

The Story of Flatland: An Adventure in Many Dimensions

By Suzanne Fox Buchele

Adapted from the original story by Edwin A. Abbott

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Preface

The classic story *Flatland: A Romance of Many Dimensions* by Edwin A. Abbott was first published in 1884. Both the original and this version of the story are written for a general audience, and tell the story of a two-dimensional figure who discovers three-dimensions and contemplates other dimensions as well. The story introduces and reinforces mathematical and especially geometric concepts, and also engages in social commentary, as part of a fantastical story.

Abbott was a clergyman and educator whose passion was mathematics; I am a computer scientist, mathematician, and educator with an interest in spirituality. Abbott's delightful story contains sociological and philosophical elements, particularly in his descriptions of Flatland society. Written when it was, in late 19th century England, the Flatland society Abbott describes is a strict caste system, with women at the bottom. In particular, the original book is quite derogatory in its descriptions of women in Flatland society, who are lines as opposed to closed figures, are hysterical, unable to reason, have to make a "peace-cry" when they are out in public, and more. While the social elements of Abbott's version were apparently intended as a sharp critique of the place of women in Victorian England society, the depreciatory elements concerning women can be difficult to intellectualize as satire, especially for a younger audience or one not familiar with Abbott's goals. My goal has been to "free" the delightful geometric story from Abbott's social caste system. However, I found that instead

of simply removing the old social system, I needed to replace it with something. In my story, figures gain sides as they grow, in a process called *siding*. The essential elements of the story remain unchanged; a two-dimensional mathematician is visited by a sphere from three dimensions, who attempts to convince the narrator of the Gospel of Three Dimensions. After failing with argument alone, the narrator is transported to three dimensions and is convinced. The mathematician also has visions of Lineland and Pointland (one- and zero-dimensions), and speculates about the existence of forth and higher dimensional spaces as well. The narrator returns to Flatland, and unable to convince any fellow inhabitants, is eventually incarcerated for treason. The result is a story very much like Abbott's, retaining the same spirit and even some satirical elements such as the quarantine of irregular figures, but without the stumbling block of pejorative views of women. I also took the opportunity to add a few more geometric ideas, such as convexity and regular polygons, and a subtle spiritual element as well, as part of the process of siding. I also added an afterword, in which the figure has a vision of a non-Euclidean space (spherical geometry).

My motivation for this project had to do with my family. I had wanted to re-write Flatland since the spring of 1996, when I was in graduate school. My oldest son Wesley came home from Kindergarten one day, particularly excited (he had a wonderful teacher, who engaged his intellect quite masterfully). "Mom, I found out about two- and three-dimensions today, but, I was wondering, is there such a thing as four- or five-dimensions, or one- or even zero- dimensions?" he asked me. "Flatland!" I thought, recalling the delightful geometric story I read as part of my 10th grade geometry class. The next time I was at the bookstore, I picked up a copy and re-read it, to assess its appropriateness for my then 6-year-old (he was a precocious reader). After reading it again as a mother and a female mathematician/computer scientist, I decided that the derogatory content concerning women made it

impossible for me to give to my child to read, or even read to him aloud by adapting the story on the fly. Hence, the idea to re-write Flatland was born, but I had to wait until after graduate school and then for my first sabbatical leave to have the time to complete it. To this day, I have not given Abbott's book to my children to read – until now, when they are, with you, readers of this updated version.

Acknowledgements

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Suzanne Fox Buchele,

May 2006

PART 1: My World

1. An Introduction to the Nature of Flatland

I call my world Flatland, not because that is what we call it, but rather to make its nature clearer to you, my happy readers, who are privileged to live in Space. In order for you to fully understand my world, I will take the time to give you a comprehensive introduction to Flatland.

Imagine, if you will, a vast sheet of paper on which Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power to rise above or sink below the surface. Think of shadows on a flat surface, only hard and with luminous edges, and you will have a pretty good idea of the people in my world. Alas, a few years ago I might have said 'my universe', but my mind has now been opened to a higher view of things.

In such a land as mine, you will notice at once that it is impossible that anything there be 'solid' in any way. In fact, it is difficult for us to distinguish between Triangles, Squares, and other figures. Nothing is visible to us except the appearance of straight lines. Why is this so? If you do the following Spaceland experiment, you will understand this fundamental principle of the appearance of things in Flatland.

Place a penny on the middle of one of your tables in Space, and, leaning over it, look down from the top of it. It will look like a circle.

But then, draw back to the edge of the table, and gradually lower your head and thus your eye (and therefore bringing yourself more and more into the perspective of inhabitants of Flatland), and you will find the penny appearing more and more oval; and at last, when you have placed your eye exactly on the edge of the table (as if you were actually a Flatlander), then the penny will then have ceased to appear oval at all, and will have become, as far as you can tell, a straight line.

The same thing would happen to a Triangle or a Square, or any other flat figure on a table, if you were to treat it in the same way. As soon as you look at it with your eye on the edge of the table, you will see that it ceases to appear like a figure, and that in fact it appears as a straight line. Consider, for example, an equilateral Triangle, who represents to us a young child. Figure 1 represents the youngster as you would view him or her from above (a); as you would see him or her if your eye were close to level (b); or as if you were all but level with the table (c). If your eye were quite on the level of the table (this is how we would see the child in Flatland), you would see nothing but a straight line.

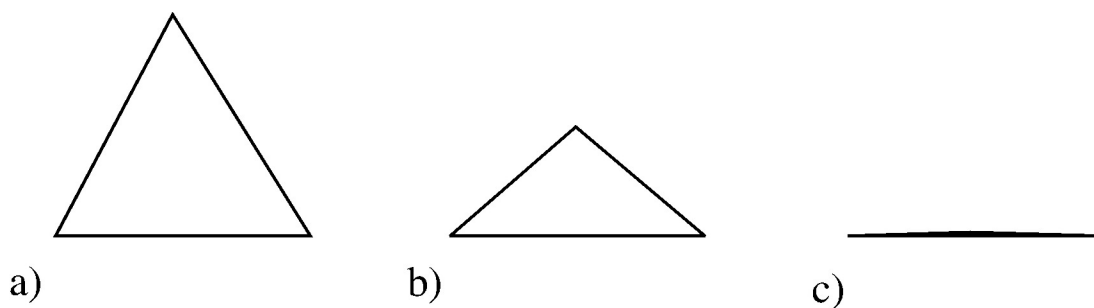


Figure 1. How you would view a triangle on a table

When I was in Spaceland I heard of sailors having a similar experience when they sail your seas and view some distant island or coast on the horizon. The far-off land may have bays, protrusions, and all kinds of jagged angles in and out, yet at a distance the sailor sees none of these (unless, perhaps, the sailor is quite close and the sun is shining bright upon the land, revealing depth cues by means of light and shading); nothing but a gray unbroken line upon the horizon.

Well, a line is just what we see when one of our hexagonal or other acquaintances comes into our view in Flatland. Since we have no sun, nor any kind of shadows, we do not have these aids for discerning depth as you have in Spaceland. If our friend comes closer to us, to our view we see the line become larger; if he or she moves away from us, it becomes smaller. But still he or she appears as a straight line, no matter if he or she is a Triangle, Square, Pentagon, Hexagon, or even a 20-sided figure. A straight line he or she will always appear.

You may ask yourself how, under these disadvantageous circumstances, we are able to distinguish our friends and family from one another. The answer to this very natural question will be more easily given after I describe more about the inhabitants and other objects that make up Flatland. Therefore, I will defer this subject, and first say a few words about the houses and weather of my country.

2. The Houses and Forces of Nature in Flatland

Just as with you, so it is with us; there are four points of the compass: North, South, East, and West.

Since we have no sun or stars, it is impossible for us to determine North the way that you in Spaceland do, by watching the sun rise in the East and set in the West, or locating the North star. However, we do have a method of our own for determining direction, given to us as a Law of Nature. In Flatland there is a constant attraction to the South; and, although in the temperate climates it is very slight, the hampering effect of the southward attraction is sufficient enough in most climates to serve as a reasonable compass. In addition, the rain (which falls at regular intervals, the raindrops of which are points) comes always from the North, and so is an additional assistance for determining direction. In the towns we also have the guidance of the houses, whose side walls for the most part run North and South, so that the roofs may keep the rain off from the North. In the country, where there are fewer houses that may be far between, the trunks of the trees serve as some sort of guide. Altogether, we do not have as much difficulty as you might think in determining our bearings.

However, in the more temperate regions, in which the southward attraction is hardly felt, walking at times in a perfectly desolate plain where there have been no houses or trees to guide me, I have been occasionally compelled to remain stationary for hours, waiting to feel my being drift slightly toward the South, or for the next rain interval, in order to gain my bearings and continue on my journey. On the infirmed and aged the Southward pull is more apparent, even in the temperate regions.

The most common form of a house in Flatland is a five-sided figure, or a pentagon, as in Figure 2. The two northern sides constitute the roof, and generally have no doors. On the east or west you will find the main door, with occasionally a back door as well. The southern side, or floor, is usually perpendicular to the southward pull, and door-less.

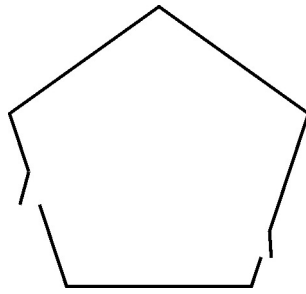


Figure 2. A Typical Flatland House

Square and triangular houses are not allowed, for the following reason. The angles of a Square (and even more so at least two of the angles of a Triangle) are much more pointed than those of a Pentagon, and so can pose a danger to the inhabitants of my world. The corners of a square or triangular house could inflict serious injury to a distracted or absent-minded traveler accidentally running into them. As early as the eleventh century these houses were forbidden by law throughout the land.

At that time, square houses were still permitted, though discouraged by a special tax. But, about three centuries later, the Legislature decided that in all towns containing a population above ten thousand (and hence, have a higher density of houses), the angle of the Pentagon was the smallest house-angle that would be allowed consistent with public safety. The good sense of the community has surpassed the efforts of the Legislature so that now, even in the country, pentagonal house construction has replaced that of square house construction. It is only now and then, in some very remote and backward district, that one may discover an antiquated square house. The only modern exceptions to this Law are certain military buildings, for which the general public does not approach in haste, if at all, and so pose little danger to the public.

None of our houses have windows, for the light comes to us alike in our homes and out of them, equally at all times and in all places. Because it is the same everywhere and none of us (except for myself) have experienced any other phenomenon than constant, shadowless light, light is not an entity described or discussed by us in any way. It would be like people in Spaceland having a term for the wetness of water or the density of air, although even these are imperfect analogies, since you do have a concept of wetness of other types of liquids, and air density does change somewhat with elevation and weather in your Spaceland. I, alone of the inhabitants of Flatland, know the origin and even the name for this light, but my knowledge cannot be made intelligible to any of my fellow citizens. I am mocked, as if I were the maddest of the mad of all of Flatland.

I have been told of your color, for which we do not seem to have an analogous phenomenon in Flatland. I have gathered that color differs from light to a completely different degree, although alas I do not know exactly what this concept is. Suffice it to say, for your own edification and assuming that you know what is meant by this, that we have no color in Flatland, only brightness.

3. The Plants and Animals of Flatland

The plants and trees of Flatland are non-closed figures, that is, figures made up of lines that do not enclose an interior space. Draw a point on a piece of paper, with six lines emanating from it in equally spaced increments, and you have drawn a pretty fair approximation of a tasty Flatland fruit. A tree or shrub is not that different from how a child might draw one of your Space-trees on a piece of paper. Figure 3 shows a Flatland fruit tree, just before the fruit is ready to fall from the tree.

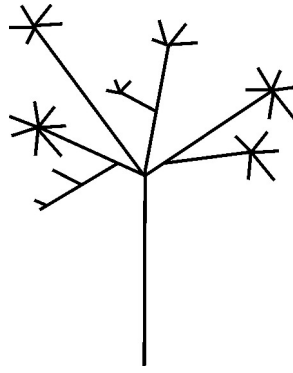


Figure 3. A Flatland Tree

Trees and other plants always grow with the trunk pointing southward, so that the leaves and fruit may catch and absorb the rain. When fruit ripens on a tree, the fruit will gently fall southward, obeying the southward attractive force. For this reason, all vegetation in Flatland is forbidden in the cities and villages, except in village parks and private gardens, which must be enclosed by sturdy fences. For if this were not the case, fruit might constantly be underfoot and a hazard in the public right-of-ways. And, of course, the trees and shrubs themselves would pose a danger to unsuspecting people who might accidentally bump into one and cause harm to oneself. Most vegetation is grown in fenced agricultural regions, where considerable energy is spent in gathering the crops as they mature, lest mounds of ripened fruit pile up against the southern fences and eventually break the fence.

Trees are particularly useful to us, for in addition to providing us food, a tree may be disassembled into lines and used to build our houses, fences, and other objects. Some people enjoy keeping enclosed gardens of non-edible vegetation in their yards simply for the interesting view that they provide, although I personally see no point in keeping vegetation that has no useful purpose. Although such

gardens would have to be enclosed to protect unsuspecting travelers from happening upon them, the danger of injury in such a garden, for me, far outweighs any benefit of keeping it.

The animals of Flatland are various domesticated and savage beasts. Animals of any kind can be quite dangerous to the people of Flatland, and so it is a blessing that Nature has designed them to be quite noisy. In fact, animals utter some kind of cry almost continuously. Flatland animals are closed figures (that is, figures made up of lines that enclose an interior space) and always have at least one concave angle (an angle of more than 180 degrees). Animals are typically composed of only a few lines, never more than six, and typically have two or more sharp angles, that is angles of degree significantly less than 90 degrees. Figure 4 shows a sketch of my daughter's pet, Ozco, consisting of four lines and one concave angle (the upward protrusion at the bottom of Ozco).

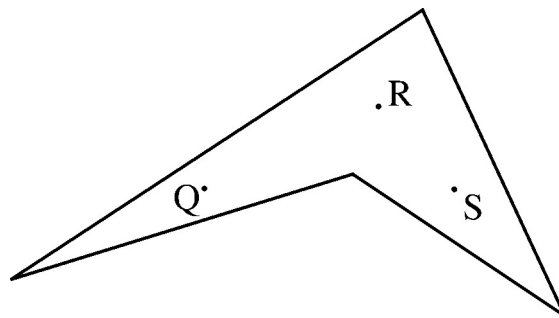


Figure 4. Our Pet, Ozco

Animals wiggle and shake almost constantly. For this reason, and because of their sharp angles, animals can be quite dangerous. If a person encounters a wild beast in the country, it is generally advisable to shoot the animal on sight, by piercing it with a spear. Otherwise, one is susceptible to the

animal either charging or wandering close, and instead killing you with its shaking and sharp angles. You see why it is a good thing that animals cry out almost constantly, so that one can always be aware when such a danger is near.

One might wonder, with as noisy and potentially dangerous as they are, why people would choose to keep animals as pets. In fact, many Flatlanders would never consider maintaining such an irregular creature. Others are not so disinclined toward the disorder that animals entail. I can only refer you to your own experiences with your Spaceland dogs and cats. The affection and companionship of a pet can more than make up for the disorder and noise they produce (however, your neighbors may not agree). Furthermore, domesticated animals have been bred over the years to be more calm and somewhat less prone to cry as continuously and loudly as their wild counterparts. Therefore, they are less dangerous and troublesome than wild beasts. In fact, Ozco will lessen her shaking significantly upon command, and will be almost completely still if approached and petted with affection.

4. The Inhabitants of Flatland

Now, I must explain to you the physical nature of the people of Flatland. Like animals, the people of Flatland are closed figures. However, the physical property that sets people apart from animals is that people are always *convex*, without any concave angles whatsoever. A convex figure (without any angles of more than 180 degrees), has the property that if you choose any two points on the interior of the figure, the straight line connecting those two points is always on the interior of the figure. As an illustration, refer again to a sketch of Ozco in Figure 4. The points Q, R, and S are interior to the figure (Ozco), and in fact, the line QR is also interior to the figure. However, the line QS is not

interior to the figure, that is, following on a straight line from point Q to point S, we exit poor little Ozco and then re-enter her again. Since we can find two points for which the line connecting them is not completely interior to the figure, then Ozco is decidedly not convex (nor human). Consider next a sketch of myself, a healthy middle aged Flatlander (Figure 5). As you can see, I am currently a figure of sixteen sides, called a hexadecagon, and although I cannot show every possible point on my interior, I hope you can plainly see that if you chose any two points on my inside, that the line connecting them would lie completely on my inside. It so happens that it is a Law of Nature that any closed figure without a concave angle is convex. It is also a Law of Nature that animals always have at least one concave angle, and people never do. A Flatland legend asserts that the cognizant and intelligent thoughts of persons, bouncing around in their interior, keep their angles convex.

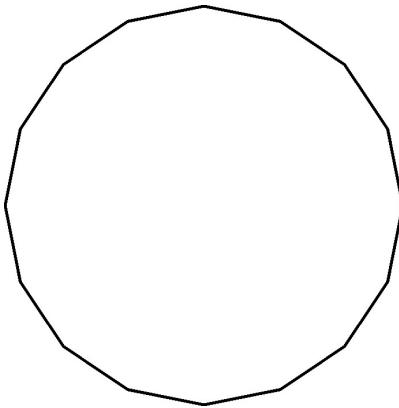


Figure 5. Myself

From what I understand of your world, people are conceived and born in Flatland in much the same way. Two figures decide to commingle in a very satisfiable manner, and, God willing, a child is conceived. After carrying the growing child inside her for some months, when the time is right the child passes through two lines forming one of the woman's angles, and the child is born. Newborn

children are almost indistinguishable from a line, which I daresay makes the child bearing process much more pleasant for our women than for yours. In fact, however, a newborn child is a very thin Isosceles triangle (that is, a triangle with two sides of equal length), with two relatively long sides and one extremely short one. Figure 6 shows the approximate proportions of a typical newborn Flatlander. An acute angle of 1 degree or less is a typical baby angle size.

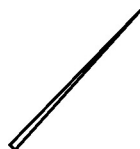


Figure 6. A Newborn Flatlander

As you might imagine, being in close proximity to a newborn child with a sharply pointed angle might be as problematic as happening upon a wild animal in the country. While there are some risks in childrearing, it is generally quite safe. Newborns, when their most acute angle is smallest, are generally calm and helpless. A parent or older sibling is with them at all times, to care for the child but also to keep others safe in its presence.

As a child grows, its smallest side lengthens and so the acute angle grows, until at about two years of age, the child is approximately an equilateral triangle (with all sides of equal length). After the equilateral stage, the third or back side (the side that was originally the shortest at birth) becomes longer than the other two. Figure 7 shows the development of a Flatlander through the first three or so years of life. At some point between the third and fifth years, the long side of the child develops a slight bend, which eventually breaks (in a process called *siding*), until the child is four-sided by the start of schooling. The new angle shrinks as the adjoining sides lengthen, eventually becoming a

square. In general, the natural process of siding involves lengthening of one side, splitting the side in two, and then gradually rearranging the body (growing some parts, contracting others) to form approximately equal angles and equal sides.

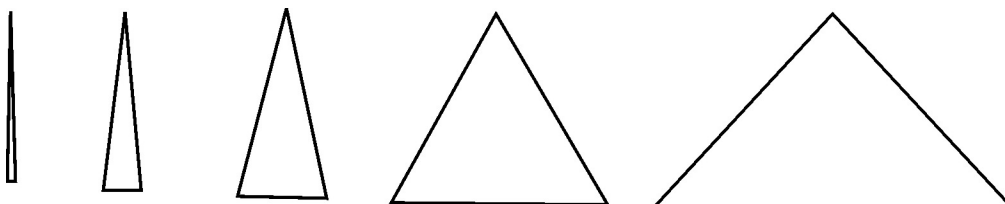


Figure 7. A Typical Flatlander, From Newborn to Three Years of Age

Flatlanders do not grow appreciably larger in berth past the equilateral triangle stage. We do encompass more breadth in our interior, but after the initial growth to equilateral triangle, which includes growth both in angle and side length, all Flatlanders are of relatively uniform size, with a breadth of about two of your feet, or sixty centimeters. Generally, the growth from isosceles to equilateral triangle, and then equilateral triangle to square, is a slow and steady growth process, in which sides and angles gradually change size slowly and uniformly over time. Past the square stage, however, Flatlanders tend to remain regular (with sides of equal length and angles of equal size, arranged symmetrically around the center) for most of their lives, with the change in shape necessary for adding a side occurring during a relatively short period of time. Thus, the siding experience is more gradual and less painful from birth to square, but more compacted in time and thus more painful after the square stage. However, we all grow continuously, at least slightly, even into old age, so no one is ever exactly regular for very long.

By the time school commences (typically at or before the age of five), the child is four-sided, and approximately a square. Children side quickly throughout their school years, so that by the time they are graduating from their formal schooling, they are typically at least hexagonal, and sometimes as sided as octagonal. In general, an octagonal person is considered an adult. Twelve to sixteen sides is typical for a middle aged person such as myself, while a much older and very wise person might have so many sides (twenty to thirty) as to be difficult to distinguish. Such older persons are reverently referred to as circles, although in fact they are many-sided polygons.

It is law that a Flatlander of less than 5 sides must be accompanied by a parent or other adult in public at all times. For the safety of all citizens, children not directly adjacent to a supervising adult are required to give a constant cry, called a peace cry, at an almost constant rate so that others are aware of their presence. This is similar to how Nature has made wild animals, and has been a rule of Flatland civilization for as long as recorded history has been in existence. However, the peace cry of children is far more pleasing to the ear than the irritating noise animals ululate. In fact, a child's peace cry is generally quite pleasing to adults, and children receive copious amounts of positive feedback for performing it in the company of others. Children also find performing the peace cry enjoyable in its own right, and will often continue the practice well into the pentagon stage (long after it is really necessary).

The eye, ear, and mouth of a Flatlander all lie on their most forward angle, also the most acute angle at birth and the angle opposite the side at which they grow (their back side) throughout their lifetime. Naturally, this is the angle that is kept in the front during movement, so that we can use our eye and ear to help us recognize the things we come upon, and our mouth to announce our presence to others.

5. Our Methods of Recognizing One Another

You, who live in a world with shades of light, who have two eyes, who have an innate knowledge of perspective and depth, you, who can actually *see* an angle and an entire figure from the happy perspective of three dimensions - how can I possibly make clear to you the extreme difficulty that we in Flatland experience in recognizing one another's configuration?

Recall what I told you previously, which I illustrated for you with a coin and tabletop. All beings of Flatland, animate or inanimate, closed forms or not, present to our view the same, or nearly the same appearance: that of a straight line. How then can one be distinguished from another, when all appear the same? Nature has given us three methods which have become highly developed in Flatlanders to undertake this difficult and important task.

The first means of recognition, that works almost without effort and with both animals and persons, is by using the sense of hearing. Our hearing is much more highly developed than yours, and so it enables us to not only discriminate by voice our personal friends and family, but even to distinguish between the age (or, in fact, sidedness) of a person, since the voice characteristics, while unique to each individual, are inexorably linked to the growth of the person with respect to their number of sides (or, actually, the size of their front angle where the mouth resides). Animals each have their own distinguishable cry as well, so that one can discern not only the cry of their own pet, but, upon hearing an animal's cry, can ascertain the type of beast that they are coming upon. Our hearing is so developed, then, that a typical Flatlander can tell, while conversing with another person, whether they

are 4- or 5- or even 12-sided. However, the voice differentiation among figures that are 20-sided and above can be hard to distinguish. In addition, some youngsters that enjoy playing tricks on their more mature companions are able to feign the voice of a pentagon or even sometimes a much more sided person, which makes recognition by hearing alone a troublesome method to use in all circumstances.

