol{e}_{\alpha}\,.$$

Using (A.2.3), one obtains

$$a^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\,a^{\alpha}\,. \tag{A.2.6}$$

It follows from definition (A.1.8) of the matrix $m_{\alpha\beta}$ and (A.2.2) that

$$m_{\alpha'\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}}\frac{\partial x^{\beta}}{\partial x^{\beta'}}\,m_{\alpha\beta}\,. \tag{A.2.7}$$

is the transformation law of this matrix.

Formula (A.1.9) permits to derive the transformation law of covariant components of a vector $\boldsymbol{a}$:

$$a_{\alpha'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} a_{\alpha} . \tag{A.2.8}$$

Since $a_{\alpha'}\boldsymbol{e}^{\alpha'} = a_{\alpha}\boldsymbol{e}^{\alpha}$, one has

$$\boldsymbol{e}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \boldsymbol{e}^{\alpha} \tag{A.2.9}$$

and

$$m^{\alpha'\beta'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} m^{\alpha\beta} . \tag{A.2.10}$$

Consider equations (A.2.6) and (A.2.8). It is seen that the vector $\boldsymbol{a}$ is characterized by the fact that the set of its components (contravariant or covariant) transforms for transition from coordinates x^{α} to coordinates $x^{\alpha'}$ according to (A.2.6) or (A.2.8). The inverse statement is also true: If in every system of coordinates there is defined a set of three numbers which transform during transition from one coordinate system to another according to (A.2.6) or (A.2.8), then this set of three numbers may be considered to be the components of some vector, since $a^{\alpha'}\boldsymbol{e}_{\alpha'} = a^{\alpha}\boldsymbol{e}_{\alpha}$ or $a_{\alpha'}\boldsymbol{e}^{\alpha'} = a_{\alpha}\boldsymbol{e}^{\alpha}$.

Thus, the fundamental property of invariance of a vector with respect to choice of coordinate system has been expressed successfully in terms of its components each of which depends naturally on the choice of this or another coordinate system.

It must be emphasized that in the law (A.2.6) the matrix $A^{\alpha'}_{.\alpha}$ is multiplied by the vector a^{α}, and in the law (A.2.8) the transposed matrix $A^{\alpha}_{.\alpha'}$ by the vector a_{α}. Comparison of (A.2.6) and (A.2.8) with (A.2.2) yields an explanation of the terminology *contravariant* (i.e., transformation different from that of the basis vector $\boldsymbol{e}_{\alpha}$) and *covariant* (i.e., transformation like that of the basis $\boldsymbol{e}_{\alpha}$).

Example. Let φ be a scalar function. Consider the set of three numbers $\partial\varphi/\partial x^{\alpha}$. One has

$$\frac{\partial\varphi}{\partial x^{\alpha'}} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial\varphi}{\partial x^{\alpha}} .$$

Hence it is seen that the three numbers $\partial\varphi/\partial x^{\alpha}$ transform according to (A.2.8) and that one may introduce a vector

$$\nabla\varphi = \frac{\partial\varphi}{\partial x^{\alpha}} \boldsymbol{e}^{\alpha} .$$

Thus it will be convenient to select covariant or contravariant components of a vector depending on the problem under consideration.

A.3 TENSORS

The concepts of scalar and vector are basic in physics; however, it is not possible to limit consideration to these quantities. For example, it is known from hydrodynamics that the vector of the surface forces $\boldsymbol{F}$ acting on an arbitrary area element $d\sigma$ at a point M depends linearly and homogeneously on the vector of the normal $\boldsymbol{n}$ to this area. In the coordinate system x^α, this statement may be written as

$$F^\alpha = p^{\alpha}_{.\beta} n^\beta \, .$$

Since the character of the link between the vectors $\boldsymbol{F}$ and $\boldsymbol{n}$ does not depend on the choice of coordinate system, the set of nine numbers $p^{\alpha}_{.\beta}$ transforms on transition from a system of coordinates x^α to a system of coordinates $x^{\alpha'}$, by (A.2.6), according to the law

$$p^{\alpha'}_{.\beta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} p^{\alpha}_{.\beta} \, .$$

The converse is also true: If a vector $\boldsymbol{F}$ depends in one coordinate system on the vector $\boldsymbol{n}$ linearly and homogeneously and the set of numbers $p^{\alpha}_{.\beta}$ transforms only in accordance with the prescribed law, then the linear and homogeneous link between these vectors does not depend on the choice of a concrete coordinate system.

Generalizing the example under consideration, the concept of *tensor* will now be introduced. A tensor $Q^{\alpha\beta}_{..\gamma}$ of third order, twice contravariant and once covariant, will be said to exist, if there is defined in every coordinate system x^α a set of 27 numbers $Q^{11}_{..1}$ $Q^{12}_{..1}, \dots$ which transform from coordinates x^α to coordinates $x^{\alpha'}$ in accordance with the law

$$Q^{\alpha'\beta'}_{..\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} \frac{\partial x^\gamma}{\partial x^{\gamma'}} Q^{\alpha\beta}_{..\gamma} \, . \tag{A.3.1}$$

The numbers $Q^{\alpha\beta}_{..\gamma}$ are called the components of the tensor.

The general definition of a tensor of arbitrary order and type may be constructed in an analogous manner. In the sequel, relation (A.3.1) will be referred to as the transformation law for the components of tensors of any kind. Obviously, scalars, vectors and matrices $p^{\alpha}_{.\beta}$ are tensors (for example, the set of contravariant components of a vector $\boldsymbol{a}$ forms a contravariant first order tensor a^α).

Note that each tensor component in a coordinate system $x^{\alpha'}$ is a linear homogeneous function of all its components in a system x^α. It must be emphasized that the nature of a tensor is determined by the number and arrangement of its indices which are given as Greek letters, where in each vertical there is written down only one index. As a consequence, in general, tensors $Q^{\alpha}_{.\beta}$ and $Q_{\beta}^{.\alpha}$ will differ.

On the basis of (A.2.5), it is not difficult to derive from (A.3.1) the inverse transformation law (proceeding from coordinates $x^{\alpha'}$ to coordinates x^{α}).

Two tensors of the same type and order are equal if all corresponding components of these tensors in each coordinate system are equal.

A tensor is said to be symmetric with respect to pairs of upper (or lower) indices, if its components do not change on transposition of the indices; for example, $A^{\alpha\beta} = A^{\beta\alpha}$. If, as a consequence of such a transposition, the components change sign, but not their absolute values, a tensor is said to be anti-symmetric, for example, $B^{\alpha\beta} = -B^{\beta\alpha}$. It is to be emphasized that it follows from (A.3.1) that, if the property of symmetry (or antisymmetry) exists in one coordinate system, then it exists in any coordinate system.

There exists a class of *admissible* operations on tensors which again generate tensors. Functional relations which include only such operations are said to be tensorial. By strength of the invariance of such relations with respect to choice of coordinate system, it is sufficient to verify their truth only in a single coordinate system. For example, if in one coordinate system all tensor components vanish, then they vanish in any coordinate system and one has a zero tensor. Consequently, it may be said about tensors and tensor relations that they do not refer to a particular coordinate system. This is the basic idea of tensor calculus.

The following operations will be defined by means of concrete examples:

(1) The sum of two tensors $A^{\alpha}_{.\beta\gamma}$ and $B^{\alpha}_{.\beta\gamma}$ of identical type and order is a tensor $C^{\alpha}_{.\beta\gamma}$ of the same type and order the components of which are equal to the sums of corresponding components of the summand tensors:

$$C^{\alpha}_{.\beta\gamma} = A^{\alpha}_{.\beta\gamma} + B^{\alpha}_{.\beta\gamma} \,. \tag{A.3.2}$$

(2) The product of two tensors $A^{..\gamma}_{\alpha\beta}$ and $B^{\nu\mu}_{..\omega}$ is the tensor with components

$$C^{..\gamma\nu\mu}_{\alpha\beta...\omega} = A^{..\gamma}_{\alpha\beta} B^{\nu\mu}_{..\omega} \,. \tag{A.3.3}$$

In general, the operation of multiplication is not commutative.

(3) The operation of contraction. Let there be given a tensor $A^{..\alpha\beta\gamma}_{\nu\mu}$. Select any superscript (for example, α) and any subscript (for example, μ) and sum all components with the same values of these indices. The resulting sum will be components of the new tensor

$$A^{.\beta\gamma}_{\nu} = A^{..\alpha\beta\gamma}_{\nu\alpha} \,. \tag{A.3.4}$$

(4) Tensor criterion. Let it be known that for an arbitrary tensor $B^{\beta\gamma}$ the operation

$$A^{..\alpha}_{\beta\gamma} B^{\beta\gamma} = C^{\alpha} \tag{A.3.5}$$

always yields a tensor C^{α}. Then $A^{..\alpha}_{\beta\gamma}$ is likewise a tensor.

The generalization of rules 1—4 to tensors of any kind is obvious.

Consider a tensor $Q^{\alpha}_{.\beta}$. Side by side with this tensor, one may consider the tensors

$$Q^{\alpha\beta} = m^{\alpha\gamma} Q^{\beta}_{.\gamma}\,, \qquad Q_{\alpha\beta} = m_{\alpha\gamma} Q^{\gamma}_{.\beta}\,, \qquad Q^{.\beta}_{\alpha} = m^{\gamma\beta} Q_{\alpha\gamma}\,. \tag{A.3.6}$$

Operations (A.3.6) are referred to as raising and lowering of indices and the tensors $Q^{\alpha}_{.\beta}$, $Q^{\alpha\beta}$, $Q^{.\beta}_{\alpha}$ and $Q_{\alpha\beta}$ are said to be associated. In essence, they are all different representations of one and the same physical characteristic [cf. (A.1.9)]. If consideration is limited to rectangular Cartesian coordinates, then the difference between associated tensors vanishes. Obviously, classes of associate tensors may be formed for tensors of any kind.

A.4 EXAMPLES OF SIMPLE TENSORS

It is readily verified that $\delta^{\alpha}_{.\beta}$ is a tensor. In fact, it follows from (A.2.6) that

$$\frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \delta^{\beta}_{.\alpha} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} = \delta^{\beta'}_{.\alpha'}\,. \tag{A.4.1}$$

The tensor character of $m_{\alpha\beta}$ and $m^{\alpha\beta}$ follows from (A.2.7) and (A.2.10). The last equality (A.1.10) yields that $m_{\alpha\beta}$, $m^{\alpha\beta}$ and $\delta^{\alpha}_{.\beta}$ are associate tensors. The tensor $m_{\alpha\beta}$ is referred to as *metric* tensor.

Let the determinant of the matrix $m_{\alpha\beta}$ be denoted by m: $|m_{\alpha\beta}| = m$. Equation (A.2.7) now renders

$$m' = \left| \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \right|^2 m, \tag{A.4.2}$$

where $|\partial x^{\alpha}/\partial x^{\alpha'}|$ is the determinant of the matrix $\partial x^{\alpha}/\partial x^{\alpha'}$ (α referring to rows, α' to columns).

The transformation law for the quantity m differs from the general law (A.3.1) by the presence of the factor $|\partial x^{\alpha}/\partial x^{\alpha'}|^2$. In such cases one speaks of a pseudo-scalar of weight 2.

Let y^1, y^2, y^3 be a right-handed Cartesian rectangular coordinate system. Then

$$(\boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_3) = \left| \frac{\partial y^{\alpha}}{\partial x^{\beta}} \right| = \frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, x^3)}\,. \tag{A.4.3}$$

It follows from this relation and the property of Jacobians that $|\partial x^{\alpha}/\partial x^{\alpha'}| > 0$, if the orientations of the vectors $\boldsymbol{e}_{\alpha}$ and $\boldsymbol{e}_{\alpha'}$ are identical, and $|\partial x^{\alpha}/\partial x^{\alpha'}| < 0$, if their orientations differ.

In each coordinate system x^{α}, define the sign of the quantity $\sqrt{m}$: $\sqrt{m} > 0$, if $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ form a right-handed triple of vectors and $\sqrt{m} < 0$ if this triple is left-handed. Taking into consideration the sign of $|\partial x^{\alpha}/\partial x^{\alpha'}|$, one obtains, by (A.4.2), the transformation law for $\sqrt{m}$

$$\sqrt{m'} = \left| \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \right| \sqrt{m} \, . \tag{A.4.4}$$

In this manner, the pseudo-scalar $\sqrt{m}$ of weight 1 has been introduced.

Consider now sets of numbers $e_{\alpha\beta\gamma}$ and $e^{\alpha\beta\gamma}$ which are antisymmetric in any pair of the indices, defining their components in an arbitrary coordinate system by

$$e^{\alpha\beta\gamma} = \begin{cases} 0 \text{ if two indices are equal,} \\ 1 \text{ if } \alpha,\beta,\gamma \text{ form an even permutation of } 1,2,3, \\ -1 \text{ if } \alpha,\beta,\gamma \text{ form an odd permutation of } 1,2,3, \end{cases}$$

and analogously for $e_{\alpha\beta\gamma}$.

It follows from the theory of determinants that

$$e_{\alpha'\beta'\gamma'} = \left| \frac{\partial x^{\omega'}}{\partial x^{\omega}} \right| \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} e_{\alpha\beta\gamma} \, . \tag{A.4.5}$$

$$e^{\alpha'\beta'\gamma'} = \left| \frac{\partial x^{\omega}}{\partial x^{\omega'}} \right| \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \frac{\partial x^{\gamma'}}{\partial x^{\gamma}} e^{\alpha\beta\gamma} \, , \tag{A.4.6}$$

It is seen that the transformations of $e^{\alpha\beta\gamma}$ and $e_{\alpha\beta\gamma}$ likewise differ from the general law (A.3.1) by the presence of the factor $|\partial x^{\omega}/\partial x^{\omega'}|^{k}$ (k equal to +1 or −1). As a rule, it is said that $e^{\alpha\beta\gamma}$ is a pseudo-tensor of weight +1 and $e_{\alpha\beta\gamma}$ a pseudo-tensor of weight −1.

It follows from (A.4.4)—(A.4.6) that $\epsilon^{\alpha\beta\gamma}$ and $\epsilon_{\alpha\beta\gamma}$, defined as

$$\epsilon^{\alpha\beta\gamma} = \frac{1}{\sqrt{m}} e^{\alpha\beta\gamma} \, , \qquad \epsilon_{\alpha\beta\gamma} = \sqrt{m} \, e_{\alpha\beta\gamma} \, , \tag{A.4.7}$$

are true tensors; it is readily shown that $\epsilon^{\alpha\beta\gamma}$ and $\epsilon_{\alpha\beta\gamma}$ are associate tensors.

An expression for the vector product of two vectors from the right $\boldsymbol{a} \times \boldsymbol{a}$ will now be found. By (A.1.5), (A.4.3) and (A.4.4), one has

$$\boldsymbol{a} \times \boldsymbol{b} = (a^{\alpha}\boldsymbol{e}_{\alpha}) \times (b^{\beta}\boldsymbol{e}_{\beta}) = \epsilon_{\alpha\beta\gamma} a^{\beta} b^{\gamma} \boldsymbol{e}^{\alpha} \, . \tag{A.4.8}$$

It is seen that the vector product of two vectors from the right (and only such a vector product will be considered here) is a vector the covariant components of which are given by (A.4.8). Analogously, one has

$$\boldsymbol{a} \times \boldsymbol{b} = \epsilon^{\alpha\beta\gamma} a_{\beta} b_{\gamma} \boldsymbol{e}_{\alpha} \, . \tag{A.4.9}$$

The triple product of vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is a scalar and may be written in the form

$$(\boldsymbol{abc}) = \epsilon_{\alpha\beta\gamma} a^{\alpha} b^{\beta} c^{\gamma} = \epsilon^{\alpha\beta\gamma} a_{\alpha} b_{\beta} c_{\gamma} \, . \tag{A.4.10}$$

The following formulae are important:

$$\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\nu\mu} b_{\nu\mu} = b_{\beta\gamma} - b_{\gamma\beta} \, , \qquad \epsilon^{\alpha\beta\gamma}\epsilon_{\alpha\nu\mu} b^{\nu\mu} = b^{\beta\gamma} - b^{\gamma\beta} \, , \tag{A.4.11}$$

where $b_{\beta\gamma}$ is an arbitrary tensor.

Consider an anti-symmetric tensor $b_{\alpha\beta}$; it has only three independent components. Introduce the vector

$$u^{\alpha} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}b_{\beta\gamma} . \tag{A.4.12}$$

If the vector u^{α} is known, then, by (A.4.11), one has

$$\epsilon_{\alpha\beta\gamma}u^{\alpha} = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\nu\mu}b_{\nu\mu} = \tfrac{1}{2}(b_{\beta\gamma} - b_{\gamma\beta}) = b_{\beta\gamma} . \tag{A.4.13}$$

Thus, specification of an anti-symmetric tensor of second order $b_{\alpha\beta}$ is equivalent to specification of a vector u^{α}, and conversely.

The vector product from the right $\boldsymbol{a} \times \boldsymbol{b}$ could be introduced as an anti-symmetric second order tensor $a_{\alpha}b_{\beta} - a_{\beta}b_{\alpha}$. By the results above, the tensor $a_{\alpha}b_{\beta} - a_{\beta}b_{\alpha}$ is equivalent to the vector defined by (A.4.8) and (A.4.9).

A.5 ISOTROPIC AND AXISYMMETRIC TENSORS

A tensor is said to be isotropic if its components do not change for all possible coordinate transformations leading to rotation about an axis passing through a point M and mirror reflection with respect to a plane through M.

For example, consider an isotropic tensor $Q_{\alpha\beta}$ and find its general form. For this purpose, introduce the bilinear function $F(\boldsymbol{a}, \boldsymbol{b}) = Q_{\alpha\beta}a^{\alpha}a^{\beta}$ (where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary vectors). Then $Q_{\alpha\beta} = F(\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta})$. Since the tensor $Q_{\alpha\beta}$ is isotropic, the scalar function $F(\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta})$ turns out to be invariant with respect to the group of rotations and mirror reflections and, according to the theory of invariants of the group, to depend only on the scalar products $\boldsymbol{e}_{\alpha}\boldsymbol{e}_{\beta}$. Taking into account the bilinearity of the function $F(\boldsymbol{e}_{\alpha}, \boldsymbol{e}_{\beta})$, one finds that the general expression for the isotropic tensor $Q_{\alpha\beta}$ depends on the single scalar a

$$Q_{\alpha\beta} = a(\boldsymbol{e}_{\alpha}\boldsymbol{e}_{\beta}) = am_{\alpha\beta} .$$

This method permits to determine the general form of a covariant isotropic tensor of any order. Limiting consideration to tensors of order not higher than 4, one has

$$Q_{\alpha} = 0 , \qquad Q_{\alpha\beta} = am_{\alpha\beta} , \qquad Q_{\alpha\beta\gamma} = 0 ,$$
$$Q_{\alpha\beta\nu\mu} = a_1 m_{\alpha\beta}m_{\nu\mu} + a_2 m_{\alpha\nu}m_{\beta\mu} + a_3 m_{\alpha\mu}m_{\beta\nu} , \tag{A.5.1}$$

where a, a_1, a_2 and a_3 are arbitrary scalars.

Next, consider the definition of the axisymmetric tensor. Let $\boldsymbol{k}$ be a unit vector which specifies the axis of symmetry at a given point. A tensor will be said to be axisymmetric, if its components do not change for all possible coordinate transformations which induce rotation about the axis $\boldsymbol{k}$ and mirror reflection in places containing $\boldsymbol{k}$ or perpendiculars to $\boldsymbol{k}$.

As an example, consider an axisymmetric tensor $Q_{\alpha\beta}$ and find its general form. Introduce again the bilinear function $F(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) = Q_{\alpha\beta}$. This function turns out to be invariant with respect to the group of rotations and reflections; it does not disturb the configuration of the system of vectors $\boldsymbol{k}, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and its general form is given by

$$Q_{\alpha\beta} = F(\boldsymbol{e}_\alpha \boldsymbol{e}_\beta) = a_1(\boldsymbol{e}_\alpha \boldsymbol{e}_\beta) + a_2(\boldsymbol{k}\boldsymbol{e}_\alpha)(\boldsymbol{k}\boldsymbol{e}_\beta) = a_1 m_{\alpha\beta} + a_2 k_\alpha k_\beta \ ,$$

where a_1 and a_2 are arbitrary scalars.

Applying the method above, one obtains the general form of covariant axisymmetric tensors of any order. Limiting consideration to tensors of order not higher than 4, one finds

$$Q_\alpha = 0 \ , \qquad Q_{\alpha\beta} = a_1 m_{\alpha\beta} + a_2 k_\alpha k_\beta \ , \qquad Q_{\alpha\beta\gamma} = 0 \ ,$$
$$Q_{\alpha\beta\nu\mu} = b_1 m_{\alpha\beta} m_{\nu\mu} + b_2 m_{\alpha\nu} m_{\beta\mu} + b_3 m_{\alpha\mu} m_{\beta\nu} + b_4 m_{\alpha\beta} k_\nu k_\mu + b_5 m_{\alpha\nu} k_\beta k_\mu$$
$$+ b_6 m_{\alpha\mu} k_\beta k_\nu + b_7 m_{\nu\mu} k_\alpha k_\beta + b_8 m_{\beta\mu} k_\alpha k_\nu + b_9 m_{\beta\nu} k_\alpha k_\mu + b_{10} k_\alpha k_\beta k_\nu k_\mu \ , \tag{A.5.2}$$

where $a_1, a_2, b_1, \ldots, b_{10}$ are arbitrary scalars.

A.6 DIFFERENTIATION OF TENSORS

In the preceding sections, tensors and tensor operations at a fixed point have been studied. Next, consider tensor fields and investigate the problem of the change of a tensor during transition from one point M to a closeby point M'.