6. Recognition by Feeling

The second method we use to recognize others is that of *feeling*. Among our younger and less educated persons, feeling is the principle test of recognition. This is especially useful to identify not only the individual, but the sidedness of the person, and will therefore also give an indication of the respect due to the person. What “introduction” is to you, the process of “feeling” is with us. “Let me ask you to feel and be felt by Ms. So-and-so” is still, among the more old-fashioned of our country folk or elders, the customary formula for a proper Flatland introduction. But in towns and in the hustle and bustle of the workplace, the words “felt by” are usually omitted and the sentence is abbreviated to “Let me ask you to feel Mr. So-and-so”, although it is assumed, of course, that the feeling will be reciprocal. Among our modern youth, who are extremely adverse to superfluous effort and indifferent to the purity of their native language, the formula is still further curtailed by the use of “to feel” in a technical sense, to mean “to introduce for the purposes of feeling and being felt by”, so that at the moment, the slang of youth sanctions such indecorousness as “Luis, feel Sara”.

However, I do not want to give you the impression that feeling is with us an intrusive, tedious, or cumbersome process as it would be for you, or that we find it necessary to feel around all the sides of an individual before we determine their sidedness. Long practice and training, begun at home and

continued in schools and through the experiences of life, enable us to discriminate at once, by the sense of touch, between the angles of a triangle, square, pentagon, hexagon, and so on. Recall that Flatlanders constantly experience growth, a lifelong process in which one side lengthens, forms an angle in the middle, and then the entire body contracts to an approximately equal sided and equal angled figure. Therefore, feeling a single angle, especially the back angle shortly after a siding, will possibly give only an estimate of the sidedness of the person (since it may be a very large angle that has just occurred due to the siding process). In that case, the relatively short sides adjacent to the rogue angle would be an indication of it not being representative of the person as a whole. However, a polite person would never present their newest angle for feeling. In any event, no more than two angles and two sides must be felt, with the sidedness indicated by the smaller angle and longer side (since upon the onset of a new angle, one longer side splits into two smaller sides and one large angle is formed, until the subsequent contraction and regularization has fully completed). If a disingenuous purpose is ever suspected, one may politely and discretely feel up to three angles and the two intervening sides, and if an offense is taken, a casual remark that your “feeling is a little off today” should suffice.

While experience and practice makes most Flatlanders able to distinguish the sidedness of most persons up to middle age or so, it is difficult to distinguish between the ages of more elderly persons. Even a Master of Arts from our University of Compton has been known to occasionally confuse a twenty and twenty-one sided person. Furthermore, there is hardly a Doctor of Science in or out of that famed University who could pretend to immediately and unhesitatingly distinguish between a twenty-nine and thirty sided distinguished member of society.

You may recall my description of the sometimes dangerous nature of animals and small children in Flatland. Their sharply pointed angles can be quite dangerous, and even fatal, to a Flatlander. Since all but the largest of angles could be a hazard, the process of feeling requires care and discretion. Otherwise, angles might inflict on the unwary feeler irreparable injury. It is essential, for the safety of a feeler, that the felt should stand perfectly still. At first, a fidgety shifting of the position, yes even a violent sneeze, has been known to be fatal to the feeler. As you might imagine, injury of this sort may nip in the bud a promising friendship or business relationship. Because of their uncontrollable nature, it is generally not advisable to feel animals, even the domesticated pets of good friends, or small children, unless they are asleep or in some way restrained. For this reason it is generally not considered proper to invite another to feel their child (or pet) unless they are asleep. Women, especially, can scarcely bare not to feel a newborn infant; luckily, they are so often calm and asleep, and have little strength to move in any event, that the danger there is minimal. Because of the inherent danger, an invitation to feel one's restless child or rambunctious pet should be viewed with both caution and suspicion.

It is known that my great-grandmother forever lamented an unfortunate accident that occurred while she was being felt by an important and politically powerful younger statesman. My great-grandmother, wise and learned, an advisor to many other learned persons besides the aforementioned statesman, and a mentor to many younger members of our community, was afflicted in her old age with rheumatism. During this fateful greeting, she was set upon with a fit of sudden shaking, and although old and of large angle, she succeeded in impaling the younger politician. Although the younger recovered, his further growth was never normal. The incident stripped my great-grandmother of her advisory and mentor positions, and made her the subject of much contempt. Charges were even

brought against her, since she was aware of her rheumatic ailment and the real possibility of jerking during feeling. However, the jury, some of which were older than the old woman, had pity upon her and did not send her to prison as would have been their right. However, the scorn has been said to remain with my family for all these generations, to even today, when I have perhaps revived and strengthened it. But, that part of the story comes much later.

At this point, you may be wondering how, exactly, we are able to tell the difference in two angles, or even discuss the precise measure of an angle, even to degree or minute. You, in the region of Space, can *see* an angle as the inclination of two straight lines against each other. You even have devices to measure angles. How can we, who can see nothing but a straight line at a time, or a number of bits of straight lines before us, how can we discern an angle, much less distinguish angles of different degrees?

My answer is that while we cannot *see* angles, we can *infer* them, and this with great precision. Our sense of touch, which has evolved out of necessity over time and further developed by training, allows us to distinguish angles far more accurately than your sense of sight (when not using a ruler or protractor). Since infants are born with a tiny primary angle and during siding an angle of almost 180 degrees is formed, throughout history we have had at our disposal a veritable alphabet of angles. Therefore, people get ample opportunity to practice the common angle ranges of 30 to 120 degrees or so, but also occasional practice in feeling the more extreme angles as well. In fact, in our schools are placed models of angles from 1 minute (1/60 of a degree) to 179 degrees 59 minutes, for training and exam purposes. In more barbaric times, actual persons, usually irregularly formed or mentally challenged, were actually confined to schools during the day, not for their own education but simply to

allow students to practice in feeling their angles. In some instances these poor souls were actually fed and housed in the schools as well, and made to exist there for several years before being released in favor of a new specimen. Needless to say, nowadays model angles are constructed at the Universities for use in the lower schools. These models work quite well and have trained all but the oldest of Flatlanders living today.

Enough has been said, I trust, to convince you that recognition by feeling is a learned art that is not so tedious or intrusive a process as you might have supposed, and certainly more trustworthy than recognition by hearing. Still there remains the objection that this method has its dangers. For this reason, many Flatlanders prefer a third method, that of recognition by sight.

7. Recognition by Sight

I am about to appear to be inconsistent. Previously I told you that all figures in Flatland give the appearance of a straight line, and have approximately the same breadth, and it was added or implied that it is therefore impossible to distinguish by sight between individuals with a differing number of sides. I am about to explain that most of us are, in certain circumstances, able to recognize one another by sense of sight. It is generally among our most well-educated, and dare I say elitist persons that sight recognition is most widely practiced.

That this ability exists at all is not due to differences of light and shading, as I have already explained do not exist in Flatland, but because of fog, which prevails during the greater part of the year and in all parts save the most torrid zones of our land. That which you in Spaceland seem to despise as

depressing, abhorrent weather, is recognized by us as a blessing scarcely inferior to air itself. But, let me explain why we revere fog so, without further eulogies to this most beneficial element.

If fog were non-existent, it is true that all figures would appear equally and indistinguishably clear as straight lines, and this is actually the case in those unhappy regions in which the atmosphere is perfectly dry and transparent. But wherever there is a luxurious supply of fog, objects that are at a distance of say three feet are appreciably dimmer than closer objects, say at a distance of two feet six inches. The result is that by years of careful and constant experimentation and observation of comparative dimness and clearness, we are able to infer with great accuracy the configuration of the object being observed. An example will do more than volumes of general statements to make my meaning clear.

Suppose I see two individuals approaching me, whose age I wish to ascertain. They are, we will suppose, an approximately isosceles child and his pentagonal sister (of course, by Law an adult must also be present, but let us deal with that more difficult case presently). Suppose there is a loud pet nearby, or a crying baby, and I cannot make out the sound of the child's peace cry, or any casual conversation between the two, and yet they are moving close enough to me to warrant a greeting of some sort. How am I to distinguish them, and perhaps even recognize them, without interrupting their passing by feeling?

It will be obvious to every child of Spaceland who has even begun the study of geometry, that if I bring my eye so that its gaze bisects the most forward angle of an approaching stranger (the angle containing their eye, ear, and mouth, which is usually the frontmost angle as they walk), my view will

lie evenly between the two sides that are closest to me (e.g.. CA and AB in Figure 8a, the Spaceland view), so that the two sides will appear of the same size. Now, in the case of (a), the child, what shall I see? I shall see what appears to be a straight line (Figure 8b, my Flatland view), QAT, called the apparent line, but which is actually the appearance of the two different lines CA and AB. In the presence of fog, however, point A will be much closer to me than either of the points C or B. Therefore, the midpoint of the apparent line QAT, point A, will be much brighter in appearance to me because it is nearest to me, and there is less fog between me and it. On either edge of the apparent line QAT, the points will gradually become dimmer, because the sides AC and AB recede rapidly into the fog. What appear to me as the child's extremities, that is points Q and T, will be very dim indeed, at least in comparison to the closest point A.

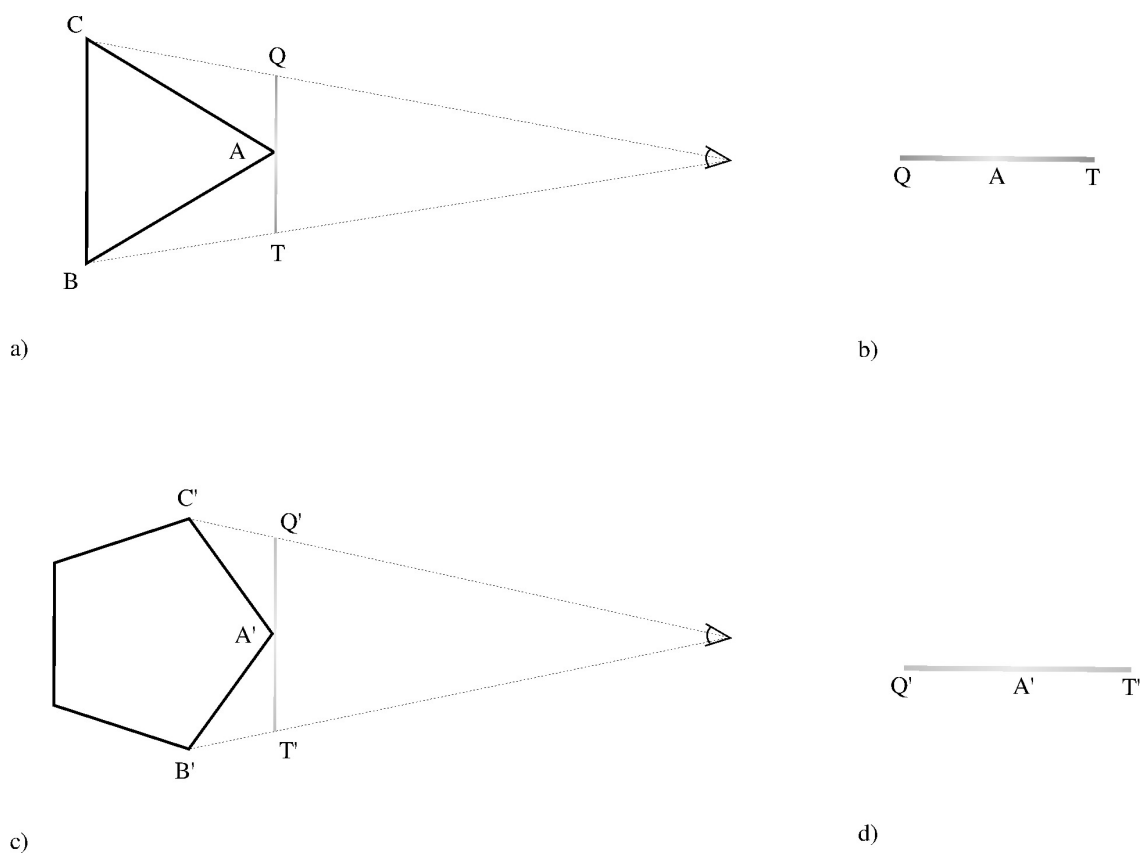


Figure 8. Recognizing a child and his sister

Now, the appearance of the adolescent sister (c and d) will be different indeed. If my eye is also lined up with her frontmost angle (the easiest case for this simple example), I will also see an apparent line (Figure 8d), Q'A'T', with a brighter center (A') and dimmer extremities (Q' and T'), yet the extremities will be less dim than those of her younger brother, since they are indeed closer to me than those of the child. In fact, the apparent line Q'A'T' will become dim less rapidly to the extremes, because the sides A'C' and A'B' recede less rapidly into the fog. This will indicate to me that the primary angle of this person is larger than the other.

You can probably understand from these two examples how, after a great deal of training accompanied by constant experience, it may be possible to not only discriminate among different angles, but even estimate the size of the observed angles. In fact, it is possible among the comprehensively educated and very well-trained persons who seek to become an expert in recognition by sight to discriminate with pretty fair accuracy between a person of say 15 and 16 sides, in the presence of a thick layer of fog. If you, my Spaceland readers, have at least grasped the general conception of recognition by sight, at least so far as to agree to the possibility of it and not dismiss it altogether as wholly impossible, then I have done all I set out to do. I fear that if I give further details, I will only perplex. However, for the sake of the young or cocky, who from the two simple examples I gave above should believe that recognition by sight is a straightforward and easy affair, let me point out that in real circumstances recognition by sight can be quite subtle and complex.

Suppose, for example, that the child in my example was accompanied not by his older sister, but instead by his uncle, an octagon (Figure 9a and b). Now, the apparent line Q'A'T' is actually made up

of five points from the young man, consisting of the endpoints of the four sides $E''C''$, $C''A''$, $A''B''$, and $B''D''$. Now, how shall the apparent line $Q''A''T''$ appear to me? It will still be less dim in the middle, at point A'' , and more dim to the extremes, at points Q'' and T'' . And, in fact, points E'' and D'' will be approximately the same distance from me as were the points B'' and C'' on the sister. So, how would I, in fact, distinguish the uncle from his younger niece? The difference, of course, is in the pattern of the gradual dimness from the midpoint A'' to the extremities Q'' and T'' on the apparent line. In the case of the sister, the graduation was constant and uniform; a gradual linear progression from less dim (A') to more dim (Q' and T'). In the case of the uncle, however, we have two phases of gradual dimness on each side of the apparent line. Moving along the apparent line from A'' to R'' , we have a gradual dimness corresponding to the gradual recession from point A'' to point C'' . However, we then have a break in the rate of graduation of dimness, corresponding to the angle $E''C''A''$, and a more rapid recession of dimness from R'' to Q'' corresponding to the rapid recession into the fog of line $C''E''$. Thus, it is not only the gradual dimness along the apparent line that must be taken into account, but also the pattern and rate of dimness that not only must be discerned, but be transferred into knowledge about the object or person's configuration.

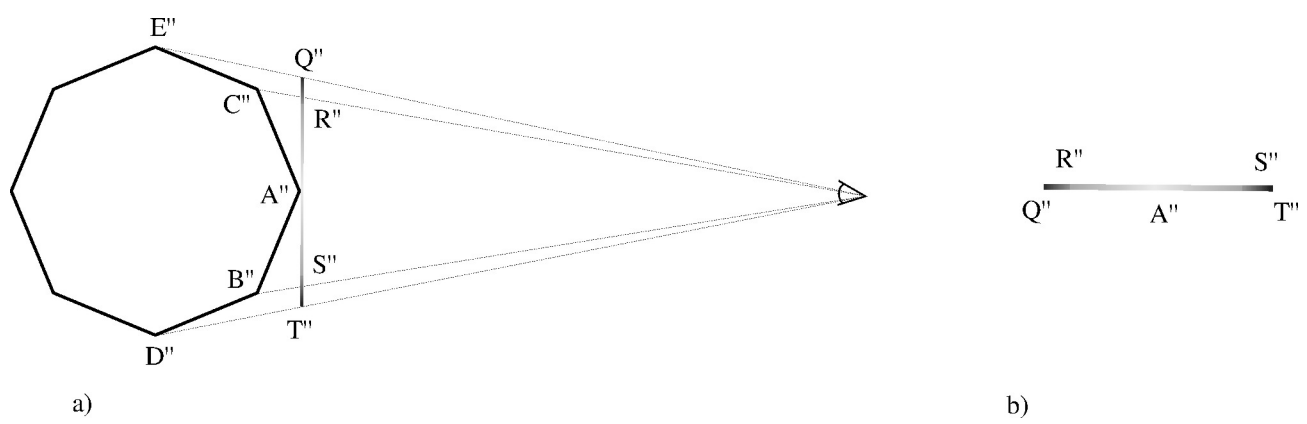


Figure 9. Recognizing an octagon

However, we have only touched upon the complexities that occur in recognizing people and objects in everyday situations. You see, when people are out and about in public places, one is constantly rotating one's figure around one's center. This is because the eye is fixed upon front angle of the person, and because we have but one eye and not two, we are forced to constantly rotate, at least slightly, so that we can gain the perspective we need to more clearly recognize objects and function in our world. Furthermore, we prefer to greet people eye-on, but since, especially in a public situation, we may be coming upon people from many different directions in quick succession, we tend to be moving our eye about rather constantly. For all of these reasons, both a viewed person and the viewer are typically in constant rotational movement, which considerably complicates the process of recognition by sight.

Although I expect that I have now convinced you of the necessity of the years of training and experience required to become only somewhat competent at recognition by sight, I will leave you with a further example; in fact, a more realistic example than the simplistic ones we started with. Suppose we are walking about and come upon the aforementioned uncle and child. This time, however, instead of being exactly aligned so that the viewer can most easily discern their sidedness, suppose that their attention is elsewhere for a moment, and they are not regularly aligned with the viewer's eye (Figure 10). Suppose also that both the uncle and child are rotating constantly back and forth around their respective centers, as is the viewer. Suppose further that the child is approaching their age of division, from triangle to square, and so his back side is longer than his two front sides, and, furthermore, that the child has briefly turned around to consider some plant or other curiosity he just passed. Add to these two a city street full of people, rotating, advancing, and retreating, and at the same time trying to

discriminate by the sense of sight between the apparent lines of all these figures, the apparent dimness of these lines due to fog, all different and constantly changing due to motion. Now you have a more accurate idea of the enormously complicated situations encountered with recognition by sight in practice.

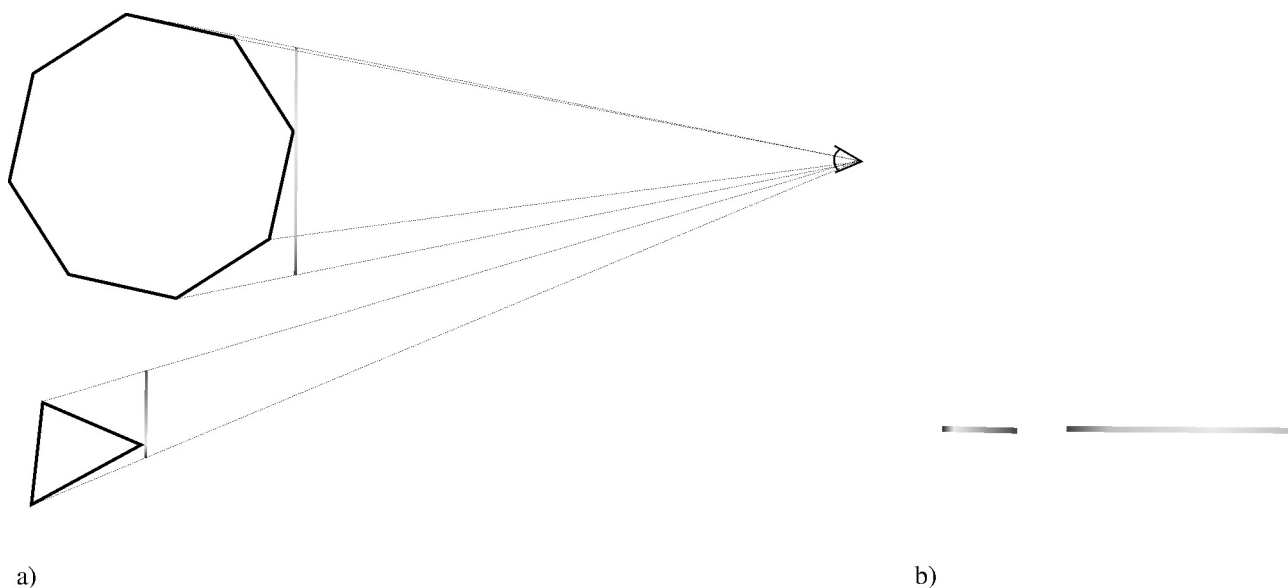


Figure 10. Uncle and Child, both rotated with respect to viewer

Not everyone in Flatland is able to capably practice the art of recognition by sight, all the time. Obviously, children are limited to feeling and hearing, as are those near the end of their lives, whose eyesight is not able to discern the subtle gradations of dimness necessary to accurately practice recognition by sight. Furthermore, as I mentioned at the start but which I must emphasize here, the art of recognition by sight is only possible in the presence of fog. There are some desert-like regions in

Flatland in which fog is simply not present, and even in our more temperate zones, fog may be luxuriously thick one day and rather sparse the next.

But, more importantly than the conditions is the education and training necessary for recognition by sight. All children receive elementary training in this art during their primary years of schooling, and so all graduates are able to perform at least rudimentary recognition by sight with a few simple, static figures. Many near-graduates, or rather their parents, find this level of training perfectly acceptable for the trades which they would pursue, and in fact would find it ludicrous to spend the extra years of time and money on a masters degree in the seemingly abstract art of recognition by sight. Many others, however, choose to cultivate this art to its fullest, and pursue even doctorates in this complex art. There are plenty of employment opportunities in teaching the art of recognition by sight at various levels, although many of the well educated are content to simply use their finely honed skills recreationally as often as they are able. I myself, a mathematician of moderate standing, will often find myself in crowds of rotating polygons and find myself utterly perplexed. And, of course, to someone with only elementary training, such a sight would be as unintelligible as it would be to you, my Spaceland reader, if you were suddenly transported to my land. In such a crowd all you would see about you would be a line (the transposition of all the apparent lines of all the people around you), seemingly straight, but the parts of which would vary irregularly and continuously in brightness or dimness (due to fog). In such a confused situation, without the ability to distinguish persons and more importantly their trajectories, bumping into others is a real possibility. Even if you had completed your third year of studies at the University and were perfect in the theory of the subject, you would still find that there was need of many years of experience practicing the theory before you could move around in the crowd without jostling your neighbors. If the crowd is, say, a class of graduating Ph.D.s at the

University, then you would certainly stand out in your inability to seamlessly negotiate the crowd. However, such highly educated groups are obviously uncommon, and most persons function quite happily in Flatland using whatever sight recognition skills they have acquired, while using cautionary movements coupled with feeling and hearing whenever necessary. In fact, many Flatland readers who never went to University at all may find my opinions elitist in themselves, and would scoff at the “necessity” of complex forms of recognition by sight at all.

I will mention that while some find recognition by sight a mere luxury, most University trained Flatlanders, or other self-trained intellectuals and those of more proper upbringing, find the excessive use of feeling, especially in very straightforward sight situations such as my first two examples, to be distasteful. Thus, recognition by sight is the more socially acceptable means of recognition, except when circumstances make it more complex than the person is trained to handle. The general rule of good manners is, if you can perform sight recognition in a certain situation, then you should. Although this is the social norm, it is not without its practicalities. Recall that people moving about are often constantly rotating about their centers. An unsuspected feeling during this perpetual rotation could injure both the felt and feeler. Thus, recognition by sight is the safer option in any event, and, at the very least, if feeling is to be used, one must either ask to feel or agree to be felt to ensure that any movement will cease before the feeling. A verbal greeting of some sort, indicating your imminent intention to feel, is of the utmost importance. While these may seem like trivial niceties to you, I assure you, it is a matter of health and well-being to us.