First of all, the change of local basis vectors will be determined. One has

$$\frac{\partial \boldsymbol{e}_\alpha}{\partial x^\beta} = \frac{\partial^2 \boldsymbol{OM}}{\partial x^\beta \partial x^\alpha} = \Gamma^\nu_{\alpha\beta} \boldsymbol{e}_\nu \ . \tag{A.6.1}$$

Relation (A.6.1) defines the new object $\Gamma^\nu_{\alpha\beta}$ referred to as Christoffel symbol of the second kind. It follows from (A.6.1) that $\Gamma^\nu_{\alpha\beta} = \Gamma^\nu_{\beta\alpha}$. The following formula may be derived for $\Gamma^\nu_{\alpha\beta}$:

$$\Gamma^\nu_{\alpha\beta} = \tfrac{1}{2} m^{\nu\kappa} \left(\frac{\partial m_{\kappa\alpha}}{\partial x^\beta} + \frac{\partial m_{\kappa\beta}}{\partial x^\alpha} - \frac{\partial m_{\alpha\beta}}{\partial x^\kappa} \right) . \tag{A.6.2}$$

It is not difficult to find now the transformation law for $\Gamma^\nu_{\alpha\beta}$ and to show that it is not a tensor.

Thus, on transition to the neighbouring point $M'(x_\alpha + \mathrm{d}x_\alpha)$, the local basis changes to $\boldsymbol{e}'_\alpha = \boldsymbol{e}_\alpha + \mathrm{d}\boldsymbol{e}_\alpha$, so that

$$\boldsymbol{e}'_\alpha = a^{\beta}_{.\alpha} \boldsymbol{e}_\beta \ , \qquad a^{\beta}_{.\alpha} = \delta^{\beta}_{.\alpha} + \Gamma^\beta_{\kappa\alpha} \mathrm{d}x^\kappa \ . \tag{A.6.3}$$

Consider now the solution of the fundamental problem, for example, in the case of the tensor $Q^{\alpha}_{.\beta}$. How is one to compare $Q^{\alpha}_{.\beta}(M)$ and $Q^{\alpha}_{.\beta}(M')$? Difficulties encountered in such a comparison arise from the fact that algebraic operations may only be performed on tensors given at one point.

Proceed in the following manner. Transpose the basis vectors e_i', parallel to themselves from the point M' to the point M and together with the basis *transfer* the tensor $Q^{\alpha}_{.\beta}(M')$. In other words, consider at the point M a new basis e_α' and the tensor $Q'^{\alpha}_{.\beta}(M)$ given with respect to this basis with components equal to the components of $Q^{\alpha}_{.\beta}(M')$. Since now both tensors $Q'^{\alpha}_{.\beta}(M)$ and $Q^{\alpha}_{.\beta}(M)$ are given at the same point (however, in different bases) they may be compared (once they have been referred to one and the same basis).

The tensor $Q'^{\alpha}_{.\beta}$ will now be written in terms of the basis e_α. This problem is readily resolved with the aid of (A.2.4), (A.2.5) and (A.3.1). Denoting the components of $Q'^{\alpha}_{.\beta}$ to the basis e_α by $\tilde{Q}^{\alpha}_{.\beta}$, one obtains

$$a^{\gamma}_{.\beta}\tilde{Q}^{\alpha}_{.\gamma}(M) = a^{\alpha}_{.\gamma}Q'^{\gamma}_{.\beta}(M) .$$

Disregarding squares of dx^α, one finds

$$\tilde{Q}^{\alpha}_{.\beta} = Q^{\alpha}_{.\beta} + \frac{\partial Q^{\alpha}_{.\beta}}{\partial x^{\gamma}}\,dx^{\gamma} + \Gamma^{\alpha}_{\kappa\gamma}Q^{\gamma}_{.\beta}dx^{\kappa} - \Gamma^{\kappa}_{\gamma\beta}Q^{\alpha}_{.\kappa}dx^{\gamma} . \tag{A.6.4}$$

Denote now the absolute differential of the tensor $Q^{\alpha}_{.\beta}$ by $DQ^{\alpha}_{.\beta}$:

$$DQ^{\alpha}_{.\beta} = \tilde{Q}^{\alpha}_{.\beta} - Q^{\alpha}_{.\beta} = \left(\frac{\partial Q^{\alpha}_{.\beta}}{\partial x^{\kappa}} + \Gamma^{\alpha}_{\kappa\gamma}Q^{\gamma}_{.\beta} - \Gamma^{\gamma}_{\kappa\beta}Q^{\alpha}_{.\gamma}\right) dx^{\kappa} . \tag{A.6.5}$$

The preceding reasoning may be considered as a *guide*; one may simply define $DQ^{\alpha}_{.\beta}$ by (A.6.5) and then verify that the result is a tensor. By the tensor criterion, the set

$$\nabla_{\kappa}Q^{\alpha}_{.\beta} = \frac{\partial Q^{\alpha}_{.\beta}}{\partial x^{\kappa}} + \Gamma^{\alpha}_{\kappa\gamma}Q^{\gamma}_{.\beta} - \Gamma^{\gamma}_{\kappa\beta}Q^{\alpha}_{.\gamma} \tag{A.6.6}$$

will be a tensor; it is called the covariant derivative of the tensor $Q^{\alpha}_{.\beta}$.

It is readily verified that $\partial Q^{\alpha}_{.\beta}/\partial x^{\kappa}$ is not a tensor. In fact, on transition to a nearby point, there is superimposed on the change of the set yet a change caused by choice of a definite coordinate system (the change of the basis e_α from point to point depends, of course, on the choice of the coordinate system). Thanks to introduction of additional terms into (A.6.5) or (A.6.6) with factors $\Gamma^{\alpha}_{\kappa\gamma}$, one has succeeded, as has been shown, in separating out the invariant part of the change of $Q^{\alpha}_{.\beta}$. Note that in affine coordinates all $\Gamma^{\nu}_{\alpha\beta} = 0$ and there do not arise additional terms in (A.6.5) and (A.6.6).

It is clear from the example under consideration how one can define the absolute derivative and covariant derivative for a tensor of any kind. For example, let there be given a tensor $Q^{\alpha\beta}_{..\nu\mu}$. For every superscript (for example, α), construct the additional term with plus sign of the type $\Gamma^{\alpha}_{\kappa\sigma}Q^{\sigma\beta}_{..\nu\mu}dx^{\kappa}$, and for each subscript (for example, ν) the additional term with minus sign of the type $\Gamma^{\sigma}_{\kappa\nu}Q^{\alpha\beta}_{..\sigma\mu}dx^{\kappa}$. The number of such additional terms is equal to the order of the tensor (in general, there are no such terms for a scalar). Thus

$$DQ^{\alpha\beta}_{..\nu\mu} = \left(\frac{\partial Q^{\alpha\beta}_{..\nu\mu}}{\partial x^\kappa} + \Gamma^\alpha_{\kappa\sigma} Q^{\sigma\beta}_{..\nu\mu} + \Gamma^\beta_{\kappa\sigma} Q^{\alpha\sigma}_{..\nu\mu} - \Gamma^\sigma_{\kappa\nu} Q^{\alpha\beta}_{..\sigma\mu} - \Gamma^\sigma_{\kappa\mu} Q^{\alpha\beta}_{..\nu\sigma}\right) \mathrm{d}x^\kappa \,. \tag{A.6.7}$$

Note that the absolute differential of a tensor turns out to be a tensor of the same type and order, while in the covariant derivative of a tensor order and number of subscripts are increased by unity.

A.7 INVARIANT DIFFERENTIAL OPERATORS

In affine coordinate systems, when all $\Gamma^\nu_{\alpha\beta} = 0$, the following rules are readily verified.

(1) The covariant derivative of a sum of tensors is equal to the sum of covariant derivatives of each of the terms.

(2) The general rule for evaluation of the covariant derivative of a product of two tensors follows from the example:

$$\nabla_\kappa (A^\alpha_{.\beta} B^\nu_{.\mu}) = (\nabla_\kappa A^\alpha_{.\beta})\, B^\nu_{.\mu} + A^\alpha_{.\beta} (\nabla_\kappa B^\nu_{.\mu}). \tag{A.7.1}$$

(3) The covariant derivatives of the tensors $m_{\alpha\beta}$, $m^{\alpha\beta}$, $\delta^\alpha_{.\beta}$, $\epsilon_{\alpha\beta\gamma}$ and $\epsilon^{\alpha\beta\gamma}$ are equal to zero:

$$\nabla_\kappa m_{\alpha\beta} = 0 \,, \qquad \nabla_\kappa m^{\alpha\beta} = 0 \,, \qquad \nabla_\kappa \delta^\alpha_{.\beta} = 0 \,, \qquad \nabla_\kappa \epsilon_{\alpha\beta\gamma} = 0 \,, \qquad \nabla_\kappa \epsilon^{\alpha\beta\gamma} = 0 \,. \tag{A.7.2}$$

For parallel-translation of a vector $\boldsymbol{a}$, obviously, $\mathrm{D}\boldsymbol{a} = 0$. By (A.1.11), (A.4.8) and (A.7.2), scalar and vector products of parallel-translated vectors do not change.

From (A.7.2) follows the fact that the operations of covariant differentiation and contraction are commutative:

$$\delta^\beta_{.\alpha} (\nabla_\kappa A^\alpha_{.\beta\omega}) = \nabla_\kappa (\delta^\beta_{.\alpha} A^\alpha_{.\beta\omega}) = \nabla_\kappa A^\alpha_{.\alpha\omega} \,. \tag{A.7.3}$$

and likewise the rule for differentiation of associate tensors

$$\nabla_\kappa (m^{\alpha\beta} A_\beta) = m^{\alpha\beta} \nabla_\kappa A_\beta \,. \tag{A.7.4}$$

Writing out the identity $\nabla_\kappa \epsilon^{\alpha\beta\gamma} = 0$ in an arbitrary coordinate system x^ω, one arrives at the useful formula

$$\Gamma^\nu_{\nu\alpha} = \frac{1}{\sqrt{m}} \frac{\partial}{\partial x^\alpha} (\sqrt{m}) \,. \tag{A.7.5}$$

Consider now the most frequently encountered differential tensor operators (invariant operators):

(1) The scalar operator $\nabla_\alpha a^\alpha$ is called the divergence of the vector $\boldsymbol{a}$ and denoted by div $\boldsymbol{a}$. By (A.7.5), one has

$$\operatorname{div} a = \nabla_\alpha a^\alpha = \frac{\partial a^\alpha}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\beta} a^\beta = \frac{1}{\sqrt{m}} \frac{\partial}{\partial x^\alpha} (\sqrt{m}\, a^\alpha) . \tag{A.7.6}$$

The following vector is referred to as the divergence of the symmetric tensor $a^{\alpha\beta}$:

$$\nabla_\alpha a^{\alpha\beta} = \frac{\partial a^{\alpha\beta}}{\partial x^\alpha} + \Gamma^\kappa_{\kappa\alpha} a^{\alpha\beta} + \Gamma^\beta_{\kappa\nu} a^{\kappa\nu} = \frac{1}{\sqrt{m}} \frac{\partial}{\partial x^\alpha} (\sqrt{m}\, a^{\alpha\beta}) + \Gamma^\beta_{\nu\mu} a^{\nu\mu} . \tag{A.7.7}$$

Again, (A.7.5) has been used in writing down this result.

(2) The vector $\nabla_\alpha \psi$ is called the gradient of the scalar function ψ and denoted by $\nabla\psi$:

$$\boldsymbol{\nabla} \psi = \nabla_\alpha \psi \boldsymbol{e}^\alpha = m^{\alpha\beta} \frac{\partial \psi}{\partial x^\alpha} \boldsymbol{e}_\beta . \tag{A.7.8}$$

The second-order tensor $\nabla_\alpha a_\beta$ is called the gradient of the vector function $\boldsymbol{a}$ and often denoted by $\boldsymbol{\nabla a}$.

(3) The scalar $\Delta\psi = m^{\alpha\beta}\nabla_\alpha\nabla_\beta\psi$ is called the Laplace operator of the scalar function ψ. One has

$$\Delta \psi = m^{\alpha\beta} \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} - m^{\alpha\beta} \Gamma^\kappa_{\alpha\beta} \frac{\partial \psi}{\partial x^\kappa} = \frac{1}{\sqrt{m}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{m}\, m^{\alpha\beta} \frac{\partial \psi}{\partial x^\beta} \right) = \operatorname{div}(\boldsymbol{\nabla} \psi) . \tag{A.7.9}$$

Formulae (A.7.6)–(A.7.8) and the identity $\nabla_\alpha m^{\alpha\beta} = 0$ have been used to derive this result.

(4) Consider the anti-symmetric tensor $\nabla_\alpha a_\beta - \nabla_\beta a_\alpha$ which, by (A.4.12) and (A.4.13), may be represented in the form

$$\omega^\alpha = \tfrac{1}{2} \epsilon^{\alpha\beta\gamma} (\nabla_\beta a_\gamma - \nabla_\gamma a_\beta) = \epsilon^{\alpha\beta\gamma} \nabla_\beta a_\gamma . \tag{A.7.10}$$

or

$$\omega^1 = \frac{1}{\sqrt{m}} \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} \right), \qquad \omega^2 = \frac{1}{\sqrt{m}} \left(\frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1} \right),$$

$$\omega^3 = \frac{1}{\sqrt{m}} \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) . \tag{A.7.10$'$}$$

The vector ω^α is referred to as the vorticity (or rot) of the vector field $\boldsymbol{a}$ and denoted by rot $\boldsymbol{a}$.

(5) The vector $m^{\alpha\beta}\nabla_\alpha\nabla_\beta a^\kappa$ is called the Laplace operator of the vector function $\boldsymbol{a}$ and denoted by $\Delta\boldsymbol{a}$:

$$\Delta \boldsymbol{a} = \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{a}) - \operatorname{rot}(\operatorname{rot} \boldsymbol{a}) . \tag{A.7.11}$$

This formula is readily verified in rectangular Cartesian coordinates.

(6) Let there be given a continuous medium moving with velocity $\boldsymbol{v}(M, t)$. Consider first some scalar field ψ. By definition, the expression

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial\psi}{\partial t} + v^\alpha \nabla_\alpha \psi \tag{A.7.12}$$

is referred to as individual derivative of the scalar ψ with respect to time t. If a vector field $\boldsymbol{a}$ is given, then its individual derivative with respect to time is given by

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \left(\frac{\partial a^\alpha}{\partial t} + v^\beta \nabla_\beta a^\alpha\right) \boldsymbol{e}_\alpha\ . \tag{A.7.13}$$

The physical significance of these definitions is easily explained in a rectangular Cartesian coordinate system: An individual derivative characterizes the change of ψ (or $\boldsymbol{a}$) in unit time for fluid particles moving with velocity $\boldsymbol{v}$.

(7) Next, the derivative with respect to time of the integral $\int_v a \mathrm{d}V$ will be derived, where a is a scalar field and V an individual volume of a continuous medium (i.e., a volume which consists of the same moving particles). Introducing the external normal $\boldsymbol{n}$ to the surface Σ bounding the volume V, one has

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V a \mathrm{d}V = \int_V \frac{\partial a}{\partial t} \mathrm{d}V + \oint_\Sigma a(\boldsymbol{v}, \boldsymbol{n})\, \mathrm{d}\Sigma \tag{A.7.14}$$

or, by the Gauss-Ostrogradskii theorem,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V a \mathrm{d}V = \int_V \left[\frac{\partial a}{\partial t} + \operatorname{div}(a\boldsymbol{v})\right] \mathrm{d}V\ . \tag{A.7.15}$$

A.8 CURVATURE TENSOR

The following question will be treated briefly. Selecting a definite curvilinear coordinate system x^α and writing down in it corresponding invariant relations, one might forget that the space is Euclidean and there exists in it a rectangular coordinate system. How does one formulate this important fact in terms of the coordinate system x^α?

In order to answer this question, introduce the curvature tensor $R_{\nu\mu\alpha}^{\cdot\cdot\cdot\beta}$. Let $\boldsymbol{a}$ be an arbitrary vector. Then, by definition,

$$\nabla_\nu(\nabla_\mu a_\alpha) - \nabla_\mu(\nabla_\nu a_\alpha) = R_{\nu\mu\alpha}^{\cdot\cdot\cdot\beta} a_\beta\ , \tag{A.8.1}$$

where

$$R_{\nu\mu\alpha}^{\cdot\cdot\cdot\beta} = \frac{\partial \Gamma^\beta_{\nu\alpha}}{\partial x^\mu} + \Gamma^\beta_{\mu\sigma}\Gamma^\sigma_{\nu\alpha} - \frac{\partial \Gamma^\beta_{\mu\alpha}}{\partial x^\nu} - \Gamma^\beta_{\nu\sigma}\Gamma^\sigma_{\mu\alpha}\ . \tag{A.8.2}$$

By the tensor criterion, it follows from (A.8.1) that $R_{\nu\mu\alpha}^{\cdot\cdot\cdot\beta}$ is a tensor.

The following formula is readily derived:

$$R_{\nu\mu\alpha\beta} = m_{\beta\sigma}R_{\nu\mu\alpha}^{\cdot\cdot\cdot\sigma} = \frac{1}{2}\left(\frac{\partial^2 m_{\nu\beta}}{\partial x^\mu \partial x^\alpha} - \frac{\partial^2 m_{\nu\alpha}}{\partial x^\mu \partial x^\beta} - \frac{\partial^2 m_{\mu\beta}}{\partial x^\nu \partial x^\alpha} + \frac{\partial^2 m_{\mu\alpha}}{\partial x^\nu \partial x^\beta}\right)$$

$$+ m_{\kappa\sigma}(\Gamma^\kappa_{\nu\beta}\Gamma^\sigma_{\mu\alpha} - \Gamma^\kappa_{\mu\beta}\Gamma^\sigma_{\nu\alpha}) . \tag{A.8.3}$$

It yields the result that the tensor $R_{\nu\mu\alpha\beta}$ is symmetric:

$$R_{\nu\mu\alpha\beta} = -R_{\mu\nu\alpha\beta}, \qquad R_{\nu\mu\alpha\beta} = -R_{\nu\mu\beta\alpha} , \qquad R_{\nu\mu\alpha\beta} = R_{\alpha\beta\nu\mu} . \tag{A.8.4}$$

These formulae show that $R_{\nu\mu\alpha\beta}$ has only six independent components; for example: R_{1212}, R_{2323}, R_{3131}, R_{1213}, R_{2321}, R_{3132}.

If the space is Euclidean, then there exists in it an affine coordinate system and the tensor $R_{\nu\mu\alpha\beta}$ vanishes identically. It follows from (A.8.1) that in Euclidean space $\nabla_\alpha\nabla_\beta a_\kappa = \nabla_\beta\nabla_\alpha a_\kappa$. It may be shown that, if the tensor $R_{\nu\mu\alpha\beta}$ vanishes identically in a region V, then there exists in this region a coordinate system such that its matrix $m_{\alpha\beta}$ will be the unity matrix at all points of the region V (rectangular Cartesian coordinate system). Hence the initial coordinates x^α may be interpreted as curvi-linear coordinates in Euclidean space.

Consider an orthogonal coordinate system: $m_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j = 0$ for $i \neq j$ and $m_{ii} = \boldsymbol{e}_i\boldsymbol{e}_i = h_i^2$. The parameters h_i (lengths of the basis vectors $\boldsymbol{e}_i$) are referred to as Lamé parameters of the given coordinate system. Then

$$m^{ij} = 0 \qquad \text{for } i \neq j , \qquad m^{ii} = \frac{1}{h_i^2} . \tag{A.8.5}$$

Furthermore, according to (A.6.2), one finds

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{\partial \ln h_i}{\partial x^j} ; \qquad \Gamma^i_{jj} = -\frac{h_j}{h_i^2}\frac{\partial h_j}{\partial x^i} , \qquad i \neq j ; \qquad \Gamma^i_{jk} = 0 , \text{ all } i, j, k$$

$$\text{different} . \tag{A.8.6}$$

Now it is not difficult to write down

$$R_{1212} = -h_1 h_2\left[\frac{\partial}{\partial x^1}\left(\frac{1}{h_1}\frac{\partial h_2}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{1}{h_2}\frac{\partial h_1}{\partial x^2}\right) + \frac{1}{h_3^2}\frac{\partial h_1}{\partial x^3}\frac{\partial h_2}{\partial x^3}\right] , \tag{A.8.7}$$

with the components R_{2321} and R_{3132} obtained by cyclic transposition of indices and

$$R_{1213} = h_1\left[-\frac{\partial^2 h_1}{\partial x^2 \partial x^3} + \frac{\partial h_1}{\partial x^2}\frac{\partial \ln h_2}{\partial x^3} + \frac{\partial h_1}{\partial x^3}\frac{\partial \ln h_3}{\partial x^2}\right] , \tag{A.8.8}$$

with the components R_{2321} and R_{3132} obtained by cyclic transposition of indices.

Setting the independent curvature tensor components equal to zero, one arives at six necessary and sufficient conditions for a space to be Euclidean.

A.9 BASIC FORMULAE

Side by side with the local basis $\boldsymbol{e}_\alpha$ of a given coordinate system x_α, consider the normalized basis $\boldsymbol{u}_\alpha$:

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{m_{11}}} \boldsymbol{e}_1 , \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{m_{22}}} \boldsymbol{e}_2 , \qquad \boldsymbol{u}_3 = \frac{1}{\sqrt{m_{33}}} \boldsymbol{e}_3 . \tag{A.9.1}$$

The components $\hat{a}_\alpha$ of a vector $\boldsymbol{a}$ with respect to the basis $\boldsymbol{u}_\alpha$ will be called physical components:

$$\boldsymbol{a} = \sum_{i=1}^{3} \hat{a}_i \boldsymbol{u}_i , \qquad \hat{a}_1 = \sqrt{m_{11}} a^1 , \qquad \hat{a}_2 = \sqrt{m_{22}} a^2 , \qquad \hat{a}_3 = \sqrt{m_{33}} a^3 . \tag{A.9.2}$$

In an analogous manner, the physical components of any contravariant tensor, for example, $Q^{\alpha\beta}$ may be defined by

$$\hat{Q}_{11} = \sqrt{m_{11} m_{11}} Q^{11} , \qquad \hat{Q}_{12} = \sqrt{m_{11} m_{22}} Q^{12} , \qquad \text{etc.} \tag{A.9.3}$$

If a tensor has subscripts, one must first step over to the associate tensor which has only superscripts and then use formulae of the type (A.9.3).

An advantage of the physical components of a tensor is that they have the same dimensions as physical characteristics. However, they do not transform according to the law (A.3.1) and therefore it is significantly more difficult to formulate in terms of them the property of invariance of a parameter with respect to choice of coordinate system. Note that in rectangular Cartesian coordinates, corresponding covariant, contravariant and physical components of a tensor are identical.