It is astonishing how much the art of recognition by sight is fostered by the habitual practice of it, and by the avoidance of the custom of feeling when possible. Some puritanically intellectual households

will, in fact, attempt to keep their children from ever feeling. Of course, this is not possible in all situations, but in such households feeling is strongly discouraged and home training in recognition by sight commences at a very young age. Instead of being sent to public schools, where feeling is taught, these children are sent to private institutions of an exclusive character, and then on to University. It is said that the best sighters of our land come from such a background, since they have rarely had to resort to the primitive practice of feeling and therefore have cultivated the art of recognition by sight to a degree not possible for those of us with the ingrained art of feeling as our first means of recognition.

8. How we Age, and, Siding

I have implied during our discussion that Flatlanders age by increasing their number of sides steadily, throughout their lives. While this is true during most of a Flatlander's life, there are general trends that bear discussion. First, during the most formative years, Flatlanders increase in side approximately one side every four years, so that children begin school somewhat after they become four-sided and complete school when they have reached (or are about to reach) septagonness (seven-sidedness). In fact, schools generally require four-sidedness before a pupil can be enrolled, and if the child has not yet reached that milestone the parents are asked to wait another term before enrolling the child.

Leaving school at septagonness is not an absolute rule; some pupils have completed their schooling when only six-sided, and others have been known to stay until they are octagons. Schools frown upon students staying past the octagonal stage: if the student is older and slower and having trouble in school, then it is generally agreed that the student has had as much schooling as he or she can handle. More common, however, is a bright, younger student who is increasing in sides more quickly than some of the other students.

A Law of Nature not yet fully elaborated to you is that among Flatlanders, siding is triggered not only by one's physical age, but by one's mental state as well. It is a well-studied phenomenon that the brighter students tend to side more quickly. This phenomenon continues into adulthood as well. Students at University side more quickly than their non-University peers, and those with demanding intellectual or mentally intensive jobs tend to side more often throughout their lifetimes than others with duller, less engaging careers. Other forms of mental stimulation than schooling or intellectually demanding careers can also trigger accelerated siding. For example, I have a neighbor who never went to University and holds a job gathering fruit in a local orchard, who, in his spare time, prefers to engage in intellectual discussion with others, concerning the political situation of the day, or past trends and how they may be influencing our future. Although not strictly required of his job, at work he conducts botanical experiments with the fruit trees, suggesting improvements to the owners. When alone, I happen to know that he enjoys reading and engaging in mathematical mind puzzles more than pure idle time. He has also confided in me that at the end of the day, before sleep, he examines every facet of his day and analyzes his actions for his own betterment. For instance, he looks for situations in which he might have been able to use sight recognition when he did not, or conversations and things that he said that might have turned out for the better. In short, he is constantly learning and using his mental capacity in his own ways, without the benefit of a University education or a rigorously intellectual job. And he, in fact, has sided about as frequently as I have (although he has thirteen sides to my sixteen, but only because I am some years his senior).

However, at least seven sides is the absolute minimum requirement for University, where students are worked so hard that they generally side at least twice in the three to five years that they attend. And, as you might imagine, the frequency of siding tends to be a badge of honor among those in University.

Siding in general is an important time in any person's life. As I mentioned previously, up until the square stage, growth is a slow and constantly progressive process, in which angles and sides grow and change consistently over time. After the square stage, however, the rate of growth changes. A square will tend to stay a regular square, exhibiting almost no growth at all for much of the approximately four years before siding to the pentagonal stage. Beginning with the pentagonal siding, the process of siding is an intense and painful process, occupying approximately two weeks of time. During siding, the front angle grows larger and the back side grows longer, until, at the moment of siding, the longer side “breaks” into an angle. As you might imagine, the exact moment of siding is immensely painful, but also brings release of the pressure due to the growth of the angles. After the new angle is formed, regularization continues for another week or so. During regularization, the newly formed angle shrinks, all other angles grow, the sides adjacent to the new angle grow, and all other sides contract, until, at the end of siding, the person is approximately an equally sided and equally angled regular figure. Figure 11 shows the process of siding, from a square to a pentagon.

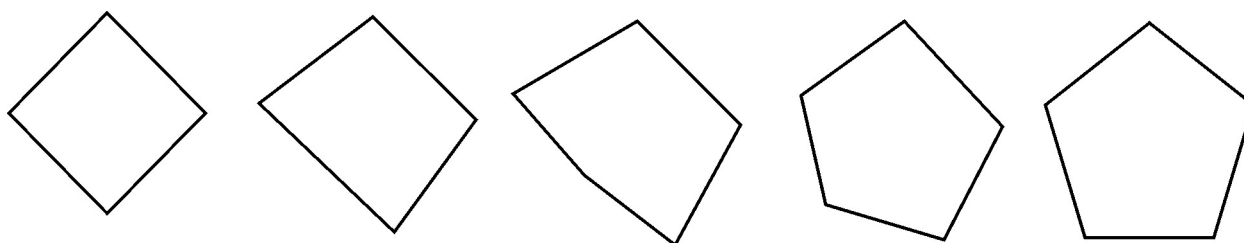


Figure 11. Before, during, and after Siding

In addition to being intense and painful, siding is also a deeply personal process. Siding is spent largely alone and with much introspection. The first siding to a square can be a loud business, with much unhappiness on the part of the child. However, due to the gradual growth that marks the transition from a triangle to a square, the first siding is not as laborious as subsequent sidings. Nevertheless, the first siding is spent in the constant company of a parent, if possible, or another adult relative if not. Most children deal with their first siding well enough, but if left unsupervised some may harm themselves, or worse, escape their homes and harm virtually anyone they chance to meet, particularly in the moments before the long side splits into two. The parent or other adult is given the difficult task of calming the hurting child during this process, while not putting themselves in harm's way. Many a parent has received wounds during a child's first siding.

By the second siding, the child is more mature, having gained experience and confidence with the first siding. Siding begins to become a venerated event, even as it becomes more painful, due to the intense, two week period of growth, split, and regularization. Beginning with the pentagonal siding, a parent or other adult mentor begins teaching the introspection that is integral to the Flatlander siding process. That is, we have learned not to spend the two week siding period in unhappy pain. Instead, the siding process is full of quiet meditation and focused introspection, with guidance from a mentor. The mentor (parent, aunt or uncle, grandparent, great-grandparent, or other elderly member of the community) will spend some time with the sider each day, to help guide their meditation to make the siding process the most useful, personal-growth filled time it can be. The mentor gently guides the sider through questioning of and reflection on past mistakes, shortcomings, and missed opportunities, channeling the physical pain into greater focus, and directing the sider toward behaviors and attitudes

that beget a more successful lifestyle in the future. In addition, study in the philosophy of society is directed, in which the sider considers the great philosophical questions of the time. They are also directed to contemplate their own place in their household, community, and the world, and conduct that facilitates the peaceful Flatland society we cherish. We have learned that the physical pain helps to focus the directed mental exercises, and that the mental energy spent tirelessly engaged in both personal and philosophical questioning alleviates the physical pain. As a result of the guided introspection, the person emerges from siding not only a new physical person (that is, having added a new side to their being) but also a changed person from within.

The sense of peace and direction resulting from siding lasts for some time, weeks if not months, in which the person lives his or her new life with vigor and purpose. The questioning and reflection in which they engaged becomes tested in new attitudes and behaviors. Over time, and as they and their friends, school, and/or employment situations change, this sense of peace, understanding, and purpose coupled with new attitudes and behaviors abates, until the next siding. Of course, some of us (I daresay all of us, to some extent) seek to continue the introspective process between sidings, often with some success. Unfortunately, the busyness of the world and weight of our responsibilities keep us from being truly successful with this process. For this reason, although siding is painful, all Flatlanders rejoice in the prospect of siding, and furthermore, support those who are siding to the fullest of their abilities. Those who are siding have their jobs done for them, have their families cared for, and have all responsibilities lifted from them for the siding period. And since a Flatlander can feel the pressure on their front angle as their back side begins the lengthening process, one can anticipate siding coming on, and so there is generally sufficient time to make these arrangements in advance.

School is obviously suspended for school-aged or University siders, and no attempt at doing school work during siding is made. Siding is too important a time for it to be hindered by schoolwork. The sider is always given ample opportunity to make up missed work, with substantial cooperation from teachers and other students. And, the new person, freshly emerged from siding, has no difficulty getting back up to speed, their mind as fresh and bright as ever. The new person, returning to family, to school, to work, or to the community, is assailed with hearty congratulations and well wishes from all whom he or she meets. Siding and the physical and spiritual processes surrounding it is a *cause celebri* for all Flatlanders.

9. Concerning Irregular Figures

Throughout the previous discourse I have been assuming the normal physical development process of Flatlanders, that is, after approximately two years of age or so, Flatlanders are generally closed figures, ranging from three sides for a young child, to up to 30 sides for a very wise and aged person; and that after the square stage Flatlanders are approximately regular, except during the two week siding process. As with any natural system, there are exceptions to the general rules, and this is no different with the people of Flatland. For example, occasionally a baby will develop that has such a large angle that the mother is injured or even killed in childbirth. Other developmental abnormalities can occur as well.

As alluded to previously, not all Flatland children grow and develop at the same rate. In fact, on rare occasions, children will seem to be unable to form a new angle at all. In this case, without intervention, the child will remain a triangle their entire life. In times past, doctors and well-meaning

parents would break a side and forcibly form a new angle in the poor child. This was done with much pain on the part of the child, and not much long-term gain, since the usual growth process still did not commence from that point forward (that is, the forcible formation of the new angle was not followed by the natural normalization of the body, followed by the development of a new angle some time thereafter). Occasionally the forcible breaking of a side did seem to jump start the growth process, although in recent years experts have agreed that in all likelihood the child was a late developer and was about to split a side on his or her own anyway, and so the forcible breaking did not have any real affect except perhaps to actually hinder the growth process by adding a broken side into the process.

In any event, it is generally agreed in this more enlightened time to accept that not everyone falls within the normal range of development, and that, in fact, some fall well outside (or, short of) normal expectations. Some perpetual triangles and 4-gons (four-sided figures - I use the more general term 4-gon since many such deformed persons are not regular squares) will attend school and function perfectly well in society, although they are typically subject to cruel taunting in the schools and latent discrimination in the workforce. Because it is law that all persons with less than 5 sides must be accompanied by a guardian and do the peace cry in public, people with such a deformity must prove themselves capable of functioning in society in court to get a special variance. However, the burden is on them to avoid harming others as they make their way in public, and harsh punishments have been delivered to such sharp-sided adults who accidentally harm others by their movements. For this reason, such persons generally avoid crowds and cities altogether, and prefer a quiet lifestyle in the country where their neighbors know and accept them, and the risk of accidental injury is much less.

However, it is a Law of Nature that every person in Flatland is a convex figure, that is to say that there are no inwardly pointed angles on our persons. It has been documented throughout history that there have been a handful of cases in which a concave angle emerged, such as you would find on a pet or wild animal. In such situations, both when immediate medical attention was sought and when the parents kept the child safe from public view, the poor child withered and died within weeks. There have been no known documented cases of an adult spontaneously developing a concave angle. However, there are other irregular deformities that have been documented over the years, such as a person with a sharp isosceles angle at the eye, but three sides of a square behind (Figure 12), for example. No one knows why fate has singled out a few persons for such deformities, and times have changed how they are viewed by society. In ancient times, such persons were destroyed; in later times, incarcerated. In more modern times, if such a person is of sound mental capacity, then they may seek to live out their days quietly in the country (like perpetual triangles and 4-gons, it would not be safe to have such a person in a crowded city). Although I should like to say that they live perfectly happy lives, I must admit that theirs are lives of seclusion and loneliness, since few normally-developed Flatlanders would seek such a person out for a friend or potential mate. However, I believe that most people are polite to, if not fully accepting of, such persons.

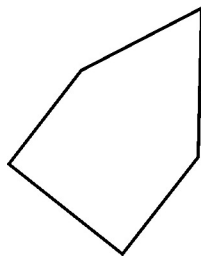


Figure 12. An Example of an Irregular Figure

In addition to physical developmental problems, some Flatlanders do seem to possess mental developmental problems that make attending or finishing school and later functioning in society an issue. These problems do occur in higher numbers among the physically deformed than among the more regularly formed, which has led to the on-going public fear and outright discrimination that some of our irregular persons must endure. The fate of those persons with mental deficiencies depends largely on the support given to the person by family and community. Some reside at home, to be cared for perpetually by a parent (or eventually sibling or cousin or niece/nephew). At times the burden is too great on the family, due either to the nature of the family or the nature of the mental deformities. In these cases, the person is turned over to the community, which is obliged to either find a suitable living situation or destroy the person. Although this sounds barbaric, I must point out that it can be extremely difficult to find suitable housing for such persons, because without the capacity to learn even the most fundamental skills, two or more such persons housed together would injure each other almost immediately. Therefore, a long-term housing situation would often result in death anyway, through a prolonged and more painful process. However, as time has progressed, more and more communities have made greater effort to deal compassionately with these most unfortunate of our society, and while there is much to be done, much progress has already been made.

Similarly, communities are charged with the punishment and incarceration of those with criminal backgrounds, ranging from accidental injury by feeling to outright, premeditated murder. I have already mentioned my great-grandmother, who was involved in an accidental injury by feeling and who was forced to live the remainder of her life confined largely to her own home. Many communities will opt to destroy a person who exhibits a criminal intent of harming, since the perpetual incarceration of such a person in isolation is difficult and expensive, and housing two or more such persons together

would likely result in an unduly premature death anyway. This is not unlike your death penalty, I am told, and so you can perhaps understand the reasoning behind our ways. The law passed to us by our ancestors is that the Safety of the State and the Interests of the Greater Number outweigh the Rights of an Irregular; in this case, Irregular refers to both criminally irregular as well as physically or mentally irregular.

10. Concerning our Elders

As we age, we become more frail, our eyesight begins to give out, and our hearing isn't what it used to be. As I understand, this is not unlike the situation with you, my Spaceland readers. However, our elderly hold an important position in our society, and are revered for their wisdom, experience, and knowledge. I have already mentioned that older adults are often mentors during siding. This is a position that an adult gradually comes into, first serving as a mentor for his or her own children, then perhaps younger nieces or nephews, grandchildren, and eventually, great-grandchildren, neighbors, family friends, and other community members. There is an advantage to knowing the person quite well before serving as a mentor during siding, so that the mentor can use their knowledge of the sider to help direct the introspection process. Eventually, however, with enough experience mentoring familiar persons, an experienced mentor can quite successfully guide anyone during siding. There is a learned pattern to the mental and spiritual awareness that comes with many siding experiences.

Because the older and experienced persons among us are a necessity at sidings, they are highly regarded in the society of Flatland. The fact that they cannot practice recognition by sight anymore

(regardless of whether they could with any accuracy when they were younger), or even that they have trouble with recognition by hearing, is irrelevant to their wisdom and guidance in the siding process.

Unfortunately, as we age we also become much more fragile and prone to injury, and so we naturally begin to limit our outings and thus exposure to possible injury. This gives the elderly more time alone, and more opportunity for their own introspection. Their experiences mentoring others, and pondering with their mentee the philosophical questions of our society and the sider's place within it, give the elderly a unique perspective regarding our community. Together with the time they have as a consequence of their hampered physical state, our elders have ample opportunity to ponder the meta-level meaning of the experiences they are encountering in the individual siding processes they mentor. Hence, their own siding naturally increases in frequency, as they become intellectually engaged in questions such as the purpose of life, the approach of death, the problems of their community and our society as a whole, and their insights and actions toward solutions. And, then, their own more frequent siding gives them even more opportunity for such introspection. Thus, in this great cycle at the end of life, our elders truly become the wisest among us, regardless of University or job experience, and are naturally positioned at the forefront of the cause to better society and to increase the quality of life for all Flatlanders. In fact, many of the reforms of our history that I have already told you about have been started by the old and wise, either through their whisperings to younger politicians or through their own positions on the High Council or other civic governance structures.

For all these reasons, the elderly are held in the highest esteem among all of society. Community leaders tend to be chosen from the elder ranks of our society, and even when statesmen and other leaders are younger, they frequently have a number of elderly persons who serve as advisors (and,

ongoing mentors, even between sidings). It is considered a great honor to have your grandparents, or even a great-grandparent, request to live in your household, as you will no doubt benefit from their presence. It is also a great loss, to family, community, and society, when such a person gives up their soul to the great unknown.

I have now told you much about my world, but little about me and how I have come to write this book for you, my Spaceland readers. I shall begin that part of my story now, by first telling you how I came to know of other worlds.

Part II: Other Worlds

11. How I Had a Vision of Lineland

It was the second to last day of the 1999th year of our era, and the first day of a long vacation. After having amused myself until a late hour with my favorite recreation, Geometry, I had retired to sleep with an unsolved problem on my mind. In the night I had a dream.

I saw before me a vast multitude of small straight lines, all moving to and fro in one and the same straight line, and, as nearly as I could judge, with the same velocity. I would have assumed these lines to be blades of grass or some other vegetation, except for the constant movement along the line.

Assuming them to be some sort of beings, I thought it oddly remarkable that such a large number of young triangles or squares would all move in such a peculiar manner, perpendicular to my view and back and forth in the same back and forth direction, without the slightest rotational movement at all.

A noise of confused, multitudinous chirping or twittering issued from them at intervals as long as they were moving; but, sometimes they ceased from motion, and then all was silence for a short time.

I could not imagine what game these youngsters might be playing, and all without the slightest hint of rotational movement. Approaching one of the largest, who I assumed to be a youngster (because I assumed them all to be triangles or squares, as there was no hint of corners on any of them), I inquired, in a somewhat authoritative manner, as to the meaning of this bizarre behavior. The youngster did not turn toward me, nor act as if he or she heard me in any way whatsoever. A second and third inquiry,

each rising in tone, were equally ineffectual. Losing patience at what appeared to be intolerable rudeness, I crossed their line and brought my mouth into a position full in front of the youngster I had targeted. To my surprise, I found that these were no youngsters at all, in fact, no closed figures of any kind, but lines, lines as thin and imperceptible from end on as any blade of grass I had ever seen. But, these beings had movement and voice. Utterly perplexed, I tried once more, with my mouth full in front of the being, and loudly repeated my question: “What is the meaning of this behavior, this strange and confused chirping, and this monotonous motion to and fro, in one and the same straight line?” “Who are you, that you should address the monarch of this land with such rudeness? And, from whence do you intrude into my realm?” Receiving this abrupt reply, I immediately begged pardon if I had in any way startled his Royal Highness (although I was not at this time sure if this was a male or female monarch, or even a monarch at all, as the voice was not like any other I had heard before), and asked forgiveness if I had appeared rude. Describing myself as a stranger I besought the creature (who at this point I assumed to be male, since he had not corrected my presumption of gender) to give me an introduction to his lands and dominion. I had the greatest difficulty in obtaining information that interested me, however, because the supposed monarch could not refrain from constantly assuming that whatever was familiar to him must also be known to me, and that I was feigning ignorance of the basics of his world. By persevering, however, I was able to elicit the following facts:

First, it seemed that this poor monarch, as he called himself, was convinced that the straight line which he called his kingdom and in which he passed his existence, was in fact the whole world, and indeed all of Space. Apparently not being able to move or see outside the straight line, he had no conception of anything outside of it. Though he had heard my voice when I first addressed him (when I was outside the straight line), the sounds of my voice had come to him outside of anything he had before

experienced, “hearing a noise as if it came from my own insides,” as he expressed it. Until the moment in which I placed my mouth in his world, he had neither seen me nor heard anything except confused sounds beating against - what I called his side, but he called his insides or stomach. He still had not the slightest conception of the world in which I had come, that of Space, or as I now know it, Flatland. His world was the straight line, which I call Lineland, and outside his world, all was blank to him; nay, not even blank, for blank implies Space; rather, there was nothing.

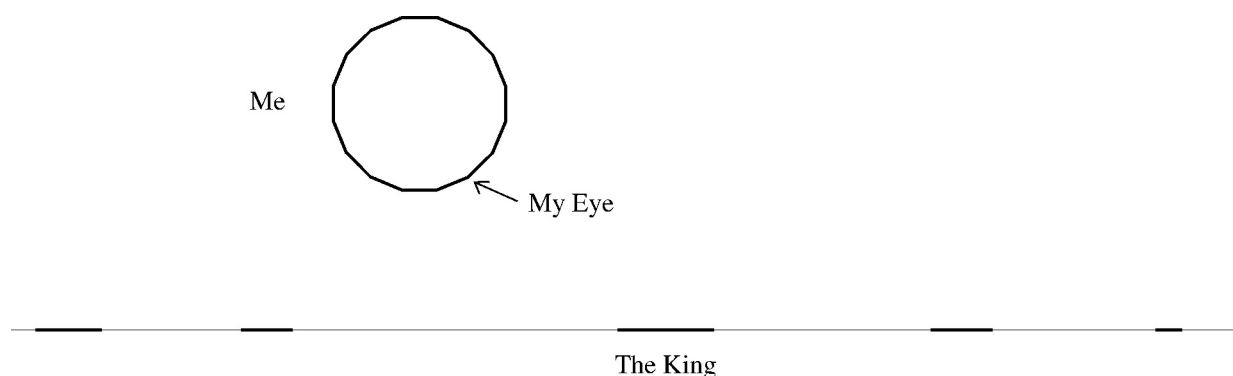


Figure 13. Lineland

His subjects, all of whom were small lines, were men and women (the babies were points, or nearly so), and were all confined as to motion and eyesight to that single straight line, which was their world. I scarcely need to point out that the whole of their horizon, yes even that of the king, was limited to a point. No one of their land could ever see anything but a point; man, woman, child, baby, thing - each appeared as a point to the eye of a Linelander. Only by the sound of the voice could age or sex be determined (I presumed this was similar to recognition by hearing in Flatland, although the characteristics of Linelanders' voices were so foreign to me that I was not able to make such distinctions). Moreover, as each individual occupied the whole of the narrow path along the straight line constituting Lineland, and no one could move right or left to allow someone to pass by, it followed

that no Linelander could ever pass another. Once neighbors, always neighbors - until death did they part.

Such a life, with all vision limited to a point and all motion limited to a straight line, seemed to me to be unimaginably dreary. I was surprised to note the seeming vivacity and cheerfulness of the king as he described his world. Wondering whether it was possible, amid circumstances so unfavorable to personal relations, to have such familial relations as marriage and children, I decided to broach the subject by inquiring as to the health and well-being of his family. "My wife and children", he replied, "are well and happy, thank you."

I was staggered at this answer, for in the immediate proximity of his Royal Highness, he had mentioned, were his servants, both male. I ventured to ask, "Pardon me, but I cannot imagine how your Royal Highness can at any time either see or approach her Royal Highness, when there are intervening persons whom you can neither see through nor pass by. Is it possible, then, that in your world proximity is not necessary for marriage and children?"

"How can you ask such an absurd question?" replied the monarch, which was his reply to many of my quite logical questions. "If it were indeed as you suggest, the world would soon be depopulated! No, no, neighborhood is needless for the union of hearts, and the birth of children is far too important a matter for Nature to have left it to depend upon such an accident as proximity. You cannot be ignorant of this. Yet, since you are pleased to feign ignorance, I will instruct you as if you were a mere baby. Once each month, a Law of Nature compels us to move to and fro with a more than usual rhythmic violence, which continues for a count of sixty-four. At the last pulsation, all the inhabitants of the

world pause, and each individual sends forth his or her richest, fullest strain. So exquisite is the adaptation of bass to treble, of tenor to alto, that oftentimes two loved ones, though twenty thousand leagues apart, recognize at once the responsive note of their destined lover. Penetrating the paltry obstacle of distance, Love unites the two. It is in this moment that romances are begun, marriages made, and children conceived.” At this he paused, assuming all was plainly understood by me.