Consider again an orthogonal coordinate system x^α with Lamé parameters h_α, and, using (A.8.5) and (A.8.6), write down Formulae (A.7.6)–(A.7.13) in terms of physical components of the corresponding tensors.

It follows from definition (A.7.6) that

$$\operatorname{div} \boldsymbol{a} = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial (h_2 h_3 \hat{a}_1)}{\partial x^1} + \frac{\partial (h_3 h_1 \hat{a}_2)}{\partial x^2} + \frac{\partial (h_1 h_2 \hat{a}_3)}{\partial x^3} \right\} . \tag{A.9.4}$$

In spherical coordinates λ, φ and r, define the two-dimensional divergence of the vector $(\hat{a}_\lambda, \hat{a}_\varphi)$ by

$$\operatorname{div}_h \boldsymbol{a} = \frac{1}{h_\lambda h_\varphi} \left\{ \frac{\partial (h_\varphi \hat{a}_\lambda)}{\partial \lambda} + \frac{\partial (h_\lambda \hat{a}_\varphi)}{\partial \varphi} \right\} . \tag{A.9.4'}$$

By definition (A.7.7), the physical components $\hat{b}_i$ of the vector $\nabla_\alpha a^{\alpha\beta}$ have the form

$$\hat{b}_i = \frac{1}{h_i} \sum_{k=1}^{3} \left\{ \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x^k} \left(\frac{h_1 h_2 h_3 h_i}{h_k} \hat{a}_{ik} \right) - \hat{a}_{kk} \frac{\partial \ln h_k}{\partial x^i} \right\} . \tag{A.9.5}$$

Recall that $a^{\alpha\beta}$ is a symmetric tensor.

By definition (A.7.8), one has the expression for the physical components of the vector $\nabla\psi$

$$(\widehat{\nabla\psi})_i = \frac{\partial\psi}{h_i \partial x^i} \tag{A.9.6}$$

In spherical coordinates λ, φ and r, the two-dimensional gradient $\nabla_h \psi$ is defined by

$$\nabla_h \psi = \left(\frac{\partial\psi}{h_\lambda \partial\lambda}, \frac{\partial\psi}{h_\varphi \partial\varphi} \right) . \tag{A.9.6$'$}$$

It is now the right moment to present the formulae for the physical components $\hat{c}_i$ of the vector $b^\alpha \nabla_\alpha a^\beta$:

$$\hat{c}_i = \sum_{k=1}^{3} \frac{\hat{b}_k}{h_k} \frac{\partial \hat{a}_i}{\partial x^k} - \sum_{k=1}^{3} \frac{\hat{b}_k \hat{a}_k}{h_i h_k} \frac{\partial h_k}{\partial x^i} + \sum_{k=1}^{3} \frac{\hat{b}_i \hat{a}_k}{h_i h_k} \frac{\partial h_i}{\partial x^k} . \tag{A.9.7}$$

Formulae (A.9.6) and (A.9.7) permit to write down expressions for individual derivatives of a scalar ψ and a vector $\boldsymbol{a}$. One has, by (A.7.12) and (A.7.13),

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\partial\psi}{\partial t} + \sum_{k=1}^{3} \frac{\hat{v}_k}{h_k} \frac{\partial\psi}{\partial x^k} , \tag{A.9.8}$$

$$\widehat{\left(\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} \right)}_i = \frac{\partial \hat{a}_i}{\partial t} + \sum_{k=1}^{3} \frac{\hat{v}_k}{h_k} \frac{\partial \hat{a}_i}{\partial x^k} - \sum_{k=1}^{3} \frac{\hat{v}_k \hat{a}_k}{h_i h_k} \frac{\partial h_k}{\partial x^i} + \sum_{k=1}^{3} \frac{\hat{v}_i \hat{a}_k}{h_i h_k} \frac{\partial h_i}{\partial x^k} . \tag{A.9.9}$$

Definition (A.7.9) yields

$$\Delta\psi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial x^1} \left(\frac{h_2 h_3}{h_1} \frac{\partial\psi}{\partial x^1} \right) + \frac{\partial}{\partial x^2} \left(\frac{h_3 h_1}{h_2} \frac{\partial\psi}{\partial x^2} \right) + \frac{\partial}{\partial x^3} \left(\frac{h_1 h_2}{h_3} \frac{\partial\psi}{\partial x^3} \right) \right\} . \tag{A.9.10}$$

In spherical coordinates λ, φ and r, define the twodimensional Laplace operator of the scalar function ψ

$$\Delta_h \psi = \frac{1}{h_\lambda h_\varphi} \left\{ \frac{\partial}{\partial\lambda} \left(\frac{h_\varphi}{h_\lambda} \frac{\partial\psi}{\partial\lambda} \right) + \frac{\partial}{\partial\varphi} \left(\frac{h_\lambda}{h_\varphi} \frac{\partial\psi}{\partial\varphi} \right) \right\} . \tag{A.9.11}$$

No consideration will be given to (A.7.11).

The physical components of the vector rot a are given by (A.7.10′):

In a right-handed coordinate system:

$$(\widehat{\mathrm{rot}}\, \boldsymbol{a})_1 = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial x^2}(h_3 \hat{a}_3) - \frac{\partial}{\partial x^3}(h_2 \hat{a}_2)\right\},$$

$$(\widehat{\mathrm{rot}}\, \boldsymbol{a})_2 = \frac{1}{h_1 h_3}\left\{\frac{\partial}{\partial x^3}(h_1 \hat{a}_1) - \frac{\partial}{\partial x^1}(h_3 \hat{a}_3)\right\},$$

$$(\widehat{\mathrm{rot}}\, \boldsymbol{a})_3 = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x^1}(h_2 \hat{a}_2) - \frac{\partial}{\partial x^2}(h_1 \hat{a}_1)\right\}; \qquad \text{(A.9.12)}$$

In a left-handed coordinate system:

$$(\widehat{\mathrm{rot}}\, \boldsymbol{a})_1 = \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial x^3}(h_2 \hat{a}_2) - \frac{\partial}{\partial x^2}(h_3 \hat{a}_3)\right\},$$

$$(\widehat{\mathrm{rot}}\, \boldsymbol{a})_2 = \frac{1}{h_1 h_3}\left\{\frac{\partial}{\partial x^1}(h_3 \hat{a}_3) - \frac{\partial}{\partial x^3}(h_1 \hat{a}_1)\right\},$$

$$(\widehat{\mathrm{rot}}\, \boldsymbol{a})_3 = \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x^2}(h_1 \hat{a}_1) - \frac{\partial}{\partial x^1}(h_2 \hat{a}_2)\right\}. \qquad \text{(A.9.13)}$$

By (A.4.8) or (A.4.9), the physical components of the vector $\boldsymbol{a} \times \boldsymbol{b}$ have the form:

In a right-handed coordinate system:

$$(\boldsymbol{a} \hat{\times} \boldsymbol{b})_1 = \hat{a}_2 \hat{b}_3 - \hat{a}_3 \hat{b}_2 ,$$

$$(\boldsymbol{a} \hat{\times} \boldsymbol{b})_2 = \hat{a}_3 \hat{b}_1 - \hat{a}_1 \hat{b}_3 ,$$

$$(\boldsymbol{a} \hat{\times} \boldsymbol{b})_3 = \hat{a}_1 \hat{b}_2 - \hat{a}_2 \hat{b}_1 . \qquad \text{(A.9.14)}$$

In a left-handed coordinate system:

$$(\boldsymbol{a} \hat{\times} \boldsymbol{b})_1 = \hat{a}_3 \hat{b}_2 - \hat{a}_2 \hat{b}_3 ,$$

$$(\boldsymbol{a} \hat{\times} \boldsymbol{b})_2 = \hat{a}_1 \hat{b}_3 - \hat{a}_3 \hat{b}_1 ,$$

$$(\boldsymbol{a} \hat{\times} \boldsymbol{b})_3 = \hat{a}_2 \hat{b}_1 - \hat{a}_1 \hat{b}_2 . \qquad \text{(A.9.15)}$$

In conclusion, formulae for the physical components of the strain rate tensor $e_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha$ will be written down:

$$\hat{e}_{ik} = \frac{h_i}{h_k}\frac{\partial}{\partial x^k}\left(\frac{\hat{v}_i}{h_i}\right) + \frac{h_k}{h_i}\frac{\partial}{\partial x^i}\left(\frac{\hat{v}_k}{h_k}\right), \qquad i \neq k ,$$

$$\tfrac{1}{2}\hat{e}_{ii} = \frac{\partial}{\partial x^i}\left(\frac{\hat{v}_i}{h_i}\right) + \sum_{k=1}^{3} \frac{\hat{v}_k}{h_k}\frac{\partial \ln h_i}{\partial x^k}. \qquad \text{(A.9.16)}$$

In formulae (A.9.4)—(A.9.16), repeated indices do not imply summation.

COMMENT ON APPENDIX A

The general handbooks [54;53;72;100;108, Chapters II and IV] have been employed in the preparation of this section.

§§ A.1 and A.2: Statement of general material.

§ A.3: Proof of Rules 1—4 (cf. [100, Chapter I]).

§ A.4: For proofs, cf. [72].

§ A.5: Cf. [2,4,96], and likewise [108, Appendix I].

§ A.6: These results follow from [100, Chapter VIII].

§ A.7: Proofs of Rules 1—3, cf. [100, Chapter VII]; a more detailed derivation of (A.7.6)—(A.7.16) is given in [53], and likewise in [72].

§ A.8: Cf. [100, Chapter IX; 72], where also the proofs are given.

§ A.9: Detailed derivations of (A.9.4)—(A.9.16) are given in [53, Chapter IV].

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APPENDIX B

ON MATCHING OF ASYMPTOTIC EXPANSIONS

First consider a simple example. Let there be given on the interval $0 \leqslant x \leqslant 1$ the function

$$y(x) = \cos x - \frac{\epsilon}{x + \epsilon}, \qquad \epsilon > 0, \tag{B.1}$$

where ϵ is a small parameter. Construct an asymptotic representation of this function, expanding it in terms of $1, \epsilon, \epsilon^2, \ldots$. One finds immediately that for finite x

$$y(x) = \sum_0^\infty \epsilon^k u_k(x), \tag{B.2}$$

where

$$u_0 = \cos x, \qquad u_m(x) = (-1)^m (1/x^m), \qquad m = 1, 2, 3, \ldots .$$

The modulus of the remainder term for a partial sum of n terms of expansion (B.2) is less than $(\epsilon/x)^n$; therefore (B.2) has the character of an asymptotic expansion for $b_e \leqslant x \leqslant 1$, where $b_e = \epsilon^\nu$, $\nu < 1$. For any given accuracy ϵ^n and parameter $\nu < 1$, selecting $N_e > n/(1-\nu)$, one finds

$$y(x) = \sum_0^{N_e} \epsilon^k u_k(x) + O(\epsilon^n), \qquad b_e \leqslant x \leqslant 1 .$$

Hence $\inf N_e \to \infty$ for $\nu \to 1$, i.e., the closer the left-hand boundary of the region of validity of (B.2) is to the point $x = 0$, the larger is the number of terms of the asymptotic expansion which is required to attain a given accuracy. However, the boundary b_e cannot approach the point $x = 0$ more closely than to order ϵ.

In the neighbourhood of the point $x = 0$, it is natural to go to the variable $\zeta = x/\epsilon$; for finite values of ζ, one obtains

$$y = \sum_0^\infty \epsilon^k v_k(\zeta), \tag{B.3}$$

where

$$v_0(\zeta) = 1 - 1/(1+\zeta); \qquad v_{2m-1}(\zeta) = 0, \qquad v_{2m}(\zeta) = (-1)^m \zeta^{2m}/(2m)!,$$

$$m = 1, 2, 3, \ldots .$$

The modulus of the remainder term for the partial sum of n terms of (B.3) is less than $(1/n!)x^n$; therefore (B.3) represents an asymptotic expansion on $0 \leqslant x \leqslant b_i$, where $b_i = \epsilon^\nu$, $\nu > 0$. For any given accuracy ϵ^n and parameter $\nu > 0$, selecting $N_i > n/\nu$, one has

$$y(x) = \sum_0^{N_i} \epsilon^k v_k(x/\epsilon) + O(\epsilon^n), \qquad 0 \leqslant x \leqslant b_i .$$

Hence $\inf N_i \to \infty$ for $\nu \to 0$, i.e., the further away is the right-hand boundary of the region of validity of (B.3) from $x = 0$, the larger is the number of terms of the asymptotic expansion which is required to attain a given accuracy. However, the boundary b_i could not be moved on a finite distance from the point $x = 0$.

Most important of all is the fact that for any $\epsilon \to 0$, the boundaries b_i and b_e may be chosen such that the regions of validity of expansions (B.2) and (B.3) will overlap.

Finally, the solution of the problem under consideration may be presented in the form

$$y(x) = \sum_0^{N_i} \epsilon^k v_k(x/\epsilon) + O(\epsilon^n), \qquad 0 \leqslant x \leqslant b ,$$

$$y(x) = \sum_0^{N_e} \epsilon^k u_k(x) + O(\epsilon^n), \qquad b \leqslant x \leqslant 1 ,$$

where

$$b = \epsilon^\nu, \qquad 0 < \nu < 1 \tag{B.4}$$

and the numbers N_e and N_i depend on the parameters n and ν.

For the sequel, it will be convenient to have a formulation of the fact that the regions of validity of expansions (B.2) and (B.3) overlap in the form of some equality. For example, writing down both expansions for $x = b$, one finds

$$\sum_0^{N_i} \epsilon^k v_k(b/\epsilon) = \sum_0^{N_e} \epsilon^k u_k(b) + O(\epsilon^n), \tag{B.5}$$

where b is given by (B.4), while N_e and N_i depend on n and ν.

Since, by (B.4), one has $b \to 0$ as $\epsilon \to 0$, and $b/\epsilon \to \infty$, the asymptotic behaviours of the functions $u_k(x)$ for $x \to 0$ and the functions $v_k(\zeta)$ for $\zeta \to \infty$ appear to be interlinked. For example, if $n = \frac{1}{2}$ and $\nu = \frac{1}{2}$, one may select $N_e = N_i = 0$ and write down (B.5) in the form

$$v_0(\epsilon^{-1/2}) = u_0(\epsilon^{1/2}) + O(\epsilon^{1/2}),$$

whence

$v_0(\infty) = u_0(0)$.

In a definite sense, the example under consideration is typical. Very frequently, an asymptotic expansion of Type (B.2) of a solution $y(x)$ of some boundary value problem for a differential equation turns out, due to one or the other physical reasons, to be inapplicable in the neighbourhood of some point (say, $x = 0$). As a rule, this signifies that near this point the solution $y(x)$ has a different asymptotic expansion of Type (B.3); usually, the regions of validity of the two expansions overlap and a condition of Type (B.5) appears to be valid.

In an analysis of concrete problems, a difficulty arises from the fact that the initial equation and boundary conditions permit usually to find the coefficients $u_k(x)$ and $v_k(\zeta)$ of asymptotic expansions (B.2) and (B.3) only apart from some unknown constants. For the determination of these constants, the property of overlap of the regions of validity of expansions (B.2) and (B.3) is basic. As it has been seen, the asymptotics of the functions $u_k(x)$ for $x \to 0$ and $v_k(\zeta)$ for $\zeta \to \infty$ are interrelated, by (B.5), whence follow the required supplementary conditions for the determination of the unknown constants.

Direct application of (B.5) leads often to very complicated constructions. Hence consider the following convenient heuristic procedures where the parameters n, ν, N_e and N_i do not appear explicitly.

PROCEDURE 1

(1) Let there be given some integer M: $M = 0, 1, 2, \ldots$. Consider the partial sum $\Sigma_0^M \epsilon^k u_k(x)$ and write it in the form $\Sigma_0^M \epsilon^k u_k(\epsilon\zeta)$.

(2) For fixed ζ, expand this expression in powers of ϵ: $1, \epsilon, \epsilon^2, \ldots$ and select the sum of its first M terms. One finds $\Sigma_0^M \epsilon^k c_k^{(M)}(\zeta)$.

Matching condition: Beginning with a sufficiently large value of M, the functions $c_k^{(M)}(\zeta)$ are the asymptotics of $v_k(\zeta)$ for $\zeta \to \infty$.

In a similar manner, one can formulate

PROCEDURE 2

(1) Let there be given some integer M: $M = 0, 1, 2, \ldots$. Consider the partial sum $\Sigma_0^M \epsilon^k v_k(\zeta)$ and write it in the form $\Sigma_0^M \epsilon^k v_k(x/\epsilon)$.

(2) For fixed x, expand this expression in powers of ϵ: $1, \epsilon, \epsilon^2, \ldots$ and select the sum of its first M terms. One finds $\Sigma_0^M \epsilon^k d_k^{(M)}(x)$.

Matching condition: Beginning with a sufficiently large value of M, the functions $d_k^{(M)}(x)$ are the asymptotics of $u_k(x)$ for $x \to 0$.

Successive application of these procedures permits, as a rule, to determine the unknown constants in the expressions for $u_k(x)$ and $v_k(\zeta)$.

At the base of the methods formulated, there lies a regrouping of terms on both sides of (B.5) by which a simpler comparison of terms of equal order is achieved. Therefore, if in the execution of at least one of these procedures the matching condition is actually fulfilled, then this must guarantee apparently also the truth of (B.5).

As an example, consider the following problem: For small positive ϵ, find an asymptotic expansion of the solution of the problem

$$\epsilon y'' + y' = \cos x, \qquad y(0) = y(1) = 0 \tag{B.6}$$

for $0 \leqslant x \leqslant 1$ (in essence, this problem is an analogue of one worked out in § 5.6).

As usual, seek the solution of the problem in the form

$$y(x) = \sum_0^\infty \epsilon^k u_k(x) \,. \tag{B.7}$$

Substitute this series into (B.6) and equate to zero the coefficients of different powers of ϵ to find for the functions $u_k(x)$ the equations:

$$u_0' = \cos x, \qquad u_1' = -u_0'', \qquad u_2' = -u_1'', \qquad u_3' = -u_2'', \qquad \ldots .$$

Integrating these equations, one finds

$$u_0 = \sin x + A_0, \qquad u_1 = -\cos x + A_1, \qquad u_2 = -\sin x + A_2,$$
$$u_3 = \cos x + A_3, \qquad \ldots ,$$

where the constants A_k are as yet unknown.

Generally speaking, there must be boundary layers near the points $x = 0$ and $x = 1$, since the functions $u_k(x)$ may not satisfy simultaneously the boundary conditions for $x = 0$ and $x = 1$. Consider first the neighbourhood of the point $x = 0$. Introduce the variable $\zeta = x/\epsilon$ and seek $y(x)$ in the form

$$y = \sum_0^\infty \epsilon^k v_k(\zeta) \,. \tag{B.8}$$

Expanding $\cos \epsilon\zeta$ in a power series for fixed ζ and substituting (B.8) into (B.6) and the boundary condition $y(0) = 0$, one obtains

$$\begin{array}{ll} v_0'' + v_0' = 0, & v_0(0) = 0, \\ v_1'' + v_1' = 1, & v_1(0) = 0, \\ v_2'' + v_2' = 0, & v_2(0) = 0, \\ v_3'' + v_3' = -\zeta^2/2, & v_3(0) = 0, \\ \ldots\ldots & \ldots\ldots \end{array}$$

TABLE B.I

$d_k^{(M)}$	$M = 0$	$M = 1$	$M = 2$
$d_0^{(M)}$	$-(1-x)\cos 1$	$-(1-x)\cos 1 - \frac{(1-x)^2}{2!}\sin 1$	$-(1-x)\cos 1 - \frac{(1-x)^2}{2!}\sin 1 - \frac{(1-x)^3}{3!}\cos 1$
$d_1^{(M)}$		$-(1-x)\sin 1$	$-(1-x)\sin 1 + \frac{(1-x)^2}{2!}\cos 1$
$d_2^{(M)}$			$(1-x)\cos 1$
...	...	...	...

TABLE B.II

$c_k^{(M)}$	$M = 0$	$M = 1$	$M = 2$	$M = 3$
$c_0^{(M)}$	$-\sin 1$	$-\sin 1$	$-\sin 1$	$-\sin 1$
$c_1^{(M)}$		$-1 + \cos 1 + \zeta$	$-1 + \cos 1 + \zeta$	$-1 + \cos 1 + \zeta$
$c_2^{(M)}$			$\sin 1$	$\sin 1$
$c_3^{(M)}$				$1 - \cos 1 - \frac{\zeta^3}{6} + \frac{\zeta^2}{2} - \zeta$
...	...	...	...	...

Hence, on integration, one finds

$$\begin{aligned}
v_0 &= B_0(1 - e^{-\zeta}),\\
v_1 &= B_1(1 - e^{-\zeta}) + \zeta,\\
v_2 &= B_2(1 - e^{-\zeta}),\\
v_3 &= B_3(1 - e^{-\zeta}) - \zeta^3/6 + \zeta^2/2 - \zeta,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

where B_k are arbitrary constants.

Analogously, expanding $y(x)$ near $x = 1$ in the series

$$y = \sum_0^\infty \epsilon^k w_k(\xi), \qquad \xi = \frac{1-x}{\epsilon}, \tag{B.9}$$

and $\cos(1 - \epsilon\xi)$ in a power series for fixed ξ, and substituting (B.9) into (B.6) and the condition $y(1) = 0$, one arrives at

$$\begin{aligned}
w_0 &= C_0(1 - e^{\xi}),\\
w_1 &= C_1(1 - e^{\xi}) - \xi \cos 1,\\
w_2 &= C_2(1 - e^{\xi}) - (\xi^2/2 + \xi) \sin 1,\\
w_3 &= C_3(1 - e^{\xi}) + (\xi^3/6 + \xi^2/2 + \xi) \cos 1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

where C_k are arbitrary constants.

The unknown constants A_k, B_k and C_k may be determined only from the matching conditions of corresponding asymptotic expansions. Start with expansions (B.7) and (B.9). By Procedure 1 the $w_k(\xi)$ can only behave for $\xi \to \infty$ like some polynomial in ξ, whence $C_k = 0$, $k = 0, 1, \ldots$ and all coefficients $w_k(\xi)$ of expansion (B.9) have been found. Next, apply successively Procedure 2 and, for the sake of clarity, present the results in the form of Table B.I.