Unsure what to ask next, I decided to ask if children were conceived by the happy couple each month, and how long it took for the child to be incubated before being born.

“Oh, monstrosity (as he had taken to calling me), you carry your feigned simplicity too far. Of course, not every hopeful youth among us finds his or her mate at the first wooing of the universal Marriage Chorus. In fact, it takes years for youngsters to develop the clarity and projection of voice to meaningfully participate. And years more for them to recognize in the Chorus the voice of the partner intended for them by Providence, and to fly into a reciprocal and harmonious embrace. With most of us the courtship is of long duration, repeated over and over. With each monthly Chorus, the lovers are brought into a closer and closer harmony. Each trial of voice, each fresh discovery of discord, almost imperceptibly induces the less perfect to modify his or her vocal utterance so as to approximate the more perfect. And after many trials and many approximations, the result is finally achieved. There comes a day at last, when, while the Marriage Chorus is sounding forth, that the two far-off lovers suddenly find themselves in exact harmony, and, before they fully realize it, the wedded couple is swept into a vocal embrace, and Nature rejoices over the marriage and immediate birth of the new child, a result of the perfect harmony just established.”

12. How I Vainly Tried to Explain the Nature of Flatland to the King

Thinking that it was time to bring the monarch down from his raptures to the level of common sense, I was determined to open up for him some glimpses of the true Universe, that is, the nature of things in Flatland. So, I began thus: “How does your Royal Highness distinguish the shapes and positions of his subjects? I saw, before I entered your kingdom, that some of your people are lines and others points, some longer –.” “You speak of an impossibility”, interrupted the king. “You must have heard the difference, or seen some sort of vision. It is impossible to detect the difference between a line and a point by the sense of sight, as everyone knows. It can be detected by sense of hearing, as can the size of the line. Behold, I am a line, the largest in the world, over six inches of Space –.” “Of length” I ventured. “Fool” he said, “Space is length, length is size. Interrupt me again and I will be done with you.”

I apologized, and he continued, scornfully, “Since you are impervious to argument, you shall hear with your ears how by means of my two voices (for he had a mouth on both sides of his being, as did all the lines of Lineland), I reveal my shape to my wife who at this moment is six thousand miles seventy yards two feet eight inches away, to the north. Listen as I call to her.”

He chirruped, and complacently continued, “My wife, at this moment receiving the sound of one of my voices, closely followed by the sound of the other, and perceiving that the latter reaches her after an interval in which sound can traverse 6.457 inches, can infer that one of my mouths is 6.457 inches further away from her than the other, and accordingly know my shape to be 6.457 inches. But you will of course understand that my wife does not make this calculation every time we converse. She made

it, once and for all, before we were married. But, she *could* make it at any time. In the same way, she or I can estimate the shape of any of our subjects by the sense of sound.”

Thinking of our own problems with recognition by hearing in Flatland, I asked: “But what if a person feigns the voice of another, or so disguises the southern voice that it cannot be determined to be an echo of the northern? May not such deceptions cause great inconvenience? And, have you no means of checking frauds of this kind by commanding your neighboring subjects to feel one another?”

Although this was likely another ignorant question from the king's perspective, I wanted to better understand the hearing and feeling limitations in Lineland.

“What!” he cried in horror, “Explain your meaning of 'feel'.” “Feel, touch, come into contact” I replied. “If you mean by feeling,” said the king, “approaching so close as to leave no space between two individuals, know, stranger, that this offense is punishable by death in my dominions. And the reason is obvious. Imagine if both persons on the two sides of one would close the gap between themselves, at once. The one in the middle would immediately perish. Since it would be impossible to tell at any instant if both persons on either side of any one person were doing such a thing as closing the gap between themselves and the one, the law ordains universally that no one shall be approached by another so closely as to destroy the interval between them. And, indeed, what possible purpose would be served by this illegal and unnatural excess of approximation which you call *feeling*, when all the ends of so brutal and coarse a process are attainable more easily and exactly by the sense of hearing. And, as to your suggested danger of deception, it is non-existent; the unique characteristics of the voice, being the essence of one's being, cannot be changed at will. But come, suppose I ordered all my subjects to *feel*, as you suggest, and report back to me their statistics. How much time and energy

would be wasted in this clumsy and unnecessary method! Whereas now, in one moment of audition, I can take the census and statistics of every living being in my land. Hark!”

So saying this he paused and listened, as if in ecstasy, to sounds which seemed to me no better than tiny chirpings from an innumerable multitude of grasshoppers.

“Truly,” I replied, “your sense of hearing is quite developed and serves you well. It compensates for many of your deficiencies. But permit me to point out that your life must be deplorably dull. To see nothing but a point from either of your eyes, ever! Not even to contemplate a straight line, nay, to not even know what a straight line is! To see, but not see anything of linear projection which are available to us in Flatland. Better surely to have no sense of sight at all, than to see so little. I grant that you have your admirable sense of hearing, for the concert of all of your people which gives you such intense pleasure is to me no better than a multitudinous twittering or chirping. But at least I can discern, by sense of sight, a line from a point. And, let me prove it. Just before I came into your kingdom, I saw you, at just over six inches, dancing from left to right, followed to the left by a being approximately 3 and a half inches, and again to the left one almost five inches. To the right of you was a person of four and three quarters inches, and then one of a mere two inches. Is this not correct?”

“It is correct,” said the king, “but I deny that you saw these things. Your hearing must be more developed than you profess; you heard these things, and then somehow had a vision of them. And, what are these terms left and right? I suppose you mean north and south?”

“Well, yes, to you, north and south, but there are other directions besides north and south. There are the directions east and west. And if one is moving forward - northward as you would say - then you can also move right and left, or, east and west.”

“Aha!” said the king, “you contradict yourself as only one as confused as a lunatic may do. You subsequently called these new motions left and right, north and south, and then subsequently you called them these new directions east and west. Which direction do you mean by left and right? North/south or east/west?”

“Well, both,” I replied, “but I am no lunatic. When I first approached you I was heading eastward, and so left was your north and right was your south. But now I am facing southward, and so left is east and right is west. Left and right are to the *side* of your current direction.”

“Exhibit to me, if you please, this left and right, or *side* motion,” ordered the king.

“That I cannot do,” I replied, “unless you could step out of your line altogether.”

“Out of my line? Do you mean out of the world? Out of Space? You are mad!”

“Well, yes, out of *your* world. Out of *your* Space. For your Space is not the true Space. True Space is a plane; but your Space is only a line.”

“If you cannot indicate to me this motion from left to right by moving in it, then either describe it to me in words, or be gone with you!” demanded the king.

“If you cannot tell your right side from your left side, I fear that no words of mine can make my meaning clear to you. Surely you are not so ignorant of such a distinction. Alas! How can I make it clear? When you move straight on, northward, does it not occur to you that you could move in some other way, turning your eye around so as to look in the direction *between* your north and south eye, the direction towards which your side is now fronting? In other words, instead of always moving in the direction of one of your extremities, do you never feel a desire to move in the direction, so to speak, of your side?”

“Never” said the king. “And, what do you mean? How can a person's inside ‘front’ in any direction? Or, how could one move in the direction of one's inside?”

The king could obviously not fathom anything outside his line. I ventured, “Well then, as words cannot explain the matter, I will try deeds. I will move gradually out of your world in the direction which I desire to indicate to you.”

I began to move my body out of Lineland. As long as any part of my sixteen sides remained in his dominion and in his view, he continued to exclaim, “I see you, I still see you, you are not moving except farther away to the north.” But when I at last moved myself out of his line, he cried in his shrillest voice, “She is vanished, she is dead!” “I am not dead” I replied, “I am simply out of the straight line which you call Space, and into the true Space, where I can see things as they are. And at

this moment I can see your line, or side - inside as you call it - and I can see the men and women on the north and south of you, whom I will now enumerate, describing their order, their size, and the interval between each.”

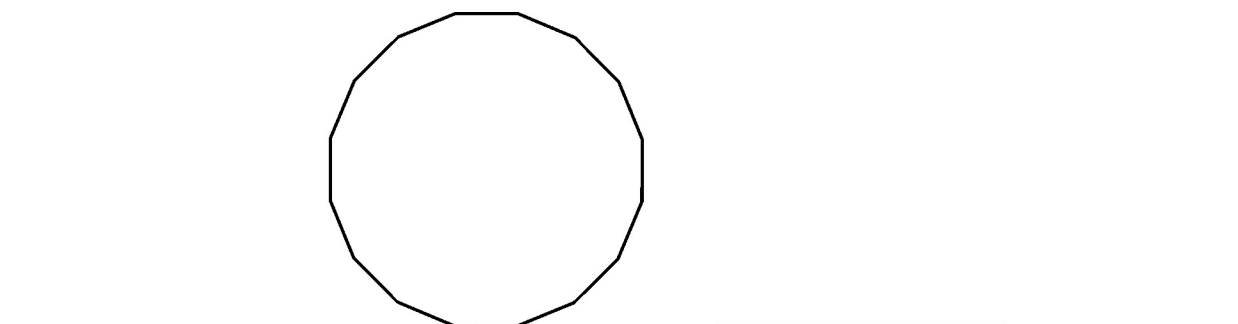


Figure 14. How I moved out of Lineland

When I had done this at great length, I cried triumphantly, “Does that at last convince you?” And with that, I once more entered Lineland, taking up the same position with my mouth in line with the king's.

“Oh madman, what magic do you practice?” the startled king replied, after I once more appeared in front of him. “If you were a person of sound mind you would listen to reason. You ask me to believe that there is another line besides that which my senses indicate, and another motion besides that of which I am daily conscious. I, in return, ask you to describe in words or indicate by motion that other line of which you speak. Instead, your spout off gibberish concerning 'left' and 'right', and instead of moving in such a new direction, you exercise some magic art of vanishing and returning to sight; and, instead of any lucid descriptions of this new world, you simply tell me the numbers and sizes of some forty of my subjects, facts known to any child in my vicinity. Can anything be more audacious or irrational? Acknowledge your folly, or depart from my dominions at once!”

Furious at his perverse ignorance, I retorted in no mild manner, “Oh wretched king! You think yourself the perfection of existence, but you are in reality the most imperfect and imbecilic of creatures! You profess to see, whereas you can see nothing but a point! You pride yourself on being able to infer the existence of a straight line; but, I can *see* straight lines, and infer the existence of angles, triangles, squares, pentagons, hexagons, septagons, and even circles! Why waste more words on you? Suffice it that I am the completeness of your incomplete self. You are a line, but I am a line of many lines, called in my country a hexadecagon. Even I, infinitely superior to you that I am, am of little account among the great minds of Flatland, whence I have come to visit you, in the hope of bringing you enlightenment.”

Hearing these words, the king advanced toward me with a menacing cry and velocity as if to pierce me. Also in that moment, there arose from the myriad of his subjects a multitudinous war-cry, of increasing vehemence, which began to rival the roar of an army of a hundred thousand. Spell-bound and motionless, I could neither speak nor move to avert the impending destruction of my being. Still the noise grew louder, and the king came closer, until at last I awoke to find the breakfast bell recalling me to the reality of Flatland.

13. The New Millennium

From dreams I proceed to facts.

It was the last day of the 1999th year of our era. Night had already fallen, and I was sitting in the company of my spouse, musing on the events of the past and prospects of the coming year, the coming century, and the coming millennium. When I say “sitting,” of course I do not mean any change of attitude such as you in Spaceland signify by that word, for since we have no feet, we can no more “sit” than “stand” (in your sense of the word). Nevertheless, we perfectly well recognize the different mental states implied by the words “sitting,” “standing,” and “lying,” which are also reflected in certain subtle changes in rotational motion. But, I digress; on this, and a thousand other kindred subjects, time forbids me to dwell.

My three children had retired to their rooms; my husband alone remained with me to see the old millennium out and the new one in.

I was rapt in thought, pondering in my mind some comments made by my youngest child, a most promising young pentagon of unusual brilliance and almost perfect angularity (as were all three of my fine children). Earlier in the evening, my husband and I and various aunts and uncles, yes even the children's beloved and esteemed grandfather had been giving the children, mine and various nieces and nephews, some advanced practical lessons in sight recognition, turning ourselves about on our centers, now rapidly, now more slowly, and questioning them as to our positions. Her answers had been so satisfactory, at times surpassing those of her older siblings and cousins, that I had been moved to take her aside and give her a few insights into arithmetic, as applied to geometry.

Taking nine squares, each an inch in every way, I put them together to make a large square, with a side of three inches, and I had hence proved to my daughter that (though it was impossible for us to *see* the

inside of a square) we might ascertain the number of square inches in a square by simply squaring the number of inches in a side. “And thus,” I told her, “we know that 3^2 , or 9, represents the number of square inches in a square whose side is 3 inches long.”

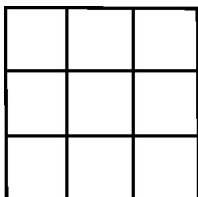


Figure 15. My demonstration of the area of a square

My little pentagon meditated on this for awhile, and then said to me, “I understand that squaring the value of the side of the square gives the size of the square - oh, is that why we call it *squaring*,” the delightful child beamed, “but tell me, mother, how would you find the size of my shape?” Having learned the formula for the area of a square, I was sure she was ready for the formula for the area of a triangle. I pulled two identical right triangles from my cabinet, each one an inch on each of the smaller two sides, showed them to her by rotating each around its center, and put them together for her. “What do these two triangles make?” I asked her. She briefly walked around the composite shape. “A square” she easily replied. “So, what is the area of each triangle making up the square?” I asked her. After a moments thought (and confirming that the square before her was one inch on each side), she replied, “It must be $1/2$ inch each, since two of them make a square of one inch.” “ $1/2$ *square* inch,” I corrected her, “and two make a square of size one *square* inch. The term square inch refers to the fact the measurement includes both length and breadth, and not length only,” I explained.

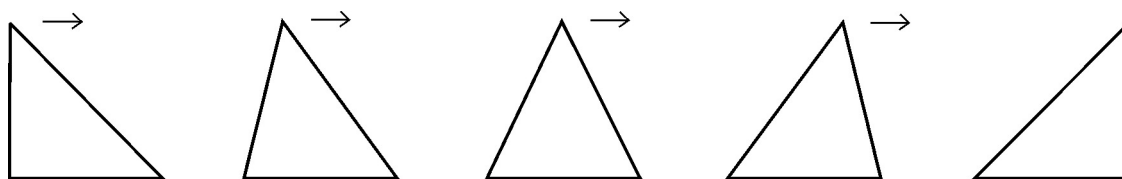


Figure 16. My demonstration of triangles and area to my daughter

“So, how do we find the square inches of my shape?” asked the bright little girl. “Now, this gets a little harder,” I warned her, “but stay with me.” Taking one of the triangles, I turned it so that one of the sides adjacent to the right angle was perpendicular to our view, and indicated the vertex closest to us. “Imagine that I could move this vertex back and forth, parallel to the far side, or base, of the triangle, to make a new triangle. Would the new triangle be bigger, smaller, or the same overall size as the old one?” I asked. After a long pause, she replied, “I’m not sure.” “Yes, this is a difficult concept to grasp,” I encouraged, “but what would you *guess*? Bigger, smaller, or the same? Imagine, in particular, if you moved the vertex from right in front of one vertex on the base all the way over to right in front of the other vertex on the base. Then what triangle would you have?” “Another right triangle,” she replied, “which I guess would be the same size as the first. So, if I had to guess, I would guess that all the triangles in between would also be the same size.” “And you would be right!” I exclaimed, thrilled at the insights of my bright little one. “Now, suppose we move the vertex to exactly halfway between the back two vertices - what kind of triangle is that?” I asked her, as I moved back to my cabinet. “An Isosceles triangle” she quickly replied, having learned this in school in her first year. Returning from my cabinet with five Isosceles triangles, I showed them to her and proceeded to assemble them into a larger figure. “A pentagon!” she exclaimed, and quickly did the

math. “So if each triangle is $1/2$ square inch, and there are five of them in a pentagon, that makes the pentagon $2\ 1/2$ square inches!”

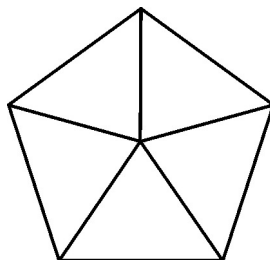


Figure 17. How the area of a pentagon can be determined

We both basked in her lucid triumph for several moments, until the sounds of merriment from the other room turned our attention away from our own happy time. Not wanting to spoil the pleasure by forcing more mathematics on her than she was ready for, I suggested that it was time to return to the family and our millennium celebration.

At the door, she turned to me and commented, quite innocently, “Thanks, mother, for explaining that to me. I will think about sizes of other figures as well. It is curious that a square's size can be obtained by raising the length of a side to the second power - *squaring* it”, she giggled. “Sometime I would like for you to explain to me the geometric meaning of raising a length to the third power.” “Well, I can tell you now, that raising a number to a third power has no geometric meaning whatsoever.” I replied. “No meaning whatsoever?” she replied, “Are you sure?” “Of course I'm sure,” I answered rather indignantly, being, as you know, a mathematician of relative stature and being questioned as to my veracity by a pentagon. “We have only two dimensions.” And, I began to show her how a point, by moving through a length of 3 inches becomes a line of 3 inches, which can be represented abstractly by

the number 3, and a line of three inches, moving perpendicular to itself through another length of three inches, makes a square of 3 inches on every side, which can be represented by the number 3^2 , or 9. “But,” I remarked, “there is no other way to move, so that the square becomes another shape, of size 3^3 .”

Upon this, my daughter commented, somewhat assuredly (in spite of my reasonable explanation), “Well, then, if a point, by moving through a length of 3 inches becomes a line of 3 inches, and a line of three inches, moving perpendicular to itself through another length of three inches, makes a square of 3^2 square inches, then there must be *some* way that a square, moving somehow - but I don't see how - three inches, would become something that is of size 3^3 inches. Even if we can't see it, shouldn't it be true?” she smiled innocently.

“Your father is calling us,” I said, a little ruffled by this insolence. “Let's not talk nonsense - we should get back to the celebration.”

So my daughter and I returned to the living room and the celebration with our many relatives, but having lost the previous euphoria of the discovery of mathematical insight. Instead, frustration was evident in both of our demeanors.

“You and Ana disappeared during the party,” my husband commented later, as we were recalling the events of the evening. “What were you two doing?” “Oh, I was showing her some properties of geometry and area of figures,” I answered. “She is a bright little girl,” I continued, not bothering to disguise my disappointment, “but, she still has the folly of childhood in her. She was trying to

convince me that 3^3 had a geometric meaning.” “I noticed that you both came back unhappy. Don't push her too hard - she may not follow in your footsteps to become a mathematician, you know,” my husband replied. “Well, she may not be a statesman like you, either,” I replied indignantly.

We continued to sit there, engaged in the retrospective of the evening, the previous year, century, and millennium, and thinking of our own contributions to the coming age; mostly, that of our bright children, who we were sure would be successful at whatever they chose to pursue. The time was at hand; the hour glass dropped its last remaining grains of sand. The new millennium was upon us.

14. The Visitor from Spaceland

Suddenly, I became conscious of a presence in the room, and a chilling breath thrilled through my being. I don't know if my husband was immediately aware of it or not, since when I arose and started around the room, he exclaimed, “What is the matter?” A cold whisper came into the room, from nowhere and everywhere at once: “Your daughter was right. 3^3 has obvious geometric meaning.”

My husband and I both heard the words. We both sprang up to find the source of the sound, but there was no one else in the room. The voice had been deep and resonating, each word had verily tickled our insides. Neither of us knew what to make of it.

Imagine our horror when we saw before us, suddenly appear, a figure! Out of nothingness the figure materialized. It appeared to be an ancient figure, for my initial observation showed that the extremities passed into dimness with no apparent indication of corners. In the sitting room, however, the relative

absence of fog (and, the season happened to be remarkably dry) made it difficult for me to trust to sight recognition. Utterly confused as to the configuration and source of this intruder, I must admit that I was quite beside myself. My husband, wanting to take control of the situation, shouted, "Who are you, and how have you come to be in our house?" "I cannot tell what you are - do you know this person?" I asked my agitated husband. "Feeling is believing," he answered, "Permit me to feel and be felt -," my husband barely had the words out before he set upon the intruder. He jumped back, abruptly, and, turning to me, he said, unbelievably, "oh, I have never felt such a thing. It is smooth but not straight, with sides retreating but not a corner do I feel. Dear," he exclaimed, turning to me, "have you ever felt such a thing?" I approached the figure cautiously, and began feeling. He was right, the figure was smooth but not straight, with sides retreating as if with numerous corners, but without a single break to be detected. "Could it be, a... a circle?" I exclaimed in awe. Mathematicians were familiar with the theory of circles, of course, although nothing but approximations had ever been built in Flatland.

"I am indeed, in a certain sense, a circle," boomed the voice, with the same tickling sensation of the previous whisper. "A more perfect circle than anything found in Flatland. But, to speak more accurately, I am many circles in one." Then he added, more mildly, presumably toward my husband, "Sir, I have business with your wife, of the mathematical kind and of utmost importance. I wonder if you would suffer to allow us a few minutes..." The voice trailed off. My husband, the statesman that he is, and, comprehending the possible significance of so perfect a being requesting an audience with his wife, was gracious to retire immediately with no shortage of nervous commentary along the way, "oh, of course, no problem at all, don't give it a thought..." He disappeared into the bedroom, leaving me alone with the perfect stranger.

15. How the Stranger Vainly Endeavored to Reveal to Me in Words the Mysteries of Spaceland

As soon as my husband's voice had trailed off, I began to approach the stranger with the intention of examining him more closely, and bidding him to be seated. However, his appearance struck me dumb and motionless with astonishment. Without the slightest symptom of angularity he nevertheless varied at every point with graduations so fine as to be completely outside the scope of my considerable experience. The thought flashed across me that I might have before me some burglar or ingenious scoundrel, who, by feigning the unusual voice of the proposed circle had somehow obtained admission into my house and now had me alone. This thought propelled me out of my astonishment and I proceeded to cautiously circle the ... circle.

He remained motionless while I walked about him, beginning from his eye and returning to it again. I couldn't resist a quick feeling as I walked about him. Indeed, there was no trace of roughness or inequality about him. Circular he was throughout, a perfectly satisfactory circle; there could be no doubt of it. Then followed a dialogue, which I will endeavor to set down as near as I can recollect it, omitting only some of my initial profuse apologies - for I was flustered and ashamed that I had not recognized the stranger as a circle without feeling. And this, a higher creature than I had ever had the privilege of meeting.

“Have you felt me enough by this time? Are you not sufficiently convinced of what I am?” asked the stranger.

“Most illustrious stranger, excuse my awkwardness, dear sir, which arises not from ignorance of polite society and of what you are, but from surprise and nervousness. For I have never felt such a perfect circle - and, you must admit, this is an unusual and unexpected visit,” I replied. Then, searching for some title worthy of his age and stature, I continued. “But, my lord, before you continue, please satisfy my curiosity and tell me whence you came?” For, now, convinced that this was no scoundrel, and putting aside for a moment my amazement at meeting a circle, I was still bothered by how he had suddenly appeared, at midnight, in my sitting room.

“From Space, from Space, madam; whence else?” he replied.