In this manner, all A_k are found from the matching condition:

$$A_0 = -\sin 1, \qquad A_1 = \cos 1, \qquad A_2 = \sin 1, \qquad \ldots .$$

Finally, the constants B_k must still be found. For this purpose, expansions (B.7) and (B.8) must be matched. Applying Procedure 1, one arrives at Table B.II.

The matching condition now yields

$$B_0 = -\sin 1, \qquad B_1 = -1 + \cos 1, \qquad B_2 = \sin 1, \qquad B_3 = 1 - \cos 1, \qquad \ldots .$$

The validity of this asymptotic representation is readily established if one will invoke the exact solution of Problem (B.6) which in the case under consideration has a simple form. Thus, simple examples have permitted to study important features of the method of matched asymptotic expansions. In analyses of more complicated problems, one may be guided by the results above.

COMMENT ON APPENDIX B

The method of matched asymptotic expansions is the topic, for example, of the monographs [125] and [7], the survey [57] and a number of other publications. Basically, the present treatment follows the work of Il'in and coauthors [37,38].

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BIBLIOGRAPHY

1 Barcilon, V. A simple model of the thermocline in a bounded ocean. J. Phys. Oceanogr., 1971, Vol. 1, No. 1, 7—11.

2 Batchelor, G.K. The theory of axi-symmetric turbulence. Proc. R. Soc. London, 1946, A186, No. 1007, 480—502.

3 Carrier, G.F. Boundary layer problems in Applied Mechanics. Adv. Appl. Mech., Vol. III.

4 Chandrasekhar, S. The theory of axi-symmetric turbulence. Philos. Trans. R. Soc. London, 1950, A242, No. 855, 557—577.

5 Charney, J.G. The generation of oceanic currents by wind. J. Mar. Res., 1955, Vol. 14, No. 4, 477—498.

6 Charney, J.G. The Gulf Stream as an inertial boundary layer. Proc. Natl. Acad. Sci. U.S.A., 1955, Vol. 41, No. 10, 731—740.

7 Cole, J.D. Perturbation Methods in Applied Mathematics. Blaisdell, Waltham, Mass., 1968, p. 260.

8 Cox, R.A. and Smith, N.D. The specific heat of sea water. Proc. R. Soc. London, 1959, A252, No. 1268, 51—62.

9 De Groot, S.R. and Masur, P. Non-equilibrium Thermodynamics. North-Holland, Amsterdam, 1962.

10 De Groot, S.R. Thermodynamics of Irreversible Processes. North-Holland, Amsterdam, 1952.

11 Dikii, L.A. Theory of Oscillations of Earth's Atmosphere. Gidrometeouzdat, Leningrad, 1969.

12 Doetsch, G. Anleitung zum praktischen Gebrauch der Laplace-Transformation und der Z-Transformation. R. Oldenbourg, München, 1957.

13 Eckart, C. Hydrodynamics of Oceans and Atmospheres. Pergamon Press, Oxford, 1960.

14 Eckart, C. The equations of motion of sea water. In: The Sea, 1962, Vol. 1, Physical Oceanography, ed. M.N. Hill, Wiley, Interscience, New York, 31—41.

15 Ekman, V.W. On the influence of the Earth's rotation on ocean currents. Ark. Mat. Astron. Phys., 1905, Vol. 2, No. 11, 1—53.

16 Ekman, V.W. Über Horizontalzirkulation bei winderzeugten Meeresströmungen. Ark. Mat. Astron. Phys., 1923, Vol. 17, No. 26, 1—74.

17 Ertel, H. Ein neuer hydrodynamischer Wirbelsatz. Meteorol. Z., 1942, Vol. 59, 277—281.

18 Fel'zenbaum, A.I. Theoretical foundations and methods of calculation of stationary ocean currents. Izd-vo AN SSSR, Moscow, 1960, pp. 127.

19 Fel'zenbaum, A.I. Dynamics of ocean currents (Survey). In: Summaries of Science, Hydromechanics, 1968, Izd-vo VINITI AN SSSR, Moscow, 1970, pp. 97—338.

20 Fel'zenbaum, A.I. On the method of total flows of Stockmann-Sverdrup-Munk. DAN SSSR, 1968, 179, No. 5, pp. 1094—1097.

21 Fofonoff, N.P. Dynamics of ocean currents. In: The Sea, Vol. 1, ed. M.N. Hill, Wiley, Interscience, New York, 1962, pp. 323—395.

22 Fofonoff, N.P. Steady flow in a frictionless homogeneous ocean. J. Mar. Res., 1954, Vol. 13, No. 3, 254—262.
23 Fofonoff, N.P. Interpretation of oceanographic measurements. Thermodynamics. In: Physical and chemical properties of sea water. Publication 600, National Academy of Sciences-National Research Council, 1959, 38—66.
24 Fofonoff, N.P. Physical properties of sea water. In: The Sea, Vol. 1, Physical Oceanography, ed. M.N. Hill, Wiley, Interscience, New York, 1962.
25 Fridman, A.A. Experience of hydromechanics of a compressible fluid, GITTL, Leningrad, 1934.
26 Gibbs, J.W. The collected works of J. Willard Gibbs, Vol. 1, Thermodynamics, Longmans and Green, Harlow, 1928.
27 Gill, A.E. Ocean models. Philos. Trans. R. Soc. London, 1971, A270, No. 1206, 391—413.
28 Guggenheim, E.A. Thermodynamics. An advanced treatment for chemists and physicists. North-Holland, Amsterdam, 1950.
29 Haase, R. Thermodynamik der irreversiblen Prozesse. Steinkopff, Darmstadt, 1973.
30 Hartman, P. Ordinary Differential Equations. Wiley, New York, 1964.
31 Hellerman, S. Computations of wind stress fields over the Atlantic Ocean. Mon. Weather Rev., 1965, Vol. 93, No. 4, 239—244.
32 Hough, S.S. On the application of harmonic analysis to the dynamical theory of the tides. Part II. On the general integration of Laplace's dynamical equations. Philos. Trans. R. Soc. London, 1898, A191, 139—185.
33 Il'in, A.M. On the asymptotic of the solution of a boundary problem. Matem. zametki, 1970, Vol. 8, No. 3, 273—284.
34 Il'in, A.M. and Kamenkovich, V.M. On the influence of friction on ocean currents. DAN SSSR, 1963, 150, No. 6, 1274—1277.
35 Il'in, A.M. and Kamenkovich, V.M. On the structure of the boundary layer in the two-dimensional theory of ocean currents. Okeanologiya, 1964, 4, No. 5, 756—769.
36 Il'in, A.M., Kamenkovich, V.M., Zhugrina, T.G. and Silkina, M.M. On the computation of total flows in the world ocean (stationary problem). Izv. AN SSSR. Physika atmosferi i okeana, 1969, 5, No. 11, 1160—1172.
37 Il'in, A.M., Gor'kov, Yu.P. and Lelikova, E.F. On a method of matching of asymptotic expansions, Dokl. AN SSSR, 1974, 217, No. 5, 1033—1036.
38 Il'in, A.M., Gor'kov, Yu.P. and Lelikova, E.F. Asymptotic solutions of elliptic equations with a small parameter at the highest derivative in the neighbourhood of special characteristics of the limit equation. Proceedings of a seminar in memory of I.G. Petrovskii, Moscow State University, State University Publishing House, 1975, No. 1, 75—133.
39 Ince, E.L., Ordinary Differential Equations. Longmans and Green, Harlow, 1926.
40 Iselin, C.O'D. A study of the circulation of the Western North Atlantic, Pap. Phys. Oceanogr. Meteorol., 1936, 4, No. 4, 101.
41 Kamenkovich, V.M. On certain simplified equations of the dynamics of stationary currents in a baroclinic sea. DAN SSSR, 1958, 119, No. 6, 1134—1137.
42 Kamenkovich, V.M. On the integration of the equations of the theory of ocean currents in multiply connected regions. DAN SSSR, 1961, 138, No. 5, 1076—1079.
43 Kamenkovich, V.M. On the theory of the Antarctic circular current. Trudy IOAN, 1962, 56, 241—293.
44 Kamenkovich, V.M. On the theory of inertial-viscous boundary layers in a two-dimensional model of ocean currents. Izv. AN SSSR, Fisika atmosferi i okeana, 1966, 2, No. 12, 1274—1295.
45 Kamenkovich, V.M. On the coefficients of turbulent diffusion and viscosity for large-scale motions of the ocean and atmosphere. Izv. AN SSSR, Fisika atmosferi i okeana, 1967, 3, No. 12, 1326—1333.

46 Kamenkovich, V.M. On normal oscillations of a multi-layer rotating fluid. Izv. AN SSSR, Fisika atmosferi i okeana, 1967, 3, No. 3, 284—290.
47 Kamenkovich, V.M., and Odulo, A.B. On the theory of free oscillations in a stratified compressible ocean of constant depth. Izv. AN SSSR, Fisika atmosferi i okeana, 1972, 8, No. 11, 1187—1201.
48 Kamenkovich, V.M. and Reznik, G.M. On separation of the boundary current from a shore due to the effect of bottom relief (linear barotropic model). DAN SSSR, 1972, 202, No. 5, 1061—1064.
49 Kamenkovich, V.M. and Tsybaneva, T.B. Analysis of Laplace's tidal equations in short-wave approximation (large longitudinal wave number). Okeanologiya, 1975, 15, No. 3, 397—404.
50 Khinchin, A.Ya. Mathematical Foundations of Statistical Mechanics. GITTL, Moscow, 1943.
51 Knauss, J.A. A note on the transport of the Gulf Stream. Deep Sea Res., 1969, suppl. to 16, 117—124.
52 Kochergin, V.P. Introduction to the Theory and Methods of Computation of Ocean Currents. Novosibirsk, 1971.
53 Kochin, N.E. Vector calculus and introduction to tensor analysis. Moscow, Izd-vo AN SSSR, 1951.
54 Korn, G.A. and Korn, T.M. Mathematical Handbook. 2nd Ed. McGraw-Hill, New York, 1968.
55 Kozlov, V.F. Lectures on the theory of stationary ocean currents. Vladivostok, 1969.
56 Kubo, R. Statistical Mechanics. North-Holland, Amsterdam, 1965.
57 Lagerstrom, P.A. and Casten, R.G. Basic concepts underlying singular perturbation techniques. SIAM Review, 1972, 14, No. 1, 63—120.
58 Lamb, H. Hydrodynamics. Cambridge University Press, 6th ed., 1932.
59 Landau, L.D. and Lifshitz, E.M. Mechanics of Continuous Media. Moscow, FITTL, 1953.
60 Landau, L.D. and Lifshitz, E.M. Statistical Physics. Moscow, Nauka, 1964.
61 Levich, V.G., Vdovin, Yu.A. and Myamlin, V.A. Course of Theoretical Physics. Vol. 2, Nauka, 1971.
62 Levitan, B.M. and Sargsyan, I.S. Introduction to Spectral Theory. Self-adjoint ordinary differential operators. Moscow, Nauka, 1970.
63 Lighthill, M.J. Dynamic response of Indian Ocean to onset of the South West Monsoon. Philos. Trans. R. Soc. London, 1969, A265, No. 1159, 45—92.
64 Lighthill, M.J. Time-varying currents. Philos. Trans. R. Soc. London, 1971, A270, No. 1206, 371—390.
65 Lineikin, P.S. On the determination of the thickness of the baroclinic boundary layer. DAN AN SSSR, 1955, 101, No. 3, 461—464.
66 Lineikin, P.S. Fundamental Questions of the Dynamic Theory of the Baroclinic Layer of the Ocean. Leningrad, Gidrometeoizdat, 1957.
67 Lineikin, P.S. Hydrodynamics of Ocean Currents. Obninsk, 1969.
68 Longuet-Higgins, M.S. Planetary waves on a rotating sphere. Proc. R. Soc. London, 1964, A279, No. 1379, 446—473.
69 Longuet-Higgins, M.S. Some dynamical aspects of ocean currents. Q. J. R. Meteorol. Soc., 1965, Vol. 91, No. 390, 425—451.
70 Longuet-Higgins, M.S. The eigenfunctions of Laplace's tidal equations over a sphere. Philos. Trans. R. Soc. London, 1968, A262, No. 1132, 511—607.
71 Lorenz, E.N. Available potential energy and the maintenance of the general circulation. Tellus, 1955, Vol. 7, No. 2, 157—167.
72 McConnel, A.J. Applications of Tensor Analysis. Dover, New York, 1957.