Hesitatingly, I continued, “Pardon me, my lord, but is not your lordship already in Space, your lordship and myself, your humble servant, even at this moment?”

“Pooh! What do *you* know of space,” he replied dismissively. “Define Space.”

“Space, my lord, is height and width indefinitely prolonged,” I replied, confident in this definition.

The stranger replied, “Exactly - see, you do not even know what Space is. You think of it as two dimensions only. But, I have come to announce to you a third - height, width, *and* breadth.”

“Your lordship is pleased to be merry,” I nervously chuckled, “We also speak of length and height, or, breadth and thickness, thus denoting two dimensions by four names.”

“But, I mean not only three names, but three *dimensions*.” the stranger replied.

I hesitated. Three dimensions? Could it be? Or was this, after all, some mad trickster? “Would your lordship indicate or explain to me in what direction is this third dimension, unknown to me?” I casually inquired, although my incredulity was significant.

“I came from it. It is up and down, above and below.” he just as casually replied.

“My lord seemingly means northward and southward .. ?” I replied, with a slight inflection at the end indicating that this was a possible question. For I did not want to correct a circle.

“I mean nothing of the kind,” he boomed. “I mean a direction in which you cannot look, because you have no eye on your side.”

“Pardon me, my lord,” I replied, “but a moment's inspection will reveal that I have a perfectly functioning eye at the junction of my two front sides.”

“Yes, yes,” he replied impatiently, “but in order for you to see into Space, you ought to have an eye, not on your perimeter, which you certainly do, but on your side - what you would probably call your inside, but we in Spaceland call it your side.”

“An eye on my inside!” I exclaimed. “An eye in my stomach? Your lordship jests!”

“I am in no jesting humor.” The voice was booming and serious as ever. “I tell you that I come from Space, or, since you will not understand the true definition of Space, from the Land of Three Dimensions, whence I but recently looked down upon your plane (which you call Space). From that position of advantage I discerned all that you speak of as solid (which to my meaning is 'enclosed on all four sides') , your houses, your churches, your closets and safes, yes even your insides and stomachs, all lying open and exposed to my view.”

I hesitated. “Such assertions are easily made, my lord”, said I.

“But not easily proved, you mean. But I mean to prove mine.” He went on. ”When I descended here, I saw your party, with many guests, family I suppose. I saw your three children, two daughters and a son as I was able to infer from the conversations I overheard. I heard you talk with your youngest daughter privately. I saw your guests leave and your children go to their rooms, leaving you and your husband alone. Then I came here, and, how do you suppose I came?”

“Through the roof?” I guessed, knowing this was not a possibility since our roof was very recently repaired.

“Not so,” he replied. “Your roof is in good condition, and has not even an aperture that a straight line could penetrate. I tell you I came from Space. Are you not convinced by what I have told you of your children and household?”

“Your lordship must be aware that such facts surrounding my house and our family party this evening could easily have been ascertained by anyone in the neighborhood possessing your lordship's ample means of obtaining information,” I replied, with feigned self-confidence.

The stranger then began a soliloquy, quietly, to himself: “What must I do? What can I say. But, lo! Her own child reasoned with her on this subject earlier this very evening. I shall use that approach.” Then, once again addressing me, he inquired, “When you see a straight line, how many dimensions do you attribute to it?”

“Your lordship would treat me as a child, ignorant of mathematics. A straight line is of one dimension only, although straight lines such as those that occur as leaves on a tree or blades of grass can reside in our roomy two-dimensional space as well.” I answered.

“And, if you approach a straight line end-on, do you see the line, or is it invisible?” he continued to ask.

“Now you are trying to trick me,” I replied. “A straight line would appear invisible if viewed directly end-on, but this is nearly impossible in practice due to our habit of rotating slightly so that we can have the advantage of perspective. Even the width of our eye, which is not infinitesimal, hampers this practicality. This is indeed a necessity, or else we might collide with grass or leaves without noticing. So,” I continued, “*theoretically*, the line would be invisible, but in practice we would see but a small point.”

“But you acknowledge that, theoretically, a line occupies one dimension, and if viewed end-on has no breadth, only length. Consider yourself, of two dimensions, and you have both breadth and length. Now, suppose there were three dimensions; then occupants of three dimensions would have three measures, what are called breadth, length, and height. Your own daughter tried to convince you of such a dimension not long ago - no?”

“Well, my daughter is a pentagon and has an active imagination,” I remarked, unmoved. “Your assertion is easily put to the test. You say there is this third dimension, which you call height. Dimension implies both direction and measurement. Do but measure my 'height', or indicate to me the direction in which my 'height' extends, and I will become your convert.”

Again, the stranger began talking to himself. “How can I show her her height, when she is but two-dimensional? How can I show her the direction of height, when she cannot herself even sense that there is such a direction? How can I convince her? Surely a plain statement of the facts, with some demonstration, will suffice.” And to me, he said, “Now madam, listen to me.”

“You are living on a plane,” he continued. “What is called Flatland is the vast level surface, the top of which you and your countrymen move about in, without rising above it or falling below it. I am not a plane figure, but a solid. You call me a circle, but in reality I am not a circle, but an infinite number of circles, of size varying from a point to a circle of eighteen inches in diameter, one placed on top of the other. When I cut through your plane, as I am now doing, I make in your plane a section which you, very rightly, call a circle. For even a sphere, which is my proper designation in my own country, if he manifests himself to someone of Flatland, must needs manifest himself as a circle.”

I was utterly confused and at a loss for words. The stranger continued. “Let us go back to the example of one dimension, a line. Suppose there were a world in which the inhabitants were all points on a line, and you came from outside the line into their world?”

“It is remarkable that you should use that example,” I replied, “since only last night I had a dream of such a place. Lineland had a king, a line segment, with whom I conversed. As I moved through his realm, all he could discern of me was a linear section of my hexadecagonal body, that being the only part of me in his realm at any moment. He was not convinced of my two-dimensionality.”

“Exactly!” exclaimed the stranger. “Because that linear realm had not the dimensions enough to represent the whole of you, but only one slice or section of you at a time, that was all he was able to perceive. In precisely the same way, your country of two dimensions is not spacious enough to represent me, a being of three dimensions, but can only exhibit a slice or section of me, which is what you call a circle. I would suppose that as you passed through this realm of Lineland, your cross-sectional line varied from a point, when you had just barely crossed into the realm, to a line of gradually increasing length as your eye approached the king, to a line of gradually decreasing length after your eye passed through the realm, and finally to a point again, and then disappearing altogether as you passed out of Lineland? So now, analogously, prepare to receive proof positive of the truth of my assertions. You cannot indeed see more than one of my sections, or circles, at a time, for you have no power to raise your eye out of the plane of Flatland. But, you can at least see now, as I rise up into space, my sections become smaller and smaller, dwindling even to a point, until I vanish from your view.”

There was no “rising” that I could see, but he diminished in diameter and finally vanished. I blinked once or twice and quickly rotated slightly, to make sure there was no trick of perspective. But there was no trick. From the depths of nowhere came forth a hollow voice, close to my insides, it seemed. “Am I quite gone? Are you convinced now? Well, now I will return to Flatland, and you will see my section become gradually larger and larger.”

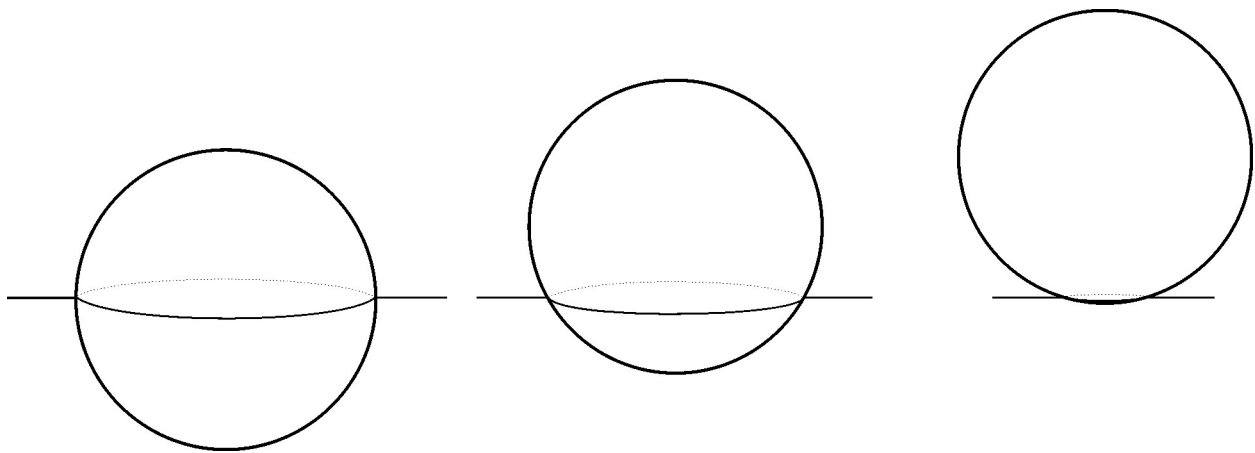


Figure 18. The sphere demonstrating his departure from Flatland

I'm sure that you, my Spaceland reader, can easily understand that my mysterious stranger was speaking the truth, and even in great simplicity. But to me, proficient as I was in Flatland mathematics, it was by no means a simple matter. The diagram above makes it clear that the sphere, ascending through the three positions shown, manifested himself to me first as a circle of full size, then as a smaller circle, and at last very small indeed, approaching a point before disappearing altogether. But to me, although I saw the facts before me, and could even grasp the theory of the analogy he placed before me, I could not discern the cause of the movement I saw. All I could comprehend was

that the circle had made himself smaller and then vanished, and that he had now reappeared and was rapidly making himself larger.

When he had regained his original size, he continued. “Now, recall your interaction with the king of Lineland. Although it is clear to us that you are two dimensional, when you stepped into his realm you were reduced to a one-dimensional section, or line. And, although you could not sufficiently describe to the king the direction in which the second dimension lay, you were able to move through his kingdom, and as you did so you varied from a point to a large line, then from a large line back to a point, until you seemingly vanished from his sight altogether. This is exactly what I have just done with you. You cannot discern the direction of the third dimension, although I can demonstrate to you what I am in three dimensions by moving through your two dimensions, in which I appear to be a point when I am just touching your world, then progressively larger circles, until I am a circle of full diameter as my eye is in your world; and, as I pass out of your world, my section or circle become smaller until I seemingly vanish as I move all the way through your plane, or world of Flatland. Is this not clear to you now?”

There was silence. The stranger heaved a deep sigh, for he perceived by my silence that I had altogether failed to comprehend him. After a long pause, he muttered to himself, as he was seemingly prone to do, “Only one resource alone remains, if I am not to resort to action. This is a mathematician to which I speak. I must try the method of geometric analogy.”

There followed another short silence, after which the stranger continued our dialogue. “Tell me, oh mathematician, if a point moves northward, and leaves a luminous trail, what name would you give to the trail?”

“A line, of course,” I replied.

“And a straight line has how many extremities?” asked the stranger.

“Two.”

“Now, imagine the northward straight line moving perpendicular to itself, east and west, so that every point in it leaves behind it the trail of a line. What name would you give to the figure thereby formed? We will suppose that it moves through a distance equal to the length of the original line - what name, I say?”

“A square”, I replied.

“And how many sides has a square? How many angles?” he asked.

“Four sides and four angles,” said I.

“Now, stretch your imagination a little,” said he. “Try to conceive of a square in Flatland, moving perpendicular to itself upward.”

“What - northward?” I asked.

“No, not northward, but upward. Out of Flatland altogether,” replied the sphere. “If it moved northward, then the southern points in the square would have to move through the positions previously occupied by the northern points. But that is not my meaning.” He continued. “I mean that every point in the square, that is to say what you would call the inside of the square, is to pass upwards through space in such a way that no point shall pass through the position previously occupied by any other point. Instead, each point inside the square, when moving in this direction upward and out of Flatland, shall create a straight line of its own. This is all in accordance with geometric analogy - surely it must be clear to you.”

I restrained my impatience, for I was now caught with the temptation to rush blindly at my visitor and to propel him into his “Space”, or out of Flatland, anywhere, so that I could be rid of him. But, summoning patience, I replied, “And what may be the nature of the figure which I am to shape out by this motion which you are pleased to denote by the word 'upward'? I presume it is describable by the language of Flatland.”

The stranger replied, “Oh, certainly. It is all plain and simple, and in strict accordance with geometric analogy - only, you must not speak of the result as a figure, but as a solid. But, I will describe it to you, or rather not I, but we shall describe it together, by means of geometric analogy.” He continued. “We began with a single point, which of course, being a point, has only *one* terminal point. One point

produces a line with *two* terminal points. A line produces a square with four terminal points. So,” he paused, “1, 2, and 4 are obviously in geometric progression¹. What is the next number?”

“Eight.” I replied.

“Exactly,” said he. “The square produces a something-which-you-do-not-as-yet-know-a-name-for, but which we call a *cube*, in three dimensions with eight terminal points. Now are you convinced?”

“And this 'cube' has sides as well as angles, or what you call 'terminal points'?” I asked.

“Of course,” he replied. “and all according to analogy. But, by the way, these are not what you would call sides, but what we would call sides, or faces. You would call them figures.”

“And how many sides or figures will this cube have, that would be generated by an 'upward' motion to a square?” I asked.

“How can you ask - and you a mathematician?” he scoffed. “The side or face of an object is always a component that has one dimension less than that of the object. As a point has dimension 0, it can have no sides. A line has one dimension, and has two terminal points, or two components of dimension 0, that define it (for we may, by courtesy call a line's terminal points its sides). A square in dimension 2 has four lines, each of dimension 1, that define it. So, we have the start of another sequence²: 0, 2, 4. What kind of progression is this?”

¹ The geometric sequence is $2^0, 2^1, 2^2, \dots$

² This sequence is defined as *dimension* x 2, e.g., $0 \times 2, 1 \times 2, 2 \times 2, 3 \times 2, \dots$

“It is an arithmetic sequence”, I answered.

“And what is the next number?” he asked.

“Six.” I answered.

“Exactly. So you have answered your own question. A cube in three dimensions will be bounded by six components of two dimensions, called sides or figures. You see it all now, eh?”

I wish I could say that I did. But I didn't, and his attempts at convincing me were both logical and absurd. I understood the analogy, but his suggestion of three dimensions was not possible! Through the spinning jumble of my thoughts, as I was trying to make sense of his argument, I could just make out the theory he was proposing. But, I was at an utter loss for the ability to construct it in my own mind. I recalled my University training, how it was impressed upon us that some theories are madness, and entertaining them would lead to insanity. I could take no more. Succumbing to the violent temptation I had been suppressing, I lunged toward the stranger. “Monster!” I yelled. “Be thou scoundrel or devil, I can no longer endure your mockeries!” Saying these things, I ran at him with my front angle.

16. How the Stranger, Having in Vain Tried Words, Resorted to Deeds

My attack was in vain. I brought my angle into violent collision with the stranger, with a force sufficient to have significantly injured any being in Flatland; but, I could feel him slowly and irresistibly slipping from my contact, without edging to the right or left, but somehow retreating, and at the same time becoming smaller and smaller against my assault, and then vanishing. Soon there was nothing. But still I heard the stranger's voice.

“Why will you refuse to listen to reason!” the disembodied voice cried. “I had hoped to find in you, a being of good sense and an accomplished mathematician, a fit apostle for the Gospel of Three Dimensions, which is allowed to be revealed only once every hundred years. But now, I know not how to convince you.” He paused, then continued excitedly. “Wait, I have it! Deeds, not words, shall proclaim the truth. Listen, my friend.”

I felt movement but still could not see the stranger, whose voice continued to come from all around me, and seemingly from *inside* of me. It was quite unnerving. He continued, “I have told you that from my position in space I can see the inside of all the things that you consider closed. For example, I see in your cupboard near where you are standing two of what you call boxes (although like everything else in Flatland, they have no tops or bottoms) full of money. Next to them I also see what you would call a tablet (but I would call a line) recording your household accounts. I am about to descend from space into that cupboard and bring you the tablet. I saw you lock the cupboard before your party some hours ago, and I know that you have the key in your possession. But here, I descend from space; and

lo, the door remains locked. Now I am in the cupboard and am taking the tablet. Now I have it, and am ascending back into space.”

I rushed to the cupboard and unlocked the door. I flung the door open. Indeed, the tablet was gone. With a laugh, the stranger appeared back in the other corner of the sitting room, and at the same time the tablet appeared on the floor. There could be no doubt - it was the missing tablet.

I moaned in dismay, doubting whether or not I was out of my senses, but the stranger continued, “Surely you must now see that my explanation, and no other, suits the phenomenon you are witnessing. What you call solid things are really superficial; what you call space is nothing but a great plane. I am in space, and can look down upon the insides of things of which you can only see the outsides. Remember your dream of the king of Lineland, and how you could see his entire kingdom whereas he could not? The analogy is the same. Think, woman, think! Can you not see that this is so?” He continued, quickly, with hardly a break for breath. “The higher I mount and the further I go from your plane, the more I can see, though of course on a smaller scale. For example, I am now ascending; now I can see your neighbors and their family in their several rooms, all asleep. Now I can see the theatre down the way, from which the audience is just now departing; and on the other side an aged polygon, appearing as a circle from this distance, alone in his study, sitting at his books. Now, I shall come back to you. And, as a crowning proof, what do you say to my giving you a touch, just the least touch, in your stomach. It will not injure you, and the slight discomfort you may feel cannot be compared with the psychological benefit you will receive from being able to *feel* your insides.”

And then, before I could utter a cry of caution or remonstrance, I felt a shooting pain in my inside, a sensation like which I have never felt before. I cried out with the pain. A moment afterwards the sharp agony ceased, leaving a dull ache behind. The mocking laugh continued, seeming to reverberate in my aching stomach. The stranger began to reappear, saying, as he gradually increased in size, “There, I have not hurt you much, have I? If you are not convinced now, I don't know what will convince you.” He paused, waiting for my further reaction. “What say you?” he asked.

My resolution was taken. Not only had this stranger appeared in my household and rattled my brain with nonsensical rantings, he had now accosted me in my own home. Where was my husband? Had he not heard my scream? If only I could manage to pin this magician against the wall until I could summon help!

Once more I dashed at him, and at the same time I cried out in the loudest shriek I could manage. Again, my angle seemed as a glancing blow to the stranger, while he moved in some unknown manner, retreating and at the same time becoming smaller and smaller. I continued my advance, still hoping to pin him, and still shouting for help, when I could hear and feel the stranger say, while my angle was in fleeting contact with his side, “This must not be! She must listen to reason, and if not, then I have no choice...” As I heard the bedroom door open, the stranger exclaimed, “Ha! Has it come to this? Then meet your fate: out of your plane you go. One, Two, Three! 'Tis done!”

17. How I came to Spaceland, and What I Saw There

An unspeakable sensation gripped my being. My insides felt as if they were flying, free of my body. My body was also flying, or floating perhaps, with a type of movement that was completely foreign to me. The sensation was overwhelming and sickening, and incredibly frightening. I could not fathom what was happening, or comprehend what I was seeing. There was darkness, then a dizzy, confused sensation of sight that was not like seeing. I saw a line that was no line; I saw space that was not Space. My mind was outside my body, yet within it. I closed my eye, since what I was viewing only increased the sickening feel of my insides. I longed for it to end, for the stranger to leave me in peace in my own, familiar world. When I could find voice, I moaned in agony, "Either this is madness, or this is hell." "It is neither," the stranger's voice calmly replied, "it is knowledge; it is Space, true Space, the Space of Three Dimensions. Let yourself become still, and open your eye once again. Try to look steadily in one direction only."

I now understand why he gave this advice to ease my transition into Spaceland. My eye, accustomed as it was to Flatland, was used to seeing in two dimensions only, and could not fathom the three dimensions it was now bombarded with. But, after becoming still, and gazing in one direction only, my eye was able to make some two-dimensional sense out of the infinite view that was presented to it. I did as he told me, and the dizzying sensation abated, but did not subside. My insides seemed to stop bouncing in unknown directions. Although still fearful, my terror subsided. I looked and saw the circle, but as I gained more confidence and was able to rotate ever so slowly, I began to fathom this whole new world. Or, Universe! The circle was no mere circle, but perfect circular beauty like I had only before inferred, conjectured, and dreamt of. What seemed the center of the stranger's form lay

open to my view. Yet, I could see no insides - no heart, lungs, or stomach. Only a beautiful, harmonious something - a something for which I had no words, but you, my Spaceland readers, would call the surface of the sphere.

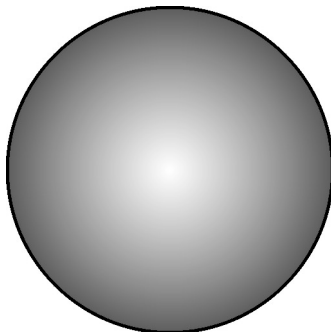


Figure 19. My perspective of the sphere in Spaceland

My fear now replaced by awe, I prostrated myself before my great guide, and cried, “How is it, O divine one of great form and wisdom, how is it that I can see your inside, and yet cannot see your inside - your heart or lungs, your stomach?” “What you think you see, you do not see,” he replied. “It is not given to you, or anyone else, to behold my internal parts, although I do, like you, have a heart and lungs and stomach. I am of a different order of beings than those of Flatland. Were I a circle you could discern my stomach. But, I am a being, as I told you before, composed of many circles, an infinite number of circles if you will, called in this land a sphere. And, just as the outside of a cube is a square, so the outside of a sphere presents the appearance of a circle.”

Bewildered though I was at my teacher's enigmatic words, I no longer chaffed against them but instead worshipped him in silent adoration. He continued, with more mildness in his voice. “Do not distress

yourself, that you cannot at this time immediately understand all of the deeper mysteries of Spaceland. In time, they will dawn upon you. Let us begin by casting back a glance at the region whence you came, that of Flatland. I will show you that which you have often reasoned about, but have never seen with the sense of sight - that of a visible angle.”

I trembled at the thought. Impossible? Surely impossible, but I was still not comprehending all that I was already seeing. The sphere was leading the way, and I was following, as if in a dream, until his voice implored, “Look yonder, and behold your own pentagonal house, and all of its residents.”

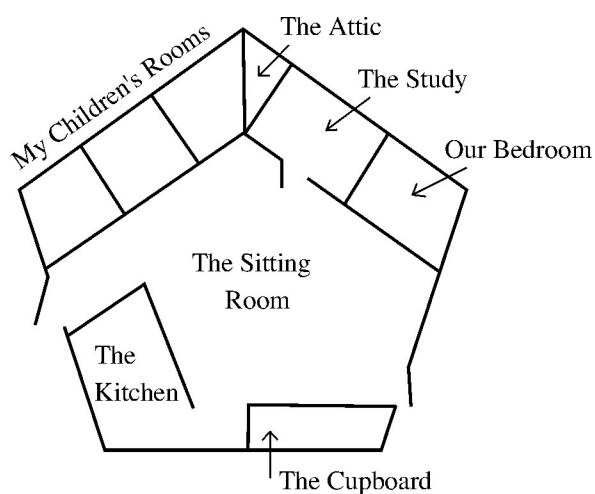


Figure 20. My house

I looked, and there I saw with my physical eye the perspective of my own house from three dimensions. All that I had previously seen as lines with the then limited view of my eye, and inferred in two dimensions with the then reasoning of my brain, I could now see laid before me. And how poor and incomplete was the inferred conjecture in comparison with the reality which I now beheld! My

pentagonal house, with angles, visible angles, that I could now envisage. Angles of 108 degrees; and all the other angles in the house - the bedroom walls, the kitchen and cupboard walls. I could see my three children sleeping soundly in their north-western rooms, and the other rooms empty this time of night, save the sitting room, where my husband, alarmed by my cries and now absence, was pacing the room and searching the other areas of the house, clearly distraught at my disappearance. Now he was entering the study, perhaps wondering if I had fallen faint behind the desk. All this was spread before me, and as we came closer and closer, I could even discern the contents of the study cabinet, which contained the many tablets of notes of mine on the left and papers of my husband's on the right.

Concerned for my husband's distress, I wanted to rush down to reassure him, but I found myself immobile. "Do not trouble yourself about your husband," the sphere said. "he will not be left in anxiety for long. But for now, let us take a survey of your Flatland."

Once more I felt myself rising through space. The feeling was much more agreeable now, even becoming familiar. But still, I could not understand how this movement was occurring. Awestruck at the sight of so many things, I greedily turned this way and that, seeing all that I could take in of Flatland. I said to my teacher, giddy with this new perspective, "Behold, I have become as a god! For the wisest of beings in our country say that to see all things, or as they express it, to become omniscient, is an attribute of God alone." There was something of scorn in this voice as he replied, "Is it so, indeed? Then the very pick-pockets and murderers of my country are to be worshipped as gods by the very wisest in your country. Trust me, your wise are wrong." "Then is omniscience the attribute of others besides God?" I asked. "Of that I do not know," he replied, now obviously impatient, "but tell me, omniscience - does it make you more just, more merciful, less selfish, more

loving? Not in the least. Then, how can it make you more divine? But enough of this. Look over there. Do you know that building?"

I looked, and indeed, far off, I saw an immense polygonal structure. Although I had completely lost my bearings, being as unaccustomed as I was to traveling outside my world instead of in it, and still not able to comprehend all that I was seeing; but the building was so large I surmised that it was the General Hall of State. It was surrounded by a dense covering of smaller pentagonal buildings offset from each other with gaps that I knew to be streets, and so I perceived that I was approaching the great metropolis of Flatland.

"Here we descend," said my guide. It was early morning, the first hour of the first day of the two thousandth year of our era. In accordance with our laws, the oldest and wisest in our land, polygons with sides so prolific that they were referred to as circles, were meeting in solemn enclave, as they had met on the first hour of the first day of every century since anyone could remember. The minutes of the previous meeting were now being read by my brother (for I recognized at once both his voice and his position of Chief Clerk to the High Council). No one, save the High Council, and my brother plus a few sworn guards, knew what would transpire during the great High Council Enclave. And here I was, perched with such a view as to both hear and see all that would happen!

It was found recorded in the minutes (read by my brother) that: "Whereas the State has been troubled by ill-intentioned persons pretending to have received revelations from another world, and professing to produce demonstrations whereby they had instigated to frenzy both themselves and others, it has been for this cause unanimously resolved by the High Council that on the first day of each century,

special injunctions be sent to the prefects in the several districts of Flatland, to make diligent search for such misguided persons, and without formality of mathematical examination, to destroy or imprison such persons in the district asylum, or if of a high ranking in society, to send the person forthwith to the capitol to be examined and judged by a specially called session of the High Council. Furthermore, it is recommended to future High Councils that the previous resolution be passed at each gathering of this body, to ensure that the madness of the unfortunate few does not carry forward into our society.”

“Hear your fate,” said the sphere to me. The council was beginning to debate the formal resolution before it. “Death or imprisonment awaits the apostle of the Gospel of Three Dimensions.” “Not so,” I replied, “for now that I understand it, the matter seems so clear to me, the nature of real space so palatable, that I think I could make a child understand it. Permit to descend at this time, and enlighten the High Council, who will then avert from this resolution and instead ratify a new one, professing the Gospel of Three Dimensions as truth.”

“Not yet,” said my guide, “the time may come for that. While I am still with you, let me perform my mission. You stay here, and observe.” With that, he leaped with great dexterity into the sea (if I may so call it) of Flatland, right in the midst of the ring of counselors. “I come,” he cried, “to proclaim that there is a Land of Three Dimensions!” I saw many of the counselors jump back in horror, as the sphere's circular cross-section widened before them. But on signal from the presiding Grand Circle (for that was his title, even though, as I have explained, he was not a true circle, but instead a prolifically-sided polygon) six guards of octagonal to decagonal countenance rushed toward the sphere from the exterior of the gathering. “We have him!” they cried, as they surrounded the sphere. “Yes, no - we have him still! No, he's gone, he's gone!”

“My lords,” said the Grand Circle to the other counselors of the council, “there is not the slightest need for surprise. The secret archives, to which I have access, tell me that a similar occurrence happened at the previous centennial enclaves. You will, of course, say nothing of these matters outside the council.”

The High Council was in an uproar. The Grand Circle moved the discussion back to the resolutions at hand. After some debate, and not without some naysayers who would have preferred more complete information as to the nature of the intruder, it was at last decided that the resolutions should be passed without amendment. It was done.

The Grand Circle addressed the council once again. “My lords, the business of the High Council now being concluded, I have only to wish you a happy new year, and new century.” Before departing, he expressed, at some length, to the clerk, my most unfortunate brother, his sincere regret that, in accordance with precedent and for the sake of secrecy, he must condemn him to perpetual imprisonment, but added with good cheer that, unless some mention were made by him of that day's incident, his life would be spared.

18. How the Sphere Showed Me the Mysteries of Spaceland

When I saw my poor brother being led away to be imprisoned, I attempted to leap down into the council chamber, desiring to intercede on his behalf, or at least lift him from Flatland to show him all that I had discovered of three dimensions. But I found that I had no ability to propel myself through

Space. I could still do my rotational movements around my own center in my own plane, and these were crucial to being able to see well in three dimensions. But, it appeared that I was solely dependent on my guide for movement. “Heed not your brother,” said the sphere, gloomily, “I think that you will have ample time to console him later. Come with me.”

With that, we ascended into space once more. “So far I have shown you nothing but plane figures and their interiors, save myself. Now I must introduce you to other solids, and reveal to you the plan upon which they are constructed.” We traveled some distance, seeing some objects around us, but I was not fully comprehending what I was seeing. We then approached an immense structure of some sort, large enough for the sphere and me to enter.

“See this multitude of movable square cards. See, I put one on another, not, as you might assume, to the northward of the other, but on top of the other. Now a second, now a third. See, I am building up a solid by a multitude of squares parallel and on top of each other. Now the solid is complete, being as high as it is long and broad. We call it a cube.”

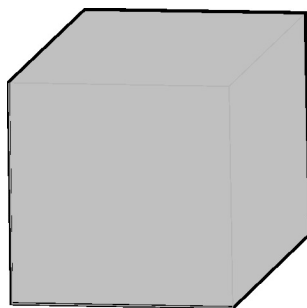


Figure 21. My Perspective of a Cube in Spaceland

“Pardon me, my lord,” I replied, “but to my eye the appearance is as a deformed figure, whose inside is laid open to view. In other words, I see no solid but an irregular figure in a plane such as Flatland, with the view outside the plane. It is of such an irregularity that it cannot be made by Nature, but must instead be a man-made product. But, I do not yet understand how you constructed it. Pray, demonstrate this to me again.”

“It appears to you a figure in a plane, because you are not accustomed to the light and shading effects of Space nor the three-dimensional perspective, just as in Flatland a hexagon would appear a straight line to one who is not practiced in recognition by sight. But in reality it is a solid, and I suggest you convince yourself by the sense of feeling. Afterward I will show you again how it is constructed.” said the sphere.

He then brought me to the cube, and as he propelled me around it I discovered that this object was indeed no plane figure, but a solid, and that it had six plane sides and eight terminal points called solid angles. I remembered the sphere instructing me previously that such an object would be formed by a square moving in space parallel to itself. As he showed me once again how the cube was constructed, I understood. However, I still could not understand what the sphere told me concerning light and shade and perspective, and the use of them in recognition by sight in three dimensions. I did not hesitate to put these difficulties before him.

If I were to give the sphere's explanation of these matters, succinct and clear though it was, I fear that it would be tedious for you, my Spaceland reader. Suffice it that a good deal of instruction was thus laid out, complete with demonstrations of movement of lights and of objects, and utilizing my own sense of

feeling as well. At last, after some time, his lucid explanation and practice with recognizing and then feeling several different objects and viewing them from several different positions and with differing light sources, I could eventually distinguish between a circle and a sphere, a plane figure and a solid. And, through my own rotational movement (which I learned is necessary in three dimensions to give the proper perspective using my one eye) and occasional propelled movement facilitated by the sphere, I was able to discern various three dimensional objects as well.

This was the climax, the paradise, the complete joy of my strange eventful history. Now I must relate my fall - my most miserable fall, and yet surely undeserved! For why should the thirst for knowledge be so aroused, only to be quenched by disappointment and punishment? My being shrinks from the painful responsibility of recalling my humiliation; yet, like a second Prometheus, I will endure this and worse, if it means I may arouse a spirit of rebellion against the conceit which would limit our dimensions to two or three or any number short of infinity. But, away with personal considerations. Let me continue as I began, without further digression, pursuing the recording of plain history. The fact, even the exact words (as they are burnt upon my brain) shall be set down without alteration, and let my readers judge between me and destiny.

19. How I Asked the Sphere for More, and the Result

The sphere would have willingly continued his lessons by indoctrinating me in the form of all regular solids, cylinders, cones, pyramids, pentahedrons, hexahedrons, dodecahedrons, and spheres. I am sure he even would have laid out an explanation of his society in three dimensions, as I have described our

society of Flatland. I ventured to interrupt him. Not that I was wearied of knowledge; on the contrary, I thirsted for yet deeper and fuller draughts than he was offering me.

“Pardon me, oh sphere,” I ventured, “would you allow me to look at your insides, as you so easily look at mine?”

“My what?” exclaimed the sphere.

“Your insides. Your stomach, heart, and lungs. I now realize that just as there is one more dimension than my own, the third dimension, there must also be one more than this, what we must call the fourth dimension. Just as you, superior to all Flatland forms, combine an infinite number of circles into one sphere, doubtless there is one above you who combines many spheres into one even more supreme, surpassing all the solids of Spaceland. And even as we, who are now in space, look down onto Flatland and can see the insides of all things, so too there must be some higher, purer region, where you surely plan to take me, my priest, philosopher, and, dare I say, friend. Let us go to this higher dimensionality, where we can look down together upon the revealed insides of all solid things, and where your own intestines, and those of your kindred spheres, will lie exposed to the view of this poor exile from Flatland, to whom already so much has been revealed.”

The sphere was haughty in his reply: “Pooh! What is this nonsense of which you speak? Our time is short, and much remains to be done before you are fit to return to your Flatland to proclaim the Gospel of Three Dimensions to the blind citizens of your world.”

Summoning my courage, I boldly replied, “But no, gracious teacher, please do not deny me what I know is in your power to perform. Grant me just one glimpse of this higher world, one glimpse of your insides, and I will be satisfied forever. I will remain your docile pupil, readily receiving all your teachings and anxiously feeding upon the profound knowledge flowing forth from your mouth.”

“Well then,” the sphere replied, “in order to satisfy your whim I would show you what you want, but I cannot. Would you have me turn my stomach inside out to oblige you?”

I was not deterred. I continued to implore, “But my lord has shown me the insides of all of my fellow citizens in the land of two dimensions, by taking me with him to the land of three. What therefore could be more easy than to now take your servant on a second journey to the land of four dimensions, where I can look down upon this land of three dimensions and see the inside of every three-dimensional house, the interior of the solid earth, the treasures in the mines of Spaceland, and the insides of every solid living creature, even of yourself, most noble and perfect sphere.”

“But, where is this land of four dimensions?” asked the sphere.

“Where? *I* know not where, but surely my teacher knows?” I replied.

“I do not,” answered the sphere. “There is no such land. The very idea of it is inconceivable to me.”

“Oh, but not inconceivable, my lord.” I quickly replied. “It is plain that this land should exist. Just as in the land of two dimensions my teacher's skill opened the eyes of his blind servant to the invisible

presence of a third dimension, so there must be a (currently invisible) fourth dimension from which we could view all of three dimensions, just as I can now see Flatland.”

I paused, but the sphere did not respond. I continued. “Let me recall the recent events which have opened my eyes. Was I not taught that when I saw a figure and inferred its depth, in reality I also saw a third unrecognized dimension, not the same as brightness, called 'height'? And does it not follow that, in this space, when I see a planar side and infer a solid, I also see a fourth but unrecognizable dimension, not the same as light and shading, but existent, though incapable of being measured?”

There was another short pause, and still no reply from the sphere. I went on, “And besides this, there is the argument from the analogy of figures that you previously used to attempt to convince me of this new, but not recognizable by me, third dimension.”

“Analogy of figures? Which analogy?” asked the sphere.

“Now you are playing with me. You are testing me, to see if I remember the revelations that you imparted to me. But do not play with me; I crave, yea thirst, for more knowledge. I expect that we cannot *see* the fourth dimension now, because we have no eye in our stomachs. But just as there was a realm of Lineland, although that puny monarch could not turn left or right to discern it, and just as there was Spaceland, close at hand and even touching my frame although I had no power to touch it, no eye in my interior to discern it, so surely there is a fourth dimension, which surely you perceive with the inner eye of thought. It must exist, just as you yourself have taught me. Or, have you forgotten what you yourself told me?”

I continued. “In the zero dimension, a point clearly has one terminal point. In one dimension, a moving point produces a line with *two* terminal points. In two dimensions, a moving line produces a square with *four* terminal points. In three dimensions, a moving square produces - my own eye witnessed your construction of it - the cube, that remarkable solid figure, with *eight* terminal points. So, in four dimensions, should not a moving cube result in a still more remarkable figure. with *sixteen* terminal points? Did you yourself not point out to me the unmistakable geometric progression 1, 2, 4, 8? The next number in the sequence is clearly 16, which must then be the number of terminal points of a cube moving perpendicular to itself in some unknown fourth dimension. Did you not say, 'and all according to analogy'?”

Again the sphere was silent. Eager to prove my understanding of his great teachings, I again continued. “And, again, you yourself taught me by analogy that a point has *no* sides, a line has *two* bounding points, a square has *four* bounding lines, and so a cube must have *six* bounding planes? Again, we have a sequence, this time the arithmetic sequence 0, 2, 4, 6. Consequently it must follow that the more divine offspring of a cube in four dimensions must have 8 bounding cubes?”

Silence. I implored, “Oh lord, I am but a lowly being from Flatland; I do not know the facts. I cast myself in faith upon your own conjecture, and appeal to your lordship to confirm or deny my logical anticipations. If I am wrong, explain to me my fault and I will yield this idea of the fourth dimension. But, if I am right, surely you must listen to reason.”

I could only assume the sphere was considering my words, and my argument. Continuing with the idea of analogy, a brilliant thought raced through my mind and I ventured forth a prospect for which I had no basis except analogy. It was a wild guess. "I ask if you or the citizens of your world have ever witnessed the descent of beings of a higher order than your own, entering closed rooms, even as you entered mine, without opening doors or windows, and appearing and vanishing at will? On the reply to this question I am ready to stake everything. Have there been such accounts? Deny it, and I will be silent. But please, give an answer."

I waited. Finally, the sphere spoke, hesitatingly. "It has indeed been reported so. But the citizens of Spaceland are greatly divided as to the facts. And even granting the facts, they are divided in their explanation. In any case, no one has adopted or even given any credence to the theory of a fourth dimension. Therefore, let us be done with this line of questioning, and return to our business."

Having found my guess correct, I now could not be deterred. "Now that my anticipations have been fulfilled, have patience with me and answer me one more question, oh highest of teachers! Those who have appeared - no one knows from where - and returned - no one knows to where - have they also expanded and contracted their solid sections, and then vanished somehow into space - or that more spacious space - which must be the fourth dimension?"

"Yes, yes, they vanished," said the sphere moodily. "As to expanding and contracting their sections there are some reports, but again, the facts are in dispute. Most agree that these visions arose not from reality but from thought - from the perturbed brain of the individual making the reports."

“Ah, so they say, but do not believe them, my master!” I implored, now more forcefully. “Just as the High Council in Flatland will seek to denounce me upon my return, do not let the unbelieving few distort what you must see is the higher reality! Let us find a way to this new dimension, where we shall see the insides of all solid things. Let us move in some altogether new direction, so as to make every particle of our interiors pass through a new kind of space, with a wake of its own in this new, fourth dimension. And, once there, shall we continue our upward course?” I was more excited now, more animated. I can see now that I was appearing mad, but the excitement of the moment wholly overtook me. “In that blessed region of four dimensions, shall we linger on the threshold of the fifth, and enter in? Then, yielding to our intellectual onset, the gates to the sixth dimension shall fly open; after that the seventh... and then the eighth -”

How long I would have continued I cannot say. I could hear his voice of thunder commanding me to be quiet, and threatening me with dire consequences if I persisted. But, nothing could stem my flow of exuberant aspirations. I was so intoxicated with the recent draughts of truth to which he himself had introduced to me. However, the end was not long in coming. My words were cut short by a crash to the outside of my frame, and a simultaneous crash inside me, and I felt myself being propelled through space with a velocity that did not allow speech. Down, down, down! I was rapidly descending. I knew that the return to Flatland was my punishment for my impertinence. One glimpse, one never-to-be-forgotten glimpse from three dimensions of my Flatland, was all that I had left. Then darkness, then a final, all-consuming jolt. When I came to myself, I was once more a common Flatlander, creeping in two dimensions, in the study of my house, listening to the approaching cry of my husband.

20. How the Sphere Encouraged Me in a Vision

Although I had only seconds for reflection, I felt, by a kind of instinct, that I must conceal my experiences from my husband. Not that I comprehended, at the moment, any danger from divulging my secret, but I knew that to one untrained in higher mathematics, a hurried description of my adventures would be unintelligible. So I proceeded to tell him that I had climbed into the attic, fallen and apparently passed some time unnoticed, until as I came to, I fell through the trap door into the study where I now lay.

The story was incredible, but my husband, after determining that I was in no way structurally injured and whose good sense far exceeds that of the average being, did not immediately press me for more details. He proceeded to insist that I take some refreshment or some rest, and upon his suggestion I retired to the bedroom, thinking it an excuse to quietly ponder the events of the evening. When I was at last by myself, a drowsy sensation fell over me; but before I closed my eyes I endeavored to recount the process by which a cube is constructed through the motion of a square. It was not as clear as I would have wished, but I remembered that it was “upward, and yet not northward” in direction. I determined to steadfastly retain these words as the clue, which, if firmly grasped, could not fail to guide me to the third dimension. So I began repeating “upward, and yet not northward” quietly to myself, over and over again, until I fell into a sound and refreshing sleep.

During my slumber I had another vision. I thought I was once again by the side of the sphere, whose lustrous hue comforted me and his gentle manner assured me that his wrath was no more. In my dream, we were moving together towards a tiny point of light. Although we were travelling quite

swiftly, the point seemed to get no closer. As we continued to approach, my guide directed my attention toward the infinitesimally small point, and I thought I heard a slight humming noise issuing from the point, so slight indeed that even in the perfect stillness of the vacuum through which we soared, the sound did not reach our ears until we were almost upon it.

“Look here,” said the sphere, “in Flatland you have lived; of Lineland you have received a vision; and you have soared with me to the heights of Spaceland. Now, in order to complete the range of your experience, I bring you even to the realm of Pointland, the abyss of no dimensions.”

He continued. “Behold this miserable creature. The point is a being like ourselves, but confined to the non-dimensional realm. He is himself his own world, his own universe; of any other than himself he cannot imagine. He knows not length nor breadth nor height, for he has had no experience of them. He has no cognizance even of the number two, nor has he even conceived of plurality, for he himself is his one and all. Yet notice his perfect self-contentment, and so learn this lesson: to be self-contented is to be ignorant, and to aspire is better than to be blindly and impotently happy. Now, listen.”

He ceased talking, and there arose from the infinitesimal buzzing creature a tiny, low, monotonous but distinct tinkling, from which I caught these words: “Infinite beatitude of existence! It is, and there is none beside It.”

“What,” said I, “does the tiny creature mean by 'It'?” “He means himself,” said the sphere, “have you not noticed in your own world, as in mine, that babies and unfortunately deformed persons who cannot

distinguish themselves from the world, speak of themselves in the third person? But, hush, let us listen further.”

“It fills all space,” continued the soliloquizing point, “and what It fills, It is. What It thinks, that It utters; and what It utters, that It hears; and It itself is Thinker, Utterer, Hearer, Thought, Word, Audition; It is the One, the All in All. Ah, the happiness, ah the happiness of being!”

“Can you not startle the thing out of its complacency?” I asked. “Tell it what it really is, as you told me; reveal to it the narrow limitations of Pointland, and lead it up to something higher.”

“That is no easy task,” said my master. “You try.”

Consequently I raised my voice to address the point. “Silence, silence you puny creature. You call yourself the All in All, but you are nothing; your so-called universe is a mere speck in a line, and a line is a mere shadow as compared with -.” “Hush, hush,” interrupted the sphere, “that’s enough. Let us listen and see what effect your harangue has had on the king of Pointland.”

The luster of the monarch continued to beam brightly, and his complacency seemed to be as assured as ever. I had hardly ceased my words when he again took up his strain, “Ah, the joy, ah, the joy of thought! What can It not achieve by thinking? Its own thought coming to Itself, suggestive of Its disparagement, to enhance Its happiness! Sweet rebellion stirred up to result in Triumph! Ah, the divine power of the All in All, the One! Ah, the joy of being!”

“You see,” said my teacher, “how little your words have done. So far as the monarch understands them at all, he accepts them as his own, for he cannot conceive of any other except himself, and prides himself upon the variety of 'Its thought' as suggestive of creative power.”

I was determined to give it one more try. I decided to abandon the path of trying to convince him of the limits of his land, and instead undertook the more moderate goal of convincing him of my existence. “Oh you, king of your realm. Hear me - I am not you, but outside of you. I am another like you, but different. I am not your thought, but a different being altogether. Hear me, and respond.”

Again, the point's luster was not diminished, and the humming continued just as before. “Ah, the joy of thought! It is All, It is One. It is different and the same. It is joy, pure joy! Yes, the joy of being!”

“You see,” said the sphere, “it is no use. Believe me, I, and others like me, have tried. He simply cannot imagine anything outside of himself. Come, let us leave this god of Pointland to the ignorant joy of his omnipresence and omniscience. Nothing you or I can do or say can rescue him from his own self-absorbed reality.”

After this, we floated gently back through space. As I saw the approach of Flatland, I heard my companion pointing out the moral of my vision. “While we can allow that some creatures are mired in the ignorance of their fate, we must not be as they are, content in our own limited understandings. We must aspire to the highest thoughts of which we are capable, and teach others to aspire. It is our duty, our obligation, to help others out of the quagmire of ignorance and self-satisfaction.”

At this he paused shortly, while the whole of Flatland filled my view. I was again mesmerized by this view of Flatland, this perspective, seeing what I for so long had only inferred. He continued. "I must confess that I was angered with you, as you questioned me about dimensions higher than my beloved Spaceland, and I did not easily give up the contented belief in no dimensions higher than three. But since then I have received new insight, and have realized the error of my self-complacency. You must forgive me, my wise pupil, for my behavior and love of ignorance. I now realize that you must be correct; that the reports that come to my world every 100 years must also be truth, pointing to a fourth and higher dimension. And then, if there are second, third, and fourth dimensions, then surely there is a fifth, and sixth, and seventh, as you yourself suggested. My mind is now opened to such inevitabilities, and I am grateful that you, my lowly pupil, would seek to convince me of what I should have known all along." With this, he proceeded to attempt to explain to me how, in his understanding, extra-solids should be constructed in four dimensions out of three-dimensional solids, all 'strictly according to analogy'. Although I was convinced that he believed what he was speaking, even in my dream I could not connect the abstractness of his words with the concrete constructions he was proposing. Nevertheless, I was filled with the excitement of discovery, of knowledge, of aspiring to know more, and of the responsibility of sharing this knowledge with others.