73 Manabe, S. and Bryan, K. Climate and the ocean circulation. Mon. Weather Rev., 1969, Vol. 97, No. 11, 739—827.

74 Marchuk, G.I. Numerical solution of problems of the dynamics of atmosphere and ocean on the basis of the method of factorization. Novosibirsk, Nauka, Sib. Otd., 1972.

75 Margules, M. Luftbewegungen in einer rotierenden Sphäroidschale. Sitzungsber. Kaiserl. Akad. Wiss., Wien, 1893, 11A, Vol. 102, 11—56.

76 Monin, A.S. Forecasting of Weather as a Problem of Physics. Moscow, Nauka, 1969.

77 Monin, A.S. On the hydro-thermodynamics of the ocean. Izv. AN SSSR, Fisika atmosferi i okeana, 1973, 9, No. 10.

78 Monin, A.S. and Obukhov, A.M. Small oscillations of the atmosphere and adaptation of meteorological fields. Izv. AN SSSR, Ser. Geofiz., 1958, No. 4, 1360—1373.

79 Monin, A.S. and Yaglom, A.M. Statistical Hydromechanics. The mechanics of turbulence. Part I. Moscow, Nauka, 1965.

80 Montgomery, R.B. Oceanographic data. In: American Institute of Physics Handbook. Sect. 2, Mechanics. McGraw-Hill, New York, 1957, 115—124.

81 Morgan, G.W. On the wind-driven ocean circulation. Tellus, 1956, Vol. 8, No. 3, 301—320.

82 Morse, P.M. Thermal Physics. Benjamin, New York, 1964.

83 Morse, P.M. and Feshbach, H. Methods of Theoretical Physics, Part I. McGraw-Hill, New York, 1953.

84 Munk, W.H. On the wind-driven ocean circulation. J. Met., 1950, 7, No. 2, 79—93.

85 Munk, W.H. and Carrier, G.F. The wind-driven circulation in ocean basins of various shape. Tellus, 1950, 2, No. 3, 158—167.

86 Munk, W.H., Groves, G.W. and Carrier, G.F. Note on the dynamics of the Gulf Stream. J. Mar. Res., 1950, 9, No. 3, 218—238.

87 Munk, W.H. and Phillips, N. Coherence and band structure of inertial motion in the sea. Rev. Geophys., 1968, 6, No. 4, 447—472.

88 Needler, G.T. A model for thermohaline circulation in an ocean of finite depth. J. Mar. Res., 1967, 25, No. 3, 329—343.

89 Obukhov, A.M. On the dynamics of layered fluids. DAN SSSR, 1962, 145, No. 6, 1239—1242.

90 Odulo, A.B. Qualitative analysis of Laplace's tidal equations. Izv. AN SSSR, Fisika atmosferi i okeana, 1972, 8, No. 7, 752—754.

91 Oleinik, O.A. On the properties of the solutions of certain boundary value problems for equations of the elliptic type. Mat. Sb., 1952, 30 (72), No. 3, 695—702.

92 Pedlovsky, J.A. A note on the western intensification of the oceanic circulation. J. Mar. Res., 1965, 23, No. 3, 207—209.

93 Pedlovsky, J.A. An overlooked aspect of the wind-driven oceanic circulation. J. Fluid Mech., 1968, 32, Pt. 4, 809—821.

94 Phillips, N.A. Geostrophic motion. Rev. Geophys. 1963, 1, No. 2, 123—176.

95 Phillips, N.A. The equations of motion for a shallow rotating atmosphere and the "traditional approximation". J. Atmos. Sci., 1966, 23, No. 5, 626—628.

96 Phillips, N.A. Reply. J. Atmos. Sci., 1968, 25, No. 6, 1155—1157.

97 Phillips, N.A. Models for weather prediction. Annu. Rev. Fluid Mech., 1970, 2, 251—292.

98 Phillips, O.M. The Dynamics of the Upper Ocean. Cambridge University Press, 1966.

99 Platzman, G.W. The Rossby wave. Q. J. R. Meteorol. Soc., 1968, 94, No. 401, 225—248.

100 Rashevskii, P.K. Riemannian Geometry and Tensor Analysis. GITTL, Moscow, 1953.

101 Reznik, G.M. On the structure of the boundary layers in the three-dimensional linear theory of wind-driven circulation in a barotropic ocean. Izv. AN SSSR, Fisika atmosferi i okeana, 1970, 6, No. 11, 1163—1177.
102 Robertson, H.P. The invariant theory of isotropic turbulence. Proc. Cambridge Philos. Soc., 1940, 36, Pt. 2, 209—223.
103 Robinson, A.R. Oceanography. In: Research Frontiers in Fluid Dynamics. Wiley, Interscience, New York, 1965, 504—533.
104 Robinson, A.R. Boundary layers in ocean circulation models. Annu. Rev. Fluid Mech., 1970, 2, 293—311.
105 Robinson, A.R. The Gulf Stream. Philos. Trans. R. Soc. London, 1971, A270, No. 1206, 351—370.
106 Rossby, C.G. and coll. Relation between variations in the intensity of the zonal circulation of the atmosphere and the displacement of the semi-permanent centers of action. J. Mar. Res., 1939, 2, No. 1, 38—55.
107 Sarkisyan, A.S. Foundations of the Theory and Analysis of Ocean Currents. Gidrometeoizdat, Leningrad, 1966.
108 Sedov, L.I. Mechanics of continuous media. Vol. 1, Moscow, Nauka, 1970; Wolters-Noordhoff, Groningen, 1971.
109 Shuleikin, V.V. Physics of the Sea. Nauka, Moscow, 1968.
110 Smirnov, V.I. Course of Higher Mathematics. Vol. 3, Pt. 2, Gostekhizdat, Moscow, 1956.
111 Stewart, R.W. The influence of friction on inertial models of oceanic circulation. In: Studies on Oceanography, Hidaka Anniv. Vol., Tokyo, 1964, 3—9.
112 Stockmann, W.B. The equations of the total flow field caused by wind in a non-homogeneous sea. DAN SSSR, 1964, 54, No. 5, 407—410.
113 Stockmann, W.B. Equatorial Countercurrent in the Oceans. Gidrometeoizdat, Leningrad, 1948.
114 Stockmann, W.B. Selected Works on the Physics of the Ocean. Gidrometeoizdat, Leningrad, 1970.
115 Stommel, H. A survey of ocean current theory. Deep Sea Res., 1957, 4, No. 3, 149—184.
116 Stommel, H. Westward intensification of wind-driven ocean currents. Trans. Am. Geophys. Union, 1948, 29, No. 2, 202—206.
117 Stommel, H. The Gulf Stream. A physical and dynamical description. 2nd ed. Univ. of California Press, Cambridge University Press, 1966.
118 Stommel, H. and Veronis, G. Steady convective motion in a horizontal layer of fluid heated uniformly from above and cooled non-uniformly from below. Tellus, 1957, 9, No. 3, 401—407.
119 Sverdrup, H.U. Wind-driven currents in a baroclinic ocean, with application to the equatorial currents of the Eastern Pacific. Proc. Natl. Acad. Sci. U.S.A., 1947, 33, No. 11, 318—326.
120 Sverdrup, H.U., Johnson, M.W. and Fleming, R.H. The Oceans, their physics, chemistry and general biology. Prentice Hall, New York, 1961, 1087.
121 Swallow, J.C. The Aries current measurements in the Western North Atlantic. Philos. Trans. R. Soc. London, 1971, A270, No. 1206, 451—460.
122 Tolstoy, I. The theory of waves in stratified fluids including the effects of gravity and rotation. Rev. Mod. Phys., 1963, 35, No. 1, 207—230.
123 Tsybaneva, T.B. On the evaluation of thermodynamic functions of sea water. Izv. AN SSSR, Fisika atmosferi i okeana, 1971, 7, No. 7, 814—817.
124 Väisälä, V. Über die Wirkung der Windschwankungen auf die Pilotbeobachtungen. In: Commentationes Physico-Mathematical Soc. Sci. Fennica, Helsingfors, 1925, Tomus II, No. 19, 1—46.

125 Van Dyke, M. Perturbation Methods in Fluid Mechanics. Academic Press, 1964.
126 Veronis, G. Comments on Phillips' proposed simplification of the equations of motion for a shallow rotating atmosphere. J. Atmos. Sci., 1968, 25, No. 6, 1154—1155.
127 Veronis, G. On theoretical models of the thermohaline circulation. Deep Sea Res., 1969, suppl. to 16, 301—323.
128 Welander, P. Wind-driven circulation in one- and two-layer oceans of variable depth. Tellus, 1968, 20, No. 1, 1—16.
129 Welander, P. The thermocline problem. Philos. Trans. R. Soc. London, 1971, A270, No. 1206, 415—421.
130 Welander, P. Lateral friction in the oceans as an effect of potential vorticity mixing. Geophys. Fluid Dyn., 1973, 5, No. 2, 173—189.
131 Wilson, W.D. Equation for the speed of sound in sea water. J. Acoust. Soc. Am., 1960, 32, No. 10, 1357.
132 Wilson, W. and Bradly, D. Specific volume of sea water as a function of temperature, pressure and salinity. Deep Sea Res., 1968, 15, No. 3, 355—363.
133 Wimbush, M. and Munk, W. The benthic boundary layer. In: The Sea, 1970, Vol. 4, Pt. 1, ed. A. Maxwell, Wiley, Interscience, New York, 731—758.
134 Yih, C.S. Stratified flows. In: Ann. Rev. Fluid Mech., 1969, 1, 73—110.
135 Zubarev, D.N. Non-equilibrium Statistical Thermodynamics. Nauka, Moscow, 1971.
136 Zubov, N.N. Oceanological Tables. Gidrometeoizdat, Leningrad, 1957.

SUBJECT INDEX

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