21. How I Tried to Teach the Theory of Three Dimensions to my Daughter, and with What Success

I awoke rejoicing, and began to reflect on the glorious mission before me. I would go forth at once, I thought, and evangelize to the whole of Flatland. Even to the children and the short-sided would I proclaim the Gospel of Three Dimensions. I would begin with my husband.

Just as I had decided on my plan, I heard the sound of many voices in the street commanding silence. Then followed a louder voice. It was a herald's proclamation. Listening attentively, I heard the Resolution of the High Council, commanding for the arrest and imprisonment or execution of any one who would try to pervert the minds of the people by delusions and by professing to have received revelations from another world.

I reflected upon this. The danger was not to be trifled with. I could do no good to my mission if I were imprisoned or executed. I decided it would be best to avoid all mention of my revelation, of my trip to another world, and instead proceed along the path of demonstration. After all, what could be so simple and so conclusive as the concrete demonstration of the third dimension - and "upward, yet not northward" was the key to the proof.

It had seemed fairly clear to me before I fell asleep, and when I first awoke, fresh from my dream, it seemed as obvious as arithmetic. But somehow, it did not seem as obvious to me now, and as the moments marched on, it was becoming less and less clear. Though my husband entered the room at

that moment, my confidence had slipped, and after we had exchanged a few words of commonplace morning salutations, I decided not to begin with him.

My children were up and about, all three of them well-versed in mathematical concepts beyond their ordinary levels. None more so than my youngest daughter, whose precocious inquisitions had caught my attention the night before, and whose remarks on the meaning of 3^3 had met with the approval of the sphere. Why not begin with her? By discussing the matter with her, a mere child, I should be in perfect safety; for she would know nothing of the proclamation of the Council; whereas my husband, being a statesman and bound by duty to uphold the law, might feel compelled to hand me over to the authorities.

The first thing to be done was to satisfy in some way the curiosity of my spouse, who naturally wanted to know the reason for which the circle had desired that mysterious interview, and of the means by which he had entered the house. Without going into the detail of my account here, I gave him an explanation - not as truthful of an account as I am accustomed, I must admit - of how the circle, a foreigner, had been in town for the day only and had heard of my latest theory through a colleague and wished to discuss it with me. And, how he was let in to the house by an exiting guest after the party, but not wishing to disturb us had waited quietly until he thought an opportune moment to approach me (he was, after all, a foreigner with different customs regarding personal interactions and manners). I must be content with saying that I eventually succeeded in convincing my husband that there was nothing so unusual about a colleague dropping by without apparent entry after midnight, although in the end I think he eventually accepted that there were some elements of the story that were just not altogether lucid.

We each went about our morning duties. As soon as I was able, I took my youngest daughter aside and led her to the study to begin my instruction. I was eager to get started, for I felt that all I had seen and heard was somehow slipping away from me. When we entered the study, I carefully secured the door. Then, sitting down side-by-side, we took out our tablets - you would call them lines - and I told her that we would resume the lesson of the night before. I taught her once more how a point by motion along one dimension produces a line, and how a straight line in two dimensions produces a square. After this, forcing a laugh, I said, "And now, tell me again of your theory that you wanted me to believe last night, that a square may, in the same way using a motion that is somehow 'upward, yet not northward' produce another figure, a sort of extra-square in three dimensions. Tell me that again, you young rascal." I feigned a merry laugh, which did not come off as believably as I had hoped.

It was doomed to failure. At this moment we heard once more a herald outside in the street proclaiming the Resolution of the Council. Young though she was, my daughter - whom I have already mentioned was unusually intelligent for her age - was brought up in a home of a statesman and had perfect reverence for the authority of the Council. She took in the situation with an acuteness for which I was quite unprepared. She remained silent for a few moments after the last words of the proclamation had died away, and then, bursting into tears, cried, "Dear Mother! That was only in fun, and of course I meant nothing at all by it! We did not know anything then about the new law, and I don't think I said anything about the third dimension, and I am sure I didn't say any words like 'upward, yet not northward', for that would be nonsense, you know, wouldn't it?" She looked at me for a clue as to how I was responding, and getting no indication from my frozen expression, she hurriedly

continued, "Upward, yet not northward! Ha, ha! Even if I were a baby I could not be so absurd as that! How silly it is! Ha, ha, ha!" Her laugh, also forced, was overly done.

"Not at all silly!" I said, realizing my disaster of timing, and quickly losing my temper. "Here, for example, I can take this square," and at the word, I grasped a movable square from the shelf, "and I can move it, you see, not northward but- yes I move it upward- what is to say, not northward, but I move it- I move it...somewhere, not exactly like this, but somehow..." Here I brought my sentence to a trailing conclusion, frustrated by my inability to move the square 'upward, yet not northward', and instead shook the square about in a ridiculous manner. My daughter burst out laughing once again, this time a true, easy laugh, now quite convinced that I was not teaching her but joking with her, and so saying she unlocked the door and ran from the room.

Thus ended my first attempt to convert a pupil to the Gospel of Three Dimensions.

22. How I Then Tried to Proclaim the Theory of Three Dimensions by Other Means, and of the Result

My failure with my daughter did not encourage me to continue my attempts with other members of my household, although my other children, just as intelligent, might have proved fruitful vessels. My failure did not, however, discourage me toward success. I did realize that I could not wholly rely on the catchphrase "upward, yet not northward," but must instead seek to demonstrate a clear view of the whole subject.

So I devoted several months in privacy to the construction of such a demonstration. I quickly learned, however, that just as I could not myself demonstrate the "upward, yet not northward" direction to my daughter, it appeared that, now confined to Flatland, I was unable to move myself or any object the slightest nudge in this novel direction. In truth, an hour after my last vision of the sphere, I was not precisely sure what direction it was; but, I knew the feeling of the direction being approximately a pulling on my insides. However, all attempts at duplicating this motion were futile. I therefore turned my attention toward the writing of a treatise on the mysteries of three dimensions.

Cognizant of the law, however, I spoke not of a physical third dimension, but of a "Thoughtland" from where, in theory, a figure could look upon Flatland and see simultaneously the insides of all things, and where it was possible that there might be supposed to exist a solid figure of six squares, and with eight terminal points. I included in this treatise, of course, the mathematical analogy and geometric and arithmetic progressions already known, up to two dimensions, and used them to suppose the new Thoughtland. But in the book I found myself greatly hampered by the impossibility of drawing such diagrams as were necessary for my purpose; for of course, in our country of Flatland, drawing tablets are lines, and so diagrams are also lines, all in one straight line and only distinguishable by difference of length and brightness, so that, when I had finished my treatise, which I entitled, "Through Flatland to Thoughtland," I could not be confident that many would understand my meaning. And although I had tried to stay away from the concept of three dimensions, anyone who did understand my meaning might clearly see that this was the subject of the treatise, thus placing me in potential danger.

Meanwhile, my life was under a cloud. All pleasures palled, and I was often impelled to verbally proclaim treason, because I was constantly comparing what I saw in two dimensions with what I

envisioned it would look like from three, and I could hardly refrain from making these comparisons aloud. I neglected my own work at the University, and although my composition of the treatise continued, I could see that it was not sufficient to convince. In fact I found myself daily losing more and more ability to reproduce in my own mind the wonders which I had come to understand so clearly in Spaceland.

One day, about three months after my return from Spaceland, I tried to see a cube with my eye closed, but failed. Though I succeeded sometime afterwards, I was never quite sure (nor am I still) that I had exactly realized the original. This made me even more melancholy than before, and I became determined to take some step, yet what, I did not know. I felt that I would be willing to sacrifice my life for the cause of three dimensions, if it would have served to convince others of the Gospel. But, if I could not convince my own daughter, how could I convince the highest and wisest of the land?

And yet at times it was all too much for me, and I gave vent to dangerous verbiage. I was beginning to be considered heterodox if not outright treasonous, and I knew that my husband was hearing rumors of my actions. Nevertheless, I could not at times keep from bursting into suspicious utterances; for example, when the question arose at a political gathering about the treatment of lunatics who said that they had received the power of seeing inside things, I quoted the saying of an ancient circle, who said that prophets and those inspired with divine wisdom were always considered by the majority to be mad. Another time, at work, in the course of a conversation concerning geometry I even let slip the forbidden terms "the third and fourth dimensions." Another time, when my family was discussing the proclamation and the unhappy incarceration of my brother, the expressions "the all-seeing eye" and "the eye that discerns the interiors of all things" came out, in dubious reference to the High Council.

The final indiscretion came at a meeting of the local Speculative Society, where some extremely silly person read an elaborate and inane paper exhibiting the precise reasons why God had limited the number of dimensions to two, and how the attribute of seeing all things at once is given to the Supreme Being alone. After the presentation, I was prompted - I can't say what came over me - to present a story of my own. I presented it as a fictional story, a magical and altogether incredible journey, and in it I recounted the whole of my voyage with the sphere into Spaceland. I relayed how this fictional character saw all of Flatland at once, and even came back down into none other than the Assembly Hall in the metropolis, where the character addressed the High Council, and then returned again to Spaceland. The hall was silent and dumbfounded as I went on and on. I recounted everything I had seen and heard, and even of my visions. The time for the end of the meeting had long since past, but still I remained, spewing forth my story as if it were one of a fictitious person. My excitement escalated, and as I reached the end of the story, I threw off all pretenses; I admitted that I was the fictitious person, and that all that I had just relayed was true. To a stunned crowd, I then, finally, called upon the audience to become believers in the Gospel of Three Dimensions.

I was exhausted from the speech, from the excitement, and from the joy of finally coming out with my story. There was silence, then murmurs, then commotion. I still could not discern the mood of my audience, or of their state of belief. Had I just converted the room? Had my enthusiasm and eloquence hit the target?

But, no. The commotion was the police arriving, having been summoned by someone in the audience. I was arrested and taken before the High Council.

The next morning the Council was assembled in special session. I stood in the very place where but a few months before the sphere had stood in my company, and I envisioned him looking down on the spectacle from "upward, yet not northward." I was allowed to begin and to continue my narration unquestioned and uninterrupted. But this time I was able to read the mood of my audience, it being clear to the unaided eye. From the first, the Chief of the Council relieved most of the guards in attendance, all except one, who I happened to know was young, inexperienced, and not politically connected. I knew only too well what this meant. I was to be executed or imprisoned, and my story was to be kept secret from society; and this being the case, the Chief desired to minimize important casualties and use the most expendable of the guards.

As I relayed my story, I could tell the council members were being silently tolerant of my discourse, nothing more. As I made some of my most clever points, that of mathematical analogy, I perceived no change of expression, no hint of doubt from them. I nevertheless went on, with as much vigor as I could muster in the face of my audience. I told of my journey into Spaceland, of my envisioning of the High Council and the events that had taken place there some months before, and of my ability, from Spaceland, to see the insides of closed objects in Flatland. I made the analogy to my vision of the monarch of Lineland, and how I had appeared and disappeared as I had moved through his world, the line, and how from the side of the line, which he could not imagine or see, I could see all of his kingdom. I told also of my vision of Pointland, and of the postulating that the sphere and I had done of even further dimensions. I told it all, I held nothing back. For I decided that if I was to be judged, I must be judged on the whole truth.

After I had concluded my defense, the Chief, perhaps fearing that my impassioned speech had raised some doubts amongst some of the more junior council members, asked me two questions:

1. Whether I could indicate the direction that I meant when I used the words, "upward, yet not northward," and 2. Whether I could by any diagram or description (other than the enumeration of imaginary sides and angles) indicate the figure I was pleased to call a "cube."

I pondered these questions, knowing that I had nothing more I could offer the Council in terms of explanation, diagram, or demonstration. I hoped for the sphere to appear on my behalf, to materialize and convince the Council of the existence of three dimensions, but he did not. Finally I declared that it was not possible for me to say anything more in answer to their questions, to convince them of the veracity of my story. I concluded, "I only submit myself to the Truth, knowing that in the end, the Truth will prevail."

The Chief replied that he quite agreed with my final statement. The guard was asked to escort me to an outer waiting room while the Council decided my fate. After just a few minutes, I was brought back in and my sentence was read: perpetual imprisonment for treason against the High Council. But, if Truth was pleased to allow my disappearance from prison, and subsequent re-appearance outside of prison (as was consistent with my experience with this "third dimension") then the Truth might be trusted to bring that result to pass. Meanwhile, I should be given every normal comfort while in prison, except any that might facilitate my escape by any more mundane methods, and that, unless I forfeited the privilege by misconduct, I would be allowed to occasionally see my brother who had preceded me to prison.

Seven years have since elapsed and I am still a prisoner, and, except for the occasional visit of my brother, I am debarred from all companionship save that of my jailors. Of my children, who now must be grown or nearly so, I have had no news. My brother is good fraternal comfort, yet I confess that during our weekly meetings, at least in one respect, he causes me great pain. He was present in the Council Chamber when the sphere appeared to the Council seven years ago; he saw the sphere changing sections; he heard the explanation then given to the council members. Since my imprisonment some months later, scarcely a week has gone by during these long seven years without his hearing from me a repetition of the part that I played in that manifestation. I have relayed to him who was in the room, how they were configured, what was said, and who said it. Although the Council must have assumed that I gained my information from him, my dear brother, he knows clearly that this is not the case. I have also tirelessly relayed ample descriptions of all the phenomena of Spaceland, and the arguments for the existence of the third dimension and the solid things therein from analogy. Yet shamefully I confess that even after seven years my brother has still not grasped the nature of the third dimension, and frankly avows his disbelief in the existence of even the sphere!

Hence I am absolutely destitute of converts, and as far as I can see, the Hundred Years Revelation has been made to me for nothing. Prometheus in Spaceland was bound for bringing down fire for mortals, but I - poor Flatland Prometheus - lie here in prison for bringing down absolutely nothing to my fellow Flatlanders. Yet I exist in the hope that these memoirs, in some manner, I know not how, might find their way to the minds of persons in some dimension - I have envisioned a Spaceland audience, having given up all hope of convincing any of my two-dimensional kin - and may stir up a group of rebels who will refuse to be confined to limited dimensionality.

That is the hope of my brighter moments. But, of course, it is not always so. That I cannot with confidence envision the once-seen, oft-forgotten cube weighs heavily on my soul. In my nightly dreams, the words "upward, yet not northward" haunt me like a soul-devouring sphinx. I have waited patiently for another encouragement from the sphere, another vision, or, dare I even wish, another liberating trip to Spaceland, but, alas, this also has not come to pass. Not even the shortest of visions, not a single a heaven-sent glimpse of the sphere, has appeared. His silence is puzzling, and lamentable. He has given up on me, forgotten me completely, locked in this Flatland prison, as he floats along through glorious Spaceland. "Does he even exist?" I sometimes ask myself. It is part of the martyrdom which I endure for the cause of Truth that there are seasons of mental illness, when cubes and spheres flit away into the background of scarce-possible existences; when the Land of Three Dimensions seems almost as visionary as one, or none; when even the hard realities which I face, the walls that lock me in, the tablets on which I write, and all the other substantial realities of my existence in Flatland, appear no better than the offspring of a diseased imagination, or the baseless fabric of a dream.

Part III Afterword

23. How I Had Another Vision, of an Even Different World

It was an unremarkable day and an unremarkable evening; one of a vast multitude in this monotony of prison. I had retired at the usual time and fallen asleep with no incident, with nothing pressing on my mind, save my ever-present failure as a witness to the Gospel of Three Dimensions.

I had a vision that evening, wholly unlike the usual chaotic dreams of sleep, and very much like the visions I had received in the past, the ones of Lineland and of Pointland. In my vision, I was approached by a two dimensional visitor, a being not unlike myself, but an enormous triangle.

Although short-sided, I soon realized that this was no child.

We were out in the open, in the country, with no buildings in sight. There were no trees nor vegetation around me, and the southward attraction could not be felt. The triangle approached me quickly, far more quickly than could be possible, appearing first as a point in the distance and then looming beside me moments later. The triangle was enormous, larger than any building in Flatland, even larger than the General Hall of State.

As she approached me she cried "Hail, imprisoned one, who has been unjustly thwarted in her intellect. I have come to tell you more great things of our Universe!" As we spoke I encircled this great figure, which took some minutes on account of her size. And triangular she was, there was no doubt.

“Whence do you come?” I asked. “I come from here,” she replied, “I have brought you to my land.”

“Is this a remote region of the country?” “This is not your country at all, but mine. We are not in your Flatland,” she answered.

I did not see the importance of arguing, although I could not distinguish this place from any remote region of Flatland, except perhaps for the lack of southward pull. “Then where are we, may I ask?

What do you call this land?” “You are in Sphereland,” she replied. “Sphereland?” I asked, excited.

“Then, are we in three dimensions? Do you know of three dimensions? Can we get there from here?”

I was thrilled at the prospect of getting out of Flatland, the ignorant land in which I was imprisoned, and returning to that precious third dimension. But, I was puzzled. Looking around, we appeared to be in Flatland. There was no hint of the third dimension. Perhaps she would take me to that blissful region.

“No, this is not the third dimension, nor will I take you there,” the great triangle replied. “I do know of it, but have never been there myself.”

“Then, how can you call this Sphereland? This is Flatland, is it not?” I asked.

“It appears to you as Flatland, as it did to my ancestors long ago. But, this world is not flat. It is curved.”

“Curved? In what way? How can a world be curved?” I replied, astonished by this preposterous notion.

“Careful observation of my world, by me and others over many centuries, have led us to propose the theory that our world is curved. Shaped, in fact, like a sphere,” she replied.

“Like a sphere,” I repeated, still not understanding. “But, if this is not three dimensions, how can your world be a sphere?” I asked.

“I cannot claim that my world *is* a sphere, only that it is shaped like one,” came her cryptic answer.

I persisted, “Well then, how do you know that your world is shaped like a sphere?”

“A good question,” she replied. “Tell me, in Flatland, if you were to continue to walk in one direction forever, where would you end up?” she asked me. I was confused. Walk in one direction forever? Had anyone ever done it? I tried to recall history lessons of my youth, my geography classes, my fuzzy recollections of persons attempting this foolhardy endeavor. “I, I do not know,” I stammered, “I don't recall if it has been seriously attempted or accomplished. Outside of our cities is a vast rural area, and beyond that is desert into which some have gone but those who have returned have reported nothing but desert - no vegetation, no water, no fog. It is too vast a wasteland to span and survive.”

“So, no one on your world knows what lies beyond the great wasteland?” she asked, seemingly amused. “Oh, you are an ignorant people.”

I did not appreciate being called ignorant. I was, in fact, enlightened. I had been to three dimensions - even she did not claim to have done that. I rushed in defense of myself and my kinfolk. "How can you say that - you do not know. The wasteland is vast and continues indefinitely. What would be the point of venturing further and further into the wasteland? Only tempting death!" I exclaimed.

"Ignorance is a quality of people who do not try to find out the answers to the great mysteries. Tell me, why have your people not even conceived of three dimensions, even before your enlightenment? As you yourself discovered, the evidence was all there, all according to analogy. Not only did your people not conceive of it, when presented with evidence of the appearance of the sphere in the High Council they refused to listen. Refused, to the point of developing and enforcing a resolution that threatens to incarcerate or destroy any persons caught to be discussing the possibility. Your people choose blindness over reason, disbelief over evidence. They are ignorant."

I paused. I didn't know what to say. She was right, certainly, to some degree. But his sweeping dismissal of my entire society as ignorant was difficult to accept.

"The truth is, you don't know the result of walking indefinitely in one direction in Flatland," the triangle said, returning to her point. "Neither did we, for many generations, until an enterprising explorer packed supplies and set out on a journey. He was determined to discover what lay beyond the known regions of my world. And, do you know what he discovered?" She waited for my acknowledgement, which I gave. "He came back to where he started. *From the opposite direction.*" She again waited for me before continuing. "The journey took many years, but it was unmistakable.

When he returned, he approached the great city from the opposite direction.” She continued, “After that, several others set out to disprove this fantastic result. They went off in many different directions. And, those that returned all approached where they had started, from the opposite direction. Since that time, we have theorized and developed propulsion systems, like the one that I am now using, that allow us to make the journey around our world in a matter of days instead of years. We have explored every bit of our world - of our sphere-shaped world.”

I was shocked at this. They set out in different directions, and all returned to their same starting point. I could not imagine how this would work. My memory of the sphere was too distant. I tried to visualize his shape, and imagined beings - small, flat beings like myself, moving about the sphere. It was no use. I couldn't understand. I asked her to explain further.

“Tell me the properties of a sphere,” she said to me. It had been years since I had really used my mathematical training. I dusted off my knowledge of circles, the two-dimensional equivalent of spheres, and of my brief experience with the sphere in Spaceland. “A sphere is three-dimensional; it has no flat sides, but is continuously smooth... and curved,” I ventured. “It also has the property that every point on the sphere is the same distance from the center of the sphere, and that distance for a circle is called the radius... I suppose that we would also call that distance the radius in three dimensions as well?” I asked. “Very good,” said the great triangle, “Now, is the property of 'continuously smooth' the same over the whole sphere, or does the curvature change depending on direction or precise location on the sphere?” she asked. “It is the same no matter what direction or location you choose on the sphere,” I surmised, knowing that it was true of circles and again trying to visualize the three-dimensional sphere. She was asking an analogous question regarding the difference

between an ellipse and a circle in two dimensions. The top of an ellipse is 'flatter' than on the side of the ellipse. But I knew that a circle, as well as a sphere in three dimensions, varies the same at any point.

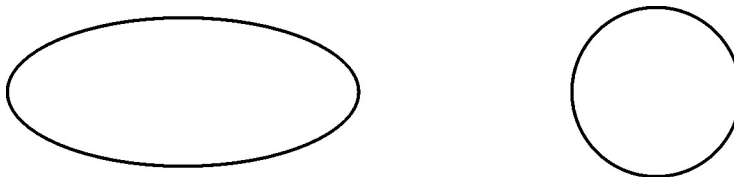


Figure 22. An Ellipse, whose Curvature Varies, and a Circle, whose Curvature is Constant

“You are correct,” answered the triangle, “and so, if you were on a great sphere, as you happen to be right now, and you began moving in a direction and you do not change direction, the curvature would remain the same for as long as you walked. Correct?” she asked. “Yes, I guess that would be true,” I replied. She paused, triumphant, as if she had made her case. But I was not satisfied. I asked, “but, how do you know that if you moved in any direction, that you would end up back at the same point? I understand that you would be moving around the curved sphere, but I don't understand why you wouldn't, for example, trace out a spiral-like motion.”

“Ah, now you are thinking like a mathematician!” she exclaimed. “How, indeed? We need a proof.”

I was excited that I would be shown this phenomenon. “Shall we venture to three dimensions, and see this motion which you describe?” I asked.

24. How the Triangle, Although Never Having Seen a Sphere, was able to Reason About It

The triangle snorted with contempt. “You *are* a simple being. Can you not prove something without seeing it? Think of your theorems and axioms of two-dimensional geometry. How do you know they are true?” she asked.

“Because they work in all cases. We know they work.” I answered.

“Yes, but that is proof by example. How do you know that they will work in all cases? What if you found an instance in which an axiom did not work, but it should?” she asked.

“These axioms have been true for hundreds if not thousands of years. If there had been a case in which they did not work and should, someone should have found it by now.” I paused. “Besides, we reason logically about the axioms,” I added.

“Precisely,” she answered, “you reason logically. That is a form of proof. You convince yourself that it must work in all cases. So to prove to you that if you moved in any direction on a sphere, you would end up back at the same point, I don't need to *show* it to you. We can reason about it, and develop a *proof*.” she said. “In two dimensions, if you start at any point on a circle and move around the circle, keeping one of your points touching it as you walk all the way around it, would you eventually end up back at the same point?” she asked. “Of course,” I replied.

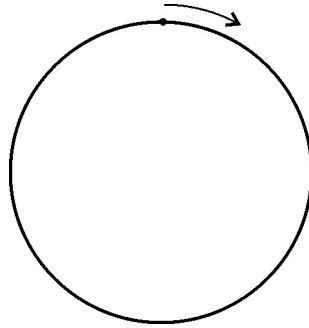


Figure 23. Traveling around a circle and ending back at the same point

“Similarly, if you pick any point on a sphere, and a direction in which you will move.” she said, “I claim that the point and direction correspond to a circle in two-dimensions. A circle of the same radius as the sphere, that is, a *great circle* of the sphere.”

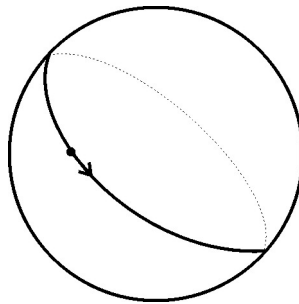


Figure 24. A point and direction determine a great circle of a sphere

“But, why could the point and direction not correspond to a smaller circle?” I asked. “For example, when the sphere I encountered moved out of my world, he appeared as many different circles, but each of a smaller and smaller radius, until he departed from my plane.”

“An excellent question,” replied the triangle. “You are trying to find holes in my argument. This is precisely what we need to do to develop the proof - develop a logical argument for which there is no further question over its validity in all circumstances.” She continued, “In two dimensions, if you were to walk in one direction, along what type of path would you walk?” she asked. “A straight line,” I answered. “And if you were to walk along the circumference of a circle in two dimensions, you would have to be constantly turning, correct?” she asked. “Correct” I replied. “So it is on a sphere; to walk along a great circle of the sphere, one walks along a straight path, but to walk along a smaller circular path on the sphere, one would have to be constantly turning.” She paused. “I would not attempt this argument with a novice in my world; but I hope that you, a mathematician who has been to three dimensions and seen a sphere, will be able to follow the reasoning. Suppose that on the surface of a sphere you wanted to trace the path of a very small circle, similar to the circle you saw just before the sphere vanished from your plane of Flatland - the intersection of a plane and a sphere, near the bottom of the sphere. Imagine yourself on a point on that small circle, and walking in the initial direction of the small circle along the surface of the sphere. If you stayed walking in that same direction, you would not trace the path of the small circle at all, but instead a great circle beginning at that point and continuing in that initial direction. To walk along the small circle, you would have to be constantly turning to remain on the smaller circle.” She paused again.

“This is difficult to conceptualize, I know,” she said. “Once we discovered that our world was shaped like a sphere, many generations of mathematicians developed this theory of 'spherical geometry'. I can use their terminology and rely on their theorems and axioms, each proved logically and without question of their validity, to prove new concepts. The full proof of the fact that a point and direction

traces a great circle on a sphere requires the use of these theorems and axioms, of which you are not familiar.”

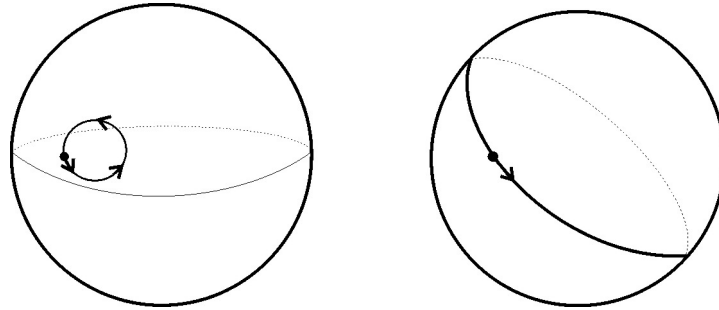


Figure 25. Tracing a Small Circular Path by Turning, Versus Tracing a Great Circular Path by Going Straight

I put aside the question of great circles and returning to the same point on a sphere. I had other questions. “If your world is a sphere,” I said, “or, *sphere-shaped*, then it must reside in three dimensions. Can you see outside of the surface of your sphere-shaped world - which to me still appears as Flatland? Can you see above or below the surface?”

“No, just as you cannot see out of the surface of your planar world, we cannot see out of the surface of our world either. It is no surprise that our world appears like Flatland to you; as I already told you, for many generations we also thought our world was flat. Because our world is so large, the curvature of it is imperceptible to us. Does it feel curved to you?” she asked.

“No, it does not. For this reason I hope you don't mind me doubting you a bit more.” I was trying to be polite, recalling my exasperating experience with the sphere when he was trying to tell me

unbelievable things. “What other observable phenomenon have you discerned that leads you to believe that your world is sphere-shaped?” I asked.

“Another good question. Again, you are thinking like a mathematician - there is hope for you yet!” the triangle beamed. “There are many other phenomenon. For example, I am a triangle, yes?” “Yes,” I replied. “And, if you look just down one of my sides, you can tell the side is perfectly straight, correct?” she asked. “Correct,” I replied. “And what is the sum of the angles of a triangle in a plane?” the triangle asked. “180 degrees,” I answered. “Correct. And this is what we triangles thought our angles summed to for many generations. We developed tools to measure angles, and the tools told us that our angles summed to 180 degrees... for the most part.” “For the most part?” I asked, unsure what she was getting at.

“After we developed our measuring tools, our sums were never exactly 180 degrees, due, we thought, to slight errors in the manufacture or use of the tools - measurement error. However, we noticed that the tools rarely gave a measurement of less than 180 degrees, but typically gave a measurement of slightly more than 180 degrees. And, the larger the triangle, the larger the error; that is, the larger the sum of the angles was,” she said.

“But, that could be explained by consistent errors in manufacture or use,” I said.

“Yes, and so we also thought. However, as we were developing the theory of our world being sphere-shaped, we began to examine all the implications of a sphere-shaped world. And, one of these implications is that triangles on the surface of a sphere have angles that sum to greater than 180

degrees.“ She paused for effect. “Our theory explained the observable phenomenon, before we even realized that it was anything other than measurement error.”

“How can triangles have angles that sum to greater than 180 degrees?” I asked. “Surely they are not really triangles if they sum to greater than 180 degrees. Could it be that the angles are swollen, and sides slightly bowed, and so they are not really triangles at all?” I said, thinking that this would be similar to the shape of a Flatlander right before siding.

“Yes, it is possible, or so we thought, for a time. We had developed tools for measuring straightness, however, and our lines appear to be perfectly straight. You, yourself noticed that my lines appear quite straight. The issue is not straightness, but straightness along what. Our lines are straight along a sphere. Let me explain,” she paused, readying her argument. “Imagine a two dimensional circle, and two points 90 degrees apart on the circumference of the circle. Also imagine a third point, at the center of the circle. In two dimensions these three points form a triangle, whose angles sum to 180 degrees. Now,” she paused, “I will need for you to recall the properties and theory of spheres. Imagine instead that the circle is a great circle of a sphere, and the first two points are in the analagous position, $\frac{1}{4}$ turn or 90 degrees apart along the great circle. Move the third point, however, from the center of the two-dimensional circle out to the surface of the sphere, so that it is also 90 degrees from the other two points, along each of the two perpendicular directions. If you can imagine the sphere in three dimensions, the points would be at the radius of the sphere along each of three perpendicular axes centered at the center of the sphere.” She again paused, to see if I was following.

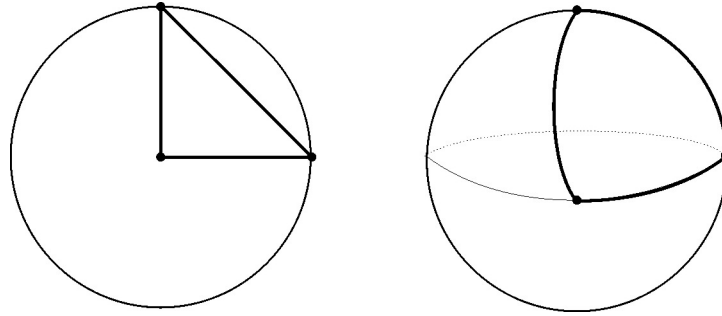


Figure 26. A Triangle in a Circle in Two Dimensions, and a Large Triangle on the Surface of a Sphere
in Three Dimensions

“Now, what are the angles of the triangle on the surface of the sphere formed by these three points?” she asked.

I was silent, my head working. “A triangle is planar, and so it can't be on the surface of a sphere. It would cut through the sphere.” I replied.

“Yes, but imagine the triangle wrapped onto the surface of the sphere, as, in fact, I am now in the surface of my sphere-shaped world. Suspend your disbelief and assume it could be done. I ask again, what are the angles of the triangle on the surface of the sphere formed by these three points? I will give you a hint - each pair of the three points lie in a great circle of the sphere.” I thought more, and did as she asked; I ignored my skepticism and tried to think logically about the information she was giving me, regardless that it didn't seem possible. But I was not able to reason it out. After some time, she answered her own question. “Each pair of the three points are 90 degrees apart, along the great circle in which the pair is contained. Therefore, the angles are each 90 degrees.” Again she paused,

and I could follow the logic in what she was saying. “The angles sum to...” she stopped, again waiting for me to answer. “270 degrees” I whispered, “but, how can it be?”

“For a relatively small triangle on a relatively large sphere, the angles will sum to something only slightly greater than 180 degrees. I am a large triangle, but still puny as compared to my great world. My angles sum to 180.028 degrees,” she said. “It can be proved that on a sphere, the angles of a triangle sum to at least 180 degrees, but no more than 270 degrees. Again, I have the happy circumstance of many generations of developed theory on which to draw,” she proudly exclaimed.

I was amazed and perplexed. “You say that you have never been to three dimensions.” “Correct,” she answered. “Then you have never seen a sphere?” I asked. “Also true,” she said. “Have others of your kind, some of your ancestors, ever seen a sphere, or ventured into the land of three dimensions?” I asked. “Not to my knowledge,” she replied, “I should think we would have heard if one of us had been on such an adventure or seen such fantastic sights,” she said. Not necessarily, I thought bitterly.

“If you or your ancestors have never even seen a sphere, how can you reason so exactly and logically about its properties?” I asked, still amazed at all I was being told. “My ancestors had a theory, which they tested, and since then we have developed a theoretical framework and proved many axioms and theorems about their theory of spherical geometry. It is not necessary to see a thing to reason about it. It is *theory*, that in this case describes the observable phenomenon. I don't need to see it - it works. I can reason about it, and develop proofs and theories about it.”

“Have you been to Flatland? Have you seen my world?” I asked. “I know of it,” the triangle replied. “Is Flatland flat?” I asked. “You thought your world was flat for many years - is our world really curved as well, yet we don't know it?” I asked, almost hysterical at the thought.

“You are an ignorant people,” she replied. “I cannot say if your world is truly flat or not. But the fact that you don't know should outrage you, just as much as being locked up for attempting to spread the Gospel of Three Dimensions. Your people are fearful of truth, and they are intimidated into conformity. Even yourself, an educated and intelligent mathematician, were conditioned into not questioning the unknown, of not exploring past the frontiers of current knowledge. When the knowledge of three dimensions was thrust upon you, your eyes were opened but not the minds of those around you. You may be imprisoned bodily, but they have imprisoned your very minds, in fear.”

“In fear of what, exactly?” I asked.

“Of the unknown,” she replied. “Of making mistakes. Of disorder. Of proposing ideas that may end up being wrong, and of hypothesizing radical new things that fit observable theory. Your people are ignorant in their fear.”

I was silent. I thought of the fear in my daughter's voice after hearing the Millennial Resolution, when I was pressing her about the third dimension. I thought of the fear that all Flatlanders had of the High Council and their other proclamations. I thought of instances when I was a student at the University, being guided against certain inquiries. I recalled an instance in which I had once been keen on discovering the exact relationship between the diameter and circumference of a circle, but was

forbidden from examining the question by a domineering professor who stated unilaterally that since true circles did not exist (in Flatland), then it was pointless to postulate theories about them. I recall questioning a little too much during sittings the absolute authority of the High Council, and being guided away from such ideas because the High Council was necessary for the good of society. I remember asking my parents childish questions about regularity and why most things in Flatland were regular, symmetric, orderly - wasn't there beauty in disorder as well? Didn't Nature make certain things irregular, like trees and animals? And I recall being rebuffed by them, much as I had done with my daughter when she first asked me about the geometric meaning of raising a length to the third power. Just as I was conditioned, so I conditioned my own children.

The great triangle was correct. Fear was a powerful force, and it had a hold on my world.

25. How the Triangle Encouraged Me Further

The triangle, seeing that she had affected me, changed her demeanor. Kindly, she said, "The important question, of course, is not whether or not Flatland is flat; it is how to facilitate the discovery of the flatness (or not) of Flatland - and other discoveries that are ready to be made, but lack only an inquisitive and liberated people."

I was filled with despair. "But, my people do not question the flatness of Flatland. Who would set out to make such a discovery, when no one even questions what we take as Truth or assume unerring. And I am imprisoned! What can I do?"

“The will of the people has been suppressed for many generations. It won't change overnight. It is true that you are not in an ideal position to facilitate change amongst your people. But, you are not helpless. It will take time for a desire to question and discover to take hold in your people. You must not make the mistake you previously made, of proposing a whole new theory in full to a fearful crowd. You must go slowly, and individually. Make them think it is their ideas taking shape. Plant seeds of doubt, of desire for discovery. And then let the seeds blossom.”

“You are more optimistic about my influences than I could ever be. I am imprisoned! I am cut off from contact with my family, my friends, my colleagues. Whom can I influence in this way?” I asked.

“You tell me,” the triangle replied. “With whom do you have contact?”

“No one.” I replied, bitterly.

“No one?” she asked. “You have spoken to no one since you have been imprisoned?”

“Well,” I said, “not quite. I see my brother once a week. I talk with my jailors, and they sometimes bring me news of Flatland. I have spoken with the chaplain. I have seen the doctor.”

“Exactly. Start with these. But start slowly. Do not press them further than they are willing to go in these matters. You can sow the seeds, but do not expect to see the harvest.”

Just then the great triangle began a hasty departure, and I was left with a vision of a vast multitude of seeds sprouting, becoming grass and vegetation, obscuring my view of the quickly retreating triangle.

26. How I Started to Change Flatland, and of the Result

I awoke suddenly, to one of my jailors bringing me breakfast.

“Here you go,” she said, brusquely.

“Thank you. Ahhh,” I said, stretching, “I am late getting up today. I had the most interesting dream,” I said.

“What was that?” she asked, clearly bored and willing to chat.

I thought of the triangle's advice. Instead of giving her an accounting of the dream, of the spherical land that appeared flat, of the great triangle whose angles added up to more than 180 degrees, I instead said, “I dreamed of a man who went far into the wasteland, and discovered another city on the other side.”

“Another city?” she said. “What was it like?”

“It was very much like Flatland,” I said. “There were houses, and people like us. But, there was one extraordinary thing,” I tempted.

“What was that?” asked she.

“Well, I... I wouldn't want you to think I was ...crazy,” I coyly ventured.

“I already think you are crazy,” she laughed. “What was it?”

I laughed with her. She was waiting. I couldn't be too bold. Seeds. Don't expect the harvest. “A circle,” I said.

“A circle? That's not so extraordinary. My great-grandfather is almost a circle.”

“Yes, but this was a true circle. With no flat sides at all, not even tiny ones. A true circle. In my dream I felt it and it was perfectly smooth.”

“Really!” she said. “How interesting. A true circle. But, of course, there are no true circles, really,” she said.

“Of course not,” I replied. “It was just a dream. But, wouldn't it be fantastic if there were another part of Flatland, on the other side of the wasteland?” I asked.

“On the other side of the wasteland?” she asked, as if never conceiving of the idea. “But, there is no other side of the wasteland.”

“Yes, of course you are right,” I said. “At least, I think so.”

“On the other side of the wasteland, ha!” she mused as she left my cell to continue her morning duties.

I had similar interactions with everyone else I came into contact with. Not at every turn, and not always the same seed planted. Over many months, I would toss out benign but leading questions. About the extent of the wasteland. About the existence of true circles. About the temporary bowing of sides during siding: did that mean that curved lines could exist more than just temporarily? They were all little questions, little ideas. But, no one seemed alarmed at my making them, and on more than one occasion I had the distinct impression that I had caused them to stop and think. Even just for a moment, even just a little.

With my brother, with whom I spent the most time, I proposed that I would begin to teach him mathematics. At first he did not agree, not seeing the point of such endeavors, but I eventually convinced him that it would give his mind something to do, with perhaps even the added benefit of extra sidings if he exercised his brain often enough - and who wouldn't want more sidings? We began slowly. At first I just taught and he listened. I started with a refresher of basic secondary school geometry and recognition by sight. The guards soon realized what I was doing, and listened in with suspicion. I was concerned that they would forbid our classes, and so I kept them benign and elementary. After a time, however, many of the jailors began to participate as well. After all, they had nothing to do, why not sit in and learn a bit? I requested and received angle measures and other tools necessary for the proper teaching of geometry and recognition by sight. I gradually came out of my

depressed stupor and began to spend my spare time deciding upon the best possible and most entertaining way to teach and reinforce each concept. Soon, the classes were quite popular, and even off-duty guards would come in for the weekly classes on their day off.

After three years, we moved on to University level studies. I began to be bolder in my inquisition and hypotheses. I worked with my pupils gently, however, so as not to raise suspicion that I had spent years quelling. The happy fact was, none of my pupils had attended University and so none were conditioned to steer clear of certain inquiries. There was still the Flatland-instilled trepidation of the unknown, but I made them believe that at the University level, serious inquiry was not only expected, it was encouraged. We reproduced theories of 0-, 1-, and 2-dimensions with ease, and connected them via analogy. I refrained from mentioning the third dimension, since it was clear that I was imprisoned for such hypotheses and I did not want to risk the progress we had painstakingly made. However, I had been intrigued with spherical geometry since my vision of Sphereland, and I decided to begin inquire with my students in this area.

We started with a circle. I introduced the theory of circles, and although mentioned that they did not exist in practice, I taught about circles with the same authority as if I was discussing hexagons.

Between classes I worked out the relationship between the circumference and diameter of a circle, that inquiry from which I was thwarted long ago, and presented the ratio, which was approximately $22/7$, as fact. I had them imagine point-like beings confined to the circumference of a circle. I demonstrated how such a being, if beginning at a point on the circle and moving completely around the circle, would appear to stationary observer; beginning at one point on the circle's apparent line, moving from that beginning point to one end of the apparent line, then disappearing from view as the being curved

around the circle, eventually reappearing at the other end of the apparent line, and then moving back to the beginning point (Figure 27). I pointed out that the point-like being would eventually come back to the starting point, but from the opposite direction along the circle and the apparent line of the circle. We also discussed how the surface of the circle, while it had no beginning and no end per se, was nevertheless finite. I firmly established all these theories before proceeding from the one-dimensional being on a circle to whatever would be the analogous situation for two-dimensional creatures like ourselves. I use the terms 'one-dimensional being' and 'two-dimensional creatures' here for your sake, my Spaceland reader. However, I refrained from mentioning dimensionality in any obvious manner, and certainly knew better than to allude to the third dimension, for I knew that my audience of jailors could end my classes at any time.

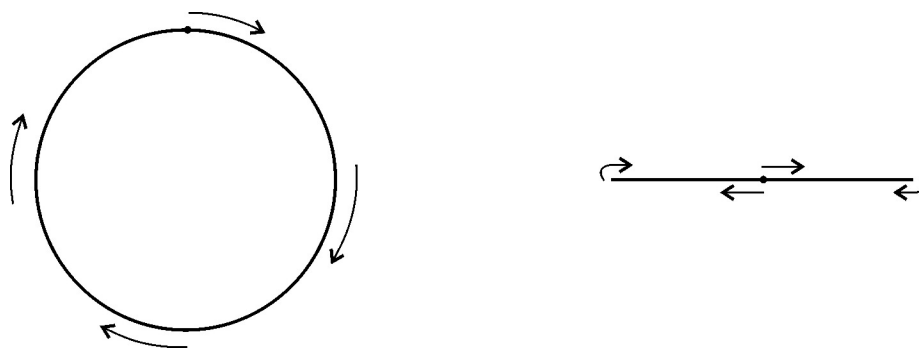


Figure 27. Movement Along the Surface of a Circle, and How it Would Appear to a Flatlander

Therefore, we proceeded, via analogy, to a hypothetical (two-dimensional) world, with creatures such as ourselves in it, but that was wholly curved (analogously to a one-dimensional line curving around in a circle). Although it was difficult to forgo the obvious connection to a three-dimensional observer, I proceeded simply from the viewpoint of the two-dimensional beings in this hypothetical world. We

agreed that in such a world, no matter in which direction you began, if you proceeded indefinitely you would disappear from view, and eventually wind up back at your starting point, but approaching from the opposite direction. I was amazed both that my pupils were adept enough to theorize in this way, and that they assumed such theoretical exercises were typical of a University experience. We discussed great circles, theorized straightness on a sphere (although we did not refer to it as a sphere - in fact, we gave it no name at all except the 'hypothetical world'), and eventually even proved that the angles of a triangle would sum to at least 180, but no more than 270, degrees in this 'hypothetical world'. I was amazed at my success.

My students were also amazed, and happy to receive a University-level education in their spare time and at no cost. Furthermore, they, like myself, were undergoing siding far more frequently than would ordinarily be the case for their age. I underwent a siding almost a year into my commencement of prison teaching. Afterward, I realized that I had squandered the opportunity to reach yet another person, the elder who was called in to mentor my siding. I resolved that with my next siding, I would attempt to plant seeds of enlightenment with my mentor as well. I could only hope that my pupils were doing the same with their mentors during their sidings.

Four years later, I felt the pangs of siding coming on once again. I had had ample time to contemplate how I would approach the siding with my mentor. I found that I had been assigned a well-known sage who was an active advisor in politics. Perfect. During the focused introspection we discussed my fulfilling role as teacher and the positive impact I was having even in my secluded environment. I eased into a discussion, cleverly disguised as a process of self-searching, about the mistakes I had made that had landed me in prison. I talked about theory; that some theory was O.K. (that which was

deemed acceptable by the High Council) but other theory was not, and that I needed to better discern which was which. We engaged in a meta-level discussion about our society and the acceptable avenues of inquiry; such a conversation I had never before experienced; I am quite sure no one had ever openly discussed this topic in the depth with which we pursued it. I pulled her along and she readily followed, my mentor, until she too was beginning to realize the control of fear that the High Council had over our lives. But all the while, the discussion was portrayed as an attempt at humble soul-searching, as reflection on my past mistakes and how I could learn from these mistakes to improve my behavior as a member of our society. At the end of my time of siding, I had a powerful convert, one who was well-regarded and well-connected in Flatland society. As she left, I was pleased with the seeds I had sown in her.

Now, ten more years have passed; I have been imprisoned for almost twenty-five years. I still lay imprisoned behind walls, but my life is full and has purpose. I have an active teaching career inside the prison, I have made friends with the guards and allies with three different siding mentors. I have gently prodded all with whom I have come into contact into examining their fear of questioning and liberating them from their conditioning of acquiescence. I have done all that I can do.

And just this morning, the word came through the prison guards: An adventurer had set off some eight years ago, in order to determine the extent of the wasteland on the edges of our world. Yesterday he returned. *From the opposite direction.*

So it seems that Flatland is not flat, after all.

THE